Turán numbers of bipartite graphs and related Ramsey-type questions

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Abstract

For a graph H and an integer n, the Turán number $ex(n, H)$ is the maximum possible number of edges in a simple graph on n vertices that contains no copy of H. H is r-degenerate if every subgraph of it contains a vertex of degree at most r. We prove that for any fixed bipartite graph H in which all degrees in one color class are at most r, $ex(n, H) \leq O(n^{2-1/r})$. This is tight for all values of r and can also be derived from an earlier result of Füredi. We also show that there is an absolute positive constant c so that for every fixed bipartite r-degenerate graph H, $ex(n, H) \leq O(n^{1-c/r})$. This is motivated by a conjecture of Erdős that asserts that for every such H, $ex(n, H) \leq O(n^{1-1/r})$.

For two graphs G and H, the Ramsey number $r(G, H)$ is the minimum number n so that in any coloring of the edges of the complete graph on n vertices by red and blue there is either a red copy of G or a blue copy of H. Erdős conjectured that there is an absolute constant c such that for any graph G with m edges, $r(G, G) \leq 2^{c\sqrt{m}}$. Here we prove this conjecture for bipartite graphs G, and prove that for general graphs G with m edges, $r(G, G) \leq 2^{c\sqrt{m}\log m}$ for some absolute positive constant c.

These results and some related ones are derived from a simple and yet surprisingly powerful lemma, proved, using probabilistic techniques, at the beginning of the paper. This lemma is a refined version of earlier results proved and applied by various researchers including Rödl, Kostochka, Gowers and Sudakov.

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1 Introduction

All graphs considered here are finite, undirected and simple. For a graph H and an integer n, the Turán number $ex(n,H)$ is the maximum possible number of edges in a simple graph on n vertices that contains no copy of H. The asymptotic behavior of these numbers for graphs of chromatic number at least 3 is well known, see, e.g., [4]. For bipartite graphs H, however, the situation is considerably more complicated, and there are relatively few nontrivial bipartite graphs H for which the order of magnitude of $ex(n,H)$ is known. Our first result here asserts that for every fixed bipartite graph H in which the degrees of all vertices in one color class are at most r, $ex(n,H) \leq O(n^{2-1/r})$. This result, which can also be derived from an earlier result of Füredi [14], is tight for every fixed r, as shown by the constructions in [18] and in [2]. Our proof is different from that in [14], and provides somewhat stronger estimates.

A graph is *r-degenerate* if every subgraph of it contains a vertex of degree at most r. An old conjecture of Erdős ([9], see also [7], [12]) asserts that for every fixed r-degenerate bipartite graph H, $ex(n,H) \leq O(n^{2-1/r})$. Here we prove that there is an absolute constant $c > 0$, such that for every such H, $ex(n,H) \leq n^{2-c/r}$.

Our technique here provides several Ramsey-type results as well. For two graphs G and H, the Ramsey number $r(G,H)$ is the minimum number n so that in any coloring of the edges of the complete graph on n vertices by red and blue there is a red copy of G or a blue copy of H. If $G = H$ we sometimes denote $r(G,G)$ by $r(G)$.

Our first Ramsey-type result is that for every graph H with h vertices, maximum degree r and chromatic number $k \geq 2$, and for every integer m, $r(H,K_m) \leq \left(\frac{100m}{\log m}\right)^{(2r-k+2)(k-1)/2} (\log m) h^r$. This is nearly tight for $k = 2$, but is probably far from being tight for large values of k.

One of the basic results in Ramsey Theory is the fact that for the complete graph G with m **edges**, $r(G) = 2^{\Theta(\sqrt{m})}$. A conjecture of Erdős (see [7]) asserts that there is an absolute constant c such that for any graph G with m edges, $r(G) \leq 2^{c\sqrt{m}}$. Here we prove this conjecture for bipartite graphs G, and prove that for general graphs G with m edges, $r(G) \leq 2^{c\sqrt{m}\log m}$ for some absolute positive constant c.

The basic tool in the proof of most of the results here is a simple and yet surprisingly powerful lemma, whose proof is probabilistic. An early variant of this lemma has first been proven in [8] and [17], and versions that are closer to the one we prove and apply here have been proved and applied in [15], [23], [20] and [3]. There is no doubt that variants of the lemma will find additional applications as well.

Our notation is mostly standard. Here are a few less conventional notations. Given a graph $G = (V,E)$, for $v \in V$ and $U \subset V$ let $N_G(v,U)$ be the set of all neighbors of v in U; $d_G(v,U) = |N_G(v,U)|$; let also $N_G(v) = N_G(v,V)$; for a subset $U \subset V$ denote $N^*_G(U) = \{v \in V : (v,u) \in E(G)$ for every $u \in U\}$ – the common neighborhood of U in G.

The rest of the paper is organized as follows. In the next section we prove our basic lemma and apply it for bounding the Turán numbers of bipartite graphs with bounded degrees on one side. In

Section 3 we bound the Turán numbers of degenerate bipartite graphs. In Sections 4 and 5 we prove the Ramsey-type results mentioned above, and in Section 6 we improve the estimate of Füredi for the Turán numbers of certain generic bipartite graphs. The final section contains some concluding remarks and open problems.

Throughout the paper we make no attempts to optimize various absolute constants. To simplify the presentation, we often omit floor and ceiling signs whenever these are not crucial. All logarithms are in the natural base e, unless otherwise specified.

2 Turán numbers of bipartite graphs of given maximum degree

We start with the following basic lemma, whose proof is probabilistic.

Lemma 2.1 *Let a, b, n, r be positive integers. Let $G = (V, E)$ be a graph on $|V| = n$ vertices with average degree $d = 2|E(G)|/n$. If*

$$\frac{d^r}{n^{r-1}} - \binom{n}{r}\left(\frac{b-1}{n}\right)^r > a - 1\,, \tag{1}$$

then G contains a subset A_0 of at least a vertices so that every r vertices of A_0 have at least b common neighbors.

Proof. Let T be a subset of r random vertices of V, chosen uniformly with repetitions. Set

$$A = N_G^*(T) = \{v \in V : T \subseteq N(v)\}\ .$$

Denote by X the cardinality of A. By linearity of expectation:

$$\begin{aligned} \mathbb{E}[X] &= \sum_{v \in V}\left(\frac{|N(v)|}{n}\right)^r = \frac{1}{n^r}\sum_{v \in V}|N(v)|^r \geq \frac{1}{n^r}n\left(\frac{\sum_{v\in V}|N(v)|}{n}\right)^r \\ &= \frac{1}{n^{r-1}}\left(\frac{2|E(G)|}{n}\right)^r = \frac{d^r}{n^{r-1}}\,, \end{aligned}$$

where the inequality follows from the convexity of $f(x) = x^r$.

Let Y denote the random variable counting the number of r-tuples in A with fewer than b common neighbors. For a given r-tuple $R \subseteq V$, the probability that R will be a subset of A is precisely $\left(\frac{|N_G^*(R)|}{n}\right)^r$. As there are at most $\binom{n}{r}$ subsets R of cardinality $|R| = r$ for which $|N_G^*(R)| \leq b - 1$, it follows that:

$$\mathbb{E}[Y] \leq \binom{n}{r}\left(\frac{b-1}{n}\right)^r\ .$$

Applying linearity of expectation once again and recalling condition (1) of the lemma, we conclude that

$$\mathbb{E}[X - Y] \geq \frac{d^r}{n^{r-1}} - \binom{n}{r}\left(\frac{b-1}{n}\right)^r > a - 1\ .$$

Hence there exists a choice for T so that for the corresponding set A we get $X - Y \geq a$. Pick such a set, and for every r-tuple from A with fewer than b common neighbors delete one vertex from A. Denote the obtained set by A_0. Then $|A_0| \geq a$, and every r-tuple of vertices of A_0 has at least b common neighbors. This completes the proof. $\square$

Theorem 2.2 *Let $H = (A \cup B, F)$ be a bipartite graph with sides A and B of sizes $|A| = a$ and $|B| = b$, respectively. Suppose that the degrees of all vertices $b \in B$ in H do not exceed r. Let $G = (V, E)$ be a graph on $|V| = n$ vertices with average degree $d = 2|E(G)|/n$. If*

$$\frac{d^r}{n^{r-1}} - \binom{n}{r}\left(\frac{a+b-1}{n}\right)^r > a - 1,$$

then G contains a copy of H.

Proof. Let $v_1, \ldots, v_b$ be the vertices of B. By Lemma 2.1 (with $a + b$ playing the role of b) there is a subset $A_0 \subset V(G)$ of cardinality $|A_0| = a$ so that every r-subset of A_0 has at least $a + b$ common neighbors in G. Next we find an embedding of H in G described by an injective function $f : A \cup B \to V(G)$. Start by defining $f : A \to A_0$ to be an arbitrary bijection. Now embed the vertices of B one by one. Suppose that the current vertex to be embedded is $v_i \in B$, $1 \leq i \leq b$. By the assumption on H, v_i has at most r neighbors in H, all of them obviously in A. Let $N_i \subseteq A$ be the set of neighbors of v_i in A, $|N_i| \leq r$. The set of images $f(N_i) = \{f(v_i) : v_i \in N_i\}$ is a subset of A_0 of cardinality at most r, and has therefore at least $a + b$ common neighbors in G. As the total number of vertices embedded so far is strictly less than $a + b$, there is a vertex $w \in V(G)$ connected to all vertices in $f(N_i)$ and not used in the embedding previously. Set $f(v_i) = w$. It is immediate from the above description that once the embedding ends, the function f produces a copy of H in G. $\square$

Corollary 2.3 *Let H be a bipartite graph with maximum degree r on one side. Then there exists a constant $c = c(H) > 0$ such that*

$$ex(n, H) \leq cn^{2-\frac{1}{r}}.$$

Note that the last corollary is tight for every value of $r \geq 2$. Indeed, by the construction in [2] (modifying that in [18]), and by the well known results of [21], for every fixed $s \geq (r-1)! + 1$ the Turán number of the complete bipartite graph $K_{r,s}$ is $\Theta(n^{2-1/r})$. Note also that the assertion of the corollary can be deduced from the main result of Füredi in [14]. An improved version of his result is proved in Section 6.

3 Turán numbers of bipartite degenerate graphs

Recall that a graph is r-degenerate if every subgraph of it contains a vertex of degree at most r. We need the following easy and well known fact.

Proposition 3.1 *Let $H = (U, F)$ be an r-degenerate graph on $|U| = h$ vertices. Then there is an ordering $(v_1, \dots, v_h)$ of the vertices of H so that for every $1 \le i \le h$ the vertex v_i has at most r neighbors v_j with $j < i$.*

The following lemma is similar to a result proved in [20].

Lemma 3.2 *For every integer $r \ge 1$ and every integer n, every graph $G = (V, E)$ with $|V| = n$ vertices and at least $n^{2-\frac{1}{10r}}$ edges contains disjoint sets $A_1, B_1 \subset V$ such that every r-tuple of vertices in A_1 has at least $n^{\frac{1}{10}}$ common neighbors in B_1, and every r-tuple of vertices in B_1 has at least $n^{\frac{1}{10}}$ common neighbors in A_1.*

Proof. Note, first, that since $n^{2-\frac{1}{10r}} < n^2/2$, $n > 2^{10r} > 1000$. Partition the vertex set V into disjoint sets A, B of cardinalities $|A| = \lceil \frac{n}{2} \rceil$, $|B| = \lfloor \frac{n}{2} \rfloor$ such that at least half of the edges of G cross between A and B. (The existence of such a partition can be proved, for example, by choosing a set A of the desired size at random and by estimating the expected number of edges between A and its complement.) Denote by G_1 the bipartite subgraph of G consisting of all edges of G between A and B. Obviously, $|E(G_1)| \ge \frac{1}{2}|E(G)| \ge \frac{1}{2}n^{2-\frac{1}{10r}}$.

Choose at random a subset $T_1 \subset B$ consisting of $4r$ (not necessarily distinct) random members of B. Thus $|T_1| \le 4r$. Denote

$$A_0 = A_0(T_1) = \{a \in A : T_1 \subseteq N_{G_1}(a)\} .$$

Let $X = |A_0|$. Let Y be the random variable counting the number of $3r$-tuples in A_0 whose common neighborhood in B has fewer than $n^{\frac{1}{10}}$ vertices. We estimate the expectations of X and Y.

$$\mathbb{E}[X] = \sum_{a \in A} \left(\frac{d_{G_1}(a, B)}{|B|} \right)^{4r} \ge |A| \left(\frac{|E(G_1)|}{|A||B|} \right)^{4r} \ge 2^{4r-1} n^{1-\frac{4r}{10r}} \ge 2n^{0.6},$$

where the first inequality follows from the convexity of $f(x) = \big(x/|B|\big)^{4r}$.

In order to estimate the expected value of Y, observe that for a fixed $3r$-tuple $R \subset A$ the probability that R will be a subset of A_0 is precisely $\left(\frac{|N^*_{G_1}(R)|}{|B|} \right)^{4r}$. As there are at most $\binom{|A|}{3r}$ subsets R of cardinality $3r$ of A for which $|N^*_{G_1}(R)| < n^{0.1}$ it follows that

$$\mathbb{E}[Y] \le \binom{|A|}{3r} \left(\frac{n^{0.1}}{|B|} \right)^{4r} \le \left(\frac{e|A|}{3r} \right)^{3r} \left(\frac{n^{0.1}}{|B|} \right)^{4r} = \left(\frac{e|A|}{3r|B|} \right)^{3r} \left(\frac{n^{0.4}}{|B|} \right)^{r} \le 1.$$

By linearity of expectation we conclude that $\mathbb{E}[X - Y] = \mathbb{E}[X] - \mathbb{E}[Y] \ge n^{0.6}$. Hence there exists a choice of T_1 for which $X - Y \ge n^{0.6}$. Choose such T_1, and for each $3r$-tuple in A_0 with fewer than $n^{0.1}$ common neighbors, delete one vertex from A_0. It follows that there is a set $A_1 \subset A$ of cardinality $|A_1| \ge n^{0.6}$ so that

$$\text{every } 3r\text{-tuple in } A_1 \text{ has at least } n^{0.1} \text{ common neighbors in } B. \tag{2}$$

Fix an A_1 as above.

Now choose at random a subset $T_2 \subset A_1$ consisting of $2r$ (not necessarily distinct) uniformly chosen members of A_1. Note that $|T_2| \leq 2r$, and set

$$B_1 = B_1(T_2) = \{b \in B : T_2 \subseteq N_{G_1}(b)\} \ .$$

We estimate the probability that B_1 contains an r-tuple R_1, whose common neighborhood in A_1 has less than $n^{\frac{1}{10}}$ vertices. As in the calculation of $\mathbb{E}[Y]$ above, this probability is at most

$$\binom{|B|}{r}\left(\frac{n^{0.1}}{|A_1|}\right)^{2r} \leq \frac{|B|^r}{r!}(n^{-0.5})^{2r} \leq \frac{n^r}{2^r r!} \cdot n^{-r} < 1 \ .$$

Hence there exists a choice of T_2 for which

$$\text{every } r\text{-tuple in } B_1 \text{ has at least } n^{0.1} \text{ common neighbors in } A_1. \tag{3}$$

We claim that the pair (A_1, B_1) fulfills the requirements of the lemma. Indeed, for B_1 the desired property holds by (3). To show it for A_1, consider an arbitrary subset $S \subset A_1$ of cardinality $|S| = r$. As $|S \cup T_2| \leq |S| + |T_2| \leq r + 2r = 3r$, by (2) the set $S \cup T_2$ has at least $n^{0.1}$ common neighbors in B. Observe, crucially, that by the definition of B_1 all common neighbors of T_2 in B belong to B_1. It follows that $N^*_{G_1}(S \cup T_2) \subseteq N^*_{G_1}(T_2) \subseteq B_1$ and $|N^*_{G_1}(S \cup T_2)| \geq n^{0.1}$. As $N^*_{G_1}(S) \cap B_1 = N^*_{G_1}(S \cup T_2)$, the statement is proven. $\square$

Theorem 3.3 *Every graph $G = (V, E)$ on $|V| = n$ vertices with $|E(G)| \geq n^{2-\frac{1}{10r}}$ edges contains every r-degenerate bipartite graph $H = (A \cup B, F)$ with $|A|, |B| \leq n^{0.1}$.*

Proof. Let $h = |A| + |B|$. Order the vertices of H in such a way that for every $1 \leq i \leq h$, a vertex $v_i \in V(H)$ has at most r neighbors preceding it. Such an ordering is possible by Proposition 3.1.

Now apply Lemma 3.2 to G to get disjoint subsets $A_1, B_1 \subset V(G)$ so that every r-tuple of vertices in A_1 has at least $n^{0.1}$ common neighbors in B_1, and every r-tuple in B_1 has at least $n^{0.1}$ common neighbors in A_1. We construct an embedding $f : V(H) \to V(G)$ by placing images of vertices from A into A_1, and images of vertices of B into B_1.

To construct the desired embedding, we proceed according to the chosen order $(v_1, \ldots, v_h)$ of the vertices of H. If the current vertex $v_i \in V(H)$ is a vertex from A, we first locate the images $f(v_j)$, $j < i$, of the already embedded neighbors of v_i in B. The set $\{f(v_j) : j < i, (v_j, v_i) \in E(H)\}$ is a subset of B_1 of cardinality at most r. It has therefore at least $n^{0.1}$ common neighbors in A_1, and obviously not all of them have been already used in the embedding. We pick one unused vertex w and set $f(v_i) = w$. If $v_i \in B$, we can repeat the above argument, interchanging the roles of A_1 and B_1. $\square$

Corollary 3.4 *For every r-degenerate bipartite graph H on h vertices and for every $n \geq n_0(H) = h^{10}$:*

$$ex(n, H) \leq n^{2-\frac{1}{10r}} \ .$$

In fact the constant 10 in this corollary can be improved to 4 as stated in the following theorem, whose proof is similar to that of one of the lemmas in [3]. This theorem also improves the estimate in Theorem 3.3, but we believe it is instructive to include the somewhat simpler proof of that theorem as well, and present the next proof separately.

Theorem 3.5 *Let H be a bipartite r-degenerate graph of order h. Then for all $n \geq h$*

$$ex(n, H) \leq h^{1/2r} n^{2-\frac{1}{4r}} \ .$$

Proof. The claim is trivial for $r = 1$ and we thus assume $r \geq 2$. Let G be a graph of order n with at least $h^{1/2r} n^{2-\frac{1}{4r}}$ edges. As described in the proof of Lemma 3.2, there is a bipartite subgraph G_1 of G with parts A and B of sizes $|A| = \lfloor n/2 \rfloor$ and $|B| = \lceil n/2 \rceil$ such that $|E(G_1)| \geq \frac{1}{2}|E(G)| \geq \frac{1}{2} h^{1/2r} n^{2-\frac{1}{4r}}$.

Choose at random an ordered subset T consisting of $2r$ (not necessarily distinct) random members of B. Denote $A' = N^*_{G_1}(T) = \{a \in A : T \subseteq N_{G_1}(a)\}$. Let $X = |A'|$. Let Y be the random variable counting the number of ordered $3r$-tuples of vertices in A' whose common neighborhood in B has fewer than h vertices. We next estimate the expectations of X^{2r} and of Y. Using the convexity of $f(x) = \big(x/|B|\big)^{2r}$ we get

$$\mathbb{E}[X] = \sum_{a \in A} \left(\frac{d_{G_1}(a, B)}{|B|} \right)^{2r} \geq |A| \left(\frac{|E(G_1)|}{|A||B|} \right)^{2r} \geq 2^{2r-2} h\, n^{1-\frac{2r}{4r}} \geq 2hn^{1/2},$$

where here we used the fact that $|A| = \lfloor n/2 \rfloor > n/4$. By Jensen's Inequality and the fact that the function z^{2r} is convex, $\mathbb{E}\big[X^{2r}\big] \geq \mathbb{E}[X]^{2r} \geq (2h)^{2r} n^r$. As explained in the proof of Lemma 3.2,

$$\mathbb{E}[Y] \leq |A|^{3r} \left(\frac{h-1}{|B|} \right)^{2r} < |A|^r h^{2r} < h^{2r} n^r.$$

By linearity of expectation we conclude that

$$\begin{aligned} \mathbb{E}\left[X^{2r} - Y - h^{2r} \binom{n}{r} \right] &= \mathbb{E}\left[X^{2r}\right] - \mathbb{E}[Y] - h^{2r} \binom{n}{r} \\ &> 2^{2r} h^{2r} n^r - h^{2r} n^r - h^{2r} \binom{n}{r} > 0. \end{aligned}$$

Hence we can fix a choice of T such that $X^{2r} - Y - h^{2r}\binom{n}{r} > 0$.

Call an ordered subset S of $2r$ (not necessarily distinct) elements of A' *bad* if:
(i) all elements of S are contained in the common neighborhood of a set $U \subset B$ of size r for which $|N^*_{G_1}(U) \cap A'| < h$, or
(ii) there exists an ordered subset R of $3r$ elements of A' whose first $2r$ members form the ordered set S, such that $|N^*_{G_1}(R)| < h$,
otherwise it is called *good*. To rephrase, S is good if: (i) for every $U \subset B$ of size $|U| = r$, for which $S \subseteq N^*_{G_1}(U)$, one has:

$$|N^*_{G_1}(U) \cap A'| \geq h\,, \tag{4}$$

and (ii) for all subsets $W \subseteq A'$ of size $|W| = r$,

$$|N^*_{G_1}(S \cup W)| \geq h\,. \tag{5}$$

Every subset $U \subset B$ satisfying $|N^*_{G_1}(U) \cap A'| < h$ creates at most $(h-1)^{2r} < h^{2r}$ bad ordered $2r$-tuples in A', and every ordered subset $R \subset A'$ of size $3r$ with $|N^*_{G_1}(R)| < h$ generates exactly one bad ordered $2r$-tuple. Therefore, the total number of bad ordered $2r$-tuples is at most $Y + h^{2r}\binom{n}{r} < X^{2r}$. It follows that there is some ordered $2r$-tuple $S \subset A'$ which is good. Fix such a good S and define $B' = N^*_{G_1}(S) = \{b \in B : S \subset N_{G_1}(b)\}$. As in the derivation of Theorem 3.3, to complete the proof it suffices to show that every r-tuple of vertices in A' has at least h common neighbors in B', and every r-tuple of vertices in B' has at least h common neighbors in A'. For B' the desired property follows directly from the fact that S is good, and from (4). To show it for A' consider an arbitrary subset $W \subset A'$ of cardinality r. Let $S \cup W$ denote the ordered $3r$-tuple of elements of A' starting with the $2r$ members of S and continuing with the r members of W. By (5), $|N^*_{G_1}(S \cup W)| \geq h$. The crucial observation is now that by the definition of B', all common neighbors of S in G_1 belong to B'. Hence $|N^*_{G_1}(W) \cap B'| = |N^*_{G_1}(S \cup W)| \geq h$. This completes the proof. $\Box$

Substituting, for example, $h = n^{1/4}$ in the last theorem, we obtain the following strengthening of Theorem 3.3

Theorem 3.6 *Every graph $G = (V, E)$ on $|V| = n$ vertices with $|E(G)| \geq n^{2-\frac{1}{8r}}$ edges contains every r-degenerate bipartite graph $H = (A \cup B, F)$ with at most $n^{1/4}$ vertices.*

As mentioned in the introduction, an old conjecture of Erdős ([9], see also [7]), asserts that for every fixed r-degenerate bipartite graph H, $ex(n, H) = O(n^{2-1/r})$. Moreover, for $r = 2$ Erdős conjectured (see [12], [13], [7]) that for any fixed bipartite graph H, $ex(n, H) = O(n^{3/2})$ if and only if H is 2-degenerate. The last theorems do not prove any of these conjectures, but do supply an estimate of a similar form, and hence provide evidence to support them. The problem of reducing the constant 4 in Theorem 3.5 all the way to 1, remains a challenging open question whose resolution seems to require some additional ideas.

4 Ramsey numbers of graphs with given maximum degree

In this section we describe an application of Lemma 2.1 in the proof of the following Ramsey-type result.

Theorem 4.1 *Let H be a graph with h vertices and chromatic number $k \geq 2$. Suppose that there is a proper k-coloring of H in which the degrees of all vertices besides possibly those in the first color class are at most r, where $1 \leq r\ (< h)$. Define $\alpha(k, r)$ to be 1 if $k > r$ and 0 otherwise. Then, for every integer $m > 1$,*

$$r(H, K_m) \leq \left(\frac{100m}{\log m}\right)^{(2r-k+2)(k-1)/2} (\log m)^{\alpha(k,r)} h^r.$$

Note that in the above theorem k is always at most $r+1$, since the graph H is r-degenerate and is thus $(r+1)$-colorable. To prove this theorem we will need the following well known bound on the independence number of a graph containing few triangles (see, e.g., Lemma 12.16 in [5], also see [1] for a more general result).

Proposition 4.2 *Let G be a graph on n vertices with maximum degree at most d such that the neighborhood of every vertex in G spans at most $t > 0$ edges. Then G contains an independent set of order at least*

$$0.1\frac{n}{d}\big(\log d - (1/2)\log t\big).$$

Proof of Theorem 4.1. We apply induction on k. Starting with $k = 2$ and $r = 1$, put $n = 100mh$ and consider a red-blue edge coloring of K_n. Note that in this case H is just a disjoint union of stars. If the red graph has average degree at least $4h$ then it contains a subgraph with minimum degree $2h$. In this subgraph one can find any union of stars of order h just greedily. Otherwise, the average degree of the red graph is at most $4h$, so by Turán's theorem it contains a blue independent set of size $100mh/(4h+1) \geq m$.

Now let $r \geq k = 2$ and consider a red-blue edge coloring of K_n with $n = \big(\frac{100m}{\log m}\big)^r h^r$. If the number of red edges is at least $\frac{1}{2}\big(\frac{100m}{\log m}\big)^{r-1} h^r\, n$, then we claim that the red graph contains a set of at least h vertices, such that every r of them have at least h common neighbors in the red graph. Indeed, by Lemma 2.1 it suffices to check that

$$\frac{\big((\frac{100m}{\log m})^{r-1}h^r\big)^r}{n^{r-1}} - \binom{n}{r}\left(\frac{h-1}{n}\right)^r \geq h^r - h^r/r! \geq h^r/2 > h-1.$$

This indeed holds, since $h \geq r \geq 2$. By the reasoning described in Section 2, this implies that the red graph contains a copy of H.

Next suppose that the red graph has at most $\frac{1}{2}\big(\frac{100m}{\log m}\big)^{r-1} h^r\, n$ edges. Then, by deleting repeatedly vertices of degree larger than $d = \big(\frac{100m}{\log m}\big)^{r-1} h^r$ we can obtain a red subgraph G with maximum degree at most d and at least $n/2$ vertices. If the neighborhood of every vertex in G spans at most $t = \big(\frac{100m}{\log m}\big)^{2r-3+1/r} h^{2r}$ edges then, by Proposition 4.2 it contains a blue independent set of size at least

$$0.1\frac{n/2}{d}\big(\log d - 1/2\log t\big) = \frac{5m}{\log m}\big(1/2 - 1/(2r)\big)\log\big(100m/\log m\big) \geq m.$$

Here we used that $1/2 - 1/(2r) \geq 1/4$ and that $\log(100m/\log m) \geq (4/5)\log m$ for all $m \geq 2$.

Otherwise, there is a subset of vertices of G of size at most d which spans at least t red edges. Then the conditions of Lemma 2.1 are satisfied again, since

$$\frac{(2t/d)^r}{d^{r-1}} - \binom{d}{r}\left(\frac{h-1}{d}\right)^r \geq (2h)^r - h^r/r! \geq h^r > h-1.$$

Therefore the red graph contains a set of at least h vertices, such that every r of them have at least h common neighbors in the red graph. As was explained earlier, this implies that the red graph contains a copy of H, showing that indeed the result holds for $k = 2$.

Assuming the result for $k-1$, we prove it for k, $k \geq 3$. Given H as in the theorem, fix a proper k-coloring of it with k color classes $V_1, V_2, \ldots, V_k$ in which the degrees of all vertices besides possibly those in V_1 are at most r. Moreover, take such a coloring in which the cardinality of V_k is as large as possible. Clearly every vertex in $V_2 \cup V_3 \ldots \cup V_{k-1}$ has a neighbor in V_k (since otherwise we can shift it to V_k, contradicting the maximality). Put $V' = V - V_k$ and let H' be the induced subgraph of H on V'. Then H' is $(k-1)$-chromatic, and it has a proper $(k-1)$-coloring in which the degrees of all vertices besides those in the first color class are at most $r-1$.

Put $n = \left(\frac{100m}{\log m}\right)^{(2r-k+2)(k-1)/2} (\log m)^{\alpha(k,r)} h^r$ and consider a red-blue edge coloring of K_n. As before, if the number of red edges is at least $e = \frac{1}{2}\left(\frac{100m}{\log m}\right)^{(2r-k+2)(k-1)/2-1} (\log m)^{\alpha(k,r)} h^r\, n$, then we claim that the red graph contains a set U of at least $\left(\frac{100m}{\log m}\right)^{(2(r-1)-k+3)(k-2)/2} (\log m)^{\alpha(k-1,r-1)} h^{r-1}$ vertices, so that any r of them have at least h common neighbors in the red graph. Note that $\alpha(k-1, r-1) = \alpha(k,r)$. Hence, by Lemma 2.1, to prove this claim it suffices to check that

$$\begin{aligned} \frac{(2e/n)^r}{n^{r-1}} - \binom{n}{r}\left(\frac{h-1}{n}\right)^r &\geq \left(\frac{100m}{\log m}\right)^{(2(r-1)-k+3)(k-2)/2} (\log m)^{\alpha(k,r)} h^r - h^r/r! \\ &\geq \left(\frac{100m}{\log m}\right)^{(2(r-1)-k+3)(k-2)/2} (\log m)^{\alpha(k-1,r-1)} h^{r-1}. \end{aligned}$$

Thus, there is a set U as claimed. By the induction hypothesis, either the induced blue subgraph on U contains a copy of K_m, in which case the desired result follows, or the induced red graph on U contains a copy of H'. In the latter case, this copy can be completed to a red copy of H in K_n, since every r vertices of U have at least h common neighbors in the red graph.

Next suppose that the red graph has at most $\frac{1}{2}\left(\frac{100m}{\log m}\right)^{(2r-k+2)(k-1)/2-1} (\log m)^{\alpha(k,r)} h^r\, n$ edges. Then, by deleting repeatedly vertices of degree larger than $d = \left(\frac{100m}{\log m}\right)^{(2r-k+2)(k-1)/2-1} (\log m)^{\alpha(k,r)} h^r$ we can obtain a red subgraph G with maximum degree d and at least $n/2$ vertices. If the neighborhood of every vertex in G spans at most $t = \left(\frac{100m}{\log m}\right)^{(2r-k+2)(k-1)-3+1/r} (\log m)^{2\alpha(k,r)} h^{2r}$ edges then, by Proposition 4.2 it contains a blue independent set of size at least

$$0.1\frac{n/2}{d}\big(\log d - (1/2)\log t\big) = \frac{5m}{\log m}\big(1/2 - 1/(2r)\big)\log\big(100m/\log m\big) \geq m.$$

Finally, we can assume that there is a subset of vertices of G of size at most d which spans at least t red edges. Then the conditions of Lemma 2.1 for getting a set U as before are satisfied again, since

$$\begin{aligned} \frac{(2t/d)^r}{d^{r-1}} - \binom{d}{r}\left(\frac{h-1}{d}\right)^r &\geq \left(\frac{100m}{\log m}\right)^{(2(r-1)-k+3)(k-2)/2} (\log m)^{\alpha(k,r)} (2h)^r - h^r/r! \\ &\geq \left(\frac{100m}{\log m}\right)^{(2(r-1)-k+3)(k-2)/2} (\log m)^{\alpha(k-1,r-1)} h^{r-1}. \end{aligned}$$

Therefore the red graph contains a set of at least $\left(\frac{100m}{\log m}\right)^{(2(r-1)-k+3)(k-2)/2} (\log m)^{\alpha(k-1,r-1)} h^{r-1}$ vertices, such that every r of them have at least h common neighbors in the red graph. As was explained earlier, using this we can either find in the red graph a copy of H or in the blue graph a copy of K_m. This completes the proof of the theorem. $\square$

An easy probabilistic argument shows that the above theorem is nearly tight when m is large and the fixed graph H is $H = K_{r,s}$ with s much bigger than r. In fact, for every $\epsilon > 0$, and every fixed r, if $s > s_0(r, \epsilon)$, then $r(K_{r,s}, K_m) > m^{r-\epsilon}$ for all $m > m_0(s)$, whereas by Theorem 4.1, $r(K_{r,s}, K_m) \le c(s)(m/\log m)^r$. See also [22] for some related results.

5 On a Ramsey-type problem of Erdős

As mentioned in the introduction, the following conjecture was raised by Erdős (see [7]).

Conjecture 5.1 *There exists an absolute constant $c > 0$ such that for every graph G with m edges and no isolated vertices,*

$$r(G) \le 2^{c\sqrt{m}}.$$

Here we first describe a very short proof of the conjecture for bipartite graphs G.

Theorem 5.2 *Let G be a bipartite graph with m edges and no isolated vertices. Then*

$$r(G) \le 2^{16\sqrt{m}+1}.$$

The order of the exponent in this estimate is asymptotically tight. Indeed, let G be the complete bipartite graph $K_{\sqrt{m},\sqrt{m}}$. Then it contains m edges and it is easy to check that almost every two-edge-coloring of the complete graph of order $2^{\sqrt{m}/2}$, where the color of every edge is chosen randomly and independently with probability $1/2$, does not contain a monochromatic copy of G. Thus $r(G) > 2^{\sqrt{m}/2}$.

Proof of Theorem 5.2. First we prove that G is $\sqrt{m}$-degenerate. Otherwise, by definition, G contains a subgraph G' with minimal degree larger than $\sqrt{m}$. Let (U, W) be the bipartition of G'. Clearly, every vertex in U has at most $|W|$ neighbors in G'. Therefore $|W| > \sqrt{m}$ and we obtain a contradiction since the number of edges in G' is

$$|E(G')| = \sum_{v \in W} d(v) > \sqrt{m}|W| > m = |E(G)|.$$

Let $n = 2^{16\sqrt{m}+1}$ and suppose that the edges of the complete graph K_n are 2-colored. Then clearly at least $\frac{1}{2}\binom{n}{2} \ge n^{2-\frac{1}{8\sqrt{m}}}$ edges have the same color. These edges form a monochromatic graph which satisfies the conditions of Theorem 3.6 with $r = \sqrt{m}$. Thus this graph contains every $\sqrt{m}$-degenerate bipartite graph of order $n^{1/4} > 2^{4\sqrt{m}} > 2m$. In particular, since the order of G is obviously bounded by $2|E(G)| = 2m$ it contains a copy of G. This completes the proof. $\square$

Theorem 5.3 *Let G be a graph with m edges and no isolated vertices. If m is sufficiently large then*

$$r(G) \le 2^{7\sqrt{m}\log_2 m}.$$

To prove this theorem we need two lemmas of Graham, Rödl and Ruciński [16]. We start with some notation. Let H be a graph with vertex set V and let U be a subset of V. Then we denote by $H[U]$ the subgraph of H induced by U and by $e(U)$ its number of edges. The *edge density* $d(U)$ of U is defined by

$$d(U) = \frac{e(U)}{\binom{|U|}{2}}.$$

Similarly if X and Y are two disjoint subset of V, then $e(X,Y)$ is the number of edges of G adjacent to exactly one vertex from X and one from Y and the density of the pair (X,Y) is defined by

$$d(X,Y) = \frac{e(X,Y)}{|X||Y|}.$$

We say that H is (ρ,γ)-*dense* if for all $U \subseteq V$ with $|U| \geq \rho|V|$, we have $d(U) \geq \gamma$. Similarly we say that H is *bi*-(ρ,γ)-*dense* if all pairs (X,Y) of disjoint subsets of V with $|X| \geq \rho|V|$ and $|Y| \geq \rho|V|$ satisfy $d(X,Y) \geq \gamma$. The following two lemmas are proved in [16].

Lemma 5.4 *Let the numbers s,β,ρ,η satisfy $0 < \beta,\rho,\eta < 1, s \geq \log_2(4/\eta)$ and $(1-\beta)^{2s} \geq 2/3$. Then if H is a $\big((2\rho)^s\beta^{s-1},\eta\big)$-dense graph on N vertices, then there exists $U \subseteq V(H)$ of size at least $\rho^{s-1}\beta^{s-2}N$ such that $H[U]$ is bi-$(\rho,\eta/2)$-dense.*

Lemma 5.5 *Let Δ and n be two integers and let a,ϵ,γ be positive numbers such that for all $0 \leq r \leq \Delta$,*

$$\Big(\gamma^{\Delta-r} - r\epsilon\Big)a \geq 1.$$

Let also G be a graph on n vertices with maximum degree at most Δ. If H is a graph of order at least $a(\Delta+1)n$ which is bi-$\big(\frac{\epsilon}{\Delta+1},\gamma\big)$-dense then H contains a copy of G.

Using these two lemmas we next prove the following statement.

Proposition 5.6 *Let m be an integer and let H be a graph with vertex set $V, |V| \geq 2^{3\sqrt{m}\log_2 m}$ such that every subset of H of size at least $4m^2$ has density at least $1/(2m)$. If m is sufficiently large, then H contains a copy of every graph G on $2m$ vertices with maximum degree at most $\frac{\sqrt{m}}{\log_2 m}$.*

Proof. Let $\eta = 1/(2m), s = \log_2(4/\eta) = \log_2 m + 3, \beta = 1/(8\log_2 m)$ and let $\rho = 2^{-2\sqrt{m}}$. Since m is sufficiently large, it is easy to check that

$$(1-\beta)^{2s} = \left(1 - \frac{1}{8\log_2 m}\right)^{2\log_2 m + 6} = e^{-(1+o(1))/4} > \frac{2}{3}.$$

Also, by assumption, every subset of H of size

$$\begin{aligned}(2\rho)^s\beta^{s-1}|V| &\geq 2^{3\sqrt{m}\log_2 m}\,2^{(-2\sqrt{m}+1)(\log_2 m+3)}\left(\frac{1}{8\log_2 m}\right)^{\log_2 m+2} \\ &= 2^{3\sqrt{m}\log_2 m}\,2^{-(2+o(1))\sqrt{m}\log_2 m} \geq 2^{(1+o(1))\sqrt{m}\log_2 m} > 4m^2\end{aligned}$$

has density at least $\eta = 1/(2m)$. Therefore, by Lemma 5.4, H contains an induced subgraph H' of order at least

$$\begin{aligned}\rho^{s-1}\beta^{s-2}|V| &\geq 2^{-2\sqrt{m}(\log_2 m+2)}\left(\frac{1}{8\log_2 m}\right)^{\log_2 m+1} 2^{3\sqrt{m}\log_2 m}\\ &= 2^{3\sqrt{m}\log_2 m}\, 2^{-(2+o(1))\sqrt{m}\log_2 m} \geq 2^{(1+o(1))\sqrt{m}\log_2 m}\end{aligned}$$

such that H' is bi-$\big(2^{-2\sqrt{m}}, 1/(4m)\big)$-dense.

Let G be a graph of order $2m$ with maximum degree at most $\Delta = \frac{\sqrt{m}}{\log_2 m}$. Set $\gamma = 1/(4m), a = 2^{3\sqrt{m}}$ and $\epsilon = (\Delta+1)2^{-2\sqrt{m}}$. Then it is easy to check that $2m(\Delta+1)a = 2^{(3+o(1))\sqrt{m}} < 2^{(1+o(1))\sqrt{m}\log_2 m} \leq |V(H')|$ and that for every $0 \leq r \leq \Delta$ we have

$$\begin{aligned}\Big(\gamma^{\Delta-r} - r\epsilon\Big)a &> \Big(\gamma^{\Delta} - \Delta\epsilon\Big)a = \left(\left(\frac{1}{4m}\right)^{\frac{\sqrt{m}}{\log_2 m}} - \frac{\sqrt{m}}{\log_2 m}\Big(\frac{\sqrt{m}}{\log_2 m}+1\Big)2^{-2\sqrt{m}}\right)2^{3\sqrt{m}}\\ &= \Big(2^{-(1+o(1))\sqrt{m}} - 2^{-(2+o(1))\sqrt{m}}\Big)2^{3\sqrt{m}} = 2^{(2+o(1))\sqrt{m}} > 1.\end{aligned}$$

In addition, we have that H' is bi-$\big(\frac{\epsilon}{\Delta+1}, \gamma\big)$-dense. Thus G and H' satisfy all the conditions of Lemma 5.5 and therefore it follows that H' contains a copy of G. $\Box$

Having finished all the necessary preparations we are now ready to prove Theorem 5.3.

Proof of Theorem 5.3. Let $G = (V, E)$ be a graph with m edges and no isolated vertices. Then, clearly, the number of vertices of G is at most $2m$. Let V_0 be the subset of $2\sqrt{m}\log_2 m$ vertices of G of largest degrees. Denote by G' the subgraph of G induced by the set $V - V_0$ and by $\Delta(G')$ its maximum degree. Note that

$$m = e(G) \geq \frac{1}{2}\sum_{v\in V_0} d(v) \geq \frac{1}{2}\Delta(G')|V_0| = \big(\sqrt{m}\log_2 m\big)\Delta(G').$$

Therefore the maximum degree of G' is bounded by $\frac{\sqrt{m}}{\log_2 m}$.

Let $n = 2^{7\sqrt{m}\log_2 m}$ and let χ be a 2-coloring of the edges of the complete graph K_n. Define, for $1 \leq i \leq 4\sqrt{m}\log_2 m$, sets of vertices U_i and elements $u_i \in U_i$ as follows. U_1 is the set of all vertices of K_n.

- Having chosen U_i, select u_i in U_i arbitrarily.
- Having selected $u_i \in U_i$, define

$$W_j = \big\{u \in U_i \mid \chi(u_i, u) = j\big\}, \quad j = 1, 2.$$

Set U_{i+1} to be the largest of the sets W_1, W_2. By definition, $|U_{i+1}| \geq (|U_i| - 1)/2$. Also, by induction, it is easy to show that

$$|U_i| \geq \frac{n}{2^{i-1}} - \sum_{t=1}^{i-1} 2^{-t} > \frac{n}{2^{i-1}} - 1 > \frac{n}{2^i}.$$

In particular we obtain that for $l = 2\sqrt{m}\log_2 m$, $|U_{2l}| \geq 2^{7\sqrt{m}\log_2 m}/2^{2l} = 2^{3\sqrt{m}\log_2 m}$.

Define a new coloring χ^* on $u_1, \ldots, u_{2l-1}$ by setting $\chi^*(u_i) = j$ (1 or 2) if $\chi(u_i, x) = j$ for all $x \in U_{i+1}$. Since this coloring splits the above $2l-1$ vertices into two parts, there is a set S of l vertices $u_{i_1}, \ldots, u_{i_l}$ all having the same color. Without loss of generality we can assume that this color is 1. Note that the vertices of S form a monochromatic clique of size $2\sqrt{m}\log_2 m$ which has color 1 and that all the edges between S and U_{2l} are also colored 1.

Let H be the graph consisting of all the edges within the set U_{2l} with color 1. First suppose that the density of every subset of H of size at least $4m^2$ is at least $1/2m$. Then, by Proposition 5.6, H contains a copy of the graph G'. It is easy to see that such a copy of G' together with the set of vertices S forms a monochromatic subgraph of K_n containing G. On the other hand if X is a subset of H of size at least $4m^2$ and density less than $1/2m$, then an easy computation shows that X spans at least

$$\left(1 - \frac{1}{2m}\right)\binom{|X|}{2} > \left(1 - \frac{1}{2m-1}\right)\frac{|X|^2}{2},$$

edges of the second color. Therefore by Turán's theorem there is a monochromatic clique of size $2m$ of the second color. This clique contains every graph on $2m$ vertices and in particular a copy of G. This completes the proof of the theorem. $\square$

Note that the proof actually shows the following, which is obviously stronger than the assertion of Theorem 5.3.

Theorem 5.7 *For every graph G with m edges and no isolated vertices, $r(G, K_{2m}) \leq 2^{7\sqrt{m}\log_2 m}$ provided m is sufficiently large.*

6 Improved bounds on a Turán-type problem

Given integers $k, t \geq 2$ and $s \geq 1$ define the graph $\mathbf{L}_t^{k,s}$ as follows. The vertex set of this graph consists of two disjoint sets X and Y of sizes $s\binom{k}{t} + 1$ and k respectively. $X = \{x_0\} \cup \{x_I^\alpha\}$ where I runs through all t-element subsets of $\{1, \ldots, k\}$ and $1 \leq \alpha \leq s$, and $Y = \{y_1, \ldots, y_k\}$. For every $1 \leq i \leq k$ we join y_i to x_0 and also to every x_I^α if $i \in I$. In particular, for $s = 1$ and $t = 2$ this graph is the induced subgraph on the first three layers of the Boolean k-cube. In this section we prove the following result.

Theorem 6.1

$$ex\big(2n, \mathbf{L}_t^{k,s}\big) \leq 2^{1+1/t}(s+1)^{1/t}kn^{2-1/t}.$$

This improves a result of Füredi [14], who proved that

$$ex\big(n, \mathbf{L}_t^{k,s}\big) = O((s+1)^{1/t}k^{2-1/t}n^{2-1/t}).$$

Note that this result, as well as that of [14], supplies an alternative proof for Corollary 2.3, as $\mathbf{L}_t^{k,k}$ contains every bipartite graph with maximum degree t on one side and at most k vertices. The dependence on k in our estimate above is essentially optimal for $k = n^{1/t}$, and the dependence on s is essentially optimal for all $s > t!$, as shown by the examples in [2].

Proof. Let $G=(V,E)$ be a graph on $2n$ vertices with at least $2^{1+1/t}(s+1)^{1/t}kn^{2-1/t}$ edges. As in Section 3, start with a partition of the vertex set V into disjoint sets V_1, V_2, each of cardinality n, such that at least half of the edges of G cross between V_1 and V_2. Denote by G_1 the bipartite subgraph of G consisting of all edges of G between V_1 and V_2. By definition, $|E(G_1)| \geq 2^{1/t}(s+1)^{1/t}kn^{2-1/t}$. Without loss of generality assume that

$$\sum_{v\in V_1}\left(d_{G_1}(v)\right)^t \leq \sum_{v\in V_2}\left(d_{G_1}(v)\right)^t.$$

Let $x_0, ..., x_{t-1}$ be a sequence of t not necessarily distinct vertices of V_1, chosen uniformly and independently at random, and denote $T=\{x_0,...,x_{t-1}\}$. Let U be the set of all the common neighbors of vertices from T in V_2, that is, $U=N^*_{G_1}(T)=\{v\in V_2 \mid T\subseteq N_{G_1}(v)\}$, and denote by X the size of U. By linearity of expectation and Jensen's inequality:

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{v\in V_2}\mathbf{Pr}\big(v\in U\big) = \sum_{v\in V_2}\left(\frac{|N_{G_1}(v)|}{|V_1|}\right)^t = \frac{\sum_{v\in V_2}\left(d_{G_1}(v)\right)^t}{n^t}\\
&\geq \frac{n\left(\frac{\sum_{v\in V_2} d_{G_1}(v)}{n}\right)^t}{n^t} = \frac{n\left(|E(G_1)|/n\right)^t}{n^t} = 2(s+1)k^t.
\end{aligned}$$

For every subset of vertices $S\subseteq V_2$ of size t define a *weight* $w(S)$ by $w(S)=\frac{1}{|N^*_{G_1}(S)|}$. Let Y be the random variable which sums the total weight of subsets of U of size t with at most $(s+1)\binom{k}{t}$ common neighbors in G_1. Note that for a given subset $S\subseteq V_2$ of size t the probability that it belongs to U is precisely $\left(\frac{|N^*_{G_1}(S)|}{n}\right)^t$. Therefore we can obtain the following bound on the expectation of Y:

$$\begin{aligned}
\mathbb{E}[Y] &= \sum_{S\subseteq V_2, |S|=t, |N^*_{G_1}(S)|\leq (s+1)\binom{k}{t}} w(S)\mathbf{Pr}\big(S\subseteq U\big) = \sum_{S\subseteq V_2, |S|=t, |N^*_{G_1}(S)|\leq (s+1)\binom{k}{t}} \frac{|N^*_{G_1}(S)|^{t-1}}{n^t}\\
&\leq \left((s+1)\binom{k}{t}\right)^{t-2}\frac{\sum_{S\subseteq V_2,|S|=t}|N^*_{G_1}(S)|}{n^t} = \left((s+1)\binom{k}{t}\right)^{t-2}\frac{\sum_{v\in V_1}\binom{d_{G_1}(v)}{t}}{n^t}\\
&\leq \frac{((s+1)\binom{k}{t})^{t-2}}{t!}\frac{\sum_{v\in V_1}\left(d_{G_1}(v)\right)^t}{n^t} \leq \frac{((s+1)\binom{k}{t})^{t-2}}{t!}\frac{\sum_{v\in V_2}\left(d_{G_1}(v)\right)^t}{n^t}\\
&\leq \frac{((s+1)\binom{k}{t})^{t-2}}{2}\mathbb{E}[X].
\end{aligned}$$

This implies that

$$E\left[X-(s+1)k^t-\frac{Y}{((s+1)\binom{k}{t})^{t-2}}\right]\geq 0. \tag{6}$$

Hence there exists a choice of T such that the random variables X and Y for the corresponding set U satisfy (6). Pick such a set U. Then, by (6), we have $|U|=X\geq (s+1)k^t$ and $Y\leq ((s+1)\binom{k}{t})^{t-2}X$.

Let U_1 be a random subset of U of size precisely k and let Y_1 be the random variable which counts the total weight of subsets S of U_1 of size t with at most $(s+1)\binom{k}{t}$ common neighbors in G_1.

Note that for every such S the probability that it lies in U_1 equals $\binom{X-t}{k-t}/\binom{X}{k}$. Thus, it is easy to see that

$$\begin{aligned}
\mathbb{E}[Y_1] &= \frac{\binom{X-t}{k-t}}{\binom{X}{k}} Y \leq \left(\frac{k}{X}\right)^t Y \leq \left(\frac{k}{X}\right)^t \left((s+1)\binom{k}{t}\right)^{t-2} X = \frac{k^t((s+1)\binom{k}{t})^{t-2}}{X^{t-1}} \\
&\leq \frac{k^t((s+1)\binom{k}{t})^{t-2}}{((s+1)k^t)^{t-1}} \leq \frac{k^t(s+1)^{t-2}\left(k^t/t!\right)^{t-2}}{(s+1)^{t-1}k^{t(t-1)}} = \frac{1}{(t!)^{t-2}(s+1)} \leq \frac{1}{s+1}.
\end{aligned}$$

This implies that there is a particular subset $U_1 \subseteq V_2$ of size k with $Y_1 \leq 1/(s+1)$. Fix such a set U_1.

Let $S_1, \ldots, S_{\binom{k}{t}}$ be all the subsets of U_1 of size t. We construct a set of distinct vertices $\left\{z_i^\alpha \in V_1 | 1 \leq i \leq \binom{k}{t}, 1 \leq \alpha \leq s\right\}$ such that for every i and α, z_i^α is adjacent to all vertices in S_i and $z_i^\alpha \neq x_0$. Arrange the sets S_i in a non-decreasing order of $|N^*_{G_1}(S_i)|$ and assign the vertices z_i^α one by one to these sets in this order. Always pick the next vertex z_i^α from $N^*_{G_1}(S_i)$ such that it is different from all previous vertices and x_0. Note that if this greedy procedure fails at step r, $1 \leq r \leq \binom{k}{t}$, then clearly the set S_r has $|N^*_{G_1}(S_r)| < rs+1 < (s+1)\binom{k}{t}$. Also by definition, we have that $|N^*_{G_1}(S_i)| \leq |N^*_{G_1}(S_r)|$ for all $i < r$. Thus we obtain a contradiction, since

$$\frac{1}{s+1} \geq Y_1 = \sum_{S_i, |N^*_{G_1}(S_i)| \leq (s+1)\binom{k}{t}} \frac{1}{|N^*_{G_1}(S_i)|} \geq \frac{r}{|N^*_{G_1}(S_r)|} > \frac{r}{sr+1} \geq \frac{1}{s+1}.$$

Finally, recall that x_0 is adjacent to all the vertices in U and hence also to all the vertices in U_1. Thus x_0 together with the vertices in U_1 and the vertices $\{z_i^\alpha\}$ form a copy of $\mathbf{L}_t^{k,s}$. This completes the proof of the theorem. □

7 Concluding remarks

- A topological copy of a graph H is any graph obtained from H by replacing each edge by a simple path, where all these paths are internally vertex disjoint. A 1-subdivision of H is the topological copy of H obtained by replacing each edge of H by a path of length 2.

  In [10] Erdős asked whether any graph on n vertices with $c_1 n^2$ edges contains a 1-subdivision of K_m with $m = c_2\sqrt{n}$ for some positive c_2 depending on c_1. We note that the results in [6], as well as those in [19], imply that any such graph contains a topological copy of $K_{c_3\sqrt{n}}$, but this copy is not necessarily a 1-subdivision.

  However, the existence of a 1-subdivision of the required size follows immediately from Theorem 6.1 with $t = 2$, $s = 1$ and $k = \Theta(\sqrt{n})$. A similar result can also be derived from the main result of [8], and can also be proved directly from the reasoning in the proof of Lemma 2.1 here. In fact, it is not difficult to show that for any fixed positive c and δ there is a positive $c' = c'(c, \delta)$ such that any graph on n vertices with at least $n^{1/2+\delta}$ vertices of degree at least cn each, contains a set of of at least $\sqrt{n}$ vertices so that each pair has at least $c'n$ common neighbors. This clearly implies that any such graph contains a 1-subdivision of K_m for any m satisfying $m + \binom{m}{2} \leq c'n$.

- As mentioned in Section 4, our estimate in Theorem 4.1 is nearly tight for some bipartite graphs H. It seems, however, that this estimate is far from being tight for graphs H with a large chromatic number. We conjecture that $r(H, K_m) \leq m^{O(r)}$ for every fixed graph H with maximum degree r and all sufficiently large m. Note that Theorem 4.1 implies merely $r(H, K_m) \leq m^{O(r^2)}$ for this case.

- The assertion of Theorem 5.2 can be extended to graphs with bounded chromatic number, combining our ideas here with the techniques in [20]. We omit the details.

- The proof of Theorem 6.1 implies a similar estimate for the Turán number of the graph $\hat{\mathbf{L}}_t^{k,s}$ obtained from $\mathbf{L}_t^{k,s}$ by replacing the vertex x_0 by an independent set of t vertices with the same neighbors. This is because the set T in the proof can be chosen with no repetitions.

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