ACYCLIC MATCHINGS

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Abstract. The purpose of this note is to give a constuctive proof of a conjecture in [1] concerning the existence of acyclic matchings.

1. Main result

Let $B,\ D \subset \mathbb{Z}^n$. Assume that $|B| = |D|$ and $0 \notin D$. A *matching* is a bijection $f : B \to D$ such that $b + f(b) \notin B$ for all $b \in B$. For any matching f, define $m_f : \mathbb{Z}^n \to \mathbb{Z}$ by $m_f(v) = \#\{b \in B \mid b + f(b) = v\}$. An *acyclic matching* is a matching f such that for any matching g such that $m_f = m_g$, we have $f = g$.

Theorem 1. *There exists an acyclic matching.*

This was first conjectured in [1]. The conjecure arises in the study of the problem, considered by Wakeford [2], of deciding which sets of monomials are removable from a generic homogeneous polynomial using a linear change of variables. For more details, see [1]. The following proof is constructive.

Proof. First totally order $\mathbb{Z}^n$ so that if $v > w$ then for any u, $v + u > w + u$ (and hence for $v > 0$, $v + u > u$.) For instance, choose a basis and order lexicographically.

Label the set B so that $b_1 < b_2 < b_3 < \cdots < b_m$.

We first consider the case where $d > 0$ for all $d \in D$.

Let $f(b_1)$ be the smallest $d \in D$ such that $b_1 + d \notin B$. Note that such a d always exists because $m = \#\{b_1 + d \mid d \in D\} > \#\{b \mid b > b_1,\ b \in B\} = m - 1$.

Next, let $f(b_2)$ be the smallest $d \in D \setminus \{f(b_1)\}$ such that $b_2 + d \notin B$. Such a d exists for virtually the same reason we are able to define $f(b_1)$.

Next, let $f(b_3)$ be the smallest $d \in D \setminus \{f(b_1), f(b_2)\}$ such that $b_3 + d \notin B$. Again, such a d exists for virtually the same reason we are able to define $f(b_1)$ and $f(b_2)$.

Continue in this manner until f is defined on all of B.

We claim that f is acyclic.

To see this, let g be a matching such that $m_f = m_g$.

If $f \neq g$, then there must be a smallest v such that

$$\{b \in B \mid b + f(b) = v\} \neq \{b \in B \mid b + g(b) = v\}.$$

Let b be the smallest element of $\{b \in B \mid b + g(b) = v\} \cap \{b \in B \mid b + f(b) = v\}^c$.

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Note that $f(b) > g(b)$ since otherwise $b+f(b) < b+g(b) = v$ contradicts our choice of v.

On the other hand, if $f(b) > g(b)$, we must have some $b' < b$ for which $f(b') = g(b)$, since otherwise f would not have been constructed according to our recipe. But since $g(b') \neq f(b')$ (because $g(b) = f(b')$), we have $b' + f(b') < v$ again contradicting our choice of v.

This impossibility implies that $f = g$.

For the general case, we partition D into D^+ and D^- so that $D^+ = \{d \in D \mid d > 0\}$. We now construct a matching f by using the above recipe twice, once for D^+ and $\{b_{|D^-|+1}, \ldots, b_m\}$ and once for D^- and $\{b_1, \ldots, b_{|D^-|}\}$. (For the latter assignment, we use the opposite ordering of $\mathbb{Z}^n$.)

We claim that f is acyclic. To see this note that any matching g with $m_f = m_g$ must satisfy $D^- = \{g(b_1), \ldots, g(b_{|D^-|})\}$. This is because $\sum_{k=1}^{|D^-|} b_k + f(b_k)$ is an absolute minimum for $\sum_{k=1}^{|D^-|} b_k + h(b_k)$ over all matchings h with equality if and only if $D^- = \{h(b_1), \ldots, h(b_{|D^-|})\}$. Acyclicity now follows from the argument given in the case where all $d \in D$ are positive.

2. Final remarks

It is worth noting that the number of matchings may be exactly one, for instance, in the case $m = 1$, or, less trivially, if $n = 1$ and $B = D = \{1, 2, 3, \ldots, m\}$.

The result in [1] is not entirely superseded by theorem 1 since in [1] a connection with hyperplanes is given.

Finally, we remark that throughout we could have replaced $\mathbb{Z}$ by $\mathbb{Q}$ or $\mathbb{R}$.

References

[1] C. K. Fan and J. Losonczy, *Matchings and Canonical Forms in Symmetric Tensors*, Advances in Mathematics, to appear.

[2] E. K. Wakeford, *On Cannonical Forms*, Proc. London Math. Soc., 2, **18** (1918-19), 403-410.

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