Transversal Numbers for Hypergraphs Arising in Geometry

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1 Introduction

Helly's theorem asserts that if $\mathcal{F}$ is a finite family of convex sets in $\mathbb{R}^d$ in which every $d+1$ or fewer sets have a point in common then $\bigcap \mathcal{F} \neq \emptyset$. Our starting point, the (p,q) *theorem*, is a deep extension of Helly's theorem. It was conjectured by Hadwiger and Debrunner and proved by Alon and Kleitman [3]. Let $p \geq q \geq 2$ be integers. A family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ is said to have the (p,q) *property* if among every p sets of $\mathcal{F}$, some q have a point in common.

Theorem 1 ((p,q) theorem, Alon & Kleitmen) *For every* $p \geq q \geq d+1$ *there exists a number* $C = C(p,q,d)$ *such that whenever* $\mathcal{F}$ *is a finite*

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family of convex sets in $\mathbb{R}^d$ with the (p, q) property then there is a set of at most C points intersecting all the sets of $\mathcal{F}$.

Note that if we are only interested in the existence of $C(p, q, d)$ and not in its precise value, it is sufficient to consider the case $q = d + 1$.

Here we consider analogues and relatives of the (p, q) theorem for other settings, both geometric and abstract. The original proof of the (p, q) theorem uses two main tools: the *fractional Helly theorem* and the *weak epsilon-nets* for convex sets. Our main result (the union of Theorem 8 and Theorem 9) shows that in an abstract setting, the appropriate fractional Helly property is sufficient to derive the existence of weak epsilon-nets and the validity of a (p, q) theorem. These notions and the precise formulation will be given in Section 3 and the theorem will be proved in Sections 4 and 5.

One consequence we derive is a "topological (p, q)-theorem". A family $\mathcal{F}$ of subsets of $\mathbb{R}^d$, whose members are either all open or all closed, is a *good cover* if $\bigcap_{F \in \mathcal{G}} F$ is contractible or empty for all $\mathcal{G} \subset \mathcal{F}$. Helly proved that his theorem continues to hold for finite good covers. Here we show

Theorem 2 *The assertion of the (p, q) theorem remains valid for all finite good covers in $\mathbb{R}^d$.*

A crucial step in the proof of this theorem is of independent interest as it gives a homological condition for the edge-cover number ρ of a hypergraph (equivalently, the simplicial complex spanned by it) to be bounded as a function of the *fractional* edge-cover ρ^*.

A simplicial complex K is called *d-Leray* if the i-th homology of K and all of its induced subcomplexes vanish when $i \geq d$.

Theorem 3 *For every $d \geq 1$ there are constants $c_1 = c_1(d)$ and $c_2 = c_2(d)$ such that for a d-Leray simplicial complex K, $\rho(K) \leq c_1(\rho^*(K))^{c_2}$.*

As a 1-Leray complex K is simply the clique complex of a chordal graph it follows that $\rho(K) = \rho^*(K)$ (since chordal graphs are perfect). For $d > 1$ our proof implies that $c_2(d) = d^{O(d)}$ but we do not have examples showing that $c_2 = 1 + \epsilon$ will not suffice. In Section 7 we describe a 2-Leray complex K which satisfies $\rho(K) = \Omega(\rho^*(K) \log \rho^*(K))$.

These topological results will be proved in Section 6.

In Section 8 we consider convex lattices sets in $\mathbb{R}^d$. Doignon proved [9] that the Helly number for convex lattice sets in $\mathbb{R}^d$ is 2^d.

Theorem 4 *For $p \geq q \geq 2^d$, the assertion of the (p, q) theorem applies to all finite families of lattice convex sets in $\mathbb{R}^d$.*

Using a theorem of Hausel we can show that planar convex lattice sets satisfy even a $(p, 3)$-theorem for every p.

Conjecture 5 *For $p \geq q \geq d+1$, the assertion of the (p, q) theorem applies to all finite families of lattice convex sets in $\mathbb{R}^d$.*

Recently, this conjecture was proved by Bárány and Matoušek [6].

Alon and Kalai [2] used the method of [3] to prove (p, q) theorems in several geometric situations, for example for piercing convex sets in $\mathbb{R}^d$ by hyperplanes. In Section 9 we provide an example showing that no (p, q) theorem or a similar property, even in a weak sense, hold for stabbing convex sets by lines in $\mathbb{R}^3$.

Proposition 6 *For every integers m_0 and k, there is a system $\mathcal{C}$ of more than m_0 convex sets in $\mathbb{R}^3$ such that every k sets of $\mathcal{C}$ have a line transversal but no $k + 4$ of them have a line transversal.*

It seems that $k + 4$ could be improved to $k + 3$, or perhaps $k + 2$, by a more careful analysis of our construction. But achieving $k + 1$ seems more challenging.

It is often asked in connection of the (p, q) theorem to give some examples where the (p, q) condition holds. The following example is useful: Let μ be a probability measure on $\mathbb{R}^d$ and consider all convex sets with measure at least δ. If $\delta > q/p$ then this family satisfies the (p, q) property. The first step in the proof of Alon and Kleitman shows that if a family satisfies the (p, q) property then it has such a form but for a much smaller value of δ.

2 Transversal numbers of hypergraphs

Transversals and matchings. Let $\mathcal{F}$ be a finite set system on a (finite or infinite) set X (so $\mathcal{F}$ can also be regarded as a hypergraph). We recall that the *transversal number* of $\mathcal{F}$, denoted by $\tau(\mathcal{F})$, is the minimum cardinality of a subset of X which intersects all $F \in \mathcal{F}$. $\tau(\mathcal{F})$ is also called the *vertex-cover* number of $\mathcal{F}$.

The *fractional transversal number* $\tau^*(\mathcal{F})$ is the minimum of $\sum_{x \in X} f(x)$ over all nonnegative functions $f\colon X \to [0, 1]$ that satisfy $\sum_{x \in F} f(x) \geq 1$ for all $F \in \mathcal{F}$. (If X is infinite we only consider functions f attaining finitely many nonzero values.) Clearly always $\tau^*(\mathcal{F}) \leq \tau(\mathcal{F})$.

Let $\nu_d(\mathcal{F})$ denote the largest size of a subhypergraph $\mathcal{M} \subset \mathcal{F}$ such that $\deg_{\mathcal{M}}(x) \leq d$ for all $x \in X$. The *matching number* of $\mathcal{F}$ is $\nu(\mathcal{F}) = \nu_1(\mathcal{F})$. Also note that the (p,q) property for the family $\mathcal{F}$ can be restated as $\nu_{q-1}(\mathcal{F}) < p$.

The fractional matching number $\nu^*(\mathcal{F})$ is the maximum of $\sum_{S \in \mathcal{F}} f(S)$ over all nonnegative real functions $f : \mathcal{F} \to [0,1]$ which satisfy: $\sum\{f(S) : S \in \mathcal{F}, x \in S\} \leq 1$, for every $x \in X$. Clearly, $\nu(\mathcal{F}) \leq \nu^*(\mathcal{F})$ and it is easy to see that $\nu_d(\mathcal{F})/d \leq \nu^*(\mathcal{F})$. Linear programming duality gives that $\tau^*(\mathcal{F}) = \nu^*(\mathcal{F})$.

There can be a large gap between the transversal number and fractional transversal number. An example to keep in mind is the family $\mathcal{M}_{m,n}$ $\binom{[m]}{n}$ of all n-subsets of a set of size m. In this case $\tau^* = m/n$ while $\tau = m-n+1$. Thus, when $m = 2n$ we get $\tau^* = 2$ and $\tau = n+1$.

The dual of the hypergraph $\mathcal{F}$ is the hypergraph $\mathcal{F}^{dual}$ whose vertices correspond to the edges of $\mathcal{F}$ and whose edges correspond to the vertices of $\mathcal{F}$ with incidence relation being reversed. The dual notion to the notion of the transversal number is the *edge-cover number*, $\rho(\mathcal{F})$, of a hypergraph $\mathcal{F}$. It is the minimal number of edges required to cover all vertices. Similarly, the fractional edge-cover number is defined by $\rho^*(\mathcal{F}) = \tau^*(\mathcal{F}^{dual})$.

Transversal numbers, fractional transversal numbers and weak ϵ-nets The relations between transversal numbers, fractional transversal numbers and matching numbers is a topic of central importance in combinatorics. Call a class of hypergraphs hereditary if it is closed under taking subhypergraphs.

Our work can be regarded as a contribution towards understanding of the following question:

Problem 7 *1. For which hereditary class* $\mathbf{F}$ *of hypergraphs* $\mathcal{F}$ *is it true that* τ *is bounded above by a function of* τ^**?*

2. Let d *be a fixed positive integer. For which hereditary class* $\mathbf{F}$ *of hypergraphs* $\mathcal{F}$ *is it true that* τ *is bounded by a function of* ν_d*?*

For a collection $\mathcal{F}$ of subsets of X and a (multi-) subset $Y \subset X$ a *weak* ϵ*-net for* Y is a set $Z \subset X$ so that every $S \in \mathcal{F}$ with $|S \cap Y| \geq \epsilon |Y|$ satisfies $S \cap Z \neq \emptyset$. (Z is called an ϵ-net if $Z \subset Y$.)

It is easy to see that for a hypergraph $\mathcal{F}$ the following conditions are equivalent (with $g(x) = f(1/x)$):

- τ is uniformly bounded by a function g of τ^* for all subhypergraphs of $\mathcal{F}$.

- There is a function f such that for every ϵ and every Y there is a weak ϵ-net of size at most $f(\epsilon)$.

We will call a hypergraph satisfying these conditions a hypergraph of *finite type* or a hypergraph with the weak ϵ-net property. (We will adopt the same notion for a class of hypergraphs (possibly all finite) when the function $f(\epsilon)$ can be chosen uniformly for all hypergraphs in the class.) The combinatorial conditions for a hypergraph to be of a finite type and the nature of the functions $f(\epsilon)$ for the size of the weak ϵ-net which can arise are not understood.

The corresponding questions for classes of hypergraphs closed under *restrictions* are well understood. (Equivalently these are the questions on the relations between ρ and ρ^* for hereditary classes of hypergraphs.) In order that τ be bounded by a function of τ^* for all restrictions of a hypergraph $\mathcal{F}$ to subsets X' of X it is necessary and sufficient that for every Y and $\epsilon > 0$ there is an ϵ-net of size at most $f(\epsilon)$ and this is equivalent to the VC-dimension of $\mathcal{F}$ being finite. (When talking about a family of hypergraphs the VC-dimension should be uniformly bounded.) Haussler and Welzl [14] proved that $f(\epsilon) = O(d(1/\epsilon)\log(1/\epsilon))$, where d is the VC-dimension, and Komlós, Pach and Woeginger [19] gave examples showing this cannot be further improved. Ding, Seymour and Winkler [8] characterized when τ is bounded by a function of ν for a hypergraph and all of its restrictions.

Having a finite VC-dimension is closed under duality. (Thus, bounded VC dimension is a necessary and sufficient condition for ρ being bounded as a function of ρ^* for a hereditary class of hypergraphs.) This is not the case for being of finite type. The class of examples $\binom{[m]}{n}$ is not of finite type but the class of their duals is.

3 The fractional Helly theorem

Fractional Helly properties. The fractional Helly theorem of Katchalski and Liu [18] states that if $F_1, F_2, \ldots, F_n \subseteq \mathbb{R}^d$ are convex sets such the number of $(d+1)$-tuples $I \subseteq [n]$ with $\bigcap_{i \in I} F_i \neq \emptyset$ is at least $\alpha\binom{n}{d+1}$ then there exists a point common to at least βn sets F_i. Here $\alpha \in (0,1]$ is a parameter and the theorem asserts the existence of a $\beta = \beta(d,\alpha) > 0$ for all α. (We will use $\beta(d,\alpha)$ to denote the best possible β for which the theorem holds.) Katchalski and Liu proved first that $\beta(d,\alpha) \geq \alpha/(d+1)$ and also presented a better bound which shows that $\beta \to 1$ when $\alpha \to 1$. Kalai [16] and Eckhoff [10] proved that $\beta(d,\alpha) = 1-(1-\alpha)^{1/(d+1)}$.

Let $\mathcal{G}$ be a (finite or infinite) family of sets. We write that $\mathcal{G}$ satisfies $\mathrm{FH}(k,\alpha,\beta)$ if for every $F_1, F_2, \ldots, F_n \in \mathcal{G}$ such the number of k-tuples $I \subseteq [n]$ with $\bigcap_{i\in I} F_i \neq \emptyset$ is at least $\alpha\binom{n}{k}$, there exists a point common to at least $\lfloor \beta n \rfloor$ of the F_i. We say that $\mathcal{G}$ has *fractional Helly number* k if for every $\alpha \in (0,1)$ there exists $\beta = \beta(\alpha) > 0$ such that $\mathrm{FH}(k,\alpha,\beta(\alpha))$ holds. If k is not important we speak of the *fractional Helly property.*[1]

It may happen that we cannot find a $\beta > 0$ for all $\alpha > 0$ but there exist *some* α and $\beta > 0$ with $\mathrm{FH}(k,\alpha,\beta)$. Then we speak of the *weak fractional Helly property.* The weakest among such properties is with $\alpha = 1$ and some $\beta > 0$. In particular, the Helly property implies $\mathrm{FH}(k,1,1)$.

In the first part of their proof of the (p,q) theorem for convex sets, Alon and Kleitman showed, using the fractional Helly theorem, that τ^* is bounded for every family of convex sets with the $(p,d+1)$-property. The proof is a simple double counting plus the linear-programming duality and it works unchanged in the abstract setting, thus showing that if $\mathcal{F}$ has fractional Helly number $d+1$ then $\tau^*(\mathcal{F})$ is bounded by a function of $\nu_d(\mathcal{F})$; this is part (i) of the following theorem. An additional observation employing the weak fractional Helly property is expressed in part (ii).

Theorem 8 (i) *For every d and p there exists an $\alpha > 0$ such that the following holds. For any finite family $\mathcal{F}$ satisfying* $\mathrm{FH}(d+1,\alpha,\beta)$ *with some $\beta > 0$ and having the $(p,d+1)$ property (i.e. $\nu_d(\mathcal{F}) < p$), we have $\tau^*(\mathcal{F}) \leq T$, where T depends only on p, d, and β.*

(ii) *For every d, p, $k \geq d+1$, and $\beta_0 > 0$ there exists an $\alpha > 0$ such that the following holds. For any finite family $\mathcal{F}$ satisfying the weak fractional Helly property* $\mathrm{FH}(d+1,1,\beta_0)$*, the fractional Helly property* $\mathrm{FH}(k,\alpha,\beta)$ *with some $\beta > 0$, and the $(p,d+1)$ property, we have $\tau^*(\mathcal{F}) \leq T$, where T depends only on p, d, k, β_0, and β.*

We give the proof in Section 4. In Section 7, we present an example showing that the $(3,2)$ property and the 2-Helly property together are not

[1] Strictly speaking, this definition only makes sense for infinite families $\mathcal{G}$, since for a finite family some $\beta(\alpha)$ depending on $|\mathcal{G}|$ always exists. When dealing with finite families, we really mean that $\beta(\alpha)$ should be independent of the size of the family.

sufficient to bound $\tau^*(\mathcal{F})$. At present we do not know whether the $(3,2)$ property plus $\mathrm{FH}(2,\alpha,\beta)$ for some $\alpha < 1$ and $\beta > 0$ are sufficient or not.

Fractional Helly and weak ϵ-nets. In the second main part of the proof of the (p,q) theorem for convex sets, the existence of weak ϵ-nets for convex sets is used. This important notion was introduced by Haussler and Welzl [14] and further studied in several papers, such as [1], [7].

As far as we know, at least three different proofs of existence of weak ϵ-nets for convex sets are known. Two are given in Alon et al [1]: a direct geometric argument, leading to a weak ϵ-net of size $O((1/\epsilon)^{-2^{d-1}})$ for every fixed d, and an argument based on a selection lemma of Bárány [5], giving a weak ϵ-net of size $O((1/\epsilon)^{d+1})$ for d fixed. Our subsequent generalization is based on this latter proof. In [1], the bound is still slightly improved, by applying a more sophisticated selection lemma, and the current best bound, due to Chazelle et al. [7], is close to $O((1/\epsilon)^d)$ and is obtained by another geometric argument. Finding the correct estimates for weak ϵ-nets is, in our opinion, one of the truly important open problems in combinatorial geometry.

The original argument about the existence of weak ϵ-nets involving Bárány's selection lemma relies on several theorems in convexity, such as Tverberg's theorem and the colorful Carathéodory theorem. Here we show that a similar conclusion can be derived from a fractional Helly property, but we have to assume it not only for $\mathcal{F}$ but also for all intersections of the sets of $\mathcal{F}$.

Theorem 9 *For every integer $d \geq 1$ there exists $\alpha > 0$ such that the following holds. Let $\mathcal{F}$ be a finite family of sets and let $\mathcal{F}^\cap = \{\bigcap \mathcal{H} \colon \mathcal{H} \subseteq \mathcal{F}\}$ be the family of all intersections of the sets in $\mathcal{F}$. If $\mathcal{F}^\cap$ satisfies* $\mathrm{FH}(d+1,\alpha,\beta)$ *with some $\beta > 0$ then we have*

$$\tau(\mathcal{F}) \leq c_1 \cdot \tau^*(\mathcal{F})^{c_2},$$

where c_1 and c_2 depend only on d and β.

Our proof yields much worse estimates for c_1 and c_2 than those known for convex sets; in fact, our exponent c_2 is exponential in d. On the other hand, in the strongest example we are aware of with the fractional Helly property for intersections, even in the abstract setting, τ is only slightly superlinear in τ^*. A lower bound concerning convex sets [21] shows that $c_1 \geq e^{\Omega(\sqrt{d})}$ is needed in the worst case.

4 The (p, q) Property and τ^*

Here we prove Theorem 8. The statement (i) can be proved exactly as in Alon and Kleitman [2]; for the reader's convenience, we outline the argument here, a little simplified but leading to slightly worse quantitative bounds.

As we already mentioned it follows from linear programming duality that for every finite hypergraph $\mathcal{F}$ we have $\tau^*(\mathcal{F}) = \nu^*(\mathcal{F})$. Recall that the fractional matching number, $\nu^*(\mathcal{F})$ is the maximum of $\sum_{F\in\mathcal{F}} g(F)$ over all functions $g: \mathcal{F} \to [0,1]$ satisfying $\sum_{F\in\mathcal{F}: x\in F} g(F) \leq 1$ for all $x \in X$. Moreover, the maximum is attained by a rational-valued function g, for which we can write $g(F) = \frac{n_F}{D}$ for integers n_F and D. Let $\{F_1, F_2, \ldots, F_n\}$ be the multiset containing n_F copies of each $F \in \mathcal{F}$ (so $n = \sum_{F\in\mathcal{F}} n_F$).

Suppose that $\nu_d(\mathcal{F})$ is bounded, i.e. $\mathcal{F}$ has a $(p, d+1)$ property. Then the multiset $\{F_1, \ldots, F_n\}$ certainly has the $(p', d+1)$ property with $p' = (p-1)d+1$ since among any p' of its sets, the same set occurs $(d+1)$-times or there are at least p distinct sets.

For brevity, call an index set $I \subseteq [n]$ *good* if $\bigcap_{i\in I} F_i \neq \emptyset$ (i.e. I is in the nerve of $\mathcal{F}$). So for every $I \in \binom{[n]}{p'}$ there is at least one good $(d+1)$-tuple $J \subseteq I$, and hence the total number of good $J \in \binom{[n]}{d+1}$ is at least $\binom{n}{p'}/\binom{n-d-1}{p'-d-1} \geq \alpha\binom{n}{d+1}$ for a suitable $\alpha = \alpha(p, d)$.

By FH$(d+1, \alpha, \beta)$, there is a point x in at least βn of the F_i. On the other hand, since the multiset $\{F_1, \ldots, F_n\}$ was defined using a fractional matching, no point is in more than $\frac{n}{\nu^*(\mathcal{F})}$ of the sets F_i, and we conclude that $\tau^*(\mathcal{F}) = \nu^*(\mathcal{F}) \leq \frac{1}{\beta}$.

In part (ii), we assume that $\mathcal{F}$ satisfies FH$(d+1, 1, \beta_0)$ and FH(k, α, β) with a suitable $\alpha > 0$ and some $\beta > 0$, and has the $(p, d+1)$ property. We define $F_1, \ldots, F_n$ using an optimal fractional matching as above, and it suffices to show that there is a point common to at least βn of the F_i.

We want to show that there are at least $\alpha\binom{n}{k}$ good index sets $K \in \binom{[n]}{k}$, with $\alpha = \alpha(p, d, k, \beta_0) > 0$; then we can use FH$(k, \alpha, \beta)$.

To this end, let $m = m(p, d, k, \beta_0)$ be a sufficiently large integer (independent of n). It suffices to prove that each index set $M \in \binom{[n]}{m}$ contains at least one good k-element K, since then the total number of good k-tuples is at least $\binom{n}{m}/\binom{n-k}{m-k} \geq \alpha\binom{n}{k}$. To exhibit a good k-tuple in a given m-tuple M, we use Ramsey's theorem.

For each $I \in \binom{M}{p'}$, we choose a good $(d+1)$-element $J = J(I) \subset I$ (here we use the $(p', d+1)$ property, where p' is as in the proof of (i)). This $J(I)$ has one of $\binom{p'}{d+1}$ types, where the type is given by the relative positions of the elements of $J(I)$ among the elements of I (in the natural ordering of I). By Ramsey's theorem, if m is sufficiently large, there exists an r-element $N \subseteq M$, with r still large, such that all $I \in \binom{N}{p'}$ have the same type. Let $i_1 < i_2 < \ldots < i_r$ be the elements of N in the increasing order, let $s = \lfloor r/p' \rfloor$, and let $L = \{i_{p'}, i_{2p'}, \ldots, i_{sp'}\}$. Now all the $J \in \binom{L}{d+1}$ are good, since for each of them we can find an $I \in \binom{N}{p'}$ with $J(I) = J$.

By FH$(d+1, 1, \beta_0)$ applied to $\{F_i : i \in L\}$, there are at least $\beta_0 s$ among the sets indexed by L sharing a common point. If $\beta_0 s \geq k$, which can be guaranteed by setting m sufficiently large, we have obtained a good k-tuple contained in M. This proves part (ii) of Theorem 8. □

5 The Fractional Helly Property and Piercing

In this section, we prove Theorem 9. Let $c\colon 2^X \to 2^X$ denote the closure operation induced by the considered family $\mathcal{F}$ given by $c(A) = \bigcap\{F : A \subseteq F \in \mathcal{F}\}$, where $c(A) = X$ if no $F \in \mathcal{F}$ contains A ($c(A)$ is an abstract analogue of the convex hull). For a multiset $\{x_1, \ldots, x_m\} \subseteq X$ and $I \subseteq [m]$, put $G_I = c(\{x_i : i \in I\})$.

Proposition 10 (A Tverberg-type theorem) *Let $\mathcal{F}$ be a finite family and suppose that $\mathcal{F}^\cap$ satisfies* FH$(d+1, \frac{1}{4}, \beta)$ *for some $\beta > 0$. Then there exist integers $a = a(d, \beta)$ and $b = b(d, \beta)$ such that for every multiset $\{x_1, \ldots, x_{ab}\} \subseteq X$ there are $d+1$ pairwise disjoint subsets $I_1, \ldots, I_{d+1} \in \binom{[ab]}{a}$ with*

$$\bigcap_{i=1}^{d+1} G_{I_i} \neq \emptyset. \tag{1}$$

That is, a sufficiently large (multi)set can be partitioned into $d+1$ parts whose closures have a common point.

Let us remark that $\alpha = \frac{1}{4}$ is used just for concreteness and it can be replaced by any other constant strictly below 1, if a and b are chosen suitably.

Proof. Let $b = \lceil d/\beta \rceil + 1$ and $a = b^d$. Let $m = \binom{ab}{a}$ and consider the multiset $\mathcal{S} = \{G_I : I \in \binom{[ab]}{a}\}$; its sets are members of $\mathcal{F}^\cap$. We want to apply fractional Helly to $\mathcal{S}$ and so we first need to show that at least $\frac{1}{4}$ of the $(d+1)$-tuples of sets in $\mathcal{S}$ intersect.

We check that, in fact, at least $\frac{1}{4}$ of all $(d+1)$-tuples $(I_1, I_2, \ldots, I_{d+1})$ of pairwise distinct a-element index sets $I_i \subset [ab]$ satisfy $\bigcap_{i=1}^{d+1} I_i \neq \emptyset$. Intuitively, this is because $d+1$ independent random a-element subsets of $[ab]$

are very likely to be all distinct and to have a point in common, since a is very large compared to b. Quantitatively, the relative fraction of intersecting $(d+1)$-tuples of distinct a-element subsets of $[ab]$ is

$$\begin{aligned} & \frac{|\{(I_1,\ldots,I_{d+1}) \in \binom{[ab]}{a}^{d+1} : I_i \neq I_j \text{ for } i \neq j \text{ and } \bigcap_{i=1}^{d+1} I_i \neq \emptyset\}|}{m(m-1)\cdots(m-d)} \\ \geq\ & \frac{|\{(I_1,\ldots,I_{d+1}) \in \binom{[ab]}{a}^{d+1} : \bigcap_{i=1}^{d+1} I_i \neq \emptyset\}|}{m(m-1)\cdots(m-d)} \\ & \qquad - \frac{m^{d+1} - m(m-1)\cdots(m-d)}{m(m-1)\cdots(m-d)} \\ \geq\ & \frac{ab\binom{ab-1}{a-1}^{d+1} - \binom{ab}{2}\binom{ab-2}{a-2}^{d+1}}{m^{d+1}} - \frac{1}{4} \geq \frac{a}{b^d} - \frac{a^2}{2b^{2d}} - \frac{1}{4} = \frac{1}{4}. \end{aligned}$$

By FH$(d+1, \frac{1}{4}, \beta)$ applied to $\mathcal{S}$, there exists an $\mathcal{H} \subseteq \binom{[ab]}{a}$ such that $\bigcap_{I \in \mathcal{H}} G_I \neq \emptyset$ and

$$|\mathcal{H}| \geq \lfloor \beta m \rfloor > \frac{d}{b}\binom{ab}{a}. \tag{2}$$

Thus $\mathcal{H}$ contains a significant fraction of all possible a-tuples of indices, and such a large system has to contain $d+1$ disjoint a-tuples. With our parameters, we can use a result of Frankl (Theorem 10.3 in [11]), according to which (2) implies the existence of pairwise disjoint $I_1, \ldots, I_{d+1} \in \mathcal{H}$ (but it is easy to derive a similar result with somewhat worse quantitative parameters). □

Bárány [5] proved the following *selection lemma*: if $P \subset \mathbb{R}^d$ is an n-point (multi)set, then there exists a point x contained in the convex hulls of at least $c_d\binom{n}{d+1}$ subsets of P of cardinality $d+1$, where $c_d > 0$ depends on d but not on n. Here we derive an abstract analogue (replacing the colored Carathéodory theorem in Bárány's argument by the fractional Helly property).

Proposition 11 (A selection lemma) *Let $\mathcal{F}$ be a finite family such that $\mathcal{F}^\cap$ satisfies* FH$(d+1, \alpha, \beta)$ *with a suitable $\alpha = \alpha(d) > 0$ and some $\beta > 0$. Then for any multiset $\{x_1, \ldots, x_n\} \subseteq X$ there exists a family $\mathcal{H} \subseteq \binom{[n]}{a}$ such that $|\mathcal{H}| \geq \lambda\binom{n}{a}$ and*

$$\bigcap_{I \in \mathcal{H}} G_I \neq \emptyset,$$

where $a = a(d, \beta)$ is as in Proposition 10 and $\lambda > 0$ depends only on d and β.

Proof. Let $\mathcal{S} = \{G_I\colon I \in \binom{[n]}{a}\}$; we want to show that a significant fraction of the $(d+1)$-tuples in $\mathcal{S}$ intersect, in order to apply fractional Helly.

Let

$$T = \Big\{ \{I_1, \ldots, I_{d+1}\} : I_i \in \tbinom{[n]}{a},\ I_i \cap I_j = \emptyset \text{ for } i \neq j \text{ and } \bigcap_{i=1}^{d+1} G_{I_i} \neq \emptyset \Big\}.$$

Proposition 10 implies that for each subset $J \in \binom{[n]}{ab}$ there exist pairwise disjoint $I_1, \ldots, I_{d+1} \in \binom{J}{a}$ such that $\bigcap_{i=1}^{d+1} G_{I_i} \neq \emptyset$, and so each J contributes a $(d+1)$-tuple in T. On the other hand, for any given $\{I_1, \ldots, I_{d+1}\} \in T$, the $a(d+1)$ indices in $I_1 \cup \cdots \cup I_{d+1}$ are contained in $\binom{n-a(d+1)}{ab-a(d+1)}$ of the ab-tuples J. Therefore

$$|T| \ \geq \ \frac{\binom{n}{ab}}{\binom{n-a(d+1)}{ab-a(d+1)}} \ \geq \left(\frac{n}{ab}\right)^{a(d+1)} \geq \ \frac{1}{(ab)^{a(d+1)}} \binom{\binom{n}{a}}{d+1}$$

and Proposition 11 follows by $\mathrm{FH}(d+1, \alpha, \beta)$ applied to $\mathcal{S}$. □

Proof of Theorem 9. The value of $\tau^*(\mathcal{F})$, being the minimum of a linear function with rational coefficients over a rational polytope, is attained for some rational-valued $f\colon X \to [0,1]$, which is nonzero only at finitely many points, say $x_1, \ldots, x_r$. We write $f(x_i) = \frac{n_i}{D}$ with integers n_i and D, and we let $Y = \{y_1, \ldots, y_n\}$ be the multiset obtained by taking each x_i with multiplicity n_i. We have $|Y| = n = \sum_{i=1}^{r} n_i = \tau^*(\mathcal{F}) \cdot D$ and $|Y \cap F| \geq D = n/\tau^*(\mathcal{F})$ for all $F \in \mathcal{F}$.

From now on, we exactly follow an argument in [1] for the existence of a weak ϵ-net. Namely, we choose a transversal Z for $\mathcal{F}$ by the following greedy algorithm. Initially, Z is empty. Having already put $z_1, \ldots, z_k$ into Z, we check if there is a D-element subset $J \subset [n]$ such that $G_J = c(\{y_i : i \in J\})$ contains none of $z_1, \ldots, z_k$. If there is no such J then the current Z intersects the closures of all D-element subsets of Y and, in particular, it is a transversal for $\mathcal{F}$. If such a J exists, we apply Proposition 11 to the set $\{y_i\colon i \in J\}$. This yields a point, which we denote by z_{k+1}, that is contained in G_I for at least $\lambda\binom{D}{a}$ a-tuples $I \subset J$. (We may assume $D \geq a$ and thus $\lambda\binom{D}{a} > 0$, for otherwise Y will do as a small transversal.) This finishes the description of the algorithm.

Call an a-tuple $I \subset [n]$ *alive* if $G_I \cap \{z_1, \ldots, z_k\} = \emptyset$ and *dead* otherwise. Initially, all the $\binom{n}{a}$ a-tuples are alive, and adding z_{k+1} to Z kills at least $\lambda\binom{D}{a}$ of the a-tuples currently alive. So the size of the transversal found by the algorithm is at most

$$\frac{\binom{n}{a}}{\lambda\binom{D}{a}} \le \frac{1}{\lambda}\left(\frac{en}{D}\right)^a \le \frac{e^a}{\lambda}\cdot\tau^*(\mathcal{F})^a.$$

□

6 The Fractional Helly Property of Leray Complexes

Next, we show that a fractional Helly property, and consequently a (p, q) theorem, are implied by a topological condition. We recall that the *nerve* $N(\mathcal{F})$ of a hypergraph $\mathcal{F}$ is the simplicial complex on the vertex set $\mathcal{F}$ whose simplices are all $\sigma \subseteq \mathcal{F}$ such that $\bigcap_{F\in\sigma} F \neq \emptyset$.

A simplicial complex K is *d-Leray* if $H_i(\mathrm{lk}(K, \sigma)) = 0$ for all $\sigma \in K$ and $i \ge d$, where H_i is the i-dimensional homology with integer coefficients and $\mathrm{lk}(K, \sigma)$ denotes the link of σ in K. Equivalently K is d-Leray iff $H_i(L) = 0$ for any induced subcomplex $L \subseteq K$ and $i \ge d$.

A hypergraph $\mathcal{F}$ is *d^*-Leray* if the nerve $N(\mathcal{F})$ is d-Leray.

Theorem 12 *Let $\mathcal{F}$ be a finite d^*-Leray hypergraph and let $\mathcal{F}^{\cap}$ be the family of all intersections of the sets of $\mathcal{F}$. Then $\mathcal{F}^{\cap}$ has fractional Helly number $d + 1$; more precisely, for all $\alpha \in (0, 1)$, $\mathcal{F}$ satisfies* $\mathrm{FH}(d + 1, \alpha, \beta(\alpha))$ *with* $\beta(\alpha) = 1 - (1 - \alpha)^{1/(d+1)}$.

The nerve of a family of subsets of $\mathbb{R}^d$ with the property that all nonempty intersections of members of the family are contractible must be d-Leray. This follows from standard nerve theorems in algebraic topology which assert that the homology of the nerve of such a family is the same as the homology of the union of the sets in the family. Theorem 2 thus follows from Theorems 8, 9 and 12. Theorems 12 and 9 imply at once Theorem 3.

Wegner [26] proved that nerves of finite families of convex sets in $\mathbb{R}^d$ satisfy the stronger d-collapsibility property. Let σ be a face of dimension at most $k - 1$ of a simplicial complex X which is contained in a *unique* maximal face τ of X. The operation $X \to Y = X - \{\eta : \sigma \subset \eta \subset \tau\}$ is called an *elementary k-collapse*. X is *k-collapsible* provided there is a sequence of elementary k-collapses

$$X = X_1 \to X_2 \to \cdots \to X_m$$

such that $\dim X_m \leq k-1$.

Since an elementary k-collapse does not effect the homology in dimensions at least k it follows that k-collapsible complexes are k-Leray. Katchalski and Liu's proof for their fractional Helly theorem uses (implicitly) only d-collapsibility. In fact, d-collapsibility (or rather the first collapse step) is implicit in Hadwiger and Debrunner early paper on the (p,q) property [12].

The main tool in the proof of Theorem 12 is the following consequence of Kalai's Upper Bound Theorem for Leray complexes, see [16, 17]. Let $f_i(L)$ denote the number of i-dimensional faces of a simplicial complex L.

Theorem 13 (Kalai) *Suppose L is d-Leray and $f_0(L) = m$. Then $f_d(L) > \binom{m}{d+1} - \binom{m-r}{d+1}$ implies $f_{d+r}(L) > 0$.*

As a consequence we obtain that $f_d(L) \geq \alpha\binom{m}{d+1}$ implies $f_{\lfloor \beta(\alpha)m \rfloor}(L) > 0$, for $\beta = 1-(1-\alpha)^{1/(d+1)}$. Note that Theorem 13 is sharp even for nerves of convex sets in $\mathbb{R}^d$ as seen by the family which consists of r copies of $\mathbb{R}^d$ and $m-r$ hyperplanes in general position.

The upper bound theorem for families of convex sets, namely the assertion of Theorem 13 for nerves of families of convex sets was conjectured by Perles and Katchalski and was settled independently by Kalai and by Eckhoff [10]. Kalai's proof applied for arbitrary d-collapsible complexes. Kalai further characterized face numbers of d-collapsible complexes which was conjectured by Eckhoff using the technique of "algebraic shifting" and extended his proof to apply for all Leray complexes where the crucial fact is that the Leray property is preserved under algebraic shifting.

This fact also follows from a recent much more general result of Aramova and Herzog [4]. As observed more recently by Kalai, d-Leray complexes with complete $(d-1)$-skeleta (and there is no loss of generality to assume this is the case) are simply Alexander-duals of Cohen-Macaulay complexes. (The Alexander duality is not the duality between hypergraphs considered above but rather it is the same as the blocker construction in combinatorial optimization. The dual of a simplicial complex K on a vertex set V is the set of all subsets S of V such that $V \backslash S \notin K$.) Since Alexander duality commutes with algebraic shifting this observation gives an easier derivation that d-Leray simplicial complexes are preserved under shifting from the corresponding fact for Cohen-Macaulay complexes. Moreover, it gives a simple derivation for the characterization of their face numbers from the corresponding characterization of f-vectors of Cohen-Macaulay simplicial complexes discovered by Stanley in 1975 [25].

We return now to the proof of Theorem 12. To apply Theorem 13, we need two auxiliary constructions. Let K be a simplicial complex on the vertex set V. For a vertex $v \in V$ and an integer l let $A_{v,l}(K)$ denote the complex obtained from K by splitting v into l vertices $v_1, \ldots, v_l$: The vertex set of $A_{v,l}(K)$ is $V' = V \setminus \{v\} \cup \{v_i\}_{i=1}^{l}$. The faces are all $\sigma' \subset V'$ such that either $\sigma' \in K$ or $\sigma' = \sigma \setminus \{v\} \cup C$ where $v \in \sigma \in K$ and $C \subset \{v_1, \ldots, v_l\}$. Let $B(K)$ denote the simplicial complex whose vertices are the non empty faces of K, and $\{\sigma_1, \ldots, \sigma_n\} \in B(K)$ if $\bigcup_{i=1}^{n} \sigma_i \in K$.

Proposition 14 *If K is d-Leray then*
(i) *$A_{v,l}(K)$ is d-Leray, and*
(ii) *$B(K)$ is d-Leray.*

Proof. For part (i), let $L \subseteq A_{v,l}(K)$ be an induced subcomplex on the vertex set $V_0 \subset V'$. The simplicial map that is the identity on $V_0 \setminus \{v_1, \ldots, v_l\}$ and that maps the vertices in $V_0 \cap \{v_1, \ldots, v_l\}$ (if any) to v is a homotopy equivalence of L onto an induced subcomplex of K, and hence $H_i(L) = 0$ for $i \geq d$.

As for part (ii), we first note that any complex L is homotopy equivalent to $B(L)$. Let $\eta = \{\sigma_1, \ldots, \sigma_p\} \in B(L)$ where $\sigma = \bigcup_{i=1}^{p} \sigma_i \in L$, and let $z = \sum_{i=1}^{p} \lambda_i \sigma_i \in |B(L)|$, where $|K|$ denotes the polyhedron of a simplicial complex K. The mapping $\phi\colon |B(L)| \to |L|$ given by

$$\phi(z) = \frac{1}{\sum_{i=1}^{p} \lambda_i |\sigma_i|} \sum_{v \in \sigma} \Big(\sum_{\{i \,:\, v \in \sigma_i\}} \lambda_i \Big) v$$

is the required retraction of $B(L)$ onto L.

Next, let $\eta = \{\sigma_1, \ldots, \sigma_p\} \in B(K)$, where $\sigma = \bigcup_{i=1}^{p} \sigma_i \in K$. Clearly

$$\mathrm{St}(B(K), \eta) = B(\mathrm{St}(K, \sigma)) = \Big\{ \{\tau_1, \ldots, \tau_q\} \in B(K) \colon \bigcup_{j=1}^{q} \tau_j \cup \sigma \in K \Big\}.$$

Therefore

$$\begin{aligned} \mathrm{lk}(B(K), \eta) \;\; = \;\; \Big\{ & \{\tau_1, \ldots, \tau_q\} \in B(K) \colon \textstyle\bigcup_{j=1}^{q} \tau_j \cup \sigma \in K, \\ & \{\tau_1, \ldots, \tau_q\} \cap \{\sigma_1, \ldots, \sigma_p\} = \emptyset \Big\}. \end{aligned}$$

We consider two cases:

(a) $\{\sigma_1, \ldots, \sigma_p\} \neq 2^\sigma \setminus \{\emptyset\}$.

Let $\emptyset \neq \tau \subset \sigma$ such that $\tau \notin \{\sigma_1, \ldots, \sigma_p\}$. Then $\mathrm{lk}(B(K), \eta)$ is a cone on τ and hence contractible.

(b) $\{\sigma_1, \ldots, \sigma_p\} = 2^\sigma \setminus \{\emptyset\}$. Then

$$\mathrm{lk}(B(K), \eta) = \Big\{\{\tau_1 \cup c_1, \ldots, \tau_q \cup c_q\} : \{\tau_1, \ldots, \tau_q\} \in B(\mathrm{lk}(K, \sigma)), c_1, \ldots, c_q \in 2^\sigma\Big\}.$$

Thus $\mathrm{lk}(B(K), \eta)$ is obtained from $B(\mathrm{lk}(K, \sigma))$ by replacing each vertex of the latter by a $(2^{|\sigma|} - 1)$-dimensional simplex.

The simplicial map $\mathrm{lk}(B(K), \eta) \to B(\mathrm{lk}(K, \sigma))$ given by

$$\{\tau_1 \cup c_1, \ldots, \tau_q \cup c_q\} \mapsto \{\tau_1, \ldots, \tau_q\}$$

is clearly a retraction. It follows that $\mathrm{lk}(B(K), \eta)$ is homotopy equivalent to $B(\mathrm{lk}(K, \sigma))$ and hence to $\mathrm{lk}(K, \sigma))$.

□

Proof of Theorem 12. By the assumption $K = N(\mathcal{F})$ is d-Leray. Suppose $\mathcal{S} = \{G_1, \ldots, G_m\}$ is a multiset in $\mathcal{F}^\cap$, and let $\sigma_1, \ldots, \sigma_k$ be distinct simplices in K such that S consists of m_i copies of $\bigcap_{F \in \sigma_i} F$ for each $i = 1, 2, \ldots, k$, $\sum_{i=1}^k m_i = m$. Then $L = N(\mathcal{S})$ is an induced subcomplex of $A_{\sigma_1, m_1} \cdots A_{\sigma_k, m_k} B(K)$. By Proposition 14, L is d-Leray, and hence Theorem 12 follows from Theorem 13. □

7 Some examples

An example with no weak ϵ-net of linear size

The first issue we would like to discuss is the following: Given an (infinite) hypergraph with the property that for every $\epsilon > 0$ and every set Y, Y admits a weak-ϵ net of size $f(\epsilon)$ what kind of behavior $f(\epsilon)$ might have.

Recall that for convex sets in R^d the known upper bounds are close to $(1/\epsilon)^d$ but no superlinear lower bound is known. In the most abstract case of the problem we do not have better insight as we do not have an answer even to the following problem:

Problem 15 *Find an example of an (infinite) hypergraph H such that $f(\epsilon)$ exists and $\frac{f(\epsilon)}{(1/\epsilon)\log(1/\epsilon)} \to \infty$.*

The fact that there are such hypergraphs for which $f(\epsilon) \geq \Omega(1/\epsilon)\log(1/\epsilon)$ follows from an example by Komlós, Pach and Woeginger [19] for the case of bounded VC-dimension. Here we present an interesting example (similar to an unpublished one found independently by Pach, who also raised Problem 16 below) which is also 2-Leray.

We first claim, without trying to optimize the absolute constants, that for every (large) prime power p there is a hypergraph whose vertices are all points of a projective plane P of order p, and whose edges, which we call *half lines*, are subsets of the lines of P, where S_L is a subset of the line L, such that the following two conditions hold:
(i) $|S_L| > \frac{1}{4p}(p^2+p+1)$ for every line L.
(ii) No subset of less than $0.1p\log p$ points of the plane intersects all half lines.

To prove this claim let each S_L be a random subset of L were each point is chosen, randomly and independently, with probability 1/2. It is easy to see that (i) holds almost surely (that is, with probability that tends to 1 as p tends to infinity). To see that (ii) holds almost surely fix a set T of $0.1p\log p$ points of P. It is easy to see that there are more than $p^2/2$ lines of P each of which contains at most $0.2\log p$ points of T. For each such line L, the probability that S_L does not intersect T is at least $(1/2)^{0.2\log p} = p^{-0.2}$ and therefore the probability that T intersects all half lines is at most

$$(1-p^{-0.2})^{p^2/2} \leq e^{-p^{1.8}/2}.$$

As the total number of choices for a set T as above is only

$$\binom{p^2+p+1}{0.1p\log p} \leq e^{O(p\log^2 p)}$$

it follows that with high probability there is no set T of at most $0.1p\log p$ points that intersects all half lines, establishing the claim.

Consider, now, the disjoint union of all the hypergraphs above (for all large prime powers p). The VC dimension of this hypergraph is clearly 2. If X is the set of all points of the projective plane of order p and $\epsilon = 1/(4p)$, then the corresponding weak ϵ-net has to intersect all half-lines of the plane and by the claim above its size has to exceed $0.1p\log p$.

To see that the nerve of this family is 2-Leray note that whenever we have a pure subcomplex of the nerve of dimension at least 2 then the set of its vertices forms a simplex. No homology beyond dimension 2 is thus possible. In fact, it is not difficult to check that this example is 2-collapsible as well.

For this example, if we close the set of edges under intersection $f(\epsilon)$ still exists (as we have only added singletons). This shows that even if we require that the hypergraph is closed under intersection the bound can be (slightly) superlinear.

Problem 16 *Can this example be realized by convex sets in* $\mathbb{R}^2$ *or perhaps in* $\mathbb{R}^4$ *or* $\mathbb{R}^{100}$*? Can the simplicial complex spanned by the lines in a finite projective plane be realized as the nerve of a family of convex sets in* $\mathbb{R}^2$ *or* $\mathbb{R}^{100}$*?*

Problem 17 *Is there a function* $d' = d'(d)$ *so that every* d*-collapsible complex (or even every* d*-Leray complex) can be realized as the nerve of a family of convex sets in* $\mathbb{R}^{d'}$*?*

In the following class of examples $\tau \geq (\tau^*)^\beta$ for $\beta > 1$, but we do not know if they are of finite type. Consider the 3^n leaves of the ternary tree of depth n. Given a set S of leaves we will define recursively a set of vertices $\bar{S} \supset S$ of the ternary tree as follows: An internal vertex belongs to $\bar{S}$ if at least two of its sons belong to S. Our hypergraph will have as vertices the leaves of the tree and as edges those subsets S of leaves such that the root of the ternary tree belongs to $\bar{S}$. In this example $\tau^* = (3/2)^n$ and $\tau = 2^n$. We do not know if this class of hypergraphs is of finite type.

A hypergraph with Helly number 2 and yet not of finite type

Next we discuss a construction, which starts with a graph G and yields a hypergraph $\mathcal{F}$ such that $\mathcal{F}^\cap$ has Helly number 2. By choosing various G, we obtain examples showing that some of the assumptions in our results cannot be removed or weakened.

Let $G = (V, E)$ be a graph, and let Ξ denote the system of all nonempty independent sets in G. We define a family $\mathcal{F}$ with Ξ as the ground set and with the sets $F_v = \{A \in \Xi \colon v \in A\}$, $v \in V$. The following properties are easy to check:

- $\mathcal{F}$, as well as $\mathcal{F}^\cap$, have Helly number 2, i.e. satisfy $\mathrm{FH}(2,1,1)$.

- If G contains no K_p as a subgraph then $\mathcal{F}$ has the $(p, 2)$ property.
- $\tau(\mathcal{F}) = \chi(G)$ (the usual chromatic number) and $\tau^*(\mathcal{F}) = \chi_f(\mathcal{F})$ (the fractional chromatic number).

Let us remark that this construction can be made "geometric": there exists a system of axis-parallel boxes in some $\mathbb{R}^m$ with the same nerve as $\mathcal{F}$. This is because every finite graph G can be represented as the intersection graph of axis-parallel boxes in a sufficiently high dimension.

First we give a result complementary to Theorem 8.

Proposition 18 *There exist hypergraphs $\mathcal{F}$ with Helly number 2 and with $\nu(\mathcal{F}) \leq 2$ (i.e. with the $(3, 2)$ property) for which $\tau^*(\mathcal{F})$ is arbitrarily large.*

Proof. In the above construction, it suffices to choose a triangle-free graph G with arbitrarily large fractional chromatic number. For the latter, it suffices that $|V(G)|/\alpha(G)$ is arbitrarily large, where $\alpha(G)$ is the independence number. There are many constructions of such graphs, both probabilistic and explicit; for example, the well-known probabilistic construction of Erdős of graphs with large girth and large chromatic number works here. □

The next example is relevant to Theorem 9.

Proposition 19 *There exist hypergraphs $\mathcal{F}$ satisfying the $(3, 2)$ property and the fractional Helly property $\mathrm{FH}(2, 0, \frac{1}{3})$ (i.e. among any n sets, at least $\frac{n}{3}$ have a common point), such that $\mathcal{F}^{\cap}$ has Helly number 2, and with $\tau^*(\mathcal{F}) \leq 3$ and $\tau(\mathcal{F})$ arbitrarily large.*

Proof. This time we let the starting graph G in the construction be a Kneser graph with the vertex set $\binom{[m]}{k}$ and with two k-tuples connected by an edge iff they are disjoint. It is well-known that the chromatic number is $m - 2k + 2$ [20], and if we set $m = 3k - 1$, it is easy to see that this G is triangle-free and $\chi_f < 3$. Finally, to verify $\mathrm{FH}(2, 0, \frac{1}{3})$ for the constructed set system, we need to check that for every multiset $\{S_1, \ldots, S_n\}$, $S_i \in \binom{[3k-1]}{k}$, there is a subsystem of at least $\frac{n}{3}$ k-tuples with a common intersection. This is because the sum of sizes of the S_i is $nk > n\,\frac{m}{3}$ and so some point is contained in at least $\frac{n}{3}$ of the S_i.

Note that in fact as $\tau^*(\mathcal{F}) \leq 3$, for every multiset of its edges there is a point in at least a 1/3 of them, that is, the property $FH(2, 0, 1/\tau^*)$ always holds. □

8 Piercing Convex Lattice Sets

A *convex lattice set* is any set of the form $C \cap \mathbb{Z}^d$, where $C \subseteq \mathbb{R}^d$ is a convex set and $\mathbb{Z}^d$ denotes the d-dimensional integer lattice. Doignon [9] proved that convex lattice sets in $\mathbb{Z}^d$ have Helly number 2^d. For a simpler proof see [23].

Let $p \geq q \geq 2^d$. The validity of the (p,q) theorem for finite families of convex lattice sets in $\mathbb{Z}^d$ (Theorem 4) is a consequence of Theorems 8 and 13 and the following

Lemma 20 *The nerve of a finite family of convex lattice sets in $\mathbb{Z}^d$ is $2^d - 1$ collapsible.*

Proof. We follow the method of Wegner [26] and Katchalski and Liu [18]. Let $\leq$ be a linear ordering on $\mathbb{Z}^d$ such that all initial segments are lattice convex sets; for example, we can choose a vector $a \in \mathbb{R}^d$ with no rational dependence among the coordinates and define $x \leq y$ iff $\langle a, x \rangle \leq \langle a, y \rangle$.

Write $k = 2^d - 1$ and let $\mathcal{F} = \{F_1, \ldots, F_n\}$ be a family of convex lattice sets in $\mathbb{Z}^d$. By intersecting the sets with a large box, we preserve their nerve but make them bounded and thus finite. For $I \subset [n]$ let $F_I = \cap_{i \in I} F_i$ and let $N(\mathcal{F}) = \{I \subset [n] : F_I \neq \emptyset\}$ denote the nerve of $\mathcal{F}$. For $I \in N(\mathcal{F})$ let $x_I = \min F_I$. Choose a subset J of minimal cardinality such that $x_J = \max\{x_I : I \in N(\mathcal{F})\}$. We claim that $|J| \leq k$. Suppose to the contrary that $|J| \geq k + 1 = 2^d$. Let $H = \{x \in \mathbb{Z}^d : x < x_J\}$ then the family $\mathcal{G} = \{F_j : j \in J\} \cup \{H\}$ has empty intersection, and so some subfamily of 2^d sets has empty intersection by the Helly property. Since H has to be one of these 2^d sets, it follows that there exists a $J_0 \subset J$, $|J_0| = 2^d - 1$ such that $x_J = x_{J_0}$, a contradiction. Clearly J is contained in a unique maximal face of N, namely $J' = \{i : x_J \in F_i\}$ hence $N \to N' = N - \{I : J \subset I \subset J'\}$ is a legal k-collapsing step. To complete the proof we note that the resulting N' is again the nerve of the family of convex lattice sets, namely $\{F_j \cap H : j \in J\} \cup \{F_i : i \notin J\}$ $\square$

Let us remark that in this case we do not really need to invoke Theorem 8 and, in fact, can get considerably better quantitative bounds by a more direct argument. Our quantitative bounds in the abstract setting are large mainly because the "Tverberg number" ab in Proposition 10 is large, but for convex lattice sets, the Tverberg number can be bounded in a much better way. For the Radon number (i.e., the number that ensures a partition into two disjoint parts with intersecting closures), the known bound is $d(2^d-1)+3$

[24]; see also Onn [22], and for an r-partition, an analogous argument of Jamison [15] yields the bound of $(r-1)(d+1)2^d+1$.

We conjectured that convex lattice sets in $\mathbb{Z}^d$ actually have fractional Helly number $d+1$ (although the bound 2^d for the Helly number is tight). As was mentioned in the introduction, this conjecture was recently proved in [6].

Hausel [13] proved a Gallai-type theorem for planar convex lattice sets: if $\mathcal{F}$ is a family of convex lattice sets in $\mathbb{Z}^2$ such that every 3 sets intersect (i.e. share a lattice point), then $\tau(\mathcal{F}) \le 2$. This implies FH$(3, 1, \frac{1}{2})$, and so by Theorem 8(ii), there is a $(p, 3)$ theorem for planar convex lattice sets.

9 No Piercing for Transversal Lines in Space

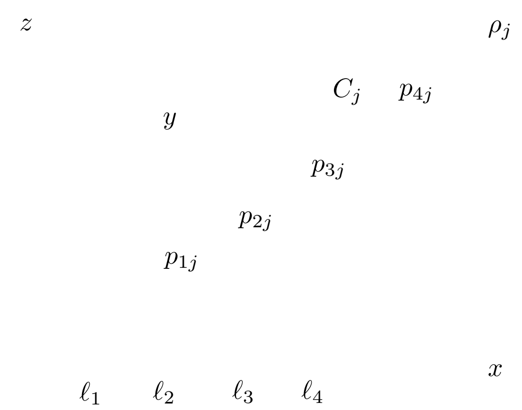
A $(p, d+1)$ theorem for hyperplane transversals for convex bodies in $\mathbb{R}^d$ was proved in [2]: if $\mathcal{C}$ is a family of convex bodies in $\mathbb{R}^d$ such that among every p of them, some $d+1$ admit a hyperplane transversal (i.e. a hyperplane intersecting all of them) then all bodies of $\mathcal{C}$ can be intersected by at most $C = C(d,p)$ hyperplanes. It is natural to ask whether a similar result could be true for piercing convex bodies in $\mathbb{R}^d$ by j-flats with $1 \le j \le d-2$. Proposition 6 formulated for the simplest case $d = 3$ and $k = 1$, shows that even quite weak results of this type cannot be expected to hold. Proposition 6 follows from the next lemma by choosing a suitable finite set system.

Lemma 21 *Let $\{S_1, S_2, \dots, S_m\}$ be a system of subsets of $[n]$. There are convex sets $C_1, C_2, \dots, C_m$ in $\mathbb{R}^3$ such that each family $\mathcal{C}_i = \{C_j : i \in S_j\}$ has a line transversal, and whenever $\{C_j : j \in J\}$ is a family possessing a line transversal, then by removing at most 3 indices from the index set J, we obtain an index set J_0 with $\bigcap_{j \in J_0} S_j \neq \emptyset$.*

Proof of Proposition 6. Choose a family $\{S_1, S_2, \dots, S_m\}$ such that every k sets intersect but no $k+1$ do; for example, set $S_i = \{I \in \binom{[m]}{k} : i \in I\}$.

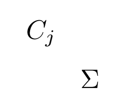
Proof of Lemma 21. The construction is based on the geometry of the hyperbolic paraboloid $z = xy$, similar to many previous examples concerning lines in $\mathbb{R}^3$, such as an example of Aronov, Goodman, Pollack and Wenger mentioned in Wenger's survey [27].

Let $\Sigma \subset \mathbb{R}^3$ be the surface with equation $z = xy$. For $i \in [n]$, let ℓ_i be the line $x = \frac{i}{n}$, $z = \frac{i}{n}y$ on Σ. Let $0 < \varepsilon_1 \ll \varepsilon_2 \ll \cdots \ll \varepsilon_m \ll 1$ be small numbers (ε_m is sufficiently small in terms of n and each ε_j is much smaller than ε_{j+1}). Let ρ_j be the vertical plane with equation $y = \frac{j}{m} + \varepsilon_j x$. So ρ_j is nearly perpendicular to the lines ℓ_i but it is tilted a little, and so its intersection with the surface Σ is a convex parabolic arc within ρ_j, with equation $z = \frac{j}{m}\,x + \varepsilon_j x^2$. We let $p_{ij} = \ell_i \cap \rho_j$, and we set $C_j = \mathrm{conv}\{p_{ij} : i \in S_j\}$. Here is an illustration (with $C_j = \{1, 3, 4\}$):



Each C_j is a very thin convex polygon. It lies vertically above Σ and below the segment connecting the points p_{0j} and p_{nj}. It can be easily calculated that the maximum vertical distance of a point of C_j from Σ is no larger than ε_j.

We divide each C_j into two regions: the *low* region consists of points at vertical distance at most $\varepsilon_j/100n^2$ from Σ, and the *high* region is the rest of C_j. Calculation shows that the low region consists of small triangle-like pieces near the points $p_{ij} \in C_j$, as is indicated in the following drawing (the low regions are drawn black):



The line ℓ_i is a transversal for the subfamily $\mathcal{C}_i$, and it remains to check the other assertion of the lemma. This is implied by the following two claims.

Claim A. If a line λ intersects at least two C_j in the low regions, then the sets met by λ in the low regions are all met by some ℓ_i.

Claim B. Any line λ meets at most 3 of the C_j in the high regions.

To prove Claim A, we note that if λ intersects the low regions of C_{j_1} and C_{j_2} near points $p_{i_1 j_1}$ and $p_{i_2 j_2}$, respectively, and $i_1 \neq i_2$, then λ cannot be almost parallel to the surface Σ and so if the ε_j are sufficiently small, no such λ can meet more than two of the C_j.

To prove Claim B, we note that if we parameterize the line λ by the y-coordinate, then the vertical distance of a point of λ from the surface Σ is a quadratic polynomial $p_\lambda(y)$. Suppose that there are 4 intersections with the high regions, and let their y-coordinates be $y_1 < y_2 < y_3 < y_4$. Let y_k correspond to the intersection with C_{j_k}; then y_k is very close to $\frac{j_k}{m}$. Since the intersections are at high regions, we have

$$\frac{\varepsilon_{j_k}}{100n^2} \leq p_\lambda(y_k) \leq \varepsilon_{j_k}. \tag{3}$$

We check that if the ε_j decrease sufficiently fast, this is impossible for a quadratic polynomial.

Namely, we show that the inequality $p_\lambda(y_4) \geq \varepsilon_{j_4}/100n^2$ is impossible if (3) holds for $k = 1, 2, 3$. Let $p_\lambda(y) = ay^2 + by + c$; then these conditions are linear inequalities for a, b, c. The coefficient vector $(y_4^2, y_4, 1)$ of the inequality $ay_4^2 + by_4 + c \geq \varepsilon_{j_4}^2/100n^2$ can be expressed as a linear combination of the vectors $(y_k^2, y_k, 1)$, $k = 1, 2, 3$. The coefficients in this linear combination can be written using Vandermonde determinants in the y_k, and so they are bounded by a polynomial function of m (since $y_{k+1} - y_k \geq \frac{1}{2m}$). It follows that the maximum value of $p_\lambda(y_4)$ is bounded by ε_{j_3} multiplied by a factor polynomial in m. Thus, if ε_{j_4} is sufficiently large compared to ε_{j_3}, we get a contradiction. □

10 Further open problems

We conclude with a few additional open problems:

Does a weak form of fractional Helly suffice?

1. Are FH$(2, \alpha, \beta)$ with some specific $\alpha < 1$ and $\beta > 0$ plus the $(3, 2)$ property, say, sufficient to bound $\tau^*(\mathcal{F})$?
2. Is FH$(2, \alpha, \beta)$ with specific $\alpha < 1$ and $\beta > 0$, assumed for $\mathcal{F}^\cap$, sufficient to bound $\tau(\mathcal{F})$ by a function of $\tau^*(\mathcal{F})$?

Is fractional Helly for distinct sets sufficient?

Our proofs use fractional Helly when some sets are repeated. Is this really necessary?

In particular, for fractional Helly number 2 we can state this problem in terms of the "non-intersection graph": suppose that a graph G is such that every k-vertex subgraph with at most $(1-\alpha)\binom{k}{2}$ edges contains an independent set of size $\beta(\alpha) \cdot k$. Is this still true if we replace each vertex of G by an independent set (maybe with smaller $\beta'(\alpha)$)?

Polytopes, Cohen-Macaulay complexes

Is ρ bounded by a function of ρ^* uniformly for all polytopes, namely, for all hypergraphs whose vertices are the vertices of some polytope and whose edges correspond to facets of the polytope ?

Is ρ bounded by a function of ρ^* uniformly for all Cohen-Macaulay complexes?

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