Restricted colorings of graphs

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Abstract

The problem of properly coloring the vertices (or edges) of a graph using for each vertex (or edge) a color from a prescribed list of permissible colors, received a considerable amount of attention. Here we describe the techniques applied in the study of this subject, which combine combinatorial, algebraic and probabilistic methods, and discuss several intriguing conjectures and open problems. This is mainly a survey of recent and less recent results in the area, but it contains several new results as well.

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1 Introduction

Graph coloring is arguably the most popular subject in graph theory. An interesting variant of the classical problem of coloring properly the vertices of a graph with the minimum possible number of colors arises when one imposes some restrictions on the colors available for every vertex. This variant received a considerable amount of attention that led to several fascinating conjectures and results, and its study combines interesting combinatorial techniques with powerful algebraic and probabilistic ideas. The subject, initiated independently by Vizing [51] and by Erdős, Rubin and Taylor [24], is usually known as the study of the *choosability* properties of a graph. In the present paper we survey some of the known recent and less recent results in this topic, focusing on the techniques involved and mentioning some of the related intriguing open problems. This is mostly a survey article, but it contains various new results as well.

A *vertex coloring* of a graph G is an assignment of a color to each vertex of G. The coloring is *proper* if adjacent vertices receive distinct colors. The *chromatic number* $\chi(G)$ of G is the minimum number of colors used in a proper vertex coloring of G. An *edge coloring* of G is, similarly, an assignment of a color to each edge of G. It is *proper* if adjacent edges receive distinct colors. The minimum number of colors in a proper edge-coloring of G is the *chromatic index* $\chi'(G)$ of G. This is clearly equal to the chromatic number of the line graph of G.

If $G = (V, E)$ is a (finite, directed or undirected) graph, and f is a function that assigns to each vertex v of G a positive integer $f(v)$, we say that G is *f-choosable* if, for every assignment of sets of integers $S(v) \subset Z$ to all the vertices $v \in V$, where $|S(v)| = f(v)$ for all v, there is a proper vertex coloring $c : V \mapsto Z$ so that $c(v) \in S(v)$ for all $v \in V$. The graph G is *k-choosable* if it is f-choosable for the constant function $f(v) \equiv k$. The *choice number* of G, denoted $ch(G)$, is the minimum integer k so that G is k-choosable. Obviously, this number is at least the classical chromatic number $\chi(G)$ of G. The choice number of the line graph of G, which we denote here by $ch'(G)$, is usually called the *list chromatic index* of G, and it is clearly at least the chromatic index $\chi'(G)$ of G.

As observed by various researchers ([51], [24], [1]), there are many graphs G for which the choice number $ch(G)$ is strictly larger than the chromatic number $\chi(G)$. A simple example demonstrating this fact is the complete bipartite graph $K_{3,3}$. If $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ are its two vertex-classes and $S(u_i) = S(v_i) = \{1, 2, 3\} \setminus \{i\}$, then there is no proper vertex coloring assigning to each vertex w a color from its class $S(w)$. Therefore, the choice number of this graph exceeds its chromatic number. In fact, it is easy to show that, for any $k \geq 2$, there are bipartite graphs whose choice number exceeds k; a general construction is given in the following section. The gap between the two parameters $ch(G)$ and $\chi(G)$ can thus be arbitrarily large. In view of this, the following conjecture, suggested independently by various researchers including Vizing, Albertson, Collins, Tucker and Gupta, which apparently appeared first in print in the paper of Bollobás and Harris ([15]), is somewhat surprising.

Conjecture 1.1 (The list coloring conjecture) *For every graph G, $ch'(G) = \chi'(G)$.*

This conjecture asserts that for *line graphs* there is no gap at all between the choice number and the chromatic number. Many of the most interesting results in the area are proofs of special cases of this conjecture, some of which are described in Sections 2, 3 and 4. The proof for the general case (if true) seems extremely difficult, and even some very special cases that have received a considerable amount of attention are still open.

The problem of determining the choice number of a given graph is difficult, even for small graphs with a simple structure. To see this, you may try to convince yourself that the complete bipartite graph $K_{5,8}$ is 3-choosable; a (lengthy) proof appears in [42]. More formally, it is shown in [24] that the problem of deciding if a given graph $G = (V, E)$ is f-choosable, for a given function $f : V \mapsto \{2, 3\}$, is Π_2^p-Complete. Therefore, if the complexity classes NP and $coNP$ differ, as is commonly believed, this problem is stricly harder than the problem of deciding if a given graph is k-colorable, which is, of course, NP-Complete. (See [27] for the definitions of the complexity classes above.) More results on the complexity of several variants of the choosability problem appear in [38], where it is also briefly shown how some of these variants arise naturally in the study of various scheduling problems.

The study of choice numbers combines combinatorial ideas with algebraic and probabilistic tools. In the following sections we discuss these methods and present the main results and open questions in the area. The paper is organized as follows. After describing, in Section 2, some basic and initial results, we discuss, in Section 3, an algebraic approach and some of its recent consequences. Various applications of probabilistic methods to choosability are considered in Section 4. A new result is obtained in Section 5 which presents a proof of the fact that the choice number of any simple graph with average degree d is at least $\Omega(\log d / \log\log d)$. Thus, the choice number of a simple graph must grow with its average degree, unlike the chromatic number. The final Section 6 contains some concluding remarks and open problems, in addition to those mentioned in the previous sections.

2 Some basic results

One of the basic results in graph coloring is Brooks' theorem [17], that asserts that the chromatic number of every connected graph, which is not a complete graph or an odd cycle, does not exceed its maximum degree. The choosability version of this result has been proved, independently, by Vizing [51] (in a slightly weaker form) and by Erdős, Rubin and Taylor [24]. (See also [40]).

Theorem 2.1 ([51], [24]) *The choice number of any connected graph G, which is not complete or an odd cycle, does not exceed its maximum degree.*

Note that this suffices to prove the validity of the list coloring conjecture for simple graphs of maximum degree 3 whose chromatic index is not 3 (known as *class* 2 graphs), since by Vizing's theorem [50], the chromatic index of such graphs must be 4.

A graph is called *d-degenerate* if any subgraph of it contains a vertex of degree at most d. By a simple inductive argument, one can prove the following result.

Proposition 2.2 *The choice number of any d-degenerate graph is at most $d + 1$.*

This simple fact implies, for example, that every planar graph is 6-choosable. It is not known if every planar graph is 5-choosable; this is conjectured to be the case in [24]- in fact, it may even be true that every planar graph is 4-choosable.

A characterization of all 2-choosable graphs is given in [24]. If G is a connected graph, the *core* of G is the graph obtained from G by repeatedly deleting vertices of degree 1 until there is no such vertex.

Theorem 2.3 ([24]) *A simple graph is 2-choosable if and only if the core of each of its connected components is either a single vertex, or an even cycle, or a graph consisting of two vertices with three even internally disjoint paths between them, where the length of at least two of the paths is exactly 2.*

Of course, one cannot hope for such a simple characterization of the class of all 3-choosable graphs, since, as observed by Gutner it follows easily from the complexity result mentioned in the introduction that the problem of deciding if a given graph is 3-choosable is NP-hard; in fact, as shown in [28], this problem is even Π_2^p-complete.

In Section 1 we saw an example of a graph with choice number that exceeds its chromatic number. Here is an obvious generalization of this construction. Let $H = (U, W)$ be a k-uniform hypergraph which is not 2-colorable; that is, every edge $w \in W$ has precisely k elements and for every 2-vertex coloring of H there is a monochromatic edge. If $|W| = n$ we claim that the complete bipartite graph $K_{n,n}$ is not k-choosable. Indeed, denote the vertices of H by $1, 2, \ldots$, and let $A = \{a_w : w \in W\}$ and $B = \{b_w : w \in W\}$ be the two vertex classes of $K_{n,n}$. For each $w \in W$, define $S(a_w) = S(b_w) = \{u \in U : u \in w\}$. One can easily check that there is a proper coloring c of the complete bipartite graph on $A \cup B$ assigning to each vertex a_w and b_w a color from its class

$S(a_w)$ ($= S(b_w)$) if and only if the hypergraph H is 2-colorable. Thus, by the choice of H, the choice number of $K_{n,n}$ is stricly bigger than k.

As shown by Erdős [23], for large values of k there are k-uniform hypergraphs with at most $n = (1 + o(1))\frac{e \ln 2}{4}k^2 2^k$ edges which are not 2-colorable, showing that there are bipartite graphs with that many vertices on each side whose choice number exceeds k. We note that this estimate is nearly sharp, as a very simple probabilistic argument shows that, if $n < 2^{k-1}$, then $K_{n,n}$ is k-choosable. Indeed, given a list $S(v)$ of k colors for each vertex v in the two vertex classes A and B, let S be the set of all the colors used in the union of all the lists and let us choose a random partition (S_A, S_B) of S into two disjoint parts, where, for each $s \in S$ randomly and independently, s is chosen to be in S_A or in S_B with equal probability. The colors in S_A will be used to color vertices in A and those in S_B to color the vertices in S_B. For a fixed vertex a in A, the probability that its coloring will fail- that is, we will not be able to color it by a color from S_A- is precisely $1/2^k$, as this is the probability that all the colors in its class $S(a)$ were chosen to be in S_B. A similar estimate holds for the members of B, and hence the probability that there exists a vertex that will fail to receive a color is at most $|A \cup B|/2^k < 1$. This estimate can be slightly improved, using the method (or the result) of Beck ([10]), but the above simple argument suffices to demonstrate the relevance of probabilistic techniques in the study of choice numbers.

The *total chromatic number* of a graph G, denoted by $\chi''(G)$, is the minimum number of colors required to color all the vertices and edges of G, so that adjacent or incident elements receive distinct colors. The following conjecture is due to Behzad [11].

Conjecture 2.4 (The total coloring conjecture) *The total chromatic number of every simple graph G with maximum degree Δ is at most $\Delta + 2$.*

There are several papers dealing with this conjecture, and the following estimates are known. If G is a simple graph on n vertices with maximum degree Δ then, as shown by Hind [33], [34]:

$$\chi''(G) \le \Delta + 1 + 2\lceil\sqrt{\Delta}\rceil,$$

and

$$\chi''(G) \le \Delta + 1 + 2\lceil\frac{n}{\Delta}\rceil.$$

Chetwynd and Häggkvist [31] showed that, if $t! > n$, then

$$\chi''(G) \le \Delta + 1 + t.$$

Note that, as observed by the authors of [15], the validity of Conjecture 1.1 (the list coloring conjecture) would imply that, for every simple graph with maximum degree Δ,

$$\chi''(G) \le \chi'(G) + 2 \le \Delta + 3.$$

Indeed, let $S = \{1, 2, \ldots, \chi'(G) + 2\}$ be our set of colors. Start with an arbitrary proper vertex coloring of G using these colors; this certainly exists, for example, by Brooks' Theorem and by the fact that $\chi'(G) \geq \Delta$. Now associate with each edge e of G a list $S(e)$ of all the colors in S except the ones appearing on its two ends. By the list coloring conjecture, there is a proper edge coloring of G using, for each edge e, a color from $S(e)$; this would give a proper total coloring of G. The fact that $|S| \leq \Delta + 3$ now follows from Vizing's theorem ([50]). It seems, however, that getting a $\Delta + O(1)$ upper estimate for the total chromatic number of a simple graph with maximum degree Δ should be much easier than getting a similar bound for the list chromatic index of such a graph.

3 An algebraic approach and its applications

An algebraic technique that, in various cases, supplies useful information on the choice numbers of given graphs, has been developed by M. Tarsi and the present author in [9]. In this section we describe this method and present some of its recent applications.

A subdigraph H of a directed graph D is called *Eulerian* if the indegree $d_H^-(v)$ of every vertex v of H is equal to its outdegree $d_H^+(v)$. Note that we do not assume that H is connected. H is *even* if it has an even number of edges, otherwise, it is *odd*. Let $EE(D)$ and $EO(D)$ denote the numbers of even and odd Eulerian subgraphs of D, respectively. (For convenience we agree that the empty subgraph is an even Eulerian subgraph.) The following result is proved in [9].

Theorem 3.1 *Let $D = (V, E)$ be a digraph, and define $f : V \mapsto Z$ by $f(v) = d_D^+(v) + 1$, where $d_D^+(v)$ is the outdegree of v. If $EE(D) \neq EO(D)$, then D is f-choosable.*

Note that the assertion of the theorem for the special case of acyclic digraphs, which implies Proposition 2.2, can be proved by a simple inductive argument. The general case seems much more difficult. To prove this theorem, we need the following simple statement.

Lemma 3.2 *Let $P = P(x_1, x_2, \ldots, x_n)$ be a polynomial in n variables over the ring of integers Z. Suppose that the degree of P as a polynomial in x_i is at most d_i for $1 \leq i \leq n$, and let $S_i \subset Z$ be a set of $d_i + 1$ distinct integers. If $P(x_1, x_2, \ldots, x_n) = 0$ for all n-tuples $(x_1, \ldots, x_n) \in S_1 \times S_2 \times \ldots \times S_n$, then $P \equiv 0$.*

Proof We apply induction on n. For $n = 1$, the lemma is simply the assertion that a non-zero polynomial of degree d_1 in one variable can have at most d_1 distinct zeros. Assuming that the lemma holds for $n - 1$, we prove it for n ($n \geq 2$). Given a polynomial $P = P(x_1, \ldots, x_n)$ and sets

S_i satisfying the hypotheses of the lemma, let us write P as a polynomial in x_n- that is,

$$P = \sum_{i=0}^{d_n} P_i(x_1, \ldots, x_{n-1}) x_n^i,$$

where each P_i is a polynomial with x_j-degree bounded by d_j. For each fixed $(n-1)$-tuple $(x_1, \ldots, x_{n-1}) \in S_1 \times S_2 \times \ldots \times S_{n-1}$, the polynomial in x_n obtained from P by substituting the values of $x_1, \ldots, x_{n-1}$ vanishes for all $x_n \in S_n$, and is thus identically 0. Thus $P_i(x_1, \ldots, x_{n-1}) = 0$ for all $(x_1, \ldots, x_{n-1}) \in S_1 \times \ldots \times S_{n-1}$. Hence, by the induction hypothesis, $P_i \equiv 0$ for all i, implying that $P \equiv 0$. This completes the induction and the proof of the lemma. $\square$

The *graph polynomial* $f_G = f_G(x_1, x_2, \ldots, x_n)$ of a directed or undirected graph $G = (V, E)$ on a set $V = \{v_1, \ldots, v_n\}$ of n vertices is defined by $f_G(x_1, x_2, \ldots, x_n) = \Pi\{(x_i - x_j) : i < j \ , \ \{v_i, v_j\} \in E\}$. This polynomial has been studied by various researchers, starting already with Petersen [44] in 1891. See also, for example, [46], [39].

For $1 \leq i \leq n$, let $S_i \subset Z$ be a set of $d_i + 1$ distinct integers. For each i, $1 \leq i \leq n$, let $Q_i(x_i)$ be the polynomial $Q_i(x_i) = \Pi_{s \in S_i}(x_i - s)$. Let $\mathcal{I}$ be the ideal generated by the polynomials Q_i in the ring of polynomials $Z[x_1, \ldots, x_n]$. It is obvious that if $f_G(x_1, \ldots, x_n) \in \mathcal{I}$, then f_G vanishes on every common zero of all the polynomials Q_i. But this means that f_G vanishes on every $(x_1, \ldots, x_n) \in S_1 \times S_2 \times \ldots \times S_n$; hence, for each assignment of values $x_i \in S_i$, there is an edge $v_i v_j$ of G with $x_i = x_j$. Therefore, there is no proper vertex coloring of G assigning to each vertex v_i a color from its set S_i. The following Nullstellensatz-type result asserts that the converse is also true.

Proposition 3.3 *Let $G = (V, E)$ be a graph on the set of vertices $V = \{v_1, \ldots, v_n\}$, and let S_i, $1 \leq i \leq n$, be sets of integers. Let $f_G = f_G(x_1, \ldots, x_n)$ be the graph polynomial of G, and let $Q_i(x_i)$ and $\mathcal{I}$ be as above. Then $f_G \in \mathcal{I}$ if and only if there is no proper vertex coloring c of G satisfying $c(v_i) \in S_i$, for all $1 \leq i \leq n$.*

Proof We have already seen that, if there is a coloring as above, then f_G is not in $\mathcal{I}$. It remains to show that if there is no such coloring, then $f_G \in \mathcal{I}$. The assumption that the required coloring does not exist is equivalent to the statement:

$$f_G(x_1, \ldots, x_n) = 0 \quad \text{for every } n\text{-tuple} \quad (x_1, \ldots, x_n) \in S_1 \times S_2 \times \ldots \times S_n. \tag{1}$$

For each i, $1 \leq i \leq n$, put

$$Q_i(x_i) = \Pi_{s \in S_i}(x_i - s) = x_i^{d_i+1} - \sum_{j=0}^{d_i} q_{ij} x_i^j.$$

Observe that,

$$\text{if } x_i \in S_i \text{ then } Q_i(x_i) = 0\text{- that is, } x_i^{d_i+1} = \sum_{j=0}^{d_i} q_{ij} x_i^j. \tag{2}$$

Let $\overline{f_G}$ be the polynomial obtained by writing f_G as a linear combination of monomials and replacing, repeatedly, each occurrence of $x_i^{f_i}$ $(1 \leq i \leq n)$, where $f_i > d_i$, by a linear combination of smaller powers of x_i, using the relations (2). The resulting polynomial $\overline{f_G}$ is clearly of degree at most d_i in x_i, for each $1 \leq i \leq n$, and satisfies $\overline{f_G} \equiv f_G$ $(mod\ \mathcal{I})$. Moreover, $\overline{f_G}(x_1, \ldots, x_n) = f_G(x_1, \ldots, x_n)$, for all $(x_1, \ldots, x_n) \in S_1 \times \ldots \times S_n$, since the relations (2) hold for these values of $x_1, \ldots, x_n$. Therefore, by (1), $\overline{f_G}(x_1, \ldots, x_n) = 0$ for every n-tuple $(x_1, \ldots, x_n) \in S_1 \times \ldots \times S_n$ and hence, by Lemma 3.2, $\overline{f_G} \equiv 0$. This implies that $f_G \in \mathcal{I}$, and completes the proof. $\square$

The special case of the last proposition, for the case in which all the sets S_i are equal, implies that, for every fixed polynomial $Q(x)$ of one variable with k distinct integer roots, a graph G is not k-colorable if and only if the graph polynomial f_G lies in the ideal generated by the polynomials $Q(x_i)$. In fact, the assumption that the roots of $Q(x)$ are integral is not essential, as the proof works equally well in the ring of polynomials $K[x_1, \ldots, x_n]$ over any field K. See [9] for more details. This result is related to a theorem of Kleitman and Lovász ([41]) , who applied a method similar to that of [39], and showed that a graph $G = (V, E)$ is not k-colorable if and only if f_G lies in the ideal generated by the set of all graph polynomials of complete graphs on $k+1$ vertices among those in V. As shown by De Loera in [19], the set of graph polynomials of complete $(k+1)$-graphs, as well as the set of polynomials $Q(x_i)$ above, are both universal Gröbner bases for the ideals they generate. See [19] for more details.

It is not too difficult to see that the coefficients of the monomials that appear in the standard representation of f_G as a linear combination of monomials can be expressed in terms of the orientations of G. For each oriented edge $e = (v_i, v_j)$ of G, define its *weight* $w(e)$ by $w(e) = x_i$ if $i < j$, and $w(e) = -x_i$ if $i > j$. The weight $w(D)$ of an orientation D of G is defined to be the product $\Pi w(e)$, where e ranges over all oriented edges e of D. Clearly $f_G = \sum w(D)$, where D ranges over all orientations of G. This is simply because each term in the expansion of the product $f_G = \Pi\{(x_i - x_j) : i < j\ ,\ \{v_i, v_j\} \in E\}$ corresponds to a choice of the orientation of the edge $\{v_i, v_j\}$ for each edge $\{v_i, v_j\}$ of G. Let us call an oriented edge (v_i, v_j) of G *decreasing* if $i > j$. An orientation D of G is called *even* if it has an even number of decreasing edges; otherwise, it is called *odd*.

For non-negative integers $d_1, d_2, \ldots, d_n$, let $DE(d_1, \ldots, d_n)$ and $DO(d_1, \ldots, d_n)$ denote, respectively, the sets of all even and odd orientations of G in which the outdegree of the vertex v_i is d_i, for $1 \leq i \leq n$. By the last paragraph, the following lemma holds.

Lemma 3.4 *In the above notation*

$$f_G(x_1, \ldots, x_n) = \sum_{d_1, \ldots, d_n \geq 0} (|DE(d_1, \ldots, d_n)| - |DO(d_1, \ldots, d_n)|)\Pi_{i=1}^{n} x_i^{d_i} . \quad \square$$

Consider, now, a fixed sequence $d_1, \ldots, d_n$ of nonnegative integers and let D_1 be a fixed orientation in $DE(d_1, \ldots, d_n) \cup DO(d_1, \ldots, d_n)$. For any orientation $D_2 \in DE(d_1, \ldots, d_n) \cup DO(d_1, \ldots, d_n)$, let $D_1 \oplus D_2$ denote the set of all oriented edges of D_1 whose orientation in D_2 is in the opposite direction. Since the outdegree of every vertex in D_1 is equal to its outdegree in D_2, it follows that $D_1 \oplus D_2$ is an Eulerian subgraph of D_1. Moreover, $D_1 \oplus D_2$ is even as an Eulerian subgraph if and only if D_1 and D_2 are both even or both odd. The mapping $D_2 \longrightarrow D_1 \oplus D_2$ is clearly a bijection between $DE(d_1, \ldots, d_n) \cup DO(d_1, \ldots, d_n)$ and the set of all Eulerian subgraphs of D_1. In the case D_1 is even, it maps even orientations to even (Eulerian) subgraphs, and odd orientations to odd subgraphs. Otherwise, it maps even orientations to odd subgraphs, and odd orientations to even subgraphs. In any case,

$$\Big| |DE(d_1, \ldots, d_n)| - |DO(d_1, \ldots, d_n)| \Big| = |EE(D_1) - EO(D_1)|$$

where $EE(D_1)$ and $EO(D_1)$ denote, as before, the numbers of even and odd Eulerian subgraphs of D_1, respectively. Combining this with Lemma 3.4, we obtain the following.

Corollary 3.5 *Let D be an orientation of an undirected graph $G = (V, E)$ on a set $V = \{v_1, \ldots, v_n\}$ of n vertices. For $1 \leq i \leq n$, let $d_i = d_D^+(v_i)$ be the outdegree of v_i in D. Then the absolute value of the coefficient of the monomial $\Pi_{i=1}^{n} x_i^{d_i}$ in the standard representation of $f_G = f_G(x_1, \ldots, x_n)$ as a linear combination of monomials, is $|EE(D) - EO(D)|$. In particular, if $EE(D) \neq EO(D)$, then this coefficient is not zero.* $\square$

Proof of Theorem 3.1 Let $D = (V, E)$ be a digraph on the set of vertices $V = \{v_1, \ldots, v_n\}$ and let $d_i = d_D^+(v_i)$ be the outdegree of v_i. Suppose that $EE(D) \neq EO(D)$. For $1 \leq i \leq n$, let $S_i \subset Z$ be a set of $d_i + 1$ distinct integers. We must show that there is a legal vertex-coloring $c : V \mapsto Z$ such that $c(v_i) \in S_i$, for all $1 \leq i \leq n$. Suppose that this is false and there is no such coloring. Then, by Proposition 3.3 and its proof, $\overline{f_D} \equiv 0$, where, as before, $\overline{f_D}$ is the reduction of the graph polynomial f_D using the relations (2). However, by Corollary 3.5, the coefficient of $\Pi_{i=1}^{n} x_i^{d_i}$ in f_D is nonzero, since, by assumption, $EE(D) \neq EO(D)$. Since the degree of each x_i in this monomial is d_i, the relations (2) will not affect it. Moreover, as the polynomial f_D is homogeneous and each application of the relations (2) strictly reduces the degree, the process of replacing f_D by $\overline{f_D}$ will not create any new scalar multiples of $\Pi_{i=1}^{n} x_i^{d_i}$. Thus, the coefficient of $\Pi_{i=1}^{n} x_i^{d_i}$ in $\overline{f_D}$ is equal to its coefficient in f_D, and is not 0. This contradicts the fact that $\overline{f_D} \equiv 0$.

Therefore, our assumption was false, and there is a legal coloring $c : V \mapsto Z$ satisfying $c(v_i) \in S_i$, for all $1 \leq i \leq n$. $\Box$

An interesting application of Theorem 3.1 has been obtained by Fleischner and Stiebitz in [26], solving a problem raised by Du, Hsu and Hwang in [21], as well as a strengthening of it suggested by Erdős.

Theorem 3.6 ([26]) *Let G be a graph on $3n$ vertices, whose set of edges is the disjoint union of a Hamilton cycle and n pairwise vertex-disjoint triangles. Then the choice number and the chromatic number of G are both* 3.

The proof is based on a subtle parity argument that shows that, if D is the digraph obtained from G by directing the Hamilton cycle as well as each of the triangles cyclically, then $EE(D) - EO(D) \equiv 2(mod\ 4\ )$. The result thus follows from Theorem 3.1. We note that the result supplies no efficient algorithm for finding a proper 3-vertex coloring of such a graph, although the methods of [3], [25] or [5] do supply an efficient algorithm for the related (easier) problem of finding a proper 4-vertex coloring of a graph on $4n$ vertices, whose set of edges is the disjoint union of a Hamilton cycle and n pairwise vertex disjoint copies of K_4. Several extensions appear in [25], [5].

Another simple application of Theorem 3.1 is the following result, that solves an open problem from [24].

Theorem 3.7 ([9]) *The choice number of every planar bipartite graph is at most* 3.

This is tight, since $ch(K_{2,4}) = 3$.

Recall that the list coloring conjecture (Conjecture 1.1) asserts that $ch'(G) = \chi'(G)$ for every graph G. In order to try to apply Theorem 3.1 for tackling this problem, it is useful to find a more convenient expression for the difference $EE(D) - EO(D)$, where D is the appropriate orientation of a given line graph. Here is a brief derivation of such an expression for line graphs of regular graphs of class 1. Let $G = (V, E)$ be a d-regular graph satisfying $\chi'(G) = d$. Observe that the line graph $L(G)$ of G is $(2d-2)$-regular, and hence has an Eulerian orientation D in which every outdegree is precisely $d-1$. Let $f_D(x_1, \ldots)$ denote the graph polynomial of D. Our objective is to compute the coefficient of the monomial Πx_i^{d-1} in the standard representation of f_D as a sum of monomials. Let us denote this coefficient by $C(D)$. Note that, by Corollary 3.5 and its proof, the absolute value of $C(D)$ is the absolute value of the difference between the number of even Eulerian orientations of $L(G)$ and the number of odd Eulerian orientations of it.

It is convenient to consider both this combinatorial interpretation and the interpretation as the appropriate coefficient. Starting with the latter, observe that the edges of the line graph $L(G)$ consist of $|V|$ edge disjoint cliques, each of size d. For every $v \in V$, there is a clique in $L(G)$

on all the d edges of G (which are vertices of $L(G)$) that are incident with v. Therefore, the graph polynomial f_D is a product of $|V|$ graph polynomials of complete graphs, each of size d. However, the graph polynomial of a complete graph is a Vandermonde determinant, and hence one can express f_D as a product of $|V|$ Vandermonde determinants. Interpreting the coefficient of the monomial Πx_i^{d-1} in this new expresion, we conclude that its absolute value is the absolute value of the difference between the number of even Eulerian orientations of $L(G)$ and the number of odd ones, where we count only (Eulerian) orientations in which each of the $|V|$ tournaments corresponding to the cliques around the vertices of G is acyclic. (It is not too difficult to show directly that the other orientations cancel each other, without using the interpretation as a product of Vandermonde determinants, but we omit the detailed argument using this direct approach.) Since an acyclic orientation of a tournament defines a permutation in the obvious way (or by expanding the Vandermonde determinants according to their definition), the last difference can be rewritten as follows.

For each vertex $v \in V$, let π_v be an arbitrary permutation of the edges of G incident with v. It is convenient to consider such a permutation as a bijection from the d edges above to the set $\{0, 1, \ldots, d-1\}$. Let SP denote the class of all sets of $|V|$ permutations $\Sigma = \{\sigma_v : v \in V\}$, where σ_v is a permutation of the edges of G incident with v so that for each edge $e = uv$, $\sigma_u(e) + \sigma_v(e) = d-1$. For each $\Sigma \in SP$, let $sign(\Sigma)$ denote the product of the signs of all $|V|$ permutations $\pi_v^{-1}(\sigma_v),\ v \in V$. Then

$$|C(D)| = |\sum_{\Sigma \in SP} sign(\Sigma)|. \tag{3}$$

Next, associate with each $\Sigma = \{\sigma_v : v\imath V\}$ a partition $P = P(\Sigma)$ of the set of edges of G into $s = \lfloor (d+1)/2 \rfloor$ classes $P_0, P_1, \ldots, P_{s-1}$, by letting P_i denote the set of all edges $e = uv$ with $\sigma_u(e) = i$ (and $\sigma_v(e) = d-1-i$). Notice that for $0 \leq i < (d-1)/2$, P_i is a 2-factor of G, whereas for odd d, the last class $P_{(d-1)/2}$ is a perfect matching. If there is some 2-factor P_i that contains an odd cycle, then let $v_0, v_1, \ldots, v_r = v_0$ be the vertices of the first such cycle in the first such 2-factor, and suppose that $\sigma_{v_j}(v_j v_{j+1}) = i$ and $\sigma_{v_{j+1}}(v_j v_{j+1}) = d-1-i$, where the indices are reduced modulo r. Then we can define $\Sigma' \in SP$ as the collection of permutations σ'_v obtained from Σ, by defining $\sigma'_{v_j}(v_j v_{j+1}) = d-1-i$ and $\sigma'_{v_{j+1}}(v_j v_{j+1}) = i$, where, in each other place, each σ'_v coincides with the corresponding σ_v. One can easily check that this is a fixed-point-free involution on the members of SP of this type that switches the sign. Therefore, these members cancel each other in equation (3). We are thus left with the members $\Sigma \in SP$ for which every class in $P(\Sigma)$ is a 2-factor of even cycles (and one class is a perfect matching, in case of odd d). From each such partition $P(\Sigma)$, with l even cycles in all its 2-factors together, one can get 2^l proper edge colorings of G with d colors, by coloring the edges of each even cycle in P_i alternately with the colors i and $d-1-i$.

It can easily be checked that this correspondence is a bijection between the remaining members of SP and the proper edge colorings. This yields the following interpretation of the coefficient $C(D)$. For every proper edge coloring c of G with the d-colors $\{0, \ldots, d-1\}$, the *sign of* c, denoted by $sign(c)$, is defined as the product of the signs of all the $|V|$ permutations $(\pi_v^{-1}c(e) : v \in e \in E(G))$. Let $EC(G)$ denote the set of all proper d edge colorings of G, and define

$$ec(G) = \sum_{c \in EC(G)} sign(c).$$

Proposition 3.8 *With the above notation*

$$|C(D)| = |ec(G)|. \quad \Box$$

This proposition is described (very briefly, and only for the special case $G = K_{n,n}$) in [9]. The case $d = 3$ of it appears in [35] (see also [46] for the case of planar cubic graphs). Combining this proposition with Theorem 3.1, we conclude.

Corollary 3.9 *The list coloring conjecture holds for any d-regular graph G with chromatic index d that satisfies $ec(G) \neq 0$.* $\Box$

Are there any interesting examples of graphs G as above, for which one can prove that $ec(G) \neq 0$? Any cubic graph with chromatic index 3, in which there is a perfect matching that appears in every proper 3-edge coloring, is such an example, and one can give infinitely many examples of this type. More interesting is the following result, observed in conversations with F. Jaeger and M. Tarsi and, independently, by M. Ellingham and L. Goddyn [22].

Corollary 3.10 *For every 2-connected cubic planar graph G, $ch'(G) = 3$.*

Proof It is known ([49]; see also [35] for a short proof) that all the 3-edge colorings of a planar cubic graph have the same sign. On the other hand, the fact that every 2-connected cubic planar graph has chromatic index 3 is well known to be equivalent to the Four Color Theorem. The result thus follows from Corollary 3.9. $\Box$

Note that the above result is a strengthening of the Four Color Theorem. The proof supplies no efficient procedure for finding a proper 3-edge coloring for a given 2-connected planar cubic graph with a list of 3 colors for each of its edges, that assigns to each edge a color from its list. As shown in [22], it is possible to extend the above proof to any d-regular planar multigraph with chromatic index d, establishing the following.

Theorem 3.11 ([22]) *The list chromatic index of any d-regular planar multigraph with chromatic index d is d.*

4 Probabilistic techniques

Probabilistic arguments have been applied by various researchers in the study of restricted colorings. We have already seen a simple example in Section 2. Here is another simple and elegant example, due to Chetwynd and Häggkvist ([31]), which deals with a variant of the restricted coloring problem, in which we color edges trying to avoid a forbidden color on each edge.

Proposition 4.1 ([31]) *Let $G = (V, E)$ be a graph with n vertices, maximum degree Δ, and chromatic index r $(\geq \Delta)$. For each edge $e \in E$, let $d(e) \in \{1, \ldots, r\}$ be a forbidden color for e. If t is the smallest integer satisfying $t! > n$, then there is a proper edge coloring $c : E \mapsto \{1, 2, \ldots, r+t\}$ satisfying $c(e) \neq d(e)$, for all $e \in E$.*

Proof Fix a proper edge coloring of G with the colors $1, 2, \ldots, r$, and consider a random permutation of these colors. For a fixed vertex v of G, the probability that there will be at least t edges incident with v that receive (after the permutation) their forbidden color, is at most

$$\binom{\Delta}{t} \frac{1}{r(r-1)\ldots(r-t+1)} \leq \frac{1}{t!} < 1/n.$$

Therefore, with positive probability, there is no vertex v in which there are at least t edges that received a forbidden color after the permutation. Let H be the subgraph of violations- that is, the subgraph of G consisiting of all edges that received their forbidden colors. We have seen that there is a permutation for which the maximum degree of H is at most $t-1$. Moreover, if there are any pairs of parallel edges in H, we can exchange the colors on such a pair and reduce the number of violations locally. Therefore, we may assume that H is simple and hence, by Vizing's Theorem [50], its edges can be properly colored with the additional t colors $r+1, \ldots, r+t$, supplying a proper edge coloring of G with the colors $1, \ldots, r+t$, as needed. $\Box$

It is shown in [31] that the same reasoning can be applied to obtain an $r+t$ upper bound for the total chromatic number of any multigraph with n vertices and chromatic index r, provided that $t! > n$. A similar result has been proved by McDiarmid ([43]).

Returning to the list coloring conjecture, observe that, by Proposition 2.2, $ch'(G) \leq 2\Delta - 1$ for any graph G with maximum degree Δ; this can be improved to $2\Delta - 2$ for $\Delta > 2$, by Theorem 2.1. Probabilistic methods are particularly powerful, when one tries to obtain asymptotic results. For the case of simple graphs G with maximum degree Δ, a $(\frac{7}{4} + o(1))\Delta$ upper bound for $ch'(G)$ has been obtained by Bollobás and Hind [16], improving a $(\frac{11}{6} + o(1))\Delta$ upper bound proved by Bollobás and Harris [15]. (Here, and in the next sentence, the $o(1)$ term tends to 0 as Δ tends to infinity.) The final asymptotic result for this problem has been obtained recently by J. Kahn

[36], who proved an asymptotically optimal $(1+o(1))\Delta$ upper bound. His proof applies delicate probabilistic arguments, which are based on the technique developed by Rödl in [45]. For upper bounds for the maximum possible value of $ch'(G)$ for *multigraphs* with maximum degree Δ, see [18], [32].

Probabilistic arguments are applied in [6] to obtain a sharp estimate for the choice numbers of complete multipartite graphs with equal color classes. For two positive integers m and r, let K_{m*r} denote the complete r-partite graph with m vertices in each vertex class. For $r=1$, K_{m*r} has no edges and hence, obviously, $ch(K_{m*1})=1$, for all m. Another trivial observation is the fact that $ch(K_{1*r})=r$, for all r. In [24] it is shown that $ch(K_{2*r})=r$ for all r. The following theorem determines, up to a constant factor, the choice number of K_{m*r} for all the remaining cases.

Theorem 4.2 ([6]) *There exist two positive constants c_1 and c_2 such that, for every $m \geq 2$ and for every $r \geq 2$,*

$$c_1 r \log m \leq ch(K_{m*r}) \leq c_2 r \log m.$$

A simple application of this theorem is the following.

Corollary 4.3 *There exists a positive constant b such that, for every n, there is an n-vertex graph G such that*

$$ch(G)+ch(G^c) \leq bn^{1/2}(\log n)^{1/2},$$

where G^c is the complement of G.

This settles a problem raised in [24], where the authors ask whether there exists a constant $\epsilon > 0$ so that, for all sufficiently large n and for every n-vertex graph G, $ch(G)+ch(G^c) > n^{1/2+\epsilon}$.

Another simple corollary of the above theorem deals with the choice numbers of random graphs. It is convenient to consider the common model $G_{n,1/2}$ (see, for example, [14]), in which the graph is obtained by taking each pair of the n labelled vertices $1,2,\ldots,n$ to be an edge, randomly and independently, with probability 1/2. (It is not too difficult to obtain similar results for other models of random graphs as well.) As proved by Bollobás in [13], almost surely (that is, with probability that tends to 1 as n tends to infinity), the random graph $G=G_{n,1/2}$ has chromatic number

$$(1+o(1))n/2\log_2 n.$$

It is also known and easy- see for example, [14], [8], that almost surely G contains no independent set of size greater than $2\log_2 n$. Therefore, for $r=n/\log_2 n$ and $m=2\log_2 n$, say, G has almost surely a proper coloring with r colors in which no color appears more than m times. It follows that G is almost surely a subgraph of K_{m*r} and hence, by Theorem 4.2, almost surely

$$ch(G) \leq ch(K_{m*r}) = O(r\log m) = O(n\frac{\log\log n}{\log n}).$$

Hence, for almost all the graphs G on n vertices, $ch(G) = o(n)$ as n tends to infinity. This solves another problem raised in [24].

A sharper estimate has been obtained by Jeff Kahn (private communication), who determined the correct asymptotic behaviour of the choice number of the random graph. Here is the result, and its surprisingly simple proof.

Proposition 4.4 *The choice number of the random graph $G_{n,1/2}$ on n vertices is almost surely $(1+o(1))n/2\log_2 n$.*

Proof As the choice number is at least as large as the chromatic number, the known estimate of the chromatic number of the random graph shows that the choice number of $G_{n,1/2}$ is almost surely at least $(1+o(1))n/2\log_2 n$. In [13], Bollobás shows that $G = G_{n,1/2}$ satisfies almost surely the following property: every set of at least $n/\log^2 n$ vertices of G contains an independent set of size $q = (1+o(1))2\log_2 n$. Suppose, therefore, that G satisfies this property, and suppose that, for each vertex v of G, we are given a set $S(v)$ of at least $n/q + n/\log^2 n$ $(= (1+o(1))n/2\log_2 n)$ colors. As long as there is a color c that appears in at least $n/log^2 n$ sets $S(v)$, take an independent set of size at least q among the vertices whose color lists contain c, color them by c, delete them from the graph, and omit c from the color lists of the other vertices. When this process terminates, every vertex still has at least $n/\log^2 n$ colors in its list, and no color appears in more than $n/\log^2 n$ color lists. Therefore, by Hall's theorem, one can assign to each of the remaining vertices a color from its list so that no two vertices will get the same color. This completes the proof. □

It is worth noting that, by applying martingales as in [47] (see also [8], pp. 84-86), one can show that, if E_n is the expectation of the choice number of $G_{n,1/2}$, then, for any $\lambda > 0$, only a fraction of at most $2e^{-\lambda^2/2}$ of the graphs on n labelled vertices have choice numbers that deviate from E_n by more than $\lambda\sqrt{n}$.

The following proposition establishes the upper bound for $ch(K_{m*r})$, asserted in Theorem 4.2. Although, as observed by J. Kahn, the proof of this proposition given in [6] can be simplified, we sketch the original proof here, since it applies an interesting splitting technique which has other applications as well (see, for example, [5]). To simplify notation, we omit all the floor and ceiling signs whenever these are not crucial. All the logarithms are to the natural base e, unless otherwise specified.

Proposition 4.5 *There exists a positive constant c such that, for all positive integers $m \geq 2$ and r, $ch(K_{m*r}) \leq cr\log m$.*

Proof Since rm is a trivial upper bound for $ch(K_{m*r})$ and since, for $c \geq 4$, say, $rm \leq cr\log m$ for all m satisfying $m \leq c$, we may assume that $m > c$ (where c will be chosen later). Let $V_1, V_2, \ldots V_r$

be the vertex classes of $K = K_{m*r}$, where $|V_i| = m$ for all i, and let $V = V_1 \cup \ldots \cup V_r$ be the set of all vertices of K. For each $v \in V$, let $S(v)$ be a set of at least $cr \log m$ distinct colors. We must show that there is a proper coloring of K, assigning to each vertex v a color from $S(v)$. Since $ch(K_{m*r})$ is a non-decreasing function of r, we may (and will) assume that r is a power of 2.

We consider two possible cases.

Case 1: $r \leq m$.
Let $S = \cup_{v \in V} S(v)$ be the set of all colors. Put $R = \{1, 2, \ldots, r\}$, and let $f : S \mapsto R$ be a random function, obtained by choosing the value of $f(c)$, randomly and independently for each color $c \in S$, according to a uniform distribution on R. The colors c for which $f(c) = i$ will be the ones to be used for coloring the vertices in V_i. To complete the proof for this case, it thus suffices to show that, with positive probability for every i $(1 \leq i \leq r)$, and for every vertex $v \in V_i$, there is at least one color $c \in S(v)$ such that $f(c) = i$.

Fix an i, and a vertex $v \in V_i$. The probability that there is no color $c \in S(v)$ such that $f(c) = i$ is clearly

$$(1 - \frac{1}{r})^{|S(v)|} \leq (1 - \frac{1}{r})^{cr \log m} \leq e^{-c \log m} \leq \frac{1}{m^c} < \frac{1}{rm},$$

where the last inequality follows from the fact that $r \leq m$ and $c \geq 4 > 2$. There are rm possible choices of i $(1 \leq i \leq r)$ and $v \in V_i$, and hence the probability that, for some i and some $v \in V_i$, there is no $c \in S(v)$ such that $f(c) = i$ is smaller than 1; this completes the proof in this case.

Case 2: $r > m$.
Here we apply a splitting trick, similar to the one used in [5]. As before, define $R = \{1, 2, \ldots, r\}$ and let $S = \cup_{v \in V} S(v)$ be the set of all colors. Put $R_1 = \{1, 2, \ldots, r/2\}$ and $R_2 = \{r/2 + 1, \ldots, r\}$. Let $f : S \mapsto \{1, 2\}$ be a random function obtained by choosing $f(c) \in \{1, 2\}$, for each $c \in S$ randomly and independently, according to a uniform distribution. The colors c for which $f(c) = 1$ will be used for coloring the vertices in $\cup_{i \in R_1} V_i$, whereas the colors c for which $f(c) = 2$ will be used for coloring the vertices in $\cup_{i \in R_2} V_i$.

For every vertex $v \in V$, put $S^0(v) = S(v)$, and define $S^1(v) = S^0(v) \cap f^{-1}(1)$ if v belongs to $\cup_{i \in R_1} V_i$, and $S^1(v) = S^0(v) \cap f^{-1}(2)$ if v belongs to $\cup_{i \in R_2} V_i$. Observe that in this manner the problem of finding a proper coloring of K in which the color of each vertex v is in $S(v) = S^0(v)$ has been decomposed into two independent problems. These are the problems of finding proper colorings of the two complete $r/2$-partite graphs on the vertex classes $\cup_{i \in R_1} V_i$ and $\cup_{i \in R_2} V_i$, by assigning to each vertex v a color from $S^1(v)$. Let $s_0 = cr \log m$ be the number of colors in each original list of colors assigned to a vertex. Using the standard tail estimates for binomial variables (see, for example, [8]), it is not too difficult to show that, for all sufficiently large c, with high probability,

$$|S^1(v)| \geq \frac{1}{2}s_0 - \frac{1}{2}s_0^{2/3}, \tag{4}$$

for all $v \in V$.

Let s_1 denote the minimum cardinality of a set $S^1(v)$, for $v \in V$. As shown above, we can ensure that

$$s_1 \geq \frac{1}{2}s_0 - \frac{1}{2}s_0^{2/3}.$$

We have thus reduced the problem of showing that the choice number of K_{m*r} is at most s_0 to that of showing that the choice number of $K_{m*(r/2)}$ is at most s_1.

Repeating the above decomposition technique, which we can repeat as long as $r/2^i > m$, we obtain, after j iterations, a sequence s_i, where $s_0 = cr \log m$ and

$$s_{i+1} \geq s_i/2 - s_i^{2/3}/2, \quad for \;\; 1 \leq i < j. \tag{5}$$

In order to show that the choice number of $K = K_{m*r}$ is at most s_0, it suffices to show that, for some i, the choice number of $K_{m*(r/2^i)}$ is at most s_i.

Let the number of iterations j be chosen so that j is the minimum integer satisfying $r/2^j \leq m$. Clearly, in this case, $r/2^j > m/2 \geq c/2$. A simple (but tedious) computation, which we omit, shows that

$$s_j \geq \frac{s_0}{2^{j+1}},$$

provided that c is sufficiently large.

To complete the proof of the proposition, observe that it suffices to show that the choice number of $K_{m*(r/2^j)}$ is at most s_j. However, $r/2^j \leq m$ and

$$s_j \geq s_0/2^{j+1} \geq \frac{c}{2}\frac{r}{2^j}\log m.$$

For a sufficiently large c, the result thus follows from Case 1. This completes the proof. □

The lower bound in Theorem 4.2 can also be derived by probabilistic arguments, which we omit. Let us, however, present the simple derivation of Corollary 4.3 from the upper bound of this theorem.

Proof of Corollary 4.3 Define $m = \sqrt{n \log n}$ and $r = n/m = \frac{\sqrt{n}}{\sqrt{\log n}}$, and let G be the graph K_{m*r}. The complement G^c of G is a disjoint union of r cliques, each of size m, and thus $ch(G^c) = m = O(\sqrt{n \log n})$. By Theorem 4.2, $ch(G) = O(r \log m) = O(\sqrt{n \log n})$. Thus,

$$ch(G) + ch(G^c) = O(\sqrt{n \log n}),$$

as required. □

For a graph $G = (V, E)$, and for two integers $a \geq b \geq 1$, G is called $(a : b)$-*choosable* if, for every assignment of sets of colors $S(v) \subset Z$, each of cardinality a, for every vertex $v \in V$, there are subsets $T(v) \subset S(v)$, $|T(v)| = b$ for all v, so that, if u and v are adjacent, then $T(u)$ and $T(v)$ are disjoint. In particular, $(k : 1)$-choosability coincides with the previous definition of k-choosability. This definition is introduced in [24], where the authors raise the following question:
Suppose G is $(a : b)$-choosable, and suppose that $c/d > a/b$, where $a \geq b$ and $c \geq d$ are positive integers. Does it follow that G is $(c : d)$-choosable as well?

S. Gutner [28] showed that the answer is "no", by establishing the following result, proved by a simple probabilistic argument.

Proposition 4.6 ([28]) *For every two integers $n \geq k$ and for every $\epsilon > 0$, there is a $c_0 = c_0(n, k, \epsilon)$ such that the following holds: for every graph G on n vertices with chromatic number k, and for every two positive integers $c > c_0$ and d satisfying $c(1 - \epsilon) \geq kd$, G is $(c : d)$-choosable.*

Proof Let us fix a proper k-coloring of $G = (V, E)$, and let $V_1, \ldots, V_k$ be the k color classes. Given sets of colors $S(v) \subset Z$ of cardinality c for each vertex v of G, let S be the union of all the sets $S(v)$, and let us split S randomly into k pairwise disjoint subsets $S_1, \ldots, S_k$, where each $s \in S$ is chosen randomly and independently as a member of one of the sets S_i, according to a uniform distribution. The colors in S_i will be used for defining the sets $T(v)$, for $v \in V_i$. To complete the proof, it suffices to check that, if c is sufficiently large and $c(1 - \epsilon) \geq kd$, then with positive probability, $|S(v) \cap S_i| \geq d$ for every $1 \leq i \leq k$ and for every $v \in V_i$. However, for each such v, $|S(v) \cap S_i|$ is a binomial random variable with expectation c/k and variance $\frac{c}{k}(1 - \frac{1}{k}) < c/k$, and hence, by Chebyshev's Inequality, the probability that there is a vertex v in V_i for some i so that $|S(v) \cap S_i| < d$, is at most

$$n \frac{ck^2}{k\epsilon^2 c^2} < 1,$$

where the last inequality holds for all sufficently large c (as a function of ϵ, n and k). (Observe that this estimate can be improved by applying the more accurate known estimates for binomial distributions.) This completes the proof. □

By the last proposition (with $n = 2m, k = 2, \epsilon = 1/3$ and $c = 3d$) for each integer m, the complete bipartite graph $K_{m,m}$ is, $(3d : d)$-choosable for all sufficiently large d, and yet it is not, say, $(100 : 1)$-choosable, if m is large enough, as mentioned in Section 2. As $100/1 > 3d/d$, this implies that the answer to the question preceding the last proposition is negative.

5 The minimum degree and choice numbers

Despite the close connection between choice numbers and chromatic numbers, these two invariants do not have the same properties. In this section, we prove a result that supplies an essential difference between the two. Let us call two graph invariants $\alpha(G)$ and $\beta(G)$ *related* if there are two functions $f(x)$ and $g(x)$, both tending to infinity as x tends to infinity, such that, for every (simple) graph G, $\beta(G) \geq f(\alpha(G))$ and $\alpha(G) \geq g(\beta(G))$. Roughly speaking, two invariants are related if they grow together, although one may grow much slower than the other. Examples of related parameters are the chromatic index of a graph and its maximum degree. Other examples of related parameters are the chromatic number of a graph and the minimum number of bipartite graphs required to cover all its edges. On the other hand, the chromatic number and the choice number of a graph are not related, as for any k there are graphs of chromatic number 2 whose choice number exceeds k.

The definition of a d-degenerate graph appears before Proposition 2.2. For a simple graph G, let $d(G)$ denote the minimum integer d so that G is d-degenerate. Equivalently, $d(G)$ is the maximum integer d such that G has a subgraph with minimum degree d. The parameter $d(G)$ arizes naturally in the study of various problems and it is not diffciult to show that it is related, in the above sense, to the arboricity of G, which is the minimum number of forests whose union covers all edges of G. By Proposition 2.2, for any graph G, $d(G) \geq ch(G) - 1 \geq \chi(G) - 1$. On the other hand, for every k there are graphs with $\chi(G) = 2$ and $d(G) > k$ as shown by the family of complete bipartite graphs. Therefore the parameters $d(G)$ and $\chi(G)$ are *not* related. On the other hand, the parameters $ch(G)$ and $d(G)$, are related. One can show that the choice number of any simple graph with minimum (or average) degree d is at least $\Omega(\log d / \log \log d)$. This is proved in the following theorem; in its statement and proof, we make no attempt to optimize the constants.

Theorem 5.1 *Let G be a simple graph with average degree at least d. If s is an integer and*

$$d > 4\binom{s^4}{s} \log(2\binom{s^4}{s}), \tag{6}$$

then $ch(G) > s$.

Proof Let G be a simple graph with average degree at least d. We first show that, as is well known, every such G contains a bipartite graph, whose minimum degree is at least $d/4$. To see this, observe that G has an induced subgraph with minimum degree at least $d/2$, since one can repeatedly delete vertices of degree smaller than $d/2$ from G, as long as there are such vertices; since this process increases the average degree, it must terminate in a non-empty subgraph G' with minimum degree

at least $d/2$. In G', one can take a spanning bipartite subgraph H with the maximum possible number of edges. If a vertex here has degree smaller than half of its degree in G', then shifting it to the other class of the bipartite graph will increase the number of edges and contradict the maximality in the choice of the bipartite graph.
Now let $H = (V, E)$ be a bipartite subgraph of G, with minimum degree $\delta \geq d/4$ and with vertex classes A and B, where $|A| \geq |B|$. Let $S = \{1, 2, \ldots, s^4\}$ be our set of colors. Our objective is to show that there are subsets $S(v) \subset S$, where $|S(v)| = s$ for all $v \in V$, such that there is no proper coloring $c : V \mapsto S$ that assigns to every $v \in V$ a color $c(v) \in S(v)$. This will imply that

$$ch(G) \geq ch(H) > s,$$

and complete the proof.

The proof is probabilistic. For each vertex $b \in B$, let $S(b)$ be a random subset of cardinality s of S, chosen uniformly and independently among all the $\binom{s^4}{s}$ subsets of cardinality s of S. Call a vertex $a \in A$ *good* if, for every subset $C \subset S$ with $|C| = s$, there is a neighbor b of a in H such that $S(b) = C$. For a fixed $a \in A$, the probability that a is not good is at most

$$\binom{s^4}{s}\left(1 - \frac{1}{\binom{s^4}{s}}\right)^{\delta} \leq 1/2,$$

where the last inequality follows from the fact that $\delta \geq d/4$ and from assumption (6). Therefore, the expected number of good vertices $a \in A$ is at least $|A|/2$, and hence there is some choice of the s-subsets $S(b)$, $b \in B$ such that there are at least $|A|/2$ good vertices in A. Let us fix these subsets $S(b)$ and choose, for each $a \in A$, a subset $S(a) \subset S$, $|S(a)| = s$, randomly and independently according to a uniform distribution on the s-subsets of S. To complete the proof, we show that with positive probability there is no proper coloring $c : A \cup B \mapsto S$ of H assigning to each vertex $v \in A \cup B$ a color from its class $S(v)$.

There are $s^{|B|}$ possibilities for the restriction $c|_B$ of c to the vertices in B so that $c(b) \in S(b)$ for all b. Fix such a restriction, and let us estimate the probability that this restriction can be extended to a proper coloring c of the desired type. The crucial observation is that, if $a \in A$ is good, then the set of all colors assigned by $c|_B$ to the neighbors of a is a set that intersects every s-subset of S, since every s-subset is $S(b)$ for some neighbor b of a. Therefore, at least $s^4 - s + 1$ distinct colors are assigned by $c|_B$ to the neighbors of a. It is thus possible to choose a proper color for a from its set $S(a)$, only if $S(a)$ contains one of the set of at most $s - 1$ colors which differ from $c(b)$ for all the neighbors b of a. The probability that the randomly chosen set $S(a)$ satisfies this is at most

$$\frac{(s-1)\binom{s^4-1}{s-1}}{\binom{s^4}{s}} = \frac{s(s-1)}{s^4} < \frac{1}{s^2}.$$

Moreover, all these events for distinct good vertices $a \in A$ are mutually independent, by the independent choice of the sets $S(a)$. It follows that, if there are g ($\geq |A|/2$) good vertices in A, the probability that a fixed partial coloring $c|_B$ can be extended to a full coloring $c : A \cup B \mapsto S$, assigning to each vertex a color from its class, is strictly less than

$$(1/s^2)^g \leq (1/s^2)^{|A|/2} \leq \frac{1}{s^{|B|}},$$

since $|A| \geq |B|$. As there are only $s^{|B|}$ possibilities for the partial coloring $c|_B$, and (as just shown) the probability that a fixed partial coloring would extend to a full coloring is strictly less than $1/s^{|B|}$, we conclude that with positive probability there is no coloring of the required type; this shows that $ch(H) > s$, completing the proof. □

As shown in Section 2, the complete bipartite graphs $K_{d,d}$ supply an example with minimum degree d and choice number $(1 + o(1)) \log_2 d$, thus showing that Theorem 5.1 is nearly tight. Note also that there is a very simple polynomial time algorithm that finds, for a given input graph G, the value of the parameter $d(G)$- the minimum d so that G is d-degenerate. In view of Theorem 5.1, this supplies a polynomial algorithm that finds, for a given input graph G, a number s so that the choice number of G lies between s and $O(s^{4s} \log s)$. Although this is a very crude approximation, there is no known similar efficient approximation algorithm for the chromatic number of a graph. In fact, it would be very interesting to find any function $f : Z \mapsto Z$ and a polynomial time algorithm that finds, for a given input graph G, a number s so that the chromatic number of G is between s and $f(s)$. Although the problem of approximating the chromatic number of a graph received a considerable amount of attention, no such algorithm is known.

6 Concluding remarks and open problems

The most interesting open problem in the area is the list coloring conjecture (Conjecture 1.1), which is wide open, although it has been verified in several cases. It is easy to see that it is true for forests and for graphs with maximum degree 2. It also holds for graphs with no cycles of length bigger than 3, as shown in [28]. As mentioned in Section 2, the assertion of the conjecture holds for graphs with maximum degree 3 and edge chromatic index 4, by Theorem 2.1. Häggkvist [30] gave an interesting proof for all complete bipartite graphs $K_{r,n}$ with $r \leq \frac{2}{7}n$ and as discussed in Section 3 it holds for every planar d-regular multigraph with chromatic index d, by the result of [22] which applies the technique of [9]. It has also been proved for $K_{3,3}$ by H. Taylor (private communication); in [9], it is derived for $K_{4,4}$ and $K_{6,6}$ from Corollary 3.9. The last three examples are special cases of the case $G = K_{n,n}$ of the list coloring conjecture, which was formulated by J. Dinitz in 1979. Although Vizing had already raised the general conjecture in 1975, the special case of Dinitz became more popular, as it has the following appealing reformulation: given an arbitrary n by n array of n-sets, it is always possible to choose one element from each set, keeping the chosen elements distinct in every row and distinct in every column. (If all the n-sets are equal, then every n by n Latin square provides a good choice, and although it does seem that in any other case one has even more freedom, nobody has been able to transform this intuitive feeling into a rigorous proof.) As mentioned above, Dinitz's Conjecture is true for all $n \leq 4$ and for $n = 6$, but is not known for any other case, despite a considerable amount of effort by various researchers. Corollary 3.9 for this special case implies the following. Define the *weight* $w(L) \in \{-1, 1\}$ of an n by n Latin square to be the product of the signs of the $2n$ permutations appearing in its rows and its columns. If the sum $\sum w(L)$ is not 0, as L ranges over all n by n Latin squares, then the assertion of Dinitz's Conjecture holds for n. It is easy to see, however, that this sum is 0 for every odd $n \geq 3$, but it is conjectured in [9] to be non-zero for all even n (and this holds for $n = 2, 4, 6$).

The total coloring conjecture (Conjecture 2.4) is another problem that has received a considerable amount of attention. It would be nice to prove a $\Delta + O(1)$ upper bound for the total chromatic number of any simple graph with maximum degree Δ.

There are many known proofs in combinatorics that supply no efficient procedures for solving the corresponding algorithmic problems; see, for example, [4] for various representative examples. Some of the results described in Section 3 also have this flavour. Thus, for example, we know by Corollary 3.10 that $ch'(G) = 3$, for every 2-connected cubic planar graph G, but the proof supplies no polynomial time (deterministic or randomized) algorithms that produce, for a given such graph $G = (V, E)$ and given lists of colors $S(e),\ e \in E$, each of size 3, a proper edge-coloring of G assigning to each edge a color from its list. Similarly, there is no known efficient procedure for solving the algorithmic problem suggested by Theorem 3.6. We note that, in contrast, one can give an efficient procedure for solving the algorithmic problem suggested by Theorem 3.7, based on Richardson's Theorem (see [12]), whose relevance to the problem has been pointed out by Bondy, Boppana and Siegel.

Another interesting problem is that of determining the largest possible choice number of a planar graph. This number is known to be at least 4 and at most 6, and in [24] it is conjectured that it is, in fact, 5.

The *Hadwiger number* $h(G)$ of a graph G is the maximum number h such that there are h pairwise vertex disjoint connected subgraphs of G with at least one edge of G between any pair of them. A well-known conjecture of Hadwiger [29] asserts that the Hadwiger number $h = h(G)$ of any graph is at least its chromatic number $\chi(G)$. This is known to be the case for all graphs with $h \leq 4$, where the case $h = 4$ is equivalent to the Four Color Theorem. Very recently, the case $h = 5$ has also been shown, by Robertson, Seymour and Thomas, to be equivalent to the Four Color Theorem. P. Seymour (private communication) has suggested that the stronger conjecture $h(G) \geq ch(G)$ may also hold. For graphs G with $h(G) \leq 3$, this can be deduced from Proposition 2.2. If it is true for $h(G) = 4$, then every planar graph has a choice number at most 4, contradicting the above mentioned conjecture of [24]. Needless to say, a proof of the general case seems beyond reach at present, but it may not be so difficult to find a counterexample, if one exists. We note that, by combining Proposition 2.2 with the known fact that the Hadwiger number of simple graphs with average degree at least d is at least $\Omega(d/\sqrt{\log d})$ (see [37], [48]), one easily concludes that, for every graph G, $h(G) \geq \Omega(\chi(G)/\sqrt{\log(\chi(G))})$.

Recall the definition of an $(a : b)$-choosable graph, given in the end of Section 4. In [24], the authors ask whether for any positive integers a, b and m, any $(a : b)$-choosable graph is $(am : bm)$-choosable as well. This is proved in [28] for the special case $a = 2, b = 1$ and $m = 2$. The general case is still open, but we can prove the following partial result.

Proposition 6.1 *For any positive integers a, b and n, there is a positive integer $f = f(a, b, n)$ such that, for every integer m which is divisible by all integers smaller than f, any graph G on n vertices which is $(a : b)$-choosable, is $(am : bm)$-choosable as well.*

The proof applies the techniques of [7]. We omit the details.

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