Graphs and Combinatorics 3, 203–211 (1987)

Graphs and Combinatorics

Large Induced Degenerate Subgraphs

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Abstract. A graph H is *d-degenerate* if every subgraph of it contains a vertex of degree smaller than d. For a graph G, let $\alpha_d(G)$ denote the maximum number of vertices of an induced d-degenerate subgraph of G. Sharp lowers bounds for $\alpha_d(G)$ in terms of the degree sequence of G are obtained, and the minimum number of edges of a graph G with n vertices and $\alpha_2(G) \leq m$ is determined precisely for all $m \leq n$.

1. Introduction

All graphs considered here are finite and simple. A graph H is *d-degenerate* if every non-null subgraph of it contains a vertex of degree smaller than d. Thus 1-degenerate graphs are graphs with no edges and 2-degenerate graphs are forests. For a graph G and for $d \geq 1$, let $\alpha_d(G)$ denote the maximum number of vertices of an induced d-degenerate subgraph of G. In this paper we study the minimum possible number of edges $e_d(n, m)$ of a graph G with a given number n of vertices and a given value m of $\alpha_d(G)$. Notice that since $\alpha_1(G)$ is just the independence number of G, the numbers $e_1(n, m)$ are determined by the well-known theorem of Turán [6], which asserts that $e_1(n, m)$ is the number of edges of a disjoint union of m cliques, whose sizes are as equal as possible and whose total size is n. This gives

$$e_1(n, m) = \sum_{i=0}^{m-1} \binom{[(n+i)/m]}{2}. \tag{1.1}$$

The situation is more complicated for $d = 2$. For $m \geq \frac{2n}{3}$, let $G^1(n, m)$ denote the graph consisting of the disjoint union of $n - m$ triangles and $3m - 2n$ isolated vertices. For even $m = 2s \leq \frac{2}{3}n$ let $G^2(n, m)$ denote the disjoint union of s cliques,

* Research supported in part by an Allon Fellowship and by a Bat-Sheva de Rothschild grant.

** Research supported in part by NSF grant MCS 8301867, and by a Sloan Research Fellowship.

whose sizes are as equal as possible, and whose total size is n. For odd $m = 2s + 1 \leq \frac{2n}{3}$ let $G^3(n,m)$ denote the disjoint union of one isolated vertex and s cliques, whose sizes are as equal as possible, and whose total size is $n - 1$. One can easily check that each of these graphs $G^i(n,m)$ has n vertices and satisfies $\alpha_2(G^i(n,m)) = m$. The following theorem asserts that any graph G with n vertices and with $\alpha_2(G) = m$ has at least as many edges as the corresponding $G^i(n,m)$.

Theorem 1.1. *Let $G = (V,E)$ be a graph with n vertices, and let $\alpha_2(G) = m$. If $m \geq \frac{2n}{3}$ then $|E| \geq |E(G^1(n,m))|$. If $m \leq \frac{2}{3}n$ is even then $|E| \geq |E(G^2(n,m))|$, and if $m \leq \frac{2}{3}n$ is odd then $|E| \geq E(G^3(n,m))$. Thus, the function $e_2(n,m)$ is determined by the following formula.*

$$\begin{cases} \text{(i)} \quad \text{For } m \geq 2n/3, & e_2(n,m) = 3(n-m). \\ \text{(ii)} \quad \text{For } m = 2s \leq 2n/3, & e_2(n,m) = \sum_{i=0}^{s-1} \binom{[(n+i)/s]}{2}. \\ \text{(iii)} \quad \text{For } m = 2s+1 \leq 2n/3, & e_2(n,m) = \sum_{i=0}^{s-1} \binom{[(n+i-1)/s]}{2}. \end{cases} \tag{1.2}$$

Turán's Theorem and Theorem 1.1 imply that for $d \leq 2$ and for all $d \leq m \leq n$, there is a graph G with n vertices, that satisfies $\alpha_d(G) = m$ and has the minimum possible number of edges, where G is a disjoint union of almost equal cliques and, possibly, some isolated vertices. This is not true for large values of d, as shown in the next Proposition.

Proposition 1.2. *As $d = 2s$ tends to infinity, $e_d(3d, \frac{3}{2}d) = (1 + o(1)) \cdot 3d^2$, whereas the minimum number of edges of a graph $G = (V,E)$ with $3d$ vertices which is a disjoint union of cliques and isolated vertices and satisfies $\alpha_d(G) = \frac{3}{2}d$ is $(1 + o(1)) \cdot \frac{25}{8}d^2$.*

The proof of Proposition 1.2 uses a random construction, which suggests that it might be hopeless to determine $e_d(n,m)$ precisely for all d, n and m. Interestingly, $e_d(n,m)$ can be determined precisely in many cases. In particular, we can determine $e_d(n,m)$ precisely for all triples (d,n,m) where $d|m$ and $m \leq n/2$. To do this we prove the following result which supplies a lower bound for $\alpha_d(G)$ in terms of the degree sequence of G.

Theorem 1.3. *Let $G = (V,E)$ be a graph and let $d(v)$ denote the degree of $v \in V$. Then*

$$\alpha_d(G) \geq \sum_{v \in V} \min\left(1, \frac{d}{d(v)+1}\right). \tag{1.3}$$

This bound is sharp for every G which is a disjoint union of cliques. Moreover, there is a polynomial time algorithm that finds in G an induced d-degenerate subgraph of this size or greater.

For $d = 1$ (1.3) reduces to $\alpha(G) \geq \sum_{v \in V} \frac{1}{d(v)+1}$. This special case was proved by Wei [7] and independently by Caro (cf. [5]) It also follows easily from a well-known result of Erdös (cf. [3, Theorem VI. 1.4], [5]). Theorem 1.3 implies the following.

Corollary 1.4. *Let $G = (V, E)$ be a graph with n vertices and average degree $\bar{d} \geq 2d - 2$. Then $\alpha_d(G) \geq \frac{nd}{1 + \bar{d}}$.*

For $d = 1$ this reduces to the well-known estimate $\alpha(G) \geq n/(1 + \bar{d})$ for the independence number of a graph G (see, e.g. [2, Corollary 2 to Theorem 13.5]). We remark that every d-degenerate graph is d-colorable and thus Theorem 1.3 and Corollary 1.4 supply lower bounds for the maximum number of vertices of an induced d-colorable subgraph of G.

Our paper is organized as follows. In Section 2 we prove a slightly strengthened version of Theorem 1.1. In Section 3 we prove Proposition 1.2 and Theorem 1.3. The final Section 4 contains a few related problems.

2. Induced Acyclic Subgraphs

In this section we prove Theorem 1.1. For convenience we split the proof into three lemmas.

Lemma 2.1. *Suppose that $m \geq 2n/3$, and let $G = (V, E)$ be a graph with n vertices, e edges and with $\alpha_2(G) \leq m$. Then*

$$e \geq 3(n - m). \tag{2.1}$$

Moreover, if $m > 2n/3$ and equality holds in (2.1), then G has at least one isolated vertex.

Proof. We first prove the inequality (2.1) for every fixed m by induction on n for $m \leq n \leq \frac{3}{2}m$. The inequality is trivial for $n = m$. Assuming it holds for $n - 1$ we prove it for n. If there is a vertex $v \in V$ whose degree $d(v)$ is at least 3, let $H = G - v$. Then since $\alpha_2(H) \leq m$ and hence, by the induction hypothesis, H has at least $3(n - 1 - m)$ edges, we deduce that $e \geq 3 + 3(n - 1 - m) = 3(n - m)$, as needed. Therefore we can assume that the maximum degree of a vertex of G is ≤ 2, i.e., G is a union of paths and cycles. Clearly, the number of cycles must be at least $n - m$ (since the graph obtained by deleting one vertex from each of them is acyclic, i.e., 2-degenerate). Since each cycle has at least 3 edges we conclude that $e \geq 3(n - m)$. This completes the proof of (2.1). Suppose, now, that $m > 2n/3$ and that equality holds in (2.1), i.e., $e = 3(n - m) < n$. Now G contains no bridges, for if f is a bridge of G then $\alpha_2(G - f) = \alpha_2(G) \leq m$, which is impossible since $|E(G - f)| < 3(n - m)$. Therefore G contains no vertices of degree 1. Since the average degree of vertices of G is smaller than 2, it follows that G has at least one isolated vertex, as needed. $\square$

Lemma 2.2. *Let s be an integer, let $m = 2s \le 2n/3$ and let $G = (V, E)$ be a graph with n vertices and e edges, and with $\alpha_2(G) \le m$. Then there are s positive integers $n_1, n_2, \ldots, n_s$ such that*

$$\sum_{i=1}^{s} n_i = n \quad \textit{and} \quad e \ge \sum_{i=1}^{s} \binom{n_i}{2}. \tag{2.2}$$

Hence

$$e \ge \sum_{i=0}^{s-1} \binom{[(n+i)/s]}{2}. \tag{2.3}$$

Proof. Obviously (2.2) implies (2.3). We prove (2.2) for every fixed $m = 2s$ by induction on n for $n \ge \frac{3}{2}m = 3s$. For $n = 3s$ (2.2) with $n_1 = n_2 = \cdots = n_s = 3$ follows from Lemma 2.1. Assuming (2.2) holds for $n - 1$ we prove it for n, $n > 3s$. Let $v \in V$ be a vertex of maximum degree $d(v) = \Delta$ in G. Put $H = G - v$. Clearly $|V(H) = n - 1$ and $\alpha_2(H) \le m \le \frac{2}{3}(n-1)$. Hence, by the induction hypothesis, there are s positive integers $l_1 \le l_2 \le \cdots \le l_s$ such that $\sum l_i = n - 1$ and $|E(H)| \ge \sum_{i=1}^{s} \binom{l_i}{2}$. In particular $|E(H)| \ge \frac{1}{2}(n-1)\cdot(l_1 - 1)$. On the other hand,

$$|E(H)| \le \tfrac{1}{2}(\Delta \cdot (\Delta - 1) + (n - 1 - \Delta)\cdot \Delta) < \tfrac{1}{2}(n-1)\cdot \Delta.$$

Indeed, by the choice of v, every vertex of H has degree at most Δ, and each of the Δ neighbors of v in G has degree at most $\Delta - 1$ in H. Hence $\frac{1}{2}(n-1)\cdot \Delta > \frac{1}{2}(n-1)(l_1 - 1)$ and $\Delta > l_1 - 1$, i.e., $\Delta \ge l_1$. Define $n_1 = l_1 + 1$ and $n_i = l_i$ for $2 \le i \le s$. Then $\sum_{i=1}^{s} n_i = n$ and

$$e = |E(G)| = \Delta + |E(H)| \ge l_1 + \sum_{i=1}^{s} \binom{l_i}{2} = \sum_{i=1}^{s} \binom{n_i}{2}.$$

This completes the proof of (2.2). □

Lemma 2.3. *Let s be an integer and suppose that $m = 2s + 1 \le \dfrac{2n+1}{3}$. Let $G = (V, E)$ be a graph with n vertices and e edges and suppose that $\alpha_2(G) \le m$. Then there are s positive integers $n_1, n_2, \ldots, n_s$ such that*

$$\sum_{i=1}^{s} n_i = n - 1 \quad \textit{and} \quad e \ge \sum_{i=1}^{s} \binom{n_i}{2}. \tag{2.4}$$

Hence

$$e \ge \sum_{i=0}^{s-1} \binom{[(n+i-1)/s]}{2}. \tag{2.5}$$

Moreover, if equality holds in (2.5) then G has an isolated vertex.

Proof. For every fixed $m = 2s + 1$ we apply induction on n for $n \ge \dfrac{3m-1}{2} = 3s + 1$. For $n = 3s + 1$, (2.4) with $n_1 = \cdots = n_2 = 3$ (which is (2.5)) follows from

Lemma 2.1. Lemma 2.1 also asserts that if equality holds here then G has an isolated vertex. Assuming the assertion of Lemma 2.3 holds for $n-1$, we prove it for n, $n > 3s+1$. Let $v \in V$ be a vertex of maximum degree $d(v) = \Delta$ in G. Put $H = G - v$. Clearly $|V(H)| = n-1$ and $\alpha_2(H) \le m = 2s+1 \le \frac{2(n-1)+1}{3}$. Hence, by the induction hypothesis,

$$|E(H)| \ge \sum_{i=0}^{s-1} \binom{l_1}{2}, \tag{2.6}$$

where $l_i = [(n+i-2)/s]$ for $0 \le i < s$. Moreover, it equality holds in (2.6) then H has an isolated vertex. By (2.6)

$$2|E(H)| \ge (n-2)(l_0 - 1) \tag{2.7}$$

and equality can hold here only if equality holds in (2.6) and $l_0 = l_{s-1}$. On the other hand, each of the Δ neighbors of v in G have degree at most $\Delta - 1$ in H, and each other vertex of H has degree at most Δ. Hence

$$2|E(H)| \le \Delta \cdot (\Delta - 1) + (n - 1 - \Delta) \cdot \Delta = (n-2) \cdot \Delta. \tag{2.8}$$

Equality can hold here only if precisely Δ vertices of H have degree $\Delta - 1$ in H and all other vertices have degree Δ in H. By (2.7) and (2.8) $\Delta \ge l_0 - 1$ and equality can hold only if equality holds both in (2.7) and in (2.8). However, if equality holds in (2.7) then equality holds in (2.6) and hence H contains an isolated vertex. But in this case the minimum degree of a vertex of H is 0, which is smaller than $\Delta - 1 \ge l_0 - 2$, since $l_0 \ge 3$ as $n \ge 3s + 2$ and $l_0 = [(n-2)/s]$. Hence in this case equality cannot hold in (2.8). We thus conclude that $\Delta > l_0 - 1$, i.e., $\Delta \ge l_0$. Define $n_0 = l_0 + 1$ and $n_i = l_i$ for $1 \le i < s$. Then $\sum_{i=0}^{s-1} n_i = n - 1$ and

$$e = |E(G)| = \Delta + |E(H)| \ge l_0 + \sum_{i=0}^{s-1} \binom{l_i}{2} = \sum_{i=0}^{s-1} \binom{n_i}{2}. \tag{2.9}$$

This implies (2.4) and hence also (2.5). It remains to show that if equality holds in (2.5) then G contains an isolated vertex. Suppose equality holds in (2.5). Then equality holds in (2.9). As $\Delta \ge l_0$, this implies that equality holds in (2.6), since otherwise inequality (2.9) is strict. But this means that H has an isolated vertex w. Thus, the degree of w in G is either 0 or 1. However, if it is 1 then G contains a bridge f, which is impossible since otherwise $\alpha_2(G - f) = \alpha_2(G) \le m$ and $G - f$ does not have enough edges. Therefore w is an isolated vertex of G. This completes the proof. □

Theorem 1.1 clearly follows from Lemmas 2.1, 2.2 and 2.3. Thus the function $e_2(n, m)$ is determined by formula (1.2).

3. Induced d-degenerate Subgraphs

Proof of Theorem 1.3. Let $G = (V, E)$ be a graph and let $d_G(v)$ denote the degree of $v \in V$. Define the *weight* $w_G(v)$ of v to be 1 if $d_G(v) < d$ and $d/(d_G(v) + 1)$ otherwise.

(Notice that the two definitions coincide if $d_G(v) = d - 1$.) The weight $w(G)$ of G is defined to be $\sum_{v \in V} w_G(v)$. In this notation inequality (1.3) is just the assertion that $\alpha_d(G) \geq w(G)$. We prove this statement by induction on the number n of vertices of G. It is trivial for $n = 1$. Assuming it holds for $n - 1$, we prove it for n. If there is a vertex $v \in V$ with $d_G(v) < d$, let $H = G - v$. Then $\alpha_d(G) = \alpha_d(H) + 1$; but since $d_H(u) \leq d_G(u)$ for all $u \in V \backslash \{v\}$, $w(H) \geq \sum_{u \in V \backslash \{v\}} w_G(u) = w(G) - 1$. By the induction hypothesis $\alpha_d(H) \geq w(H) - 1$ and hence $\alpha_d(G) = \alpha_d(H) + 1 \geq w(G)$, as needed. Thus we may assume that $d_G(v) \geq d$ for all $v \in V$, and therefore $w_G(v) = d/(d_G(v) + 1)$ for all $v \in V$. Let u be a vertex of maximum degree $d_G(u) = \Delta$ in G. Put $H = G - u$. One can easily check that $w(H) \geq w(G)$. Indeed, if $u_1, u_2, \ldots, u_\Delta$ are the Δ neighbors of u in G then

$$\begin{aligned} w(H) &= \sum_{v \in V \backslash \{u\}} \frac{d}{d_H(v) + 1} \\ &= w(G) - \frac{d}{\Delta + 1} - \sum_{i=1}^{\Delta} \frac{d}{d_G(u_i) + 1} + \sum_{i=1}^{\Delta} \frac{d}{d_G(u_i)} \\ &= w(G) - \frac{d}{\Delta + 1} + \sum_{i=1}^{\Delta} \frac{d}{d_G(u_i) \cdot (d_G(u_i) + 1)} \\ &\geq w(G) - \frac{d}{\Delta + 1} + \frac{d\Delta}{\Delta(\Delta + 1)} = w(G). \end{aligned}$$

By the induction hypothesis $\alpha_d(H) \geq w(H) \geq w(G)$, and since $\alpha_d(G) \geq \alpha_d(H)$, inequality (1.3) follows:

If $G = (V, E)$ is a disjoint union of s cliques of sizes $l_1 \leq l_2 \leq \cdots \leq l_s$ then clearly

$$\alpha_d(G) = \sum_{i=1}^{s} \min(d, l_i) = \sum_{v \in V} \min\left(1, \frac{d}{d_G(v) + 1}\right) = w(G),$$

i.e., inequality (1.3) is sharp. It is also clear from the proof that there is a polynomial time algorithm that finds in a given graph G an induced d-degenerate subgraph H with at least $w(G)$ vertices. This completes the proof of Theorem 1.3. □

Suppose, now, that $G = (V, E)$ is a graph with n vertices and e edges. One can easily check that if $e < \binom{d+1}{2}$ then $\alpha_d(G) = n$ so we may assume $e \geq \binom{d+1}{2}$. By Theorem 1.3 $\alpha_d(G) \geq w$, where w is the minimum possible value of the expression

$$\sum_{i=1}^{n} \min\left(1, \frac{d}{d_i + 1}\right) \tag{3.1}$$

subject to the constraints

$$\sum_{i=1}^{n} d_i = 2e \quad \text{and} \quad 0 \leq d_i \text{ are integers.} \tag{3.2}$$

(In fact, $d_1, \ldots, d_n$ should also be a degree sequence of a simple graph, but we will not use this fact here.) Suppose that the minimum of (3.1) subject to (3.2) is obtained for $d_i = b_i (i = 1, \ldots, n)$. Clearly if $b_i < d$ for some i we may assume $b_i = 0$, since a replacement of such a b_i by 0 and a replacement of some other b_j by $b_j + b_i$ does not increase the sum (3.1), (for $d_i = b_i$). Also, we may assume that the set of positive b_i's attains at most two consecutive values, since $\frac{1}{b_1+1} + \frac{1}{b_2-1} < \frac{1}{b_1} + \frac{1}{b_2}$ for $d \leq b_1 < b_2 - 1$. Similarly, one can easily verify the following two simple facts.

Fact 1. *If $2e \geq m \cdot (2d-2)$ for some $m \leq n$, and the number of positive b_i's is l, where $l < m$, then the sum (3.1) is not increased by changing one of the zeros to $2d-2$ and by decreasing the positive b_i's by a total of $2d-2$ in such a way that each of them is still at least $2d-2$.*

Fact 2. *If $2e \leq m \cdot (2d-1)$ for some $m < n$ and the number of positive b_i's is l, where $l > m$, then the sum (3.1) is not increased by changing the minimum of the positive b_i's from its value, say x, to 0, and by increasing the other positive b_i's by a total of x in such a way that each of them is still at most $2d-1$.*

Using the above two facts we prove the following Proposition, whose first part implies Corollary 1.4.

Proposition 3.1. *Let $G = (V, E)$ be a graph with n vertices, e edges and average degree $\bar{d} = 2e/n$. Then:*

(i) if $\bar{d} \geq 2d-2$ and $2e = nk + r$, where $0 \leq r < n$, then $\alpha_d(G) \geq \frac{d \cdot r}{k+2} + \frac{d \cdot (n-r)}{k+1} \geq \frac{d \cdot n}{1+\bar{d}}$.

(ii) if $2e$ can be written as a sum of $m \leq n$ positive integers $b_1, \ldots, b_m$, where each b_i is either $2d-2$ or $2d-1$, then $\alpha_d(G) \geq \sum_{i=1}^{m} \frac{d}{b_i+1}$.

Proof. (i) By Fact 1 and the above discussion, the minimum of (3.1) subject to (3.2) is obtained when all the b_i's are positive and are as equal as possible. This implies, by Theorem 1.3, that $\alpha_d(G) \geq \frac{dr}{k+2} + \frac{d \cdot (n-r)}{k+1}$. The last quantity is at least $d \cdot n/(1+\bar{d})$, by the convexity of the function $g(y) = 1/(y+1)$.
(ii) In this case clearly $m \cdot (2d-2) \leq 2e \leq m \cdot (2d-1)$ and hence, by Fact 1 and Fact 2, we may assume that the number of positive b_i's for which the minimum of (3.1) subject to (3.2) is obtained is m. Since the positive b_i's should be as equal as possible, the desired result follows, by Theorem 1.3. □

Proposition 3.1 is sharp whenever there is a disjoint union of cliques and isolated vertices, whose degree sequence is the sequence (b_i). This supplies the exact value of $e_d(n, m)$ in many cases, including, e.g., all (d, n, m), where d divides m and $m \leq n/2$. For this case $e_d(n, m)$ is the number of edges of the disjoint union of m/d almost equal cliques whose total size is n. Also, if G is a disjoint union of cliques and isolated vertices, where each clique is of size $2d-1$ or $2d$, then G has the minimum number of edges among all graphs with $|V(G)|$ vertices whose largest induced d-degenerate subgraph is of size $\alpha_d(G)$. Notice that G is not unique in general, since we can replace any set of d cliques of size $2d-1$ by $d-1$ cliques of size $2d$ and d isolated vertices without changing either the number of edges or the value of $\alpha_d(G)$.

We conclude this section with the proof of Proposition 1.2, which shows that for some parameters (d, n, m), the extremal graph is not a disjoint union of cliques and isolated vertices.

Proof of Proposition 1.2. By Corollary 1.4 $e_d(3d, \frac{3}{2}d) \geq \frac{3}{2}d(2d-1) = (1+o(1))3d^2$. Indeed, if G has $3d$ vertices and less than $\frac{3}{2}d(2d-1)$ edges, then its average degree $\bar{d}$ is smaller than $2d-1$ and hence G has an induced d-degenerate subgraph on more than $\frac{3}{2}d$ vertices. One can easily check that any disjoint union of cliques G on $3d$ vertices with $\alpha_d(G) = \frac{3}{2}d$ has at least as many edges as $K_{(5/2)d}$, i.e., at least $(1+o(1))\frac{25}{8}d^2$ edges.

It remains to show that for every $\varepsilon > 0$, if $d > d(\varepsilon)$ is even then $e_d(3d, \frac{3}{2}d) \leq (1+\varepsilon)3d^2$. Given $\varepsilon > 0$, let d be a large even number, and let $G = (V, E)$ be a random graph on a set V of $3d$ vertices, in which each edge is chosen, independently, with probability $\frac{2+\varepsilon}{3}$. The expected number of edges of G is $\binom{3d}{2} \cdot \frac{2+\varepsilon}{3} < \left(1+\frac{\varepsilon}{2}\right)3d^2$, and thus, by the standard estimates for binomial distribution (see, e.g., [4]), for sufficiently large d, the probability that G has more than $(1+\varepsilon)3d^2$ edges is at most 1/2. To complete the proof we show that for large d, the probability that G has an induced d-degenerate subgraph H on $\frac{3}{2}d$ vertices is smaller than 1/2. This will show that there exists a G with at most $(1+\varepsilon)3d^2$ edges and with $\alpha_d(G) \leq \frac{3}{2}d$, as needed. If H is d-degenerate, then its vertices can be linearly ordered in such a way that each vertex will have fewer than d neighbors in H among the vertices following it. Therefore, if H is a d-degenerate subgraph of G with $\frac{3}{2}d$ vertices, then the set of vertices of H can be partitioned into two disjoint sets of vertices, A and B, where $|A| = \left\lfloor \frac{\varepsilon}{10}d \right\rfloor$ and $|B| = \left\lceil \left(\frac{3}{2} - \frac{\varepsilon}{10}\right)d \right\rceil$, such that each $a \in A$ has fewer than d neighbors in B. Hence, there are A and B of the above sizes in G, and the number of edges from A to B is less than $|A| \cdot d$. The expected number of edges between A and B is $|A| \cdot |B| \cdot \frac{2+\varepsilon}{3} > \left(1+\frac{\varepsilon}{4}\right)|A| \cdot d$. By Chernoff's inequality (cf. e.g., [4]), the probability that for fixed A and B the number of $A-B$ edges will be smaller than $|A| \cdot d$ is bounded by $\exp(-c(\varepsilon)d^2)$ for some $c(\varepsilon) > 0$. Since the number of choices for A and B is bounded by

$$\binom{3d}{\frac{3}{2}d} \cdot \binom{\frac{3}{2}d}{|A|} < 2^{6d}$$

we conclude that for sufficiently large d, the probability that G will contain sets A and B with the above properties is smaller than 1/2. This completes the proof. $\square$

4. Concluding Remarks

Proposition 1.2 suggests that it might be hopeless to determine $e_d(n, m)$ precisely for all d, n and m. It might be interesting, however, to determine $e_d(n, m)$ for small values of $d(>2)$, and in particular to show that for moderate d the extremal graphs are always disjoint unions of cliques and isolated vertices.

Another interesting problem concerns the question discussed here for some restricted classes of graphs. For example, one might try and improve Theorem 1.3 and Corollary 1.4 for triangle-free graphs or for planar graphs. For the former class, the methods of Ajtai, Komlós Szemerédi [1] (who obtain a better estimate than the one given by Corollary 1.4 for $\alpha_1(G)$, if G is triangle-free) might be useful. We conclude the paper with the following interesting conjecture of Akiyama which deals with the latter class.

Conjecture 4.1 (*Akiyama*). For every planar graph G with n vertices, $\alpha_2(G) \geq n/2$.

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Received: August 6, 1986
Revised: October 10, 1986