# More on the bipartite decomposition of random graphs

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**Abstract**

For a graph $G = (V, E)$, let $bp(G)$ denote the minimum number of pairwise edge disjoint complete bipartite subgraphs of $G$ so that each edge of $G$ belongs to exactly one of them. It is easy to see that for every graph $G$, $bp(G) \le n - \alpha(G)$, where $\alpha(G)$ is the maximum size of an independent set of $G$. Erdős conjectured in the 80s that for almost every graph $G$ equality holds, i.e., that for the random graph $G(n, 0.5)$, $bp(G) = n - \alpha(G)$ with high probability, that is, with probability that tends to 1 as $n$ tends to infinity. The first author showed that this is slightly false, proving that for most values of $n$ tending to infinity and for $G = G(n, 0.5)$, $bp(G) \le n - \alpha(G) - 1$ with high probability. We prove a stronger bound: there exists an absolute constant $c > 0$ so that $bp(G) \le n - (1 + c)\alpha(G)$ with high probability.

## 1 Introduction

For a graph $G = (V, E)$, let $bp(G)$ denote the minimum number of pairwise edge disjoint complete bipartite subgraphs of $G$ (bicliques of $G$) so that each edge of $G$ belongs to exactly one of them. A well-known theorem of Graham and Pollak [7] asserts that $bp(K_n) = n - 1$, see [12], [11], [13] for more proofs, and [1], [6], [9], [10] for several variants.

Let $\alpha(G)$ denote the maximum size of an independent set of $G$. It is easy to see that for every graph $G$, $bp(G) \le n - \alpha(G)$. Indeed one can partition all edges of $G$ into $n - \alpha(G)$ stars centered at the vertices of the complement of a maximum independent set in $G$. Erdős conjectured (see [9]) that for almost every graph $G$ equality holds, i.e., that for the random graph $G(n, 0.5)$, $bp(G) = n - \alpha(G)$ with high probability (*whp*, for short), that is, with probability that tends to 1 as $n$ tends to infinity.

Chung and Peng [5] extended the conjecture for the random graphs $G(n, p)$ with $p \le 0.5$, conjecturing that for any $p \le 0.5$, $bp(G) = n - (1 + o(1))\alpha(G)$ whp. They also established lower bounds supporting this conjecture, and the one of Erdős, by proving that for $G = G(n, p)$ and for all $0.5 \ge p \ge \Omega(1)$, $bp(G) \ge n - o((\log n)^{3+\epsilon})$ for any positive $\epsilon$.

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The first author proved in [2] that Erdős' conjecture for $G = G(n, 0.5)$ is (slightly) incorrect. It turns out that for most values of $n$, and for $G = G(n, 0.5)$, $bp(G) \leq n - \alpha(G) - 1$ whp, while for some exceptional values of $n$ (that is, those values for which the size of $\alpha(G)$ is concentrated in two points, and not in one), $bp(G) \leq n - \alpha(G) - 2$ with probability that is bounded away from 0.

He also improved the estimates of [5] for $G(n,p)$ for any $c \geq p \geq \frac{2}{n}$, where $c$ is some small positive absolute constant, proving that if $\frac{2}{n} \leq p \leq c$ then for $G = G(n,p)$

$$bp(G) = n - \Theta\left(\frac{\log(np)}{p}\right)$$

whp.

In this note we establish a better upper bound for $bp(G)$ for $G = G(n, 0.5)$, as follows.

**Theorem 1.1** *There exists an absolute constant $c > 0$ so that for $G = G(n, 0.5)$,*

$$bp(G) \leq n - (2+2c)\log_2 n \leq n - (1+c)\alpha(G)$$

*with high probability.*

The proof is based on the second moment method applied to an appropriately defined random variable. We also describe another argument, based on a three-stage exposure of the edges of the random graph, which provides a simple proof of the fact that for $G = G(n, 0.5)$,

$$bp(G) \leq n - \alpha(G) - \Omega(\log\log n). \tag{1}$$

Although this is weaker than the assertion of Theorem 1.1 we believe this proof is also interesting.

The rest of this note is organized as follows. In Section 2 we describe the short proof of (1). Section 3 includes the proof of Theorem 1.1. The final Section 4 contains some concluding remarks, open problems and a brief discussion of related questions.

Throughout the rest of the note we assume, whenever this is needed, that $n$ is sufficiently large. To simplify the presentation we omit all floor and ceiling signs whenever these are not crucial. We make no attempt to optimize the absolute constants in our estimates.

## 2 Three stage exposure and the birthday paradox

In this section we give a proof of inequality (1) based on the following two facts:

(1) If $p = c$ where $c$ is a constant then $\alpha(G(n,p)) = 2\log_b n - 2\log_b \log_b n + \Theta(1)$ where $b = 1/(1-p)$ with high probability, and

(2) If we choose $a$ items uniformly and independently at random from a collection of $b$ items (with replacement) then the probability that the $a$ items are all distinct is at most $e^{-a(a-1)/2b}$.

The second fact is known as the birthday paradox.

Let $X, Y$ be an equi-partition of the vertex set of $G$. We expose the random edges in three stages: We first observe edges inside $X$, then we expose the edges between $X$ and $Y$, and finally we reveal the edges within $Y$. It follows from fact (1) that *whp* $X$ contains an independent set $I$ such that

$$|I| \geq 2\log_2 n - 2\log_2\log_2 n - O(1).$$

Let $\ell = (\log_2 n)^{1/3}$. We partition $Y$ into sets $Y_1, Y_2, \ldots, Y_\ell$ of size $n/(2(\log_2 n)^{1/3})$. Note that for every vertex $v \in Y$ the neighborhood of $v$ in $I$ is a uniform random subset of $I$. Thus it follows from fact (2) that the probability that every vertex in $Y_i$ has a different neighborhood in $I$ is at most

$$\exp\left\{-\Omega\left(\frac{n^2/(\log_2 n)^{2/3}}{n^2/(\log_2 n)^2}\right)\right\} = \exp\left\{-\Omega(\log_2 n)^{4/3}\right\} = o(1/n).$$

It follows that with high probability each set $Y_i$ contains a pair $a_i, b_i$ of distinct vertices that have the same neighborhood in $I$. Let $I_i = I \cap N(a_i) = I \cap N(b_i)$. Once this collection of pairs is fixed, we reveal the edges within $Y$. With high probability at least $\ell/3$ of the pairs $a_i, b_i$ are non-edges, and it follows from fact (1) (taking $p = 1/16$) that among these $\ell/3$ there is a collection of $\Omega(\log\log n)$ pairs $a_i, b_i$ that spans no edge. We decompose the edge set of $G$ into $n - |I| - \Omega(\log\log n)$ bicliques using the bicliques $\{a_i, b_i\} \times I_i$ for the pairs $a_i, b_i$ in this collection together with a collection of stars. $\square$

## 3 The proof of the main result

The proof of Theorem 1.1 is based on the second moment method. The crucial point here is the choice of the random variable to which it is applied.

For a (large) integer $k$ define a family $\mathcal{F}_k$ of graphs on $k$ vertices, as follows. Each graph in $\mathcal{F}_k$ is a bipartite graph with classes of vertices $A$ and $B$, where $|A| = 0.1k$ and $|B| = 0.9k$. The set $A$ is the disjoint union of $r = 0.01k$ sets $A_1, A_2, \ldots, A_r$, where $|A_i| = 10$ for each $i$. For each vertex $b \in B$ there is a binary vector $v_b = (v_b(1), v_b(2), \ldots, v_b(r))$ of length $r$. If $v_b(i) = 0$ then there are no edges between $b$ and $A_i$, and if $v_b(i) = 1$ then $b$ is connected to all members of $A_i$. We further assume (although this is not too crucial, but simplifies matters) that all the vectors $\{v_b : b \in B\}$ are distinct, and that the degree of each $a \in A$ is at least $k/3$ (that if, for each $i$, $v_b(i) = 1$ for at least $k/3$ indices $b$.) In addition we assume that for each two distinct $i, j$ corresponding to different sets $A_i, A_j$, the number of vertices $b \in B$ so that $v_b(i) \neq v_b(j)$ is at least $k/3$. The family $\mathcal{F}_k$ contains all the above graphs.

Note that each graph $F \in \mathcal{F}_k$ is a bipartite graph on $k$ vertices satisfying $bp(F) \leq r$. Indeed, the $r$ complete bipartite graphs with classes of vertices $A_i$ and $\{b \in B : v_b(i) = 1\}$, $(1 \leq i \leq r)$ form a bipartite decomposition of $F$. Let $f_k$ denote the number of graphs on $k$ labelled vertices that are members of $\mathcal{F}_k$. We claim that

$$f_k = (1 - o(1))\binom{k}{10}\binom{k-10}{10}\cdots\binom{k-10r+10}{10}\frac{1}{r!}(2^r)^{0.9k} = k^{(0.09+o(1))k}2^{0.9rk}. \qquad (2)$$

Indeed, there are

$$\binom{k}{10}\binom{k-10}{10}\cdots\binom{k-10r+10}{10}\frac{1}{r!}$$

ways to choose the disjoint sets $A_1, A_2, \ldots, A_r$. After these are chosen, there are $2^r$ possibilities to choose the edges from $b$ to the sets $A_i$, for each of the $0.9k$ vertices of $B$. For a typical choice of these edges, the degree of each $a \in \cup A_i$ is close to $0.5 \cdot 0.9k$ with high probability, no two vertices of $B$ have the same sets of neighbors, and the symmetric difference between the sets of neighbors of any two vertices of $A$ belonging to distinct sets $A_i$ is also close to $0.9k/2$. This means that indeed $1 - o(1)$ of the above choices lead to distinct members of $\mathcal{F}_k$, establishing (2). For our purpose here it suffices to note that by the above, since $r = 0.01k$,

$$f_k = 2^{0.9rk} k^{\Theta(k)} = 2^{(0.9+o(1))rk}.$$

Let $V = \{1, 2, \ldots, n\}$ be a fixed set of $n$ labeled vertices, and let $G = G(n, 0.5) = (V, E)$ be the random graph on $V$. Let $h(k) = \binom{n}{k} f_k 2^{-\binom{k}{2}}$ be the expected number of members of $\mathcal{F}_k$ that appear as induced subgraphs of $G$ and (with a slight abuse of notation) let $k$ be the largest integer such that $h(k) \geq 2^k$. It is not difficult to check that this value of $k$ satisfies

$$k = 2\log_2 n + 1.8r + O(\log k) = (1 + o(1))2\log_2 n + 0.018k$$

implying that $k = (1 + o(1))\frac{1}{0.982} 2\log_2 n$, which is slightly bigger than $2.036 \log_2 n$. Note that $r = 0.01k < 0.0204 \log_2 n$. If $G$ contains an induced copy of a member $F$ of $\mathcal{F}_k$ then

$$bp(G) \leq n - k + bp(F) \leq n - 2.036\log_2 n + 0.0204\log_2 n \leq n - 2.015\log_2 n.$$

As it is well known that $\alpha(G) = (2 + o(1))\log_2 n$ whp (see [4], [3]), it suffices to show that $G$ contains such an induced subgraph whp in order to complete the proof of the theorem. We proceed to do so using the second moment method.

For each $K \subset V$, $|K| = k$, let $X_K$ be the indicator random variable whose value is 1 iff $K$ induces a member of $\mathcal{F}_k$ in $G$. Let $X = \sum_K X_K$, where $K$ ranges over all subsets of size $k$ of $V$, be the total number of such induced members. The expectation of this random variable is $E(X) = h(k) \geq 2^k$. We proceed to estimate its variance. For $K, K' \subset V$, $|K| = |K'| = k$, let $K \sim K'$ denote that $|K \cap K'| \geq 2$ (and $K \neq K'$). The variance of $X$ satisfies:

$$\mathrm{Var}(X) = \sum_K \mathrm{Var}(X_K) + \sum_{K \sim K'} \mathrm{Cov}(X_K, X_{K'}) \leq E(X) + \sum_{K \sim K'} E(X_K X_{K'}), \tag{3}$$

where $K, K'$ range over all ordered pairs of subsets of size $k$ of $V$ satisfying $2 \leq |K \cap K'| \leq k - 1$.

For each $i$, $2 \leq i \leq k - 1$, let $h_i$ denote the contribution of the pairs with intersection $i$ to the above sum, that is

$$h_i = \sum_{|K \cap K'| = i} E(X_K X_{K'}).$$

Our objective is to show that $\sum_{i=2}^{k-1} h_i = o(h(k)^2)$.

We consider two possible ranges for the parameter $i$, as follows.

**Case 1:** $2 \leq i \leq 0.9k$. In this case

$$h_i \leq \binom{n}{k} f_k \binom{k}{i} \binom{n-k}{k-i} f_k 2^{-2\binom{k}{2}+\binom{i}{2}}.$$

Indeed, for each of the $\binom{n}{k} f_k$ choices of the set $K$ and the induced subgraph on it which is a member of $\mathcal{F}_k$, there are $\binom{k}{i}\binom{n-k}{k-i}$ ways to choose the set of vertices $K'$ and then at most $f_k$ ways to select the induced subgraph on $K'$ (here there is an inequality, as many of these choices could lead to inconsistent assumptions about the induced subgraph on $K \cap K'$, but in this case we have enough slack and this trivial inequality suffices). Therefore

$$\frac{h_i}{h(k)^2} \leq \frac{\binom{k}{i}\binom{n-k}{k-i}}{\binom{n}{k}} 2^{\binom{i}{2}} \leq k^i \left(\frac{k}{n}\right)^i 2^{\binom{i}{2}} = \left(\frac{k^2 2^{(i-1)/2}}{n}\right)^i \leq \frac{1}{n^{0.05i}}. \tag{4}$$

Here we used the facts that $k \leq 2.04 \log_2 n$ and $i \leq 0.9k$ to conclude that

$$\frac{k^2 2^{(i-1)/2}}{n} < \frac{1}{n^{0.05}}.$$

**Case 2:** $i = k - j$ where $1 \leq j \leq 0.1k$. This case is more complicated and requires a careful estimate of the number of possibilities for the induced subgraph on $K \cup K'$.This is done in the following claim.

**Claim 3.1** *Let $k$ and $r = 0.01k$ be as above, and let $K, K'$ be two sets of labelled vertices, where $|K| = |K'| = k$ and $|K \cap K'| = i = k - j$ with $1 \leq j \leq 0.1k$. Then the number of graphs $H$ on $K \cup K'$, such that the induced subgraph of $H$ on $K$ and the induced subgraph of $H$ on $K'$ are members of $\mathcal{F}_k$ is at most*

$$f_k (r + 2^r)^j 2^{0.9kj/10}$$

**Proof of Claim:** There are $f_k$ ways to choose the induced subgraph of $H$ on $K$. Fixing such a choice, we estimate the number of ways to extend it to the edges inside $K'$ (which are not inside $K$, as this part is already fixed). Let $A$ and $B$ denote the vertex classes of the member $F'$ of $\mathcal{F}_k$ in $K'$, thus $A \cup B = K'$. Let $A = A_1 \cup A_2 \ldots \cup A_r$ denote the partition of $A$ into disjoint sets of size 10 in this member. Since $|K \cap K'| \geq 0.9k$ and the degree of each $A$-vertex in $F'$ is at least $k/3$ whereas the degree of each $B$-vertex is at most $|A| = 0.1k$ it follows that any vertex $a \in A$ must have at least $k/3 - 0.1k > 0.1k$ neighbors in $K \cap K'$, and thus knowing the edges inside $K \cap K'$ reveals the fact that this is an $A$-vertex. We thus know, for each vertex in $K \cap K'$, if it is an $A$-vertex or a $B$-vertex. Moreover, since the sets of $B$-neighbors of any two $A$ vertices from distinct subsets $A_i$ differ on at least $k/3$ vertices $b \in B$, the edges inside $K \cap K'$ reveal, for each $i$ so that $A_i$ intersects $K \cap K'$, all the vertices of $A_i \cap (K \cap K')$. There are now at most $(r + 2^r)^j$ ways to choose, for each vertex in $K' - K$, if it lies in one of the sets $A_i$ (which is either represented in $K \cap K'$ or not), and if so, decide to which of the $r$ sets it belongs, and in addition, if it is a $B$-vertex, to decide to which sets $A_i$ it is connected.

Here we are over-counting, as we ignore the fact that any set $A_i$ has to be of size exactly 10, but this estimate suffices. Note that after the above choices, the identity of all vertices in each set $A_i$ is known. As each set $A_i$ is of cardinality 10, there are at most $j/10$ sets $A_i$ which are completely contained in $K' - K$. For each such set, there are at most $2^{0.9k}$ possibilities to choose the edges between the vertices of this set and the remaining vertices of $K'$. Once these choices are made, all edges inside $K'$ are determined. This completes the proof of the claim $\square$

Returning to the proof of the theorem, we proceed with the estimate of $h_i/h(k)^2$ in Case 2. By the claim, for $i = k - j, j \leq 0.1k$ we have (since $h(k) \geq 2^k > 1$):

$$\frac{h_i}{h(k)^2} \leq \frac{h_i}{h(k)} \leq \binom{k}{j}\binom{n-k}{j}(r+2^r)^j 2^{0.9kj/10} 2^{-(k-j)j}$$

$$\leq [kn2^{2r}2^{0.9k/10}2^{-(k-j)}]^j \leq n^{-0.5j},$$

with room to spare.

Combining the last inequality with (3) and (4), and using the fact that $E(X) = h(k) \geq 2^k$, we conclude that $Var(X) = o(E(X)^2)$ and hence, by Chebyshev's Inequality, $X > 0$ whp. This implies that $bp(G) \leq n - 2.015 \log_2 n$ whp, completing the proof of Theorem 1.1. $\square$

## 4 Concluding remarks and open problems

- The estimate in Theorem 1.1 is the best we can hope to get with this method, up to the constant $c$. This is because all members of $\mathcal{F}_k$ are bipartite graphs, and the random graph $G = G(n, 0.5)$ cannot contain any induced bipartite graph on more than $2\alpha(G)$ vertices.
- We have shown that for $G = G(n, 0.5)$, $bp(G) \leq n - \alpha(G) - \Omega(\log n)$ whp. It will be interesting to decide whether or not $bp(G) = n - O(\alpha(G))$ whp.
- For $p < 0.5$ and $G = G(n,p)$ it seems that both proofs we know do not give any improvement of the trivial estimate $bp(G) \leq n - \alpha(G)$. Is it true that for any fixed positive $p < 0.5$, $bp(G) = n - \alpha(G)$ whp ? (for $p > 1/2$ it is easy to get a better upper bound).

We conclude this short paper with a note regarding biclique decompositions of twin-free graphs. Vertices $u$ and $v$ in a graph $G$ are twins if they have exactly the same neighborhoods, and $G$ is twin-free if $G$ contains no such pair of vertices. Note that if a pair of vertices $u, v$ are twins in $G$ then $bp(G) = bp(G - u)$. Thus it is quite natural to consider the maximum number of vertices in a twin-free graph $G$ with $bp(G) = r$.

**Theorem 4.1** *Suppose $G$ is a twin-free graph whose edges can be decomposed into $r$ bicliques, then $|V(G)| \leq 2^{r+1} - 1$ and this bound is tight.*

**Proof:** We first construct a graph $G$ which attains this upper bound. Let $V(G)$ be a collection of vectors $v$ in $\{0,1,2\}^r$, such that $v_i = 1$ for at most one index $i$, and $v_j = 2$ for all $j > i$, $v_j \in \{0,2\}$ for all $j < i$. In other words,

$$V(G) = \cup_{k=0}^{r} \{0,2\}^k \times \{1\} \times \{2\}^{r-k-1}.$$

The number of vertices in $G$ is equal to $1 + 2 + \cdots + 2^r = 2^{r+1} - 1$. We define two vertices $u$ and $v$ to be adjacent if there exists $i$ such that $(u_i, v_i) = (1,0)$ or $(0,1)$. To show that $G$ is twin-free, suppose $u$ and $v$ are two distinct vertices of $G$. If $u_i = v_i = 1$ for some $i$, then one can find $j < i$ so that $(u_j, v_j) = (2,0)$ or $(0,2)$, then the vector $w$ with $w_j = 1$ and $w_k = 2$ for all $k \neq j$ is only adjacent to one of $u$ and $v$. If $u_i = 1$ for some $i$ and $v_i \neq 1$, then the vector $w$ with $w_i = 0$ and $w_j = 2$ for all $j \neq i$ is adjacent to $u$ but not $v$. Finally if both $u$ and $v$ are in $\{0,2\}^r$, take the coordinate $i$ such that $(u_i, v_i) = (0,2)$ or $(2,0)$, then again letting $w_i = 1$ and $w_j = 2$ for all $j \neq i$ shows that they have different neighborhoods.

The definition of $G$ naturally induces an edge decomposition into bicliques: two vertices $u$ and $v$ are adjacent in the biclique $G_i$ iff $(u_i, v_i) = (0,1)$ or $(1,0)$. To verify that this is indeed a partition, assume that the edge $uv$ belongs to two bicliques $G_i$ and $G_j$. This can only happen when $(u_i, u_j) = (0,0)$, $(v_i, v_j) = (1,1)$ or $(u_i, u_j) = (0,1)$, $(v_i, v_j) = (1,0)$ (when necessary we swap $u$ and $v$). Note that both cases are impossible since all the vectors in $V(G)$ have at most one coordinate equal to 1, and 0 never appears after 1.

Next we are going to show that $2^{r+1} - 1$ is an upper bound. For a twin-free graph $G$ with biclique partition $E(G) = \cup_{i=1}^{r} E(G_i) = \cup_{i=1}^{r} E(A_i, B_i)$, we assign a $r$-dimensional vector $v_u$ to every vertex $u$, such that $(v_u)_i = 1$ if $u \in A_i$, 0 if $u \in B_i$ and 2 otherwise. Note that two vertices associated with the same vector have common neighborhoods, so we may assume that all the vectors $v_u$ are distinct. Let $\mathcal{F} = \{v_u\}_{u \in G}$, and $\mathcal{F}_I = \{v : v \in \mathcal{F}, \{i : v_i \in \{0,1\}\} = I\}$. We claim that for all $|I| \geq 1$, $|\mathcal{F}_I| \leq 2$. This is obvious for $|I| = 1$. The case $|I| \geq 2$ follows from the observation that among any three distinct vectors in $\{0,1\}^I$, there always exists a pair differing in at least two coordinates $i$ and $j$, which contradicts the assumption that $G_i$ and $G_j$ are disjoint. Therefore

$$|\mathcal{F}| \leq 1 + \sum_{i=1}^{r} 2\binom{r}{i} = 2^{r+1} - 1.$$

$\square$

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