Playing to retain the advantage

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Abstract

Let P be a monotone decreasing graph property, let $G = (V, E)$ be a graph, and let q be a positive integer. In this paper, we study the $(1 : q)$ Maker-Breaker game, played on the edges of G, in which Maker's goal is to build a graph that does not satisfy the property P. It is clear that in order for Maker to have a chance of winning, G must not satisfy P. We prove that if G is far from satisfying P, that is, if one has to delete sufficiently many edges from G in order to obtain a graph that satisfies P, then Maker has a winning strategy for this game. We also consider a different notion of being far from satisfying some property, which is motivated by a problem of Duffus, Łuczak and Rödl [6].

1 Introduction

Let X be a finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets. In the $(p : q)$ Maker-Breaker game $(X, \mathcal{F})$, two players, called Maker and Breaker, take turns in claiming previously unclaimed elements of X, with Maker going first. The set X is called the "board" of the game and the members of $\mathcal{F}$ are referred to as

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the "winning sets". Maker claims p board elements per turn, whereas Breaker claims q. The game ends when every board element has been claimed by some player. Maker wins the game if he occupies all elements of some winning set; otherwise Breaker wins. We say that a $(p : q)$ game $(X, \mathcal{F})$ is a *Maker's win* if Maker has a strategy that ensures his win in this game against any strategy of Breaker, otherwise the game is a *Breaker's win*. Note that p, q, X and $\mathcal{F}$ alone determine whether the game is a Maker's win or a Breaker's win.

In this paper we are interested in the following family of Maker-Breaker games. Let P be a monotone decreasing graph property, let $G = (V, E)$ be a graph, and let q be a positive integer. In the $(1 : q)$ Maker-Breaker game $(E, \mathcal{F}_P)$, Maker's goal is to build a subgraph of G that *does not* satisfy the property P, that is, $\mathcal{F}_P = \{E' \subseteq E : G[E'] \notin P\}$. We are interested in properties of the graph G which guarantee that the $(1 : q)$ game $(E, \mathcal{F}_P)$ is a Maker's win. Since P is a monotone decreasing property, it is clear that Maker will lose the game $(E, \mathcal{F}_P)$ if $G \in P$. On the other hand, it seems plausible that if G is somehow "sufficiently far" from satisfying P, then Maker should be able to win the game. This notion of farness is made precise in the following definition:

Definition 1.1 *Let $\varepsilon > 0$, let P be a monotone decreasing graph property, and let $G = (V, E)$ be a graph with m edges. G is said to be ε-*far from satisfying the property P *if one has to delete at least εm edges from G in order to obtain a graph that satisfies P.*

We prove the following general result.

Theorem 1.2 *Let $\varepsilon > 0$, let P be a monotone decreasing graph property, and let $G = (V, E)$ be a graph with n vertices and $m = \Theta(n^2)$ edges. If G is ε-far from satisfying P and n is sufficiently large, then there exist positive constants c and $\alpha = \alpha(\varepsilon, P)$, such that Maker has a winning strategy for the $(1 : q)$ game $(E, \mathcal{F}_P)$ for every $q \leq cn^{\alpha}$.*

The bound on q given in Theorem 1.2 is clearly best possible up to the value of α. For some properties, such as the property P_H of being H-free, the value of α obtained in the proof of Theorem 1.2, is best possible as well. That is, one can determine the largest value of q for which the $(1 : q)$ game $(E, \mathcal{F}_{P_H})$ is a Maker's win, up to a multiplicative constant factor (we discuss this fact in more detail in Section 4). For general properties, however, this is not the case. Indeed, for the special case in which P is the property of being r-colorable, we prove the following stronger result:

Theorem 1.3 *Let r be a positive integer and let $P(r)$ be the property of being r-colorable. Let $\varepsilon > 0$ and let $G = (V, E)$ be a graph with n vertices and m edges, where n is sufficiently large. If G is ε-far from being r-colorable, then Maker has a winning strategy for the $(1 : q)$ game $(E, \mathcal{F}_{P(r)})$, for every $q \leq \frac{c\varepsilon^2 m}{n \log r}$, where $c > 0$ is an appropriate constant.*

The bound on q given here is not far from being tight, at least for fixed values of r. This is discussed further in the final section.

A different notion of farness was considered by Duffus, Łuczak and Rödl in [6]. They study the $(1 : q)$ Maker-Breaker $(E, \mathcal{F}_{P(r)})$ game, where $P(r)$ is the property of being r-colorable, and G is a graph on n vertices with chromatic number χ. Duffus, Łuczak and Rödl asked how large should χ be in order to ensure that this game is a Maker's win. They conjectured that this value of χ is independent of n. We prove the following weaker result.

Theorem 1.4 *Let q and r be positive integers. There exists a constant $c = c(q, r)$ such that, if G is a graph on n vertices and $\chi(G) \geq c \log n$, then Maker has a winning strategy for the $(1 : q)$ game $(E, \mathcal{F}_{P(r)})$.*

In fact Duffus, Łuczak and Rödl consider the somewhat different setting, in which G is a hypergraph, and the players claim **vertices** of G. However, they mention that already the case where G is a graph and $q = r = 2$ is open. In the special case where $G = (V, E)$ is a graph, they consider the game $(V, \mathcal{F}_{P(r)})$ rather than $(E, \mathcal{F}_{P(r)})$ (that is, the players claim vertices and not edges). We however feel that, for graphs, the edge version is the more natural one. Still, the result stated in Theorem 1.4, holds for the vertex version as well.

Note that for $q = 1$ one can easily obtain a stronger result than the one ensured by Theorem 1.4, by using a **strategy stealing** argument. Indeed, let G be a graph on n vertices satisfying $\chi(G) > r^2$. Let G_M and $G_B = G \setminus G_M$ denote the subgraphs of G, built by Maker and Breaker respectively during the game (played according to some strategies). Clearly $\chi(G_M)\chi(G \setminus G_M) \geq \chi(G)$. Hence, either $\chi(G_M) > r$ or $\chi(G_B) > r$. Assume for the sake of contradiction that no strategy of Maker guarantees $\chi(G_M) > r$. It follows from the above, that there exists a strategy S_B of Breaker, that ensures $\chi(G_B) > r$, regardless of Maker's strategy. However, Maker can *steal* S_B, that is, he can claim an arbitrary first edge and then play according to S_B, pretending to be the second player (whenever he is supposed to claim an edge which is already his, he claims an arbitrary free edge). It follows by the definition of S_B that $\chi(G_M) > r$ contrary to our assumption. Note that strategy stealing is a purely existential argument; we do not know of any explicit strategy for Maker, that ensures his win in the game with these parameters. On the other hand, Theorem 1.4 gives the currently best bound for any $q \geq 2$ and $r \geq 2$.

One can apply the Duffus, Łuczak and Rödl notion of farness to other graph properties besides the property of being r-colorable. Consider the property P_k of failing to be k-edge-connected. Is there a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, such that for every $f(k,q)$-edge-connected graph $G = (V, E)$, Maker has a winning strategy for the $(1 : q)$ game $(E, \mathcal{F}_{P_k})$? (Note that for this particular problem, a more natural formal setting would be for Maker to play to create a k-edge-connected graph starting from an f-edge-connected graph, that is, to try and retain high edge connectivity. However, the chosen setting, where Maker's goal is to *not* build a *not* k-edge-connected graph, fits the general setting of the paper, where Maker strives *n*ot to create a graph possessing a monotone *d*ecreasing property P.) For $q = 1$ the answer is yes. Indeed, a classical theorem of Nash-Williams [12] and of Tutte [13] asserts that $4k$-edge-connectivity ensures the existence of $2k$ pairwise edge disjoint spanning trees. These trees, in turn, ensure Maker's win by the classical theorem of Lehman [11]. Hence, $f(k, 1) = 4k$ suffices for every $k \in \mathbb{N}$. For $q \geq 2$ and relatively small values of k, the answer is no. Indeed, consider a complete bipartite graph $G = (A \cup B, E)$, where $|A| = c \log n$ and $|B| = n - c \log n$, for an appropriate constant $c > 0$. By using a Box-Game strategy (see [5]), we conclude that Breaker can isolate some vertex of B and thus win the game. For "large" values of k we prove the following:

Theorem 1.5 *Let $G = (V, E)$ be a graph on n vertices, and let $q \geq 2$ and $k = k(n) \geq \log_2 n$ be integers. If G is $(100kq \log_2 q)$-edge-connected, then Maker has a winning strategy for the $(1 : q)$ game $(E, \mathcal{F}_{P_k})$.*

There is a striking relation between the theory of Positional Games and the theory of Random Graphs, known as the *Erdős paradigm*. Roughly speaking, it asserts that, in some games, playing randomly and playing according to an optimal strategy yields the same outcome. Theorem 1.5 is an example of this paradigm. It asserts, in particular, that if $G = (V, E)$ is $(c \log_2 |V|)$-edge-connected for an appropriate constant $c > 0$, then if Maker follows his optimal strategy, he will win the $(1 : 2)$ game $(E, \mathcal{F}_{P_1})$, (that is, will succeed to construct a connected graph), regardless of Breaker's strategy. If on the other hand both players play randomly, then the graph built by Maker can be viewed as a random subgraph of G with $|E|/3$ edges. The fact that such a graph is a.s. connected follows from a result of [1]. The aforementioned assumption on edge connectivity of G, is easily seen to be tight for the random game, up to the value of c.

1.1 Notation and preliminaries

For the sake of simplicity and clarity of presentation, we do not make a particular effort to optimize the constants obtained in our proofs. We also omit floor and ceiling signs whenever these are not crucial. Most of our results are asymptotic in nature and whenever necessary we assume that n is sufficiently large. Throughout the paper, log stands for the natural logarithm, unless stated otherwise. Our graph-theoretic notation is standard and follows that of [14]. In particular, we use the following.

For a graph G, let $V(G)$ and $E(G)$ denote its sets of vertices and edges respectively, and put $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For a set $A \subseteq V(G)$, let $E_G(A)$ denote the set of edges of G with both endpoints in A, and let $e_G(A) = |E_G(A)|$. For disjoint sets $A, B \subseteq V(G)$, let $E_G(A, B)$ denote the set of edges of G with one endpoint in A and one endpoint in B, and let $e_G(A, B) = |E_G(A, B)|$. Sometimes, if there is no risk of confusion, we discard the subscript G in the above notation. For a set $S \subset V(G)$, let $\bar{S} = V(G) \setminus S$. Let $(S, \bar{S})$ denote the edge-cut that separates S from $\bar{S}$, that is, $(S, \bar{S}) = E_G(S, \bar{S})$. For a set $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G, induced on the vertices of S. For a set $F \subseteq E(G)$, let $G[F]$ denote the subgraph of G, spanned by the edges of F. For a graph H let $m_2(H) := \max\left\{\frac{e(G)-1}{v(G)-2} : G \subseteq H, v(G) \geq 3\right\}$ denote its 2-*density*. Let $G = (V, E)$ be a graph with m edges, and let $0 \leq t \leq m$ be an integer. The *random graph* $G(t)$ is the graph obtained from G by randomly deleting $m - t$ edges from G uniformly amongst all elements of $\binom{E}{m-t}$. For positive integers r and k, and for a fixed graph H, let $P(r)$, P_k and P_H denote the property of being r-colorable, the property of not being k-edge-connected, and the property of being H-free, respectively.

The following fundamental theorem, due to Beck [3], is a useful sufficient condition for Breaker's win in the $(p : q)$ game $(X, \mathcal{F})$.

Theorem 1.6 *Let X be a finite set and let $\mathcal{F} \subseteq 2^X$. If $\sum_{B \in \mathcal{F}} (1+q)^{-|B|/p} < \frac{1}{1+q}$, then Breaker (as first or second player) has a winning strategy for the $(p : q)$ game $(X, \mathcal{F})$.*

The rest of the paper is organized as follows: in Section 2 we prove Theorems 1.2 and 1.3, and in Section 3 we prove Theorems 1.4 and 1.5. Finally, in Section 4 we present some open problems.

2 Monotone properties

Proof of Theorem 1.2: Maker's strategy is to build a graph which contains a relatively small subgraph that does not satisfy P. Since P is monotone decreasing, this will ensure his win. The existence of such a subgraph is guaranteed by the following fundamental theorem of [2]:

Theorem 2.1 *Let $\varepsilon > 0$ and let P be a monotone decreasing property. Let G be a graph with n vertices and $\Theta(n^2)$ edges, which is ε-far from satisfying P. Then there exists a graph H on $h = h(\varepsilon, P)$ vertices that does not satisfy P, and a constant $\gamma = \gamma(\varepsilon, P) > 0$, such that G contains at least γn^h copies of H.*

If there exists a forest F that does not satisfy P, then Maker wins regardless of his strategy, for every sufficiently large n and $q = o(n)$. Indeed, Maker's graph will have $\omega(n)$ edges, and will thus contain every fixed forest. Hence, we assume that P contains all forests. Let H and γ be as in Theorem 2.1; by the above we can assume that H contains a cycle. We will prove that Maker can claim a copy of H in G. In order to do so, we need the following lemma:

Lemma 2.2 *Let $G = (V, E)$ be a graph with n vertices and $m \geq c_1 n^2$ edges, where $c_1 > 0$ is a constant. Let H be a fixed graph on h vertices that contains a cycle, and let $0 < \gamma < 1$ be a constant. Assume that G contains at least γn^h copies of H. Then, there exist constants $c_2 > 0$ and $0 < \delta < 1$ such that with probability at least $2/3$, a random graph $G(t)$ with $t = c_2 m n^{-1/m_2(H)}$ edges is such that every subgraph of it with at least $(1-\delta)t$ edges contains a copy of H.*

The proof of Lemma 2.2 is a straightforward adaptation of the proof of Lemma 4 from [4], where a similar result is proved for the special case $G = K_n$. For the sake of completeness we include a short sketch of the proof.

Proof We will use the following fact which can be proved through standard methods: for $t' := t/2$, there exists a constant $c' > 0$ such that $Pr(H \not\subseteq G(t')) \leq e^{-c't}$.

Let $0 < \delta < 1/2$ be a constant, small enough to satisfy $\delta - \delta \log \delta < c'$. We count pairs (F, F') such that F is a subgraph of G with t edges, and F' is a subgraph of F with $(1-\delta)t \geq t'$ edges that does not contain a copy of H. Counting from the view point of F' and using the above fact, we conclude that the number of such pairs is at most

$$e^{-c't}\binom{m}{(1-\delta)t}\binom{m-(1-\delta)t}{\delta t}.$$

Hence we conclude that the number of such pairs is at most

$$\begin{aligned}\binom{m}{t}\binom{t}{\delta t}e^{-c't} &\leq (e/\delta)^{\delta t}\, e^{-c't}\binom{m}{t} \\ &\leq \frac{1}{3}\binom{m}{t},\end{aligned}$$

where the last inequality holds for sufficiently large n. It follows that the probability that $G(t)$ contains a subgraph on $(1-\delta)t$ vertices which does not contain a copy of H is at most $1/3$, as claimed. $\square$

We are now ready to describe Maker's strategy. Let $c_2 > 0$ and $0 < \delta < 1$ be the constants whose existence is guaranteed by Lemma 2.2. In the first $t := \delta m/(2q+2)$ rounds, Maker plays randomly, that is, before the game starts Maker draws uniformly at random t edges of G. Let $\{e_i\}_{i=1}^{t}$ be an arbitrary ordering of the chosen edges; these edges are kept secret from Breaker until they are claimed. In his ith move, for $1 \leq i \leq t$, Maker claims the edge e_i. If e_i has been previously claimed by him or by Breaker, then he claims an arbitrary free edge. If e_i is claimed by Breaker, then it is declared a *failure*. In his ith move, for $i > t$, Maker claims an arbitrary free edge.

Denote the aforementioned strategy of Maker by S_M, and let S_B be an arbitrary fixed strategy of Breaker. We claim that if Maker follows S_M and Breaker follows S_B, then Maker wins with positive probability. First we prove the following lemma.

Lemma 2.3 *With probability at least* $1/2$ *there are at most* δt *failures.*

Proof During the first t rounds, both players claim together at most $t(q+1) = \delta m/2$ edges. It follows that, for every $1 \leq i \leq t$, the probability that e_i is a failure is at most $\delta/2$. Let F be the random variable that counts the number of failures, then $\mathbb{E}(F) \leq \delta t/2$. It follows by Markov's inequality that $Pr(F \geq \delta t) \leq 1/2$. $\square$

We are now ready to prove our claim. Let M_t denote the graph built by Maker during the first t rounds. The graph $M_t \cap \{e_1, \ldots, e_t\}$ can be viewed as a random graph $G(t)$ from which an adversary has removed some edges, namely, all the edges that were declared a failure. By Lemma 2.3, with probability at least $1/2$ there are at least $(1-\delta)t$ edges in M_t. But then, for $q \leq \frac{\delta}{2c_2} n^{1/m_2(H)} - 1$, it follows by Lemma 2.2 that M_t contains a copy of H with probability at least $2/3$.

Hence, with positive probability, Maker has won the game. Moreover, since S_B was arbitrary, this holds for any strategy of Breaker. Since this is a finite perfect information game with no chance moves, it follows that there exists a deterministic winning strategy for Maker to win this game against any strategy of Breaker, that is, the game is a Maker's win as claimed.

$\square$

Proof of Theorem 1.3:

Maker plays as follows. Let $\delta = \varepsilon/4$. In the first $t := \delta m/(q+1)$ rounds, Maker plays randomly, that is, before the game starts Maker draws uniformly at random t edges of G. Let $\{e_i\}_{i=1}^t$ be an arbitrary ordering of the chosen edges; these edges are kept secret from Breaker until they are claimed. In his ith move, for $1 \leq i \leq t$, Maker claims the edge e_i. If e_i has been previously claimed by him or by Breaker, then he claims an arbitrary free edge. If e_i is claimed by Breaker, then it is declared a *failure*. In his ith move, for $i > t$, Maker claims an arbitrary free edge.

Denote the aforementioned strategy of Maker by S_M, and let S_B be an arbitrary fixed strategy of Breaker. We claim that if Maker follows S_M and Breaker follows S_B, then Maker wins with positive probability. First we prove the following lemmas.

Lemma 2.4 *Let $G = (V, E)$ be as in Theorem 1.3. Let $\gamma = \delta/(q+1)$, and let $R \subseteq E$ be a set of size γm, drawn uniformly at random among all such sets. Then, with probability at least $2/3$, $G_R := (V, R)$ is $\varepsilon/2$-far from being r-colorable, that is, one has to delete at least $\varepsilon\gamma m/2$ edges from G_R in order to obtain an r-colorable graph.*

Proof Let $V = V_1 \cup \ldots \cup V_r$ be an arbitrary fixed partition of V into r parts. Since G is ε-far from being r-colorable, it follows that $f := \sum_{i=1}^r e_G(V_i) \geq \varepsilon m$. Let $X = \sum_{i=1}^r e_{G_R}(V_i)$, then X is a hypergeometric random variable with parameters m, f and γm; in particular, $\mathbb{E}(X) = \gamma f \geq \varepsilon\gamma m$. It follows by standard bounds on the tail of the hypergeometric distribution (see e.g. [8]) that

$$\begin{aligned} Pr(X \leq \varepsilon\gamma m/2) &\leq Pr(X \leq \mathbb{E}(X)/2) \\ &\leq e^{-\varepsilon\gamma m/8} \\ &\leq \frac{1}{3} r^{-n}, \end{aligned}$$

where the last inequality follows by the upper bound on q, assumed in Theorem 1.3. Since there are at most r^n such partitions, the result follows by a union bound argument. $\square$

Lemma 2.5 *The probability that there are at least $\varepsilon\delta m/(2q+2)$ failures is at most $1/2$.*

Proof During the first t rounds, both players claim together at most $t(q+1) = \delta m$ edges. It follows that, for every $1 \leq i \leq t$, the probability that e_i is a failure is at most δ. Let F be the random variable that counts the number of failures, then $\mathbb{E}(F) \leq \delta t = \delta^2 m/(q+1)$. It follows by Markov's inequality that $Pr(F \geq \varepsilon \delta m/(2q+2)) \leq 1/2$. $\square$

We are now ready to prove our claim. Let M_t denote the graph built by Maker during the first t rounds. The graph $M_t \cap \{e_1, \ldots, e_t\}$ can be viewed as a random graph $G(t)$ from which an adversary has removed some edges, namely, all the edges that were declared a failure. It follows by Lemma 2.4, that the probability that one has to delete at least $\varepsilon \delta m/(2q+2)$ edges from $G(t)$ in order to obtain an r-colorable graph is at least $2/3$. However, by Lemma 2.5, with probability at least $1/2$, a smaller number of edges were removed.

Hence, with positive probability, Maker has won the game. Moreover, since S_B was arbitrary, this holds for any strategy of Breaker. Since this is a finite perfect information game with no chance moves, it follows that there exists a deterministic winning strategy for Maker to win this game against any strategy of Breaker, that is, the game is a Maker's win as claimed.

$\square$

3 Chromatic number and edge connectivity

Proof of Theorem 1.4:

We first prove a few simple but useful facts.

Proposition 3.1 *Let q and r be positive integers and let $G = (V, E)$ be a graph on n vertices. Each of the following is a sufficient condition for Maker's win in the $(1:q)$ game $(E, \mathcal{F}_{P(r)})$.*

(a) *Maker can win the same game on some subgraph of G.*

(b) *G contains a clique of size $1000qr \log r$.*

(c) *G does not contain an independent set of size $n/(10qr \log r)$.*

Proof

(a) This is obvious as Maker can simply ignore the rest of the board.

(b) Let $s = 1000qr \log r$ and let H be a copy of K_s in G. It was proved in [7] that if $q \leq \frac{s}{1000r \log r}$, then playing a $(1 : q)$ game on the edge set of K_s, Maker can build a graph with chromatic number at least $r + 1$. The claim now follows by condition (a) of this proposition.

(c) Let $s = 10qr \log r$, and assume that G does not admit an independent set of size n/s. Maker's goal is to build a graph that does not admit an independent set of size n/r, and so, in particular, is not r-colorable. Let $A \subseteq V$ be an arbitrary set of size n/r. By our assumption $\alpha(G[A]) < n/s$. It follows by Turán's Theorem that $e(G[A]) \geq 4.5qn \log r/r$. Let $\mathcal{I}_G$ denote the family of edge-sets of all induced subgraphs of G on n/r vertices. It is clear that if Breaker can win the $(q : 1)$ game $(E, \mathcal{I}_G)$, then Maker can win the $(1 : q)$ game $(E, \mathcal{F}_{P(r)})$ by ensuring that the independence number of his graph is strictly smaller than n/r. In order to prove that Breaker can win $(E, \mathcal{I}_G)$, we apply Theorem 1.6. We have

$$\begin{aligned}
\sum_{B \in \mathcal{I}_G} 2^{-|B|/q} &\leq \binom{n}{n/r} 2^{-4.5n \log r/r} \\
&\leq (er)^{n/r} e^{-3n \log r/r} \\
&\leq \left(e^{1-2\log r}\right)^{n/r} \\
&= o(1),
\end{aligned}$$

where the last equality holds for every $r \geq 2$.

$\square$

We are now ready to prove Theorem 1.4.

Let $a = 10qr \log r$ and let $k = k(q, r)$ be the smallest positive integer such that for every graph G on $n \geq k$ vertices, satisfying $\alpha(G) \leq n/a$, Maker has a winning strategy for the $(1 : q)$ game $(E(G), \mathcal{F}_{P(r)})$ (the existence of such an integer k is guaranteed by part (c) of Proposition 3.1). Let $b = \frac{a}{a-1}$ and let n_0 be the largest integer satisfying $n_0 \leq k + \log_b n_0$. For every $i \geq 0$, let $n_i = n_0 b^i$.

Claim 3.2 *Let $i \geq 0$ be an integer, and let G be a graph on n_i vertices such that $\chi(G) \geq k + \log_b n_i$. Then there exists a subgraph $G^* \subseteq G$ such that $v(G^*) \geq k$ and $\alpha(G^*) \leq v(G^*)/a$.*

Proof We proceed by induction on i. The claim clearly holds for $i = 0$, as then we have $v(G) = n_0 \leq k + \log_b n_0 \leq \chi(G)$. It follows that G is a complete graph and thus $\alpha(G) = 1$. Hence, by choosing $k \geq a$, we conclude that $G^* := G$ satisfies the assertion of the claim. Next, let G be a graph on n_i vertices, for some $i \geq 1$, such that $\chi(G) \geq k + \log_b n_i$. If $\alpha(G) \leq n_i/a$, then $G^* := G$ satisfies the assertion of the claim. Otherwise, let $I \subseteq V(G)$ be an arbitrary independent set of size n_i/a, and let $G' = G[V(G) \setminus I]$. Note that $v(G') = n_i - n_i/a = n_i b^{-1} = n_{i-1}$ and $\chi(G') \geq \chi(G) - 1 \geq k + \log_b n_i - \log_b b = k + \log_b n_{i-1}$. Hence, by the induction hypothesis, there exists a subgraph $G^* \subseteq G' \subseteq G$ such that $v(G^*) \geq k$ and $\alpha(G^*) \leq v(G^*)/a$. □

Let G be a graph on $n > n_0$ vertices, such that $\chi(G) \geq k + \log_b n + 1$. Let i be the integer for which $n_i < n \leq n_{i+1}$. Let G' be the graph obtained from G by adding $n_{i+1} - n$ isolated vertices. Note that $v(G') = n_{i+1}$ and $\chi(G') = \chi(G) \geq k + \log_b n_{i+1}$. It follows by Claim 3.2 that there exists a subgraph $G^* \subseteq G'$ such that $v(G^*) \geq k$ and $\alpha(G^*) \leq v(G^*)/a$. Hence, by the choice of k, it follows that Maker has a winning strategy for the $(1:q)$ game $(E(G^*), \mathcal{F}_{P(r)})$. By part (a) of Proposition 3.1 we conclude that Maker also wins this game when played on G'. Since $G' \setminus G$ consists of isolated vertices, it follows that the $(1:q)$ game $(E(G), \mathcal{F}_{P(r)})$ is a Maker's win as claimed.

□

Proof of Theorem 1.5:

We will make use of the following theorem of Karger [10]:

Theorem 3.3 *Let $G = (V, E)$ be a graph on n vertices, which is r-edge-connected. Then, for every $t \geq 1$, the number of cuts of size at most rt in G is at most cn^{2t} for some positive constant c.*

Consider the following auxiliary Maker-Breaker game $(E, \mathcal{F})$, which we refer to as the *Cut game*. Two players, called CutMaker and CutBreaker, take turns in claiming free edges of $G = (V, E)$. CutBreaker, who is the first player, claims 1 edge per turn, whereas CutMaker claims q edges. The family $\mathcal{F}$ consists of all edge-sets $F \subseteq E$, for which there exists a cut $(S, \bar{S})$ of G such that $F \subseteq E(S, \bar{S})$ and $|F| = e(S, \bar{S}) - k + 1$. CutMaker wins the game if he claims all edges of some element of $\mathcal{F}$; otherwise CutBreaker wins. It is easy to see that the $(1:q)$ game $(E, \mathcal{F}_{P_k})$ is a Maker's win if and only if the $(q:1)$ Cut game is a CutBreaker's win. By Theorem 3.3 we have

$$
\begin{aligned}
\sum_{B\in\mathcal{F}} 2^{-|B|/q} &\leq \sum_{i=100kq\log_2 q}^{n^2} |\{S\subset V : e(S,\bar{S})=i\}| \binom{i}{k-1} 2^{-(i-k+1)/q} \\
&\leq \sum_{i=100kq\log_2 q}^{n^2} cn^{\frac{2i}{100kq\log_2 q}} (ei/k)^k 2^{(k-i)/q} \\
&\leq c \sum_{i=100kq\log_2 q}^{n^2} 2^{\frac{i}{50q}-\frac{i}{2q}+2k\log(i/k)} \\
&= o(1).
\end{aligned}
$$

Hence, it follows by Theorem 1.6 that the $(q:1)$ game $(E,\mathcal{F})$ is indeed a Cut-Breaker's win. $\square$

4 Concluding remarks and open problems

- In Theorem 1.2, we have proved that there exists a winning strategy for Maker in the $(1:q)$ game $(E,\mathcal{F}_P)$ for every dense graph $G=(V,E)$ on n vertices which is far from satisfying P, and for every $q \leq cn^{\alpha}$ for some constants $c>0$ and $\alpha>0$. For the special case in which $P=P(r)$ is the property of being r-colorable, Theorem 1.3 provides a stronger result; namely, the requirement that G is dense becomes redundant, and the upper bound on q is improved to $c(r)\frac{|E|}{|V|}$. Both theorems are existential in nature. It would be interesting to find explicit, efficient and deterministic winning strategies for Maker in these games, under similar conditions.

- As mentioned in the introduction, Theorem 1.2 is essentially best possible for some properties. One witness of this fact is the property P_H of being H-free for some fixed graph H. Indeed, let $G=(V,E)$ be a graph with n vertices and $\Theta(n^2)$ edges, which is ε-far from being H-free. It is well known that G contains $\gamma n^{v(H)}$ copies of H, where $\gamma>0$ is a constant. Hence, it follows from the proof of Theorem 1.2 that Maker has a winning strategy for the $(1:q)$ game $(E,\mathcal{F}_{P_H})$ for every $q\leq c_1 n^{1/m_2(H)}$, where $c_1>0$ is an appropriate constant. Now assume that $q\geq c_2 n^{1/m_2(H)}$ for some sufficiently large constant c_2. We claim that for such a q, Breaker has a winning strategy for the $(1:q)$ game $(E,\mathcal{F}_{P_H})$. Clearly it suffices to prove this for the $(1:q)$ game $(E(K_n),\mathcal{F}_{P_H})$. This, however, has been proven in [4]. On the other hand, Theorem 1.3 shows that for some properties, such as being r-colorable, the upper bound given in Theorem 1.2 is far from being best possible. It would be interesting to obtain tighter bounds for additional monotone properties.

- The bound on q given in Theorem 1.3 is not far from being tight, at least for fixed values of r. Indeed, consider for example the graph G which consists of n/d disjoint copies of K_d, where $n >> d > r = 3$. Clearly, one has to delete a constant fraction of the edges of every d-clique of G in order to obtain an r-colorable graph. Thus G is $\Omega(1)$-far from being r-colorable. However, if $q \geq 2d/(r-2)$, then Breaker can make sure that the maximum degree in Maker's graph will be at most $r-1$, and thus Maker's graph will be r-colorable. Breaker's strategy is very simple: whenever Maker claims an edge (u, v), Breaker responds by claiming $q/2$ arbitrary free edges that are incident with u and $q/2$ arbitrary free edges that are incident with v. For large r, the maximum possible value of ε is only $\Theta(1/r)$, as any graph can be made r-colorable by deleting at most an $O(1/r)$ fraction of its edges. Here, however, one can show that for $q \geq Cd/(r \log r)$, Breaker has a winning strategy for some graphs. Indeed, let d be an integer satisfying $d >> r \log r$. Let $G = G(n, d/n)$ be a random graph on n vertices. Delete from G all edges which are incident with vertices of degree higher than $2d$; denote the resulting graph by $G' = (V, E)$. One can easily show that $|V| = n$, and almost surely $|E| = \Theta(d)n$ and G' is $\Theta(1/r)$-far from being r-colorable. However, if $q \geq Cd/(r \log r)$, for an appropriate constant $C > 0$, then Breaker can make sure that Maker's graph will be triangle-free, and with maximum degree at most $\tilde{c} r \log r$, where $\tilde{c} > 0$ is sufficiently small. It follows by Johansson's Theorem [9] that Maker's graph will be r-colorable. Breaker plays as follows: in each move he first claims every free edge that closes a triangle in Maker's graph. Denote the number of such edges by q_1 and let $q_2 = q - q_1$. Since the triangles in $G(n, d/n)$ are almost surely edge-disjoint, we can assume in fact that $q_1 = 1$. Then, if the last edge claimed by Maker was (u, v), then Breaker responds by claiming $q_2/2$ arbitrary free edges that are incident with u and $q_2/2$ arbitrary free edges that are incident with v. This strategy is quite similar to the one suggested by Chvátal and Erdős in [5] (see Theorem 5.2 there).

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