ON THE DISCREPANCY OF COMBINATORIAL RECTANGLES

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Abstract. Let $\mathcal{B}_n^d$ denote the family which consists of all subsets $S_1 \times \cdots \times S_d$, where $S_i \subseteq [n]$, and $S_i \neq \emptyset$, for $i = 1, \ldots, d$. We compute the L_2–discrepancy of $\mathcal{B}_n^d$ and give estimates for the L_p–discrepancy of $\mathcal{B}_n^d$ for $1 \leq p \leq \infty$.

1. Introduction

For a family of subsets $\mathcal{H}$ of a finite set Ω, a colouring $\chi : \Omega \to \{-1, 1\}$, and $A \in \mathcal{H}$, let $\chi(A) = \sum_{a \in A} \chi(a)$. Then, for $1 \leq p < \infty$, we set

$$\operatorname{disc}_p(\mathcal{H}, \chi) = \left(\frac{1}{|\mathcal{H}|} \sum_{A \in \mathcal{H}} |\chi(A)|^p \right)^{1/p},$$

while for $p = \infty$

$$\operatorname{disc}_\infty(\mathcal{H}, \chi) = \operatorname{disc}(\mathcal{H}, \chi) = \max \left\{ |\chi(A)| \colon A \in \mathcal{H} \right\}.$$

The L_p-discrepancy $\operatorname{disc}_p(\mathcal{H})$ of $\mathcal{H}$, where $1 \leq p \leq \infty$, is defined as the minimum value of $\operatorname{disc}_p(\mathcal{H}, \chi)$ over all possible colourings $\chi : \Omega \to \{-1, 1\}$. We shall sometimes call the L_∞–discrepancy just the discrepancy and write $\operatorname{disc}(\mathcal{H})$ instead of $\operatorname{disc}_\infty(\mathcal{H})$.

In this note we study the L_p–discrepancy of the family $\mathcal{B}_n^d$ of boxes (or combinatorial rectangles,) which consists of all sets of type $S_1 \times S_2 \times \cdots \times S_d$, where $\emptyset \neq S_i \subseteq [n] = \{1, 2, \ldots, n\}$, for $i = 1, 2, \ldots, d$. We compute the L_2–discrepancy of $\mathcal{B}_n^d$ precisely and estimate $\operatorname{disc}_p(\mathcal{B}_n^d)$ for all p, $1 \leq p \leq \infty$.

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Theorem 1. *For every $d, n \geq 1$ we have*

$$\operatorname{disc}_2(\mathcal{B}_n^d) = \left[\left(\frac{2^n}{2^n-1}\right)\left(\frac{n+\frac{1}{2}(1-(-1)^{n+1})}{4}\right)\right]^{d/2}.$$

Theorem 2. *Let $d, n \geq 1$. Then, for $1 \leq p < \infty$,*

$$8^{-d/2}n^{d/2} \leq \operatorname{disc}_p(\mathcal{B}_n^d) \leq p^7 2^{-d/2}(n+1)^{d/2}\,, \tag{1}$$

for $p \geq 2$,

$$\operatorname{disc}_p(\mathcal{B}_n^d) \geq \operatorname{disc}_2(\mathcal{B}_n^d) \geq 2^{-d}n^{d/2},$$

while for the L_∞-discrepancy of $\mathcal{B}_n^d$ we have

$$8^{-d/2}n^{(d+1)/2} \leq \operatorname{disc}(\mathcal{B}_n^d) \leq 2^{-d/2+1}\sqrt{d}(n+1)^{(d+1)/2}\,. \tag{2}$$

In the special case $d = 2$, Theorem 2 improves the bound

$$\tfrac{1}{15}n^{3/2} - \tfrac{4}{5}n \leq \operatorname{disc}(\mathcal{B}_n^2) \leq 2n^{3/2} \tag{3}$$

proven in [1]. Using the method presented in this note one can get a further improvement (for large n) of the lower bound in (3) to $(1/\sqrt{8\pi} + o(1))n^{3/2}$.

2. L_2-discrepancy

Let $\mathcal{H}$ be a family of subsets of a finite abelian group G. We say that $\mathcal{H}$ is shift-invariant if for every $A \in \mathcal{H}$ and $g \in G$ we have also $g + A \in \mathcal{H}$. In this section we compute $\operatorname{disc}_2(\mathcal{H})$ for any shift-invariant family $\mathcal{H}$ of subsets of G. Since, clearly, the family of boxes $\mathcal{B}_n^d$, considered as a family of subsets of $\mathbb{Z}_n^d$, is shift-invariant, Theorem 1 will follow.

For $A \in \mathcal{H}$ and $g \in G$ we set

$$\nu_A(g) = \left|\{(e,e') \in A \times A \colon e - e' = g\}\right|,$$

and

$$\nu(g) = \sum_{A \in \mathcal{H}} \nu_A(g).$$

Lemma 3. *Let $\mathcal{H}$ be a shift-invariant family of subsets of a finite abelian group G and $\chi : G \to \{-1,+1\}$. Then*

$$\sum_{A\in\mathcal{H}} \chi^2(A) = \frac{1}{|G|}\sum_{g,g'\in G} \chi(g)\chi(g')\nu(g-g')\,.$$

Proof. Let $A \in \mathcal{H}$. Then

$$\begin{aligned}\sum_{g\in G}\chi^2(A+g) &= \sum_{g\in G}\Big(\sum_{a\in A}\chi(a+g)\Big)^2\\ &= \sum_{g\in G}\sum_{a,a'\in A}\chi(a+g)\chi(a'+g)\\ &= \sum_{g,g'\in G}\chi(g)\chi(g')\nu_A(g-g').\end{aligned}$$

Since $\mathcal{H}$ is shift-invariant, we get

$$\begin{aligned}|G|\sum_{A\in\mathcal{H}}\chi^2(A) &= \sum_{A\in\mathcal{H}}\sum_{g\in G}\chi^2(A+g)\\ &= \sum_{g,g'\in G}\chi(g)\chi(g')\sum_{A\in\mathcal{H}}\nu_A(g-g')\\ &= \sum_{g,g'\in G}\chi(g)\chi(g')\nu(g-g')\,,\end{aligned}$$

which completes the proof. □

Proof of Theorem 1. Let $\chi_0 : \mathbb{Z}_n^d \to \{-1,+1\}$ be a "chessboard colouring" of $\mathbb{Z}_n^d$, i.e., $\chi_0(x_1,\ldots,x_d) = -1$, or 1, if the sum $\sum_{i=1}^d x_i$ is odd, or even, respectively. We shall show that for an arbitrary colouring $\chi : \mathbb{Z}_n^d \to \{-1,+1\}$ of $\mathbb{Z}_n^d$,

$$\operatorname{disc}_2(\mathcal{B}_n^d,\chi) \ge \operatorname{disc}_2(\mathcal{B}_n^d,\chi_0)\,,$$

and compute

$$\operatorname{disc}_2(\mathcal{B}_n^d,\chi_0) = \operatorname{disc}_2(\mathcal{B}_n^d)\,.$$

For a given $\mathbf{g} = (g_1,\ldots,g_d) \in \mathbb{Z}_n^d$, let

$$\operatorname{ind}(\mathbf{g}) = |\{i\in[d]\colon g_i = 0\}|\,.$$

Notice that

$$\nu(\mathbf{g}) = n^d 2^{d(n-2)+\operatorname{ind}(\mathbf{g})}.$$

For every $\mathbf{h} = (h_1,\ldots,h_d) \in \mathbb{Z}_n^d$, and $I \subseteq [d]$, define

$$\begin{aligned}\mathcal{C}(\mathbf{h},I) &= \{\mathbf{h}' = (h_1',\ldots,h_d') \in \mathbb{Z}_n^d\colon h_i = h_i' \text{ for } i\in I\}\\ &= \{\mathbf{h}'\in\mathbb{Z}_n^d\colon \mathbf{h}'|_I = \mathbf{h}|_I\}\,.\end{aligned}$$

From Lemma 3 it follows that

$$\sum_{A\in\mathcal{B}_n^d}\chi^2(A) = 2^{d(n-2)}\sum_{i=0}^{d}2^i \sum_{\substack{\mathbf{g},\mathbf{g}'\in\mathbb{Z}_n^d,\\ \mathrm{ind}(\mathbf{g}-\mathbf{g}')=i}}\chi(\mathbf{g})\chi(\mathbf{g}')$$
$$= 2^{d(n-2)}\sum_{\mathcal{C}(\mathbf{h},I)}\Big(\sum_{\mathbf{g}\in\mathcal{C}(\mathbf{h},I)}\chi(\mathbf{g})\Big)^2, \tag{4}$$

where the sum is taken over all families $\mathcal{C}(\mathbf{h},I)$. Indeed, observe that every term $\chi(\mathbf{g})\chi(\mathbf{g}')$, with $\mathbf{g}\neq\mathbf{g}'$, appears in the last double sum of (4) $2^{\mathrm{ind}(\mathbf{g}-\mathbf{g}')+1}$ times, and every term $\chi^2(\mathbf{g})$ appears 2^d times. Separating the summands with $I=[d]$ we get

$$\sum_{A\in\mathcal{B}_n^d}\chi^2(A) = 2^{d(n-2)}\sum_{\mathbf{g}\in\mathbb{Z}_n^d}\chi^2(\mathbf{g}) + 2^{d(n-2)}\sum_{\substack{\mathcal{C}(\mathbf{h},I)\\ I\neq[d]}}\Big(\sum_{\mathbf{g}\in\mathcal{C}(\mathbf{h},I)}\chi(\mathbf{g})\Big)^2$$
$$= 2^{d(n-2)}n^d + 2^{d(n-2)}\sum_{\substack{\mathcal{C}(\mathbf{h},I)\\ I\neq[d]}}\Big(\sum_{\mathbf{g}|_I=\mathbf{h}|_I}\chi(\mathbf{g})\Big)^2.$$

Now let us consider two cases. If n is even, then, clearly,

$$\sum_{A\in\mathcal{B}_n^d}\chi^2(A)\geq 2^{d(n-2)}n^d.$$

On the other hand, for every $I\subseteq[d]$, $I\neq[d]$, and every $\mathbf{h}$,

$$\sum_{\mathbf{g}\in\mathcal{C}(\mathbf{h},I)}\chi_0(\mathbf{g}) = 0$$

so, if n is even,

$$[\mathrm{disc}_2(\mathcal{B}_n^d)]^2 = [\mathrm{disc}_2(\mathcal{B}_n^d,\chi_0)]^2 = \Big(\frac{2^n}{2^n-1}\Big)^d\Big(\frac{n}{4}\Big)^d.$$

If n is odd, then for every $I\subseteq[d]$ and $\mathbf{h}\in\mathbb{Z}_n^d$

$$\Big|\sum_{\mathbf{g}|_I=\mathbf{h}|_I}\chi(\mathbf{g})\Big|\geq 1.$$

Furthermore, it is not hard to see that each such sum equals one for the colouring χ_0. Hence

$$\begin{aligned}\sum_{A\in\mathcal{B}_n^d}\chi^2(A) &\geq \sum_{A\in\mathcal{B}_n^d}\chi_0^2(A) = 2^{d(n-2)}n^d + 2^{d(n-2)}\sum_{\substack{\mathcal{C}(\mathbf{h},I)\\ I\neq[d]}} 1\\ &= 2^{d(n-2)}n^d + 2^{d(n-2)}\sum_{i=0}^{d-1}\binom{d}{i}n^i = 2^{d(n-2)}\sum_{i=0}^{d}\binom{d}{i}n^i\\ &= 2^{d(n-2)}(n+1)^d.\end{aligned}$$

Consequently, for odd n we have

$$[\mathrm{disc}_2(\mathcal{B}_n^d)]^2 = \Big(\frac{2^n}{2^n-1}\Big)^d\Big(\frac{n+1}{4}\Big)^d$$

and the assertion follows. □

The above proof of Theorem 1 has a combinatorial flavour but one can explore the fact that $\mathcal{B}_n^d$ is a "product family" using an algebraic argument. Below we sketch such an alternative proof of Theorem 1 for the case in which n is even.

Let $\chi : V \mapsto \{-1, 1\}$ be a colouring of the set of vertices of a hypergraph $\mathcal{H} = (V, E)$. Denote by $B = (B_{e,v})_{e\in E, v\in V}$ the incidence matrix of $\mathcal{H}$ in which $B_{e,v} = 1$ if and only if $v \in e$. It is easy to see that then

$$[\mathrm{disc}_2(\mathcal{H}, \chi)]^2 = \frac{1}{|E|}\chi^T B^T B\chi.$$

This implies that if the smallest eigenvalue of $B^T B$ is λ and $|V| = n$, then

$$[\mathrm{disc}_2(\mathcal{H}, \chi)]^2 \geq \frac{1}{|E|}\lambda n,$$

and equality holds if and only if there is an eigenvector of $B^T B$ corresponding to the smallest eigenvalue with $\{-1, 1\}$-coordinates.

If $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_d$ are d hypergraphs, where $\mathcal{H}_i = (V_i, E_i)$, then the product of $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_d$ is the hypergraph $\mathcal{H}$ whose set of vertices is the Cartesian product $\prod_{i=1}^d V_i$ and whose set of edges are all Cartesian products $\prod_{i=1}^d e_i$, for each choice of $e_i \in E_i$. It is not difficult to check that if B_i is the incidence matrix of H_i and B is the incidence matrix of H, then $B^T B$ is the tensor product of the matrices $B_i^T B_i$. Therefore, the set of all eigenvalues of $B^T B$ is the set of all products $\prod_{i=1}^d \mu_i$ where μ_i ranges over all eigenvalues of $B_i^T B_i$. In particular, the smallest eigenvalue of $B^T B$ is $\prod_{i=1}^d \lambda_i$, where λ_i is the smallest eigenvalue of $B_i^T B_i$, and the tensor product of any d vectors v_i, where v_i is an eigenvector corresponding to the smallest eigenvalue of $B_i^T B_i$, is an eigenvector of $B^T B$, corresponding to its smallest eigenvalue. We have thus proved the following.

Lemma 4. *Let $\mathcal{H}_i = (V_i, E_i)$, $i = 1, \dots, d$ be hypergraphs, and suppose λ_i is the smallest eigenvalue of $B_i^T B_i$, where B_i is the incidence matrix of H_i. Let $\mathcal{H}$ be the product of all hypergraphs $\mathcal{H}_i$, and let n_i denote the number of vertices of $\mathcal{H}_i$. Then*

$$\mathrm{disc}_2(\mathcal{H}, \chi) \geq \left[\frac{1}{\prod_{i=1}^d |E_i|} \prod_{i=1}^d \lambda_i n_i \right]^{1/2}. \tag{5}$$

Moreover, if for each i there is an eigenvector of $B_i^T B_i$ corresponding to the smallest eigenvalue with $\{-1, 1\}$-coordinates, then (5) holds with equality. □

In particular, if for $1 \leq i \leq d$, $\mathcal{H}_i = \mathcal{B}_n^1$ is the hypergraph whose set of vertices is $[n]$ and whose set of edges is the set of all nonempty subsets of $[n]$, then $B_i^T B_i$ is an n by n matrix with each diagonal entry being 2^{n-1} and each other entry being 2^{n-2}. It follows that its smallest eigenvalue is $2^{n-1} - 2^{n-2} = 2^{n-2}$ (with multiplicity $n - 1$). Thus, by Lemma 4 (where here the product $\mathcal{H}$ is $\mathcal{B}_n^d$, $n_i = n$, $\lambda_i = 2^{n-2}$ and $|E_i| = 2^n - 1$ for all i):

$$\mathrm{disc}_2(\mathcal{B}_n^d) \geq \left[\left(\frac{2^n}{2^n - 1} \right) \left(\frac{n}{4} \right) \right]^{d/2}.$$

Moreover, equality holds for every even n, as in this case every $\{-1, 1\}$-vector of length n whose sum of coordinates is 0, is an eigenvector of the smallest eigenvalue of $B_i^T B_i$.

3. L_p-DISCREPANCY: THE LOWER BOUND

Our proofs of the lower bounds in (1) and (2) rely on the following probabilistic theorem, proved by Szarek [3].

Lemma 5. *Let $a_1, \dots, a_n$ be real numbers and let $\tilde{\epsilon}_1, \dots, \tilde{\epsilon}_n$ denote independent identically distributed random variables such that*

$$\Pr(\tilde{\epsilon}_i = 1) = \Pr(\tilde{\epsilon}_i = -1) = 1/2 \quad \text{for} \quad i = 1, 2, \dots, n\,.$$

Set $X = \sum_{i=1}^n \tilde{\epsilon}_i a_i$. Then for the expectation $\mathrm{E}\,|\tilde{X}|$ of $|\tilde{X}|$ we have

$$\mathrm{E}\,|\tilde{X}| \geq \frac{1}{\sqrt{2}} \left(\sum_{i=1}^n a_i \right)^{1/2}. \quad \square$$

Let $\tilde{R} = \tilde{R}_n$ denote the random subset of $[n]$, where each element of $[n]$ is included in $\tilde{R}$ independently with probability 1/2, or, equivalently, where each subset of $[n]$ appears as $\tilde{R}$ with probability 2^{-n}. The following corollaries are straightforward consequences of Lemma 5.

Corollary 6. *Let $a_1, \ldots, a_n$ be a sequence of real numbers and $\tilde{Y} = \sum_{i\in\tilde{R}} a_i$. Then*

$$\mathrm{E}\,|\tilde{Y}| = 2^{-n} \sum_{A\subseteq[n]} \Big|\sum_{i\in A} a_i\Big| \geq \frac{1}{\sqrt{8n}} \sum_{i=1}^{n} |a_i|\,.$$

Proof. For every vector $\mathbf{e} = (\tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_n) \in \{-1,1\}^n$ define $A_{\mathbf{e}} = \{i : \tilde{\epsilon}_i = 1\}$ and $A'_{\mathbf{e}} = \{i : \tilde{\epsilon}_i = -1\}$. Then, by the triangle inequality

$$|\sum_{i\in A_{\mathbf{e}}} a_i| + |\sum_{i\in A'_{\mathbf{e}}} a_i| \geq |\sum_{i=1}^{n} \tilde{\epsilon}_i a_i|.$$

As $\tilde{\epsilon}$ ranges over all 2^n members of $\{-1,1\}^n$, $A_{\mathbf{e}}$, as well as $A'_{\mathbf{e}}$ range over all 2^n subsets of $\{1,2,\ldots,n\}$. Thus, using Lemma 5 and Cauchy-Schwartz inequality we infer that

$$2E|\tilde{Y}| \geq E\Big|\sum_{i=1}^{n} \tilde{\epsilon}_i a_i\Big| \geq \frac{1}{\sqrt{2}}\left(\sum_{i=1}^{n} a_i\right)^{1/2} \geq \frac{1}{\sqrt{2n}} \sum_{i=1}^{n} |a_i|\,. \quad \square$$

Remark. If $|a_1| = \cdots = |a_n| = 1$, then $\tilde{X} = \sum_{i=1}^{n} \tilde{\epsilon}_i a_i$ is asymptotically a normal random variable with standard deviation $\sqrt{n}$, and hence

$$\begin{aligned}\mathrm{E}\,|\tilde{Y}| \geq \frac{1}{2}\,\mathrm{E}\,|\tilde{X}| = \frac{1}{2}(1+o(1))\sqrt{n}\int_0^\infty \frac{2}{\sqrt{2\pi}} x e^{-x^2/2} dx \\ = (1+o(1))\sqrt{n/2\pi}\,.\end{aligned} \tag{6}$$

Corollary 7. *Let $\chi : [n]^d \to \{-1,1\}$. Then for every ℓ, $0 \leq \ell \leq d$,*

$$2^{-\ell n} \sum_{x_1\in[n]} \cdots \sum_{x_{d-\ell}\in[n]} \sum_{A_{d-\ell+1}\subseteq[n]} \cdots \sum_{A_d\subseteq[n]} \Big|\sum_{x_{d-\ell+1}\in A_{d-\ell+1}} \cdots \sum_{x_d\in A_d} \chi(x_1, x_2, \ldots, x_d)\Big| \geq 8^{-\ell/2} n^{d-\ell/2}\,.$$

Proof. We use induction on ℓ. For $\ell = 0$ there is nothing to prove. In order to show the assertion for $\ell \geq 1$ it is enough to set for each

$(d-\ell)$-tuple $x_1, \dots, x_{d-\ell}$ and all $A_{d-\ell+2}, \dots, A_d \subseteq [n]$,

$$a_i(x_1, \dots, x_{d-\ell}, A_{d-\ell+2}, \dots, A_d) = \sum_{x_{d-\ell+2} \in A_{d-\ell+2}} \cdots \sum_{x_d \in A_d} \chi(x_1, x_2, \dots, x_{d-\ell}, i, x_{d-\ell+2}, \dots, x_d),$$

and apply Corollary 6. □

Proof of the lower bounds in Theorem 2. Note that for every family of sets $\mathcal{H}$ and $1 \le r \le s \le \infty$, we have

$$\mathrm{disc}_r(\mathcal{H}) \le \mathrm{disc}_s(\mathcal{H})\,. \tag{7}$$

Now it is enough to observe that Corollary 7 applied with $\ell = d$, gives the required lower bound for $\mathrm{disc}_1(\mathcal{B}_n^d)$, and thus, for $\mathrm{disc}_p(\mathcal{B}_n^d)$ with $1 \le p < \infty$. For $p \ge 2$ we get a slightly better lower bound, as in this case

$$\mathrm{disc}_p(\mathcal{B}_n^d) \ge \mathrm{disc}_2(\mathcal{B}_n^d) \ge 2^{-d} n^{d/2},$$

by Theorem 1.

In order to deal with $\mathrm{disc}(\mathcal{B}_n^d)$ note that Corollary 7 with $\ell = d-1$ gives

$$2^{-(d-1)n} \sum_{A_2 \subseteq [n]} \cdots \sum_{A_d \subseteq [n]} \sum_{x_1 \in [n]} \left|\chi(\{x_1\} \times A_2 \times \cdots \times A_d)\right| \ge 8^{-(d-1)/2} n^{(d+1)/2}.$$

Thus, there exist sets $S_2, \dots, S_d$ such that

$$\sum_{x_1 \in [n]} \left|\chi(\{x_1\} \times S_2 \times \cdots \times S_d)\right| \ge 8^{-(d-1)/2} n^{(d+1)/2}.$$

Let $S_1^{\pm}$ be the set of all $x_1 \in [n]$ for which

$$\pm\chi(\{x_1\} \times S_2 \times \cdots \times S_d) > 0\,.$$

Take as S_1 any of the sets S_1^-, S_1^+, such that

$$\sum_{x_1 \in S_1} \left|\chi(\{x_1\} \times S_2 \times \cdots \times S_d)\right| = |\chi(S_1 \times \cdots \times S_d)| \ge 8^{-(d-1)/2} n^{(d+1)/2}/2 > 8^{-d/2} n^{(d+1)/2}\,.$$

The above holds for arbitrary $\chi : [n]^d \to \{-1, 1\}$, so $\mathrm{disc}(\mathcal{B}_n^d) \ge 8^{-d/2} n^{(d+1)/2}$.

Finally, from (6) we get $\mathrm{disc}(\mathcal{B}_n^2) \ge (1/\sqrt{8\pi} + o(1)) n^{3/2}$. □

4. L_p-DISCREPANCY – THE UPPER BOUND

Proof of the upper bounds in Theorem 2. Let us divide the set $[n] = \{1, 2, \ldots, n\}$ into $m = \lceil n/2 \rceil$ subsets, setting $P_i = \{2i-1, 2i\}$ for $i = 1, 2, \ldots, \lfloor n/2 \rfloor$ and, if n is odd, $P_m = \{n\}$. Let also

$$\mathcal{P} = \{P_{i_1} \times \cdots \times P_{i_d} \colon\ 1 \le i_1, \ldots, i_d \le m\}\,.$$

Hence, the family $\mathcal{P}$ is a partition of the set $[n]^d$ into m^d boxes, each of at most 2^d elements.

Note that for each $P \in \mathcal{P}$ there exist two "natural" colourings $\chi_{\text{odd}}(P)$, $\chi_{\text{even}}(P) : P \to \{-1, 1\}$ which colour elements $(x_1, \ldots, x_d)$ of P according to the parity of $\sum_{i=1}^d x_i$, so that no two points at Hamming distance one are coloured with the same colour. Let $\tilde{\chi} : [n]^d \to \{-1, 1\}$ denote a random colouring of $[n]^d$ in which for each $P \in \mathcal{P}$ independently we choose with probability $1/2$ one of the colourings $\chi_{\text{odd}}(P)$, $\chi_{\text{even}}(P)$. Our aim is to show that with positive probability $\text{disc}_p(\mathcal{B}_n^d, \tilde{\chi})$ is small; this will imply the existence a colouring χ with small $\text{disc}_p(\mathcal{B}_n^d, \chi)$ and the assertion will follow.

Let us first find the upper bound for $\text{disc}_p(\mathcal{B}_n^d)$, where $1 \le p < \infty$. Note that from Theorem 1 and (7) it follows that for $1 \le p \le 2$

$$\text{disc}_p(\mathcal{H}) \le \text{disc}_2(\mathcal{H}) \le 2^{-d}(n+1)^{d/2} < p^7 2^{-d/2}(n+1)^{d/2},$$

so it is enough to verify (1) for $2 \le p < \infty$. Since the colouring $\tilde{\chi}$ is random, $[\text{disc}_p(\mathcal{B}_n^d, \tilde{\chi})]^p$ is a random variable with expectation

$$\begin{aligned} \text{E}[\text{disc}_p(\mathcal{B}_n^d, \tilde{\chi})]^p &= \text{E}\left[\frac{1}{|\mathcal{B}_n^d|} \sum_{B \in \mathcal{B}_n^d} |\tilde{\chi}(B)|^p\right] \\ &= \frac{1}{|\mathcal{B}_n^d|} \sum_{B \in \mathcal{B}_n^d} \text{E}\,|\tilde{\chi}(B)|^p \le \max_{B \in \mathcal{B}_d^n} \text{E}\,|\tilde{\chi}(B)|^p\,. \end{aligned} \tag{8}$$

In order to estimate the above sum we study the behaviour of the random variable $\tilde{\chi}(B)$, for $B \in \mathcal{B}_n^d$. Note that for any colouring χ of $[n]^d$,

$$\chi(B) = \sum_{P \in \mathcal{P}} \chi(P \cap B)\,.$$

Let us assume now that χ is such that for every $P \in \mathcal{P}$ we have $\chi | P = \chi_\alpha(P)$ for some $\alpha = \textit{odd}, \textit{even}$. It is not hard to see that then, for any box $B \in \mathcal{B}_n^d$,

$$|\chi(P \cap B)| \le 1\,,$$

and equality holds if and only if $|P \cap B| = 1$. Thus, for a fixed B, $\tilde{\chi}(B)$ is a sum of w independent identically distributed random variables

$\tilde\epsilon_1,\dots,\tilde\epsilon_w$, where

$$w = w(B) = \big|\{P\in\mathcal{P}\colon |P\cap B| = 1\}\big| \le m^d \tag{9}$$

and $\Pr(\tilde\epsilon_i = -1) = \Pr(\tilde\epsilon_i = 1) = 1/2$ for $i = 1,\dots,w$. Thus, using Chernoff's bounds for the tails of the binomial distribution (see, for instance, [2], Corollary A.1.2), we infer that for every $t > 0$

$$\Pr(|\tilde\chi(B)| \ge t) < 2\exp\Big(-\frac{t^2}{2w(B)}\Big) \le 2\exp\Big(-\frac{t^2}{2m^d}\Big). \tag{10}$$

Set $\tau_i = 2^i m^{d/2}$ for $i = 0,1,\dots$. Then, from (10), we get

$$\begin{aligned}\mathrm{E}\,|\tilde\chi(B)|^p &\le \tau_0^p + \sum_{i=0}^{\infty}\tau_{i+1}^p 2\exp\Big(-\frac{\tau_i^2}{2m^d}\Big)\\ &= m^{pd/2} + m^{pd/2}\sum_{i=0}^{\infty}2^{ip+p+1}\exp\big(-2^{2i-1}\big)\\ &= m^{pd/2}\sum_{j=0}^{\infty}2^{jp+1}\exp\big(-2^{2j-3}\big).\end{aligned}$$

A crude estimate of the above sum gives

$$\begin{aligned}\sum_{j=0}^{\infty}2^{jp+1}\exp\big(-2^{2j-3}\big) &\le \sum_{j=0}^{\infty}2^{jp+1-2^{2j-3}}\\ &\le 5\log_2 p\, 2^{5p\log_2 p+1} + \sum_{j\ge 5\log_2 p}2^{jp+1-p^5 2^{j-3}}\\ &\le 10p^{5p}\log_2 p + 1 \le p^{7p}.\end{aligned}$$

Hence $\mathrm{E}[\mathrm{disc}_p(\mathcal{B}_n^d,\tilde\chi)]^p \le p^{7p}m^{pd/2}$, so there exists a colouring $\chi : [n]^d \to \{-1,1\}$ such that $[\mathrm{disc}_p(\mathcal{B}_n^d,\chi)]^p \le p^{7p}m^{pd/2}$. Hence

$$\mathrm{disc}_p(\mathcal{B}_n^d) \le \big[p^{7p}m^{pd/2}\big]^{1/p} \le p^7 2^{-d/2}(n+1)^{d/2}.$$

Finally, note that (10) implies that the probability that for some set B of $\mathcal{B}_n^d$ we have $|\tilde\chi(B)| \ge t$ is at most

$$|\mathcal{B}_n^d| 2\exp(-t^2/2m^d) \le 2^{dn+1}\exp(-t^2/2m^d).$$

The above expression is strictly smaller than 1 for $t = 2\sqrt{dn}m^{d/2}$, so for some colouring χ we have $\mathrm{disc}(\mathcal{B}_n^d,\chi) \le 2\sqrt{dn}\,m^{d/2}$ and

$$\mathrm{disc}(\mathcal{B}_n^d) \le 2\sqrt{dn}\,m^{d/2} \le 2^{-d/2+1}\sqrt{d}(n+1)^{(d+1)/2}. \qquad \square$$

We conclude the section with a remark that in the proof of the upper bound in Theorem 2, instead of the random colouring $\tilde\chi$ one can use the random colouring $\tilde\chi'$, in which each element of $[n]^d$ is coloured independently with -1 or 1. Then, similarly as in the argument above, for a given $B \in B_n^d$ the random variable $\tilde{\chi}'(B)$ is a sum of independent identically distributed random variables $\tilde{\epsilon}_i$, but in this case the number of $\tilde{\epsilon}_i$'s can be substantially larger than for $\tilde{\chi}(B)$. Consequently, Chernoff's bounds we used in the paper would give a weaker estimate for $\mathrm{disc}_p(\mathcal{B}_n^d, \tilde{\chi}')$.

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