T-choosability in graphs

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Abstract

Given a set of nonnegative integers T, and a function S which assigns a set of integers $S(v)$ to each vertex v of a graph G, an *S-list T-coloring* c of G is a vertex-coloring (with positive integers) of G such that $c(v) \in S(v)$ whenever $v \in V(G)$ and $|c(u) - c(w)| \notin T$ whenever $(u, w) \in E(G)$. For a fixed T, the *T-choice number* T-ch(G) of a graph G is the smallest number k such that G has an S-list T-coloring for every collection of sets $S(v)$ of size k each. Exact values and bounds on the $T_{r,s}$-choice numbers where $T_{r,s} = \{0, s, 2s, \dots, rs\}$ are presented for even cycles, notably that $T_{r,s}$-ch$(C_{2n}) = 2r + 2$ if $n \geq r + 1$. More bounds are obtained by applying algebraic and probabilistic techniques, such as that T-ch$(C_{2n}) \leq 2|T|$ if $0 \in T$, and $c_1 r \log n \leq T_{r,s}$-ch$(K_{n,n}) \leq c_2 r \log n$ for some absolute positive constants c_1, c_2.

1 Introduction

Restricted vertex-colorings of graphs arose in connection with the *frequency assignment* problem. Hale [4] formulated several frequency assignment problems in graph-theoretic terms. Suppose that n transmitters are stationed at various locations, and we wish to assign to every transmitter a frequency over

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which it will operate. If nearby transmitters must be assigned different frequencies so as not to interfere, we can formulate this problem as a vertex-coloring problem of a graph as follows. Every transmitter is represented by a vertex, and frequencies are referred to as colors. Any pair of vertices representing close transmitters are connected by an edge.

An additional restriction stems from considering the interference that occurs in practice between adjacent transmitters: not only will they interfere when operating on exactly the same frequency, but also when the difference between their frequencies equals certain values. T-colorings of graphs deal with such a restriction, and have been surveyed by Roberts [7]. Given a set of nonnegative integers T, a *T-coloring* of a graph G is a vertex-coloring (with positive integers) of G such that the difference between any two colors assigned to adjacent vertices does not belong to T.

Another restriction is imposed when we consider the specifications of the transmitters: each transmitter may have a limited pool of frequencies on which it can operate. List-colorings of graphs deal with this restriction, and were originated independently by Vizing [10] and by Erdős, Rubin, Taylor [3]. Given a set of allowed colors $S(v)$ for each vertex v of a graph G, a *list-coloring* of G is a proper vertex-coloring of G such that the color assigned to each vertex v belongs to $S(v)$. Given a function $f : V \to N$, a graph G is *f-choosable* if a list-coloring of G exists for every collection of sets $S(v)$ such that $|S(v)| = f(v)$ for every vertex v of G. A graph G is *k-choosable* if it is f-choosable for the constant function $f(v) = k$. The *choice-number* $\mathrm{ch}(G)$ of G is the minimal number k such that G is k-choosable. See Alon [1] for a survey on list-colorings and choosability.

Combining both restrictions, *list-T-colorings* of graphs were formulated and studied extensively by Tesman [8,9]. It is worth noting that list-T-coloring is an instance of the *Constraint Satisfaction* problem; see Mackworth [5,6]. Given a set T and a function f as above, a graph G is *T-f-choosable* if a list-T-coloring of G exists for every collection of sets $S(v)$ such that $|S(v)| = f(v)$ for every vertex v of G. A graph G is *T-k-choosable* if it is T-f-choosable for the constant function $f(v) = k$. The *T-choice-number* $T\text{-ch}(G)$ of G is the minimal number k such that G is T-k-choosable.

Attention has been given mainly to the T-choice-numbers of special classes of graphs and sets T of specific form. Let $T_{r,s}$ denote the set $\{0, s, 2s, \ldots, rs\}$ where r is a nonnegative integer and s is a positive integer, and let $T_r = T_{r,1}$. From now on we always assume that $0 \in T$, unless otherwise specified. The main results concerning $T\text{-ch}(G)$ are [8,9]:

(1) $T\text{-ch}(G) \geq \mathrm{ch}(G)$ $(\geq \chi(G))$ for every set T and graph G.
(2) $T\text{-ch}(G) \leq |T| \cdot \Delta(G) + 1$ for every set T and graph G of maximal degree

$\Delta(G)$. This can be realized by coloring the vertices one by one, choosing each time a vertex having the minimal (or maximal) available color.

(3) T-ch$(G) \leq (2|T|-1)(\chi(G)-1)+1$ for every set T and chordal graph G. This can be obtained by coloring the vertices one by one according to a perfect elimination ordering, using any available color each time.

(4) $T_{r,s}$-ch$(K_n) = (r+1)(n-1)+1$ for every $r \geq 0, s > 0, n > 0$.

(5) T_r-ch$(C_{2n+1}) = 2r+3$ for every $r \geq 0, n > 0$.

(6) T_r-ch$(B_n) = \lfloor (2r+2)(n-1)/n \rfloor + 1$ for every tree B_n on $n \geq 1$ vertices and $r \geq 0$.

In this paper we continue this study, and present more general results as well. In section 2 we observe that T_r-ch$(G) = T_{r,s}$-ch(G), and present a lower bound for $T_{r,s}$ that is tight for odd cycles and cliques. In section 3 we present exact values and bounds on the T_r-choice-numbers of cycles. Additional bounds of T-ch(G) that are connected with ch(G) are presented in section 4, by using algebraic techniques. Further results concerning T-ch(G) using probabilistic methods are presented in section 5.

2 $T_{r,s}$-choice numbers

We begin this section by proving the following simple equality:

Theorem 1 T_r-ch$(G) = T_{r,s}$-ch(G) *for every* $r \geq 0$ *and* $s > 0$.

PROOF. Fix G, r and s.

Part 1: T_r-ch$(G) \leq T_{r,s}$-ch(G).

Let $k = T_r$-ch$(G)-1$. By the definition of T_r-ch(G), there exists an assignment S of a set S_v of k colors to each vertex v of G, that does not admit an S-list-T_r-coloring. In every vertex-coloring c of G such that $c(v) \in S_v$ for every vertex v, there exists an edge (u, w) of G such that $|c(u) - c(w)| \in T_r$. Let S'_v be the set obtained from S_v by replacing every color i by the color is. Clearly, $|S'_v| = |S_v| = k$. In any vertex-coloring c' of G such that $c'(v) \in S'_v$ for every vertex v, there exists an edge (u, w) of G such that $|c'(u) - c'(w)| = s \cdot |c(u) - c(w)| \in T_{r,s}$. This shows that $T_{r,s}$-ch$(G) > k$.

Part 2: T_r-ch$(G) \geq T_{r,s}$-ch(G).

Let $k = T_{r,s}$-ch$(G)-1$, and let S be an assignment of a set S_v of k colors to each vertex v of G, that does not admit an S-list-$T_{r,s}$-coloring. Denote by M the maximal color of S plus one: $M = \max\{c : c \in S_v, v \in V\}+1$. Let S'_v be the set obtained from S_v by replacing every color i by the color $i' = \lfloor \frac{i}{s} \rfloor + M(i \bmod s)$.

It is clear that $|S'_v| = |S_v| = k$. Note that if $i - j = as$ for some integer a, then $i' - j' = a$. In any vertex-coloring c' of G such that $c'(v) \in S'_v$, there exists an edge (u, w) of G such that $|c'(u) - c'(w)| = |c(u) - c(w)|/s \in T_r$. This shows that T_r-ch$(G) > k$.

This result can be generalized to the following color sets. Let $T^a_{r,s} = \{as, (a+1)s, \ldots, (a+r)s\}$, where a and r are nonnegative integers and s is a positive integer.

Proposition 2 $T^a_{r,1}$-ch$(G) = T^a_{r,s}$-ch(G) *for every graph* G *and* $r \geq 0$.

We end this section by presenting a simple bound on T_r-ch(G), which is based upon ch(G).

Theorem 3 T_r-ch$(G) > (r+1)(\text{ch}(G) - 1)$ *for every graph* G *and* $r \geq 0$.

PROOF. Let $k = \text{ch}(G) - 1$. By the definition of ch(G), there exists an assignment S of a set S_v of k colors to each vertex v of G, that does not admit an S-list coloring. In every vertex-coloring c of G such that $c(v) \in S_v$ for every vertex v, there exists an edge (u, w) of G such that $c(u) = c(w)$. Let S'_v be the set obtained from S_v by replacing every color i by the set A_i of $r+1$ colors: $A_i = \{i(r+1), i(r+1)+1, \ldots, i(r+1)+r\}$. Notice that different colors $i \neq j$ are replaced by disjoint sets $A_i \cap A_j = \emptyset$, hence for each vertex v, $|S'_v| = (r+1)k$. In any vertex-coloring c' of G such that $c'(v) \in S'_v$ for every vertex v, there exists an edge (u, w) of G such that $c'(u)$ and $c'(w)$ both belong to the same set A_i, implying that $|c'(u) - c'(w)| \in T_r$. This shows that T_r-ch$(G) > (r+1)k$.

3 The T_r-choice numbers of cycles

Tesman [8,9] proved that the T_r-choice number of any odd cycle equals $2r+3$. As mentioned by Roberts [7], the T_r-choice numbers of even cycles have not yet been found. We prove lower and upper bounds for these numbers T_r-ch(C_{2n}) which are tight for long even cycles ($n \geq r+1$), the lower bound being tight for C_4 as well.

Throughout this section we let $S_i \subset Z$ be the set of colors assigned to vertex i, and we denote by m_i and M_i the minimal and maximal colors of S_i. The set $\{i, \ldots, j\}$ is denoted by $[i, j]$ (if $i > j$ then $[i, j] = \emptyset$).

3.1 The T_r-choice numbers of C_{2n}

We begin with a simple constructive proof that $T_r\text{-ch}(C_{2n}) \leq 2r + 2$. Denote the vertices of C_{2n} by $1, 2, \ldots, 2n$ and suppose $|S_i| \geq 2r + 2$ for each i, so that $M_i - m_i > 2r$. We need the following lemma.

Lemma 4 *Either*

$$M_i - m_j > r \text{ and } M_j - m_i > r \tag{1}$$

or else

$$|m_i - m_j|, |M_i - M_j| > r \text{ and either } M_i - m_j > r \text{ or } M_j - m_i > r\,. \tag{2}$$

PROOF. If $M_i - m_j \not> r$ then $m_i + 2r < M_i \leq m_j + r < M_j - r$ and so $M_j - m_i > m_j - m_i > r$ and $M_j - M_i > r$. If $M_j - m_i \not> r$ then the situation is symmetric.

An edge (i, j) is said to be of *type* X if (1) holds and of *type* Z if (2) holds and (1) doesn't. We now state formally the result we are about to prove.

Theorem 5 $T_r\text{-ch}(C_{2n}) \leq 2r + 2$, *and if* $|S_i| \geq 2r + 2$ *for every vertex* i *then there exists a proper vertex coloring of* C_{2n} *using the minimal or maximal color of each set.*

PROOF. Let $G = (V, E)$ be a simple graph having $4n$ colored vertices, obtained from C_{2n} by splitting each vertex i into two vertices, coloring one m_i and the other M_i. Abusing the notation, we shall refer to the vertices of G by m_i and M_i for simplicity. The edges of G are defined as follows: for each edge (i, j) of C_{2n}, choose either $(M_i, m_j), (m_i, M_j)$ or $(m_i, m_j), (M_i, M_j)$ to be edges in E, according to whether the edge (i, j) is of type X or type Z, respectively. Our goal is to show that G contains a $2n$-cycle — such a cycle represents a proper vertex coloring of C_{2n}. It is evident that G is either a $4n$-cycle or the edge sum of two vertex-disjoint $2n$-cycles. In the former case, since C_{2n} is an even cycle, C_{2n} contains an edge (i, j) of type Z. Adding the edge (M_i, m_j) or (m_i, M_j) to E (according to whether $M_i - m_j > r$ or $M_j - m_i > r$) will create a $2n$-cycle in G, completing the proof.

A lower bound for $T_r\text{-ch}(C_{2n})$ can be obtained by considering a spanning tree B_{2n} of C_{2n} and using a result proved by Tesman [8]:

Theorem 6 (Tesman [8, Thm 4.2]) $T_r\text{-ch}(B_n) = \lfloor (2r + 2)(n - 1)/n \rfloor + 1$.

Therefore T_r-ch$(B_{2n}) = 2r + 2$ if and only if $n \geq r + 1$. Combining this with Theorem 5 proves the following.

Theorem 7 *If $n \geq r + 1$ then T_r-ch$(C_{2n}) = T_r$-ch$(B_{2n}) = 2r + 2$.*

Next, we wish to compute the T_r-choice-number of C_4. ¿From Theorem 5 it is clear that T_r-ch$(C_4) \leq 2r + 2$. A lower bound can be obtained by considering a spanning tree B_4 of C_4 and applying Theorem 6: T_r-ch$(C_4) \geq T_r$-ch$(B_4) = \lfloor \frac{3}{2}(r+1) \rfloor + 1$. It turns out that the precise value of T_r-ch(C_4) is $\lfloor \frac{12}{7}(r+1) \rfloor + 1$.

Theorem 8 T_r-ch$(C_4) = \lfloor \frac{12}{7}(r + 1) \rfloor + 1$ *for every $r \geq 0$.*

Proof (sketch) Our proof that $\lfloor \frac{12}{7}(r + 1) \rfloor + 1$ is an upper bound is a rather lengthy case-by-case analysis, and is therefore omitted. Following is a proof showing that this value is a lower bound.

Denote the vertices of C_4 by $(1, 2, 3, 4)$. Set $s = \lfloor \frac{12}{7}(r + 1) \rfloor$. Following is an example of four color sets S_1, S_2, S_3, S_4 of size s each which do not admit a proper coloring of C_4. First, if $r = 0$ then $s = 1$ and $S_1 = S_2 = S_3 = S_4 = \{1\}$ will do. Assuming $r > 0$ we define the following color sets:

$$S_1 = [1, s]$$

$$S_2 = [s - r, 2s - r - 1]$$

$$S_3 = [2s - 2r - 1, 3s - 2r - 2]$$

$$S_4 = [3s - 5r - 4, r + 1] \cup [3s - 3r - 2, 3r + 3] \,.$$

It is evident that $|S_1| = |S_2| = |S_3| = s$. Since $3r + 3 \geq 3s - 3r - 2 > r + 1 \geq 3s - 5r - 4$ (for $r > 0$) we have $|S_4| = 12(r + 1) - 6s$ which is at least s (if it is more than s, choose any subset of S_4 of size s instead of S_4).

To see why C_4 cannot be colored using $S_1, \ldots, S_4$, suppose that it can, with each vertex i being given a color $s_i \in S_i$. Then, $s_1 < s_2 < s_3$ and therefore $s_3 \geq s_2 + r + 1 \geq s_1 + 2r + 2$. Since $m_i \leq s_i \leq M_i$ for each i, it follows that

$$(r + 1) - r = 1 \leq s_1 \leq (3s - 2r - 2) - (2r + 2) = (3s - 5r - 4) + r$$

and

$$(3r + 3) - r = 1 + (2r + 2) \leq s_3 \leq 3s - 2r - 2 = (3s - 3r - 2) + r \,.$$

The former implies $s_4 \notin [3s - 5r - 4, r + 1]$ and the latter implies $s_4 \notin [3s - 3r - 2, 3r + 3]$, a contradiction.

We can extend the lower bound of Theorem 8 to any cycle C_n.

Theorem 9 *T_r-ch$(C_n) \geq \lfloor \frac{2n-2}{2n-1} \cdot 2(r+1) \rfloor + 1$ for any $r \geq 0$ and $n \geq 4$.*

PROOF. Denote the vertices of C_n by (1, ... ,n). Set $s = \lfloor \frac{2n-2}{2n-1} \cdot 2(r+1) \rfloor$. Following is an example of n color sets $S_1, \ldots, S_n$ of size s each which do not admit a proper coloring of C_n. First, if $r = 0$ then $s = 1$ and $S_i = \{1\}$ for every vertex i will do. Assume $r > 0$. For vertices $1 \leq i \leq n-1$ define

$$\begin{aligned} m_i &= 1 + (i-1)(s-r-1) \\ M_i &= s + (i-1)(s-r-1) \\ S_i &= [m_i, M_i] \end{aligned}$$

and for vertex n define

$$\begin{aligned} m_n &= s + (n-2)(s-2r-2) - r \\ M'_n &= 1 + r \\ m'_n &= s + (n-2)(s-r-1) - r \\ M_n &= 1 + (n-2)(r+1) + r \\ S_n &= [m_n, M'_n] \cup [m'_n, M_n]\,. \end{aligned}$$

It is evident that $|S_1| = \cdots = |S_{n-1}| = s$. Since $M_n \geq m'_n > M'_n \geq m_n$ (for $r > 0$ and $n \geq 4$) we have $|S_n| = 2(n-1)(2(r+1)-s)$ which is at least s (if it is more than s, choose any subset of S_n of size s instead of S_n).

To see why C_n cannot be colored using $S_1, \ldots, S_n$, suppose that it can, with each vertex i being given a color $s_i \in S_i$. Then, for $1 \leq i \leq n-2$, $s_{i+1} - s_i \geq r+1$, so that $s_{n-1} - s_1 \geq (n-2)(r+1)$. Since $m_i \leq s_i \leq M_i$ for each i, it follows that

$$M'_n - r = m_1 \leq s_1 \leq M_{n-1} - (n-2)(r+1) = m_n + r$$

and

$$M_n - r = m_1 + (n-2)(r+1) \leq s_{n-1} \leq M_{n-1} = m'_n + r\,.$$

The former implies $s_n \notin [m_n, M'_n]$ and the latter implies $s_n \notin [m'_n, M_n]$, a contradiction.

We have obtained a lower bound for even cycles that is tight for the shortest even cycle C_4. Fix r, and set $n = r+1$. Observe that this bound attains the value $\lfloor \frac{4r+2}{4r+3}(2r+2) \rfloor + 1 = \lfloor \frac{4r+4}{4r+3}(2r+1) \rfloor + 1 = 2r+2$ which by Theorem 7 is the exact value of T_r-ch(C_{2r+2}). We conjecture that this bound is tight for all even cycles.

Conjecture 10 $T_r\text{-ch}(C_{2n}) = \lfloor \frac{4n-2}{4n-1} \cdot 2(r+1) \rfloor + 1$ *for all* $r \geq 0$ *and* $n \geq 2$.

3.2 The T_r-choice number of C_{2n+1}

Tesman [9] proved that $T_r\text{-ch}(C_{2n+1}) = 2r+3$. Here we give a simple and constructive proof, restricting ourselves as much as possible to the extreme colors of each set.

Proposition 11 $T_r\text{-ch}(C_{2n+1}) \leq 2r+3$, *and there exists a proper vertex coloring of* C_{2n+1} *using the minimal or maximal color of each set, with at most one exception.*

PROOF. The proof uses a construction similar to the one described in the proof of Theorem 5. Indeed the only case that requires special attention is when all the edges of C_{2n+1} are of type X. In this case G is a $(4n+2)$-cycle, and an intermediate color is needed on one vertex. For each i, let s_i denote the $(r+1)$th element of S_i, so that $m_i + r < s_i < M_i - r$. Since $2n+1$ is odd, there exist three consecutive vertices i, j, k of C_{2n+1} such that $s_i \leq s_j \leq s_k$. Add a new vertex to G and color it s_j. Adding the edges (m_i, s_j) and (s_j, M_k) to G will create a cycle of length $2n+1$ in G, which clearly represents a proper coloring of C_{2n+1}.

Theorem 3 shows that this bound is tight.

4 Choosability and T-Choosability

In order to describe a result of Alon and Tarsi [2] concerning f-choosability of graphs, we need the following definitions. A subdigraph H of a digraph D is called *Eulerian* if the in-degree $d_H^-(v)$ of every vertex v of H in H is equal to its out-degree $d_H^+(v)$. H is *even* if it has an even number of edges, otherwise, it is *odd*. Let $\mathrm{EE}(D)$ and $\mathrm{EO}(D)$ denote the number of even and odd Eulerian subgraphs of a digraph D, respectively.

Theorem 12 (Alon and Tarsi [2, Thm 1.1]) *If the graph* G *has an orientation* D *such that* $\mathrm{EE}(D) \neq \mathrm{EO}(D)$ *then* G *is* f*-choosable, where* $f(v) = d_D^+(v) + 1$ *is the out-degree of the vertex* v *in* D *plus one.*

This result can be extended to T-choosability as follows. Let G^m denote the graph obtained by replacing each edge of G by m parallel edges.

Theorem 13 *If* G^{2t-1} *has an orientation* D *such that* $\mathrm{EE}(D) \neq \mathrm{EO}(D)$*, then* G *is* T*-*f*-choosable for every set* T *of size* t *containing zero, where* $f(v) = d_D^+(v) + 1$ *is the out-degree of vertex* v *in* D *plus one.*

PROOF. With every (multi-)graph G on $V = \{1, \ldots, n\}$ we associate the polynomial

$$f_G(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j)^{e_{ij}}$$

where $e_{ij} \in Z_{\geq 0}$ is the number of edges joining i and j. Notice that a vertex-coloring c of G is proper if and only if $f_G(c(1), \ldots, c(n)) \neq 0$. Expand f_G into a linear combination of monomials. Let D be any orientation of G, and $d_i = d_D^+(v_i)$. Alon and Tarsi [Corollary 2.3] showed that the coefficient of the monomial $\prod_{i=1}^n x_i^{d_i}$ in f_G is equal to $\pm(\mathrm{EE}(D) - \mathrm{EO}(D))$. They then showed that if this coefficient is nonzero, G is $(d_D^+ + 1)$-choosable. We will follow these two steps, relating to T-colorings.

Step 1:

Let $T = \{0, a_1, \ldots, a_{t-1}\}$. We associate with any (simple) graph G on $V = \{1, \ldots, n\}$ the polynomial

$$f_G^T(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j)^{e_{ij}} \prod_{k=1}^{t-1} [(x_i - x_j - a_k)(x_i - x_j + a_k)]^{e_{ij}}$$

where $e_{ij} \in \{0, 1\}$ is the number of edges joining i and j in G. Notice that a vertex-coloring c of G is a proper T-coloring if and only if $f_G^T(c(1), \ldots, c(n)) \neq 0$. Expand f_G^T into a linear combination of monomials.

A monomial $\prod_{i=1}^n x_i^{c_i}$ of f_G^T has *degree* $\sum_{i=1}^n c_i$. Let $\mathrm{coef}(m, f)$ denote the coefficient of monomial m in polynomial f. It is easy to see that if m is a monomial of f_G^T of maximum degree, then $\mathrm{coef}(m, f_G^T) = \mathrm{coef}(m, f_{G^{2t-1}})$, where

$$f_{G^{2t-1}}(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j)^{(2t-1)e_{ij}}.$$

Let D be an orientation of G^{2t-1} such that $\mathrm{EE}(D) \neq \mathrm{EO}(D)$. Denote $d_i = d_D^+(i)$ and $M = \prod_{i=1}^n x_i^{d_i}$. Since M is a monomial of maximum degee in f_G^T, $\mathrm{coef}(M, f_G^T) = \mathrm{coef}(M, f_{G^{2t-1}})$, but $\mathrm{coef}(M, f_{G^{2t-1}}) = \pm(\mathrm{EE}(D) - \mathrm{EO}(D))$ by Alon and Tarsi [2, Cor 2.3]. Hence $\mathrm{coef}(M, f_G^T) \neq 0$.

Step 2:

It remains to show that if $\mathrm{coef}(M, f_G^T)$ is nonzero, G is T-$(d_D^+ + 1)$-choosable. Assuming the contrary, there exists for each vertex i of G a set S_i of $d_i + 1$ colors, such that

$$f_G^T(x_1, \ldots, x_n) = 0 \text{ for every } n\text{-tuple } (x_1, \ldots, x_n) \in S_1 \times S_2 \times \ldots \times S_n\,.$$

For each i, $1 \le i \le n$, let $Q_i(x_i)$ be the polynomial

$$Q_i(x_i) = \prod_{s \in S_i} (x_i - s) = x_i^{d_i+1} - \sum_{j=0}^{d_i} q_{ij} x_i^j\,.$$

Observe that

$$\text{if } x_i \in S_i \text{ then } Q_i(x_i) = 0, \text{ i.e., } x_i^{d_i+1} = \sum_{j=0}^{d_i} q_{ij} x_i^j\,. \tag{3}$$

Let $\hat{f}_G^T$ be the polynomial obtained from f_G^T by replacing, repeatedly, each occurrence of $x_i^{f_i}$, $(1 \le i \le n)$, where $f_i > d_i$, by a linear combination of smaller powers of x_i, using the relations (3). $\hat{f}_G^T$ is of degree at most d_i in x_i for each i. Moreover, $\hat{f}_G^T(x_1, \ldots, x_n) = f_G^T(x_1, \ldots, x_n) = 0$ for all $(x_1, \ldots, x_n) \in S_1 \times S_2 \times \ldots \times S_n$ and hence, by Lemma 2.1 of [2], $\hat{f}_G^T \equiv 0$. Since the degree of each x_i in M is d_i, the relations (3) will not affect it. Moreover, since M is of maximum degree in f_G^T and each application of relations (3) strictly reduces degree, the process of replacing f_G^T by $\hat{f}_G^T$ will not create any new scalar multiples of M. Thus, $\mathrm{coef}(M, \hat{f}_G^T) = \mathrm{coef}(M, f_G^T) \neq 0$. This contradicts the fact that $\hat{f}_G^T \equiv 0$, and completes the proof.

Applying Theorem 13 to even cycles, let D be a cyclic orientation of $G^{2t-1} = (C_{2n})^{2t-1}$ (all parallel edges are directed the same). Then all Eulerian subgraphs are even and hence $\mathrm{EE}(D) \neq \mathrm{EO}(D)$. Since $d_D^+(v) = 2t - 1$ for every vertex v, we have obtained the following.

Corollary 14 T-$\mathrm{ch}(C_{2n}) \le 2|T|$ *whenever* $0 \in T$ *and* $n \ge 2$.

Hence, for $T = \{0, a_1, \ldots, a_{t-1}\}$, a proper T-coloring of C_{2n} exists using any collection of color sets S_v of size $2t$ each. But it may happen that no such coloring exists using only the minimal or maximal color of each set S_v, as demonstrated by taking $n = 2$; $t = 2$; $T = \{0, 3\}$; $S_v = \{1, 2, 3, 4\}$ for all v. Therefore Theorem 5 does not hold for general T, whereas Corollary 14 holds for general T but its proof is non-constructive.

Now we extend Theorem 13 to a wide class of graphs, and prove the following.

Theorem 15 *Let $p = 2t - 1$ be prime. If a graph G has an orientation D such that* $\mathrm{EE}(D) \not\equiv \mathrm{EO}(D) \pmod p$*, then G is T-f-choosable for every set T of size t containing zero, where $f(v) = (2t-1)d_D^+(v) + 1$.*

PROOF. Let D^p be the orientation of G^p obtained by directing every edge of G^p according to the direction of the corresponding edge in D. We will show that $\mathrm{EE}(D^p) \equiv \mathrm{EE}(D) \pmod p$ and $\mathrm{EO}(D^p) \equiv \mathrm{EO}(D) \pmod p$.

In order to calculate $\mathrm{EE}(D^p)$ and $\mathrm{EO}(D^p)$, we use the following observations. Denote by Φ^p the circulations of (0,1) flows in the network D^p, where every arc has capacity 1. For a flow F^p in Φ^p, let $|F^p|$ denote the total number of arcs of D^p whose flow in F^p equals 1. Clearly there is a one-to-one correspondence between Eulerian subdigraphs of D^p and circulations Φ^p. A flow F^p corresponds to an even or odd Eulerian subdigraph of D^p, according to whether $|F^p|$ is even or odd, respectively. Let Φ^p_E and Φ^p_O denote the flows F^p of Φ^p such that $|F^p|$ is even or odd, respectively.

Denote by Φ the circulations of integer flows in the network D, where every arc of D has capacity p. For a flow F in Φ, let $f_{ij} = f_{ij}(F)$ denote the flow of F along arc (i,j) of D. There is a natural mapping $R : \Phi^p \to \Phi$ as follows. For a given flow F^p in Φ^p, $R(F^p)$ is the flow obtained by setting the flow along every arc (i,j) of D to f_{ij}, where f_{ij} equals the number of arcs of flow 1 in F^p from i to j. Notice that for every flow F in Φ, $R^{-1}(F)$ contains $\prod_{(i,j)} \binom{p}{f_{ij}}$ different flows of Φ^p. Note also that if $F^p \in R^{-1}(F)$, then $|F^p| = \sum_{(i,j)} f_{ij}$. Let Φ_E and Φ_O denote the flows F in Φ such that $\sum_{(i,j)} f_{ij}$ is even or odd, respectively. Summing up, we have obtained the following equations:

$$\mathrm{EE}(D^p) = |\Phi^p_E| = \sum_{F \in \Phi_E} \prod_{(i,j)} \binom{p}{f_{ij}}$$

$$\mathrm{EO}(D^p) = |\Phi^p_O| = \sum_{F \in \Phi_O} \prod_{(i,j)} \binom{p}{f_{ij}}.$$

Let F be a flow of Φ, and let $P(F) = \prod_{(i,j)} \binom{p}{f_{ij}}$. F is called an *edge-saturating* flow if $P(F) = 1$. Denote by Φ_S the set of edge-saturating flows of Φ. There is a natural bijection between Φ_S and the Eulerian subdigraphs of D. A flow F of Φ^p which is not edge-saturating contains a flow f_{ij} such that $0 < f_{ij} < p$, and since p is prime this implies that $P(F) \equiv 0 \pmod p$. Therefore

$$\mathrm{EE}(D^p) \equiv \sum_{F \in \Phi_E \cap \Phi_S} P(F) = |\Phi_E \cap \Phi_S| = \mathrm{EE}(D) \pmod p$$

$$\mathrm{EO}(D^p) \equiv \sum_{F \in \Phi_O \cap \Phi_S} P(F) = |\Phi_O \cap \Phi_S| = \mathrm{EO}(D) \pmod{p}.$$

Since $\mathrm{EE}(D) \not\equiv \mathrm{EO}(D) \pmod{p}$ we conclude that $\mathrm{EE}(D^p) \neq \mathrm{EO}(D^p)$. This enables us to apply Theorem 13, which completes the proof.

If T does not contain zero, we can omit from f_G^T the corresponding term $(x_i - x_j)^{e_{ij}}$. Proceeding as in the proof of Theorem 13, we obtain the following.

Theorem 16 *If G^{2t} has an orientation D such that $\mathrm{EE}(D) \neq \mathrm{EO}(D)$, then G is T-f-choosable for every set T of size t not containing zero, where $f(v) = d_D^+(v) + 1$ is the out-degree of vertex v in D plus one.*

Note that there is a trivial T-coloring for T not containing zero, namely to color every vertex using the same color. But this is not necessarily true for list-T-colorings and T-choosability, since the color sets may be disjoint. Therefore it may be interesting, at least theoretically, to deal with T-choosability where T does not contain zero.

The proof of Theorem 13 does not rely upon the values of the elements of T. Hale [4] and Roberts [7, Section 9] studied cases having different separation sets T for different edges of a graph G. Here we formulate such a problem for which we can obtain results using the techniques above.

Definition 17 *Let $G = (V, E)$ be a graph. Given a function $f : V \to N$ and a function $g : E \to N$, G is g-f-*choosable *if for every collection of sets $\{S(v) : |S(v)| = f(v)\}_{v \in V}$ and every collection of sets $\{T(e) : |T(e)| = g(e), 0 \in T(e)\}_{e \in E}$ there exists a vertex-coloring c of G such that $c(v) \in S(v)$ for every vertex v, and $|c(u) - c(w)| \notin T(e)$ for every edge $e = (u, w)$.*

Denote by G^{2g-1} the multigraph obtained by replacing each edge e of G by $2g(e) - 1$ parallel edges. A slight modification of the proof to Theorem 13 yields the following.

Theorem 18 *Let $G = (V, E)$ be a graph, and $g : E \to N$. If G^{2g-1} has an orientation D such that $\mathrm{EE}(D) \neq \mathrm{EO}(D)$, then G is g-f-choosable where $f(v) = d_D^+(v) + 1$ is the out-degree of vertex v in D plus one.*

5 Probabilistic Methods

Bounds on the choice-number of graphs can be obtained by using probabilistic methods. For example, in [3] (see also [1]) it is shown that the choice-number of $K_{n,n}$ is $O(\log n)$ by using such a method. The next theorem illustrates how bounds on the T-choice-number of graphs can also be obtained using such methods.

Theorem 19 *T_r-ch$(K_{n,n}) = O(r\ \log n)$ for $r \geq 1$ and $n > 1$.*

PROOF. Let the partite sets of $K_{n,n}$ be A, B, where $|A| = |B| = n$. Suppose we are given a set $S(v)$ of k integers for each $v \in A \cup B$. Denote by $A_i = \{ir, ir+1, \ldots, ir+r-1\}$ for $i \in Z$. Let $Z = Z_A \cup Z_B$ be a random partition of the integers into two disjoint sets, obtained by assigning, for each i randomly and independently, all the elements of A_i either to Z_A or to Z_B with equal probability. For $z \in Z$, let $i(z) = \lfloor z/r \rfloor$ denote the index i for which $z \in A_i$. A color $s \in S(a)$ where $a \in A$ is called *good* if $s \in Z_A$ (i.e., $A_{i(s)} \subset Z_A$), $A_{i(s)-1} \subset Z_A$ and $A_{i(s)+1} \subset Z_A$. Otherwise s is called *bad*. A color $s \in S(b)$ where $b \in B$ is *good* if $A_{i(s)-1}, A_{i(s)}, A_{i(s)+1} \subset Z_B$, and otherwise it is *bad*. Notice that if every color set $S(v)$ contains a good color, a proper T_r-coloring of G exists using any such good colors. The probability that a fixed color s is bad, equals $\frac{7}{8}$. Let $I(v)$ denote the set of all indices i such that $S(v) \cap A_i \neq \emptyset$. Clearly, $|I(v)| \geq k/r$ for every vertex v. Let v be any vertex, and s_1, s_2 be two colors of $S(v)$. If $i(s_1) = i(s_2)$ then s_1 and s_2 are either both good or both bad. More specifically, the two events $\{s_1 \text{ is bad}\}$ and $\{s_2 \text{ is bad}\}$ are independent iff $|i(s_1) - i(s_2)| \geq 3$. In order to eliminate this dependency, let $I'(v)$ be a subset of $I(v)$ for every vertex v, such that $|I'(v)| \geq \frac{1}{3}|I(v)| \geq \frac{k}{3r}$ and $|i - j| \geq 3$ for every pair of indices $i, j \in I'(v), i \neq j$. For each vertex v and index $i \in I'(v)$, choose a color $s(v,i)$ from $S(v) \cap A_i$. Call a vertex v *bad*, if every color $s \in S(v)$ is bad. For a fixed vertex v,

$$P[v \text{ is bad}] = P[s \text{ is bad}, \forall s \in S(v)] \leq P[s(v,i) \text{ is bad}, \forall i \in I'(v)]$$

$$= \prod_{i \in I'(v)} P[s(v,i) \text{ is bad}] \leq \left(\frac{7}{8}\right)^{\frac{k}{3r}}.$$

The expected number of bad vertices is therefore no more than $2n(\frac{7}{8})^{\frac{k}{3r}}$. If $k > 3r(\log 2n)/(\log(\frac{8}{7})) = O(r \log n)$, then $2n(\frac{7}{8})^{\frac{k}{3r}} < 1$, implying that there exists a partition $Z = Z_A \cup Z_B$ with no bad vertices. Thus $K_{n,n}$ is T_r-k-choosable if $k = O(r \log n)$, as desired.

Erdős, Rubin and Taylor [3] showed that there exists a positive constant c such that the choice-number of $K_{n,n}$ is at least $c \log n$ for all $n > 1$. Applying Theorem 3 we conclude that T_r-ch$(K_{n,n}) \geq cr \log n$ for $r \geq 0$ and $n > 1$. Combined with Theorem 19 we obtain the following.

Theorem 20 *There exist two positive constants c_1, c_2 such that $c_1 r \log n \leq T_r$-ch$(K_{n,n}) \leq c_2 r \log n$ for all $r \geq 0,\ n \geq 2$.*

6 Concluding Remarks and Open Problems

(1) The constructive proofs of Theorem 5 and Proposition 11 supply simple linear-time algorithms for obtaining a T_r-coloring of an even or an odd cycle, given color sets of appropriate sizes.

(2) Determining the exact T_r-choice-number of even cycles C_{2n} for $3 \leq n \leq r$ is still open. We have narrowed the gap by presenting a lower bound of $\lfloor \frac{4n-2}{4n-1} \cdot 2(r+1) \rfloor + 1$, and an upper bound of $2(r+1)$. We believe that the lower bound is the exact value for all n, r, as mentioned in Conjecture 10.

(3) It seems that in many cases, if D is an orientation of a graph G, t is sufficiently large and D' is obtained from D by replacing each directed edge by $2t-1$ parallel copies of it, then D' has a different number of even and odd Eulerian subgraphs (i.e., $\mathrm{EE}(D') \neq \mathrm{EO}(D')$). When this is the case, it follows that G is T-f-choosable where T is any arbitrary set of t nonnegative integers including zero, and $f(v) = (2t-1)d_D^+(v) + 1$.

(4) For arbitrary separation sets T, the size of T may have no influence on the T-choice-number of a graph G. For example, if the elements of T are contained in the series $\{a_0 = 0, a_{i+1} = 3a_i + 1 \text{ for } i \geq 0\}$, then one can prove that $T\text{-ch}(B) \leq d_B + 2$ for any tree B having maximum degree d_B, regardless of the size of T (which may even be infinite). Thus it is interesting to characterize the important features of separation sets T, with respect to T-choice-numbers. As mentioned in section 4, it may be of interest to consider also separation sets which do not contain zero.

Note added in proof. We have recently found that B.Tesman, in his thesis [9], proved (independently, and before us) the assertion of theorem 5 and consequently theorem 7. His proof is similar to the one given here, but we include our proof since it contains a further restriction to the extreme colors of each color set, and since [9] is not easily accessible. The lower bound of Tesman [9, Thm. 3.9]: $T_r\text{-ch}(C_{2n}) \geq \lfloor \frac{4nr+4n-1}{2n+1} \rfloor + 1$ is strengthened by our lower bound of $T_r\text{-ch}(C_{2n}) \geq \lfloor \frac{4n-2}{4n-1} \cdot 2(r+1) \rfloor + 1$ (theorem 9).

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