Spanning subgraphs of Random Graphs
(A research problem)

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Abstract

We propose a problem concerning the determination of the threshold function for the edge probability that guarantees, almost surely, the existence of various sparse spanning subgraphs in random graphs. We prove some bounds and demonstrate them in the cases of a d-cube and a two dimensional lattice.

B. Bollobás (cf. e.g., [3]) raised the following problem:

Let G be a random graph with $n = 2^d$ vertices, in which each edge is taken randomly and independently with probability $p = 1 - \varepsilon$, where ε is a positive small constant. Is it true that for $d > d(\varepsilon)$ almost surely G contains a copy of the d-cube, Q^d? Note that Q^d has $2^{d-1}d = O(n \log n)$ edges, and is thus a relatively sparse graph.

Here we show that the answer is "yes" for every fixed $p > 1/2$ and observe that it is "no" for $p \leq 1/4$. This is a special case of the following general theorem.

Theorem 1 *Let $G = G(n,p)$ be a random graph on a set V of n labelled vertices obtained by choosing each pair of vertices to be an edge randomly and independently, with probability p. Let $H = (U,F)$ be a fixed simple graph on n vertices with maximum degree d, where $(d^2+1)^2 < n$. If*

$$p^d > \frac{10\log(\lfloor n/(d^2+1)\rfloor)}{(\lfloor n/(d^2+1)\rfloor)}, \tag{1}$$

then the probability that G does not contain a copy of H is smaller than $1/n$.

Remark 1. The number 10 can be easily improved. We make no attempt to optimize it.

Remark 2. In case H is d-regular, the expected number of copies of H in G is at most $n!p^{nd/2}$, which is $o(1)$ for $p = n^{-2/d}$. Thus for such a p almost surely G has no copy of H, whereas the theorem gives that for $p \geq c(\log n/n)^{1/d}$ almost surely G does have a copy of H.

The d-cube. In case $H = Q^d$ is the d-cube, the right hand side of (1) is $2^{-d}O(d^3)$ and hence the theorem implies that for every fixed $p > 1/2$, almost surely $G(n,p)$ has a spanning d-cube. On the other hand, Remark 2 shows that for $p \leq 1/4$ almost surely $G(n,p)$ does not contain a spanning d-cube. We strongly believe that the threshold probability, $p(Q^d)$, defined as the infimum value of the numbers p such that almost surely $G(n,p)$ contains a spanning Q^d is much closer to $1/4$ than to $1/2$. I.e., we suspect that as usual, the computation of the expectation gives the correct bound.

The two dimensional lattice. Let L_k denote the 2-dimensional lattice of size $n = k^2$, i.e. the graph with vertex set $\{(x_1,x_2) : 1 \leq x_1, x_2 \leq k\}$ in which there is an edge between the vertices (x_1,x_2) and (y_1,y_2) if and only if $|x_1 - y_1| + |x_2 - y_2| = 1$. Then the simple argument of Remark 2 gives that the threshold probability $p(L_k)$ for the existence of a spanning L_k is at least $\Omega((1/n)^{1/2})$. On the other hand, Theorem 1 shows that $p(L_k) \leq O((\log n/n)^{1/4})$. The problem of estimating $p(L_k)$ was raised by Levin and Venkatesan [5], who were motivated by the study of certain graph coloring problems which are computationally hard, even on random instances. They proved that $p(L_k) = o(1)$.

Proof of Theorem 1 (sketch). By applying a well known theorem of Hajnal and Szemerédi [4] to the square of H we obtain a partition of the vertex set U of H into $D = d^2 + 1$ pairwise disjoint sets $U_1, \ldots, U_D$ so that each U_k is an independent set in H, no two vertices of U_k have a common neighbor in H and the cardinality of each U_k is either $\lfloor n/D \rfloor$ or $\lceil n/D \rceil$. Now let us split, arbitrarily, the set of vertices V of G into pairwise disjoint sets $V_1, \ldots V_D$ so that $|V_k| = |U_k|$ for all k.

We next show that with high probability there is a one to one function $f : U \mapsto V$, which maps each U_k onto V_k and which maps H into a copy of H in G. To do so we define f on each U_k in its turn. Start with an arbitrary one to one mapping of U_1 onto V_1. Assume, by induction, that we have already defined $f : U_1 \cup \ldots \cup U_k \mapsto V_1 \cup \ldots \cup V_k$ and that f maps the induced subgraph of H on $U_1 \cup \ldots \cup U_k$ into a copy of it in $V_1 \cup \ldots \cup V_k$. We next show how to extend this f and define it on U_{k+1}. Suppose $U_{k+1} = \{u_1, \ldots, u_m\}$ and $V_{k+1} = \{v_1, \ldots, v_m\}$. Construct a bipartite graph B with classes of vertices $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_m\}$ by joining x_i by an edge to y_j if and only if we can define $f(u_i) = v_j$. More formally, x_i is joined by an edge to y_j if and only if in the graph G, the vertex v_j is joined by an edge to all the vertices $f(u)$, where u is a neighbor of u_i in H which belongs to $U_1 \cup \ldots \cup U_k$. Observe that for each i and j, the probability that u_i is adjacent to v_j is at least p^d. The crucial fact is that all these probabilities are mutually independent, since they all depend on pairwise disjoint sets of edges of G. Thus we can apply the known results on the existence of perfect matchings in graphs (see, e.g., [1]) and conclude that in view of (1) (and the fact that $D^2 < n$) the probability that there is no perfect matching in B is at most $1/(nD)$. We can now define f according to this perfect matching; if it matches x_i to y_j we define $f(u_i) = v_j$.

The probability that all these $D - 1$ matchings exist is at least $1 - 1/n$, completing the proof. $\square$

Remark 3. As pointed out by J. Spencer, an indication for the belief that the lower bound is closer to the truth than the upper bound is the fact that an edge probability which is only slightly larger than the lower bound already implies, almost surely, the existence of a large piece of the required spanning subgraph. Here is a demonstration of this fact for the grid L_k. Suppose $n = k^2$ where k is even, and let $p = c\sqrt{\log n/n}$ where c is a large positive constant. Let U_1 and U_2 be two arbitrarily chosen disjoint subsets of vertices of $G(n, p)$, each having cardinality $k^2/2$. Let A and B be the two vertex classes of the bipartite graph L_k and choose an arbitrary one to one mapping from A to U_1. We can now extend this mapping by mapping vertices of B to suitable vertices of U_2. It is easy to see that with high probability this Markov-type process breaks down only after at least $(1 - o(1))k^2$ of the vertices are mapped. In other words, $G(n, p)$ contains, almost surely, a large piece of L_k.

The difficult problem is of course the question if this process can be finished to give a full copy of L_k. When L_k is replaced by a cycle, the required last step has been established by Pósa, who proved a clever lemma that enabled him to conclude that edge probability $\Theta(\log n/n)$ suffices (and is also necessary) for the existence, almost surely, of a Hamilton cycle (cf. [1]). It would be interesting to decide if there exists an appropriate Pósa-type Lemma for the case of the grid too.

Bandwidth. The *generalized bandwidth problem* is the following. Let H and G be two graphs with the same number of vertices. Given a bijection b from $V(H)$ to $V(G)$ let $|b|$ denote the maximum, over all edges xy of H, of the distance in G between $b(x)$ and $b(y)$. Finally, let $B(H,G)$ be the minimum value of $|b|$ over all such b. Clearly, $B(H,G)=1$ if and only if H is a spanning subgraph of G.

When G is a path the parameter $B(H,G)$ is known as the *bandwidth* $B(H)$ of H. The case in which G is a multidimensional lattice and H is a random graph has been investigated by McDiarmid and Miller (see [2], and the references there). The cases in which G is a random graph (and H varies) also look interesting and difficult, but certainly there are interesting solvable special cases.

References

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