Algorithmic Aspects of Acyclic Edge Colorings

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Abstract

A proper coloring of the edges of a graph G is called *acyclic* if there is no 2-colored cycle in G. The *acyclic edge chromatic number* of G, denoted by $a'(G)$, is the least number of colors in an acyclic edge coloring of G. For certain graphs G, $a'(G) \geq \Delta(G) + 2$ where $\Delta(G)$ is the maximum degree in G. It is known that $a'(G) \leq \Delta + 2$ for almost all Δ-regular graphs, including all Δ-regular graphs whose girth is at least $c\Delta \log \Delta$. We prove that determining the acyclic edge chromatic number of an arbitrary graph is an NP-complete problem. For graphs G with sufficiently large girth in terms of $\Delta(G)$, we present deterministic polynomial time algorithms that color the edges of G acyclically using at most $\Delta(G) + 2$ colors.

1 Introduction

All graphs considered here are finite, undirected and simple. A coloring of the edges of a graph is proper if no pair of incident edges are colored with the same color. A proper coloring of the edges of a graph G is called *acyclic* if there is no 2-colored cycle in G. The *acyclic edge chromatic number* of G, denoted by $a'(G)$, is the least number of colors in an acyclic edge coloring of G. The maximum degree in G is denoted by $\Delta(G)$.

It is known that $a'(G) \leq 16\Delta(G)$ for any graph G, and that an acyclic edge coloring of G using at most $20\Delta(G)$ can be found efficiently (see [12],[3]). For certain graphs G, $a'(G) \geq \Delta(G) + 2$. It is conjectured that $a'(G) \leq \Delta(G) + 2$ for all graphs [4]. This conjecture was proven true for almost all Δ-regular graphs, and all Δ-regular graphs G whose girth (length of shortest cycle) is at least $c\Delta(G) \log \Delta(G)$ for some constant c.

It is easy to see that $a'(G) \leq 2$ iff G is a union of vertex disjoint paths. However,

Theorem 1 *It is NP-complete to determine if $a'(G) \leq 3$ for an arbitrary graph G.*

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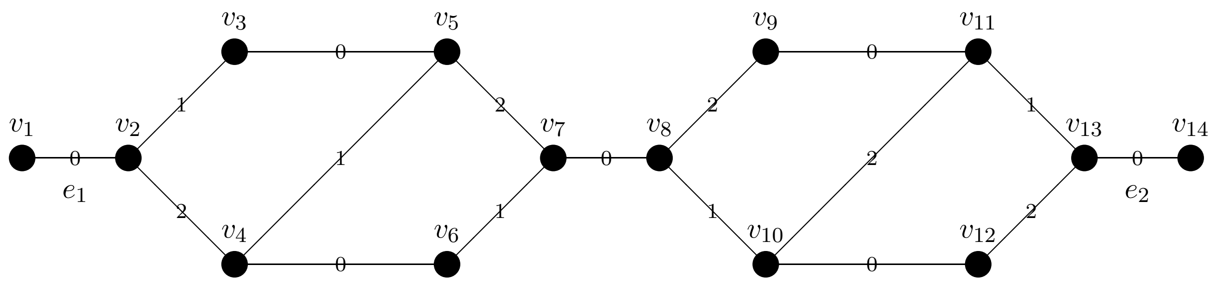
Figure 1: Graph F with an acyclic 3 coloring f



For certain graphs G which are known to have a $\Delta(G) + 2$ acyclic coloring, such a coloring can be constructed efficiently. Let $g(G)$ denote the girth of graph G.

Theorem 2 *The edges of a graph G of maximum degree d can be colored acyclically in polynomial time using $d + 2$ colors, provided that $g(G) > cd^3$ where c is an appropriate absolute constant.*

In the next sections we prove theorem 1 and theorem 2.

2 Proof of Theorem 1

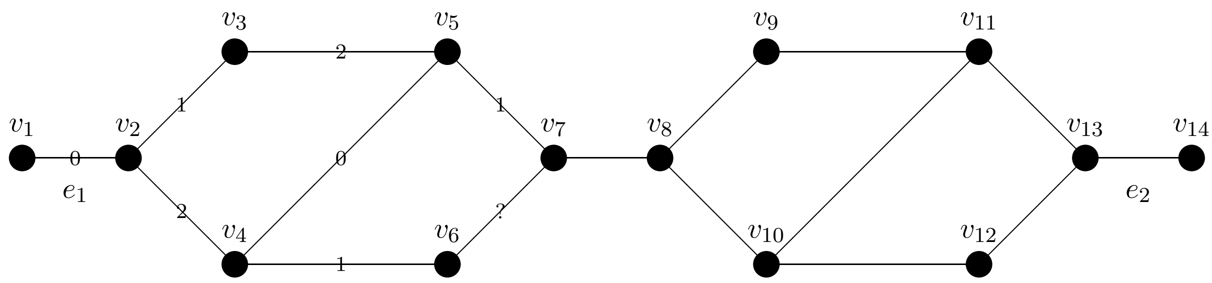
The following lemma states two useful properties of the graph F shown in figure 1.

Lemma 3 *Let F be the graph presented in figure 1, then*

1. *The edges of F can be colored acyclically using 3 colors, with no bichromatic path connecting v_1 and v_{14}.*
2. *Any acyclic coloring of the edges of F using 3 colors, colors e_1 and e_2 with the same color.*

Proof of Lemma 3. A coloring f proving the first property appears in figure 1, where the 3 colors are represented by digits $0, 1, 2$ displayed on the edges. To prove the second property, suppose $h : E(F) \to \{0, 1, 2\}$ is an acyclic coloring, having w.l.o.g $h(v_1, v_2) = 0, h(v_2, v_3) = 1$, and $h(v_2, v_4) = 2$ (similar to f in figure 1). Now we claim that $h(v_3, v_5) = 0$. Indeed, if $h(v_3, v_5) \neq 0$, then $h(v_3, v_5) = 2$, $h(v_4, v_5) = 0$ (to avoid a bichromatic cycle on v_2, v_3, v_4, v_5), $h(v_5, v_7) = 1$ and $h(v_4, v_6) = 1$, leaving no possible color for edge (v_6, v_7) (see figure 2). Therefore, $h(v_3, v_5) = 0$, which implies that $h = f$ for the following edges: $(v_4, v_5), (v_4, v_6), (v_5, v_7), (v_6, v_7)$, and in particular $h(v_7, v_8) = 0 = h(v_1, v_2)$. Using a similar argument we conclude that $h(v_{13}, v_{14}) = h(v_7, v_8) = h(v_1, v_2)$, as desired. $\square$

Figure 2: Partial 3 acyclic coloring of graph F



Proof of Theorem 1. The proof is by transformation from the chromatic index problem [7]. The chromatic index χ' of a graph G is the least number of colors in a proper edge coloring of G. Let H be a cubic (3-regular) graph. By Vizing [13], the chromatic index of H is either 3 or 4. Holyer [10] proved that it is NP-complete to determine if $\chi'(H) = 3$ or $\chi'(H) = 4$.

The transformation from edge coloring is as follows. Construct a graph G by replacing each edge $e_H = (u, w)$ of a cubic graph H with a copy of graph F, identifying u with v_1 and w with v_{14}. The size of G is clearly polynomial in the size of H, and $\Delta(G) = 3$. Therefore, $a'(G) \geq 3$.

Now we claim that $a'(G) \leq 3$ iff $\chi'(H) \leq 3$. Suppose $a'(G) \leq 3$, and let $c_G : E(G) \to \{1, 2, 3\}$ be an acyclic coloring of G. Then the edges of H can be colored properly using 3 colors, by collapsing each copy of F back to its original e_H edge, coloring it with $c_G(e_1) = c_G(e_2)$. Now suppose $\chi'(H) \leq 3$, and let $c_H : E(H) \to \{1, 2, 3\}$ be a proper coloring of H. Then c_H can be extended to an acyclic 3 coloring of G by coloring each copy of F using f, such that e_1 and e_2 are colored with $c_H(e_H)$. This completes the proof. □

Denote by $\mathcal{G}$ the family of graphs that can be constructed from cubic graphs using the construction in the proof above. Since $\Delta(G) = 3$ for $G \in \mathcal{G}$, it is easy to produce an acyclic coloring of any $G \in \mathcal{G}$ with 5 colors in polynomial time (see [4]). Moreover, it is easy to color any graph $G \in \mathcal{G}$ acyclically with 4 colors in polynomial time, by coloring the underlying cubic graph H with 4 colors (using Vizing, cf.[6],[11]) and coloring each copy of F using f. Therefore, the above proof shows that it is NP-complete to determine if $a'(G) = 3$ or $a'(G) = 4$ for $G \in \mathcal{G}$. Note also that any coloring of $G \in \mathcal{G}$ which colors each F using f, will not contain any bichromatic path of length 19.

It may be interesting to try and extend theorem 1 and prove (or disprove) that it is NP-complete to determine $a'(G)$ for k-regular graphs where $k > 3$, perhaps using the general hardness result concerning the chromatic index [8].

3 Proof of theorem 2

In this section we show how to color the edges of a graph G acyclically in polynomial time, provided the girth of G is large enough. Let g denote the girth of G (the length of a shortest cycle), and let d denote the maximum degree in G.

Proof of Theorem 2. First, color the edges of G properly using $d+1$ colors. The proof of Vizing's theorem supplies a polynomial-time algorithm for constructing such a coloring (see for example [6],[11]). If every cycle is colored with at least 3 colors we are done, so assume from now that there exist $b > 0$ bichromatic cycles $C_1, \ldots, C_b$. Each cycle contains at least g edges, and each edge belongs to at most d bichromatic cycles. Therefore by Hall's theorem there exist b disjoint sets $E_1, \ldots, E_b$ of g/d edges each, such that $E_i \subset C_i$ for every $1 \leq i \leq b$. It is possible to construct sets $E_1, \ldots, E_b$ in polynomial time using a max flow algorithm.

We now restrict our attention to the subgraph H of G containing the bg/d chosen edges $E(H) = \cup_{i=1}^{b} E_i$, and construct a graph $\bar{H}$ whose vertices correspond to the edges of H, where two vertices are connected if the corresponding edges of H are incident or at distance 1 from each other. Clearly, the maximum degree in $\bar{H}$ is less than $2d^2$.

Applying the Lovaśz local lemma [2, Proposition 5.3], we know that there exists an independent set $S \subseteq V(\bar{H})$ of graph $\bar{H}$ that contains one vertex from each E_i $(0 \leq i \leq b)$, provided that[1] $g > 2de(2d^2)$. Such a set S contains one edge from every bichromatic cycle, and no pair of edges in S are incident or at distance 1 in G. This will enable us to produce an acyclic coloring of G using $d+2$ colors, as desired, by recoloring all the edges in S using a new color. What remains to show is how to construct S efficiently.

The independent set S can be constructed in polynomial time using a coloring algorithm presented by Beck [5], provided that $g \geq cd^3$ for some fixed constant c ($c \approx 10^8$ suffices). If $g \geq d2^{2d^2}$, a simpler coloring algorithm presented by Alon [1, Proposition 2.2] can be used to produce the set S. □

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