

# Nonconvex stochastic optimization on manifolds via Riemannian Frank-Wolfe methods

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## Abstract

We study stochastic projection-free methods for constrained optimization of smooth functions on Riemannian manifolds (where there are additional constraints beyond the parameter domain being a manifold). Specifically, we introduce stochastic Riemannian Frank-Wolfe methods for both nonconvex and geodesically convex problems. We present algorithms for both stochastic optimization and finite-sum problems. For the latter, we develop variance reduced methods, including a Riemannian adaptation of the recently proposed SPIDER technique. For all settings, we recover convergence rates that are comparable to the best known rates for their Euclidean counterparts. Finally, we discuss applications to two basic tasks: computation of the Karcher mean and Wasserstein barycenters for multivariate normal distributions. For both tasks, stochastic Fw methods yield state-of-the-art empirical performance.

## 1 Introduction

We study the following constrained (possibly nonconvex) stochastic and finite-sum problems:

$$\min_{x \in \mathcal{X} \subset \mathcal{M}} \Phi(x) := \mathbb{E}_{\xi}[\phi(x, \xi)] = \int \phi(x, \xi) dP(\xi), \quad (1.1)$$

$$\min_{x \in \mathcal{X} \subset \mathcal{M}} \Phi(x) := \frac{1}{m} \sum_{i=1}^m \phi_i(x), \quad (1.2)$$

where  $\mathcal{X}$  is compact and geodesically convex and  $\mathcal{M}$  is a Riemannian manifold. Moreover, the component functions  $\{\phi_i\}_{i=1}^m$  as well as  $\Phi$  are (geodesically) Lipschitz-smooth and may be nonconvex. These problems greatly generalize their Euclidean counterparts (where  $\mathcal{M} \equiv \mathbb{R}^d$ ), which themselves are of central importance to all of machine learning.

There has been an increasing interest in solving Riemannian problems of the above form, albeit without constraints (Bonnabel, 2013; Zhang et al., 2018, 2016; Zhang and Sra, 2016; Kasai et al., 2018b, 2019; Tripuraneni et al., 2018). This interest is driven by two key motivations: (i) exploitation of Riemannian geometry can deliver algorithms that are computationally superior to standard nonlinear programming approaches (Absil et al., 2008; Udriste, 1994; Zhang et al., 2016; Boumal et al., 2014b; Boumal, 2014); and (ii) in many applications we encounter non-Euclidean data, such as graphs, strings, matrices, tensors; using a forced Euclidean representation here can be quite inefficient (Sala et al., 2018; Nickel and Kiela, 2017; Zhang et al., 2016; Billera et al., 2001; Edelman et al., 1998). These motivations have driven the recent surge of interest in the adaption and generalization of machine learning models and algorithms to Riemannian manifolds.

We solve (1.1) by introducing Riemannian stochastic Frank-Wolfe (Fw) algorithms. These methods are projection-free (Frank and Wolfe, 1956), a property that has driven much of

the recent interest in them (Jaggi, 2013). In contrast to projection-based methods, the Fw update requires solving a “linear” optimization problem that ensures feasibility while often being much faster than projection. Fw has been intensively studied in Euclidean spaces for both convex (Lacoste-Julien and Jaggi, 2015; Jaggi, 2013) and nonconvex (Lacoste-Julien, 2016) objectives. Furthermore, stochastic variants have been proposed (Reddi et al., 2016) that enable strong performance gains. As our experiments will show, our stochastic Riemannian Fw also delivers similarly strong performance gains on some applications, substantially outperforming the state-of-the-art.

### 1.1 Main contributions

- We introduce three algorithms: (i) SRFW, a fully stochastic method that solves (1.1); (ii) SVR-RFW, a semi-stochastic variance-reduced version for (1.2); and (iii) SPIDER-RFW, an improved variance-reduced variant that uses the recently proposed SPIDER technique for estimating the gradient and solves (1.2).

All three algorithms generalize various stochastic gradient tools to the Riemannian setting. In contrast to (Weber and Sra, 2017), who consider Riemannian Fw, our methods neither require full gradients nor are they limited to geodesically convex problems. Moreover, we also study the stochastic problem (1.1). For all the methods, we establish convergence rates to first-order stationary points that match the rates of their Euclidean counterparts.

- We present an application to the computation of Riemannian centroids (Karcher mean) for positive definite matrices. This task is a well-known benchmark for Riemannian optimization, and it arises in statistical analysis, signal processing, computer vision, among others; notably, a simpler version of it also arises in computing hyperbolic embeddings.
- We present an application to the computation of Wasserstein barycenters for multivariate and *matrix-variate* Gaussians. For the latter, we prove the somewhat surprising property that the Wasserstein distance between two matrix-variate Gaussians is Euclidean convex; this result might be of independent interest.

The proposed stochastic Fw methods deliver valuable improvements (see Table 1.2), both in theory and experiment. In particular, our algorithms outperform state-of-the-art batch methods such as Riemannian L-BFGS (Yuan et al., 2016) and Zhang’s majorization-minimization algorithm (Zhang, 2017); moreover, we also observe performance gains over the deterministic Rfw, which itself is known to be competitive against a wide range of Riemannian optimization tools (Weber and Sra, 2017). But more importantly, our methods significantly outperform state-of-the-art stochastic Riemannian methods RSG (Kasai et al., 2018b) and Rsvrg (Sato et al., 2017; Zhang et al., 2016).

### 1.2 Related work

Riemannian optimization has recently witnessed a surge of interest (Bonnabel, 2013; Zhang and Sra, 2016; Huang et al., 2018; Liu and Boumal, 2019). A basic overview is available in (Absil et al., 2008; Boumal, 2014). The MANOPT toolbox (Boumal et al., 2014a) implements many successful Riemannian optimization methods, serving as a benchmark.

The study of stochastic methods for Riemannian optimization has largely focused on projected-gradient methods. Bonnabel (2013) introduced the first Riemannian SGD; Zhang

Algorithm	Rfw	Srfw	Svr-Rfw	Spider-Rfw
SFO/ IFO	$O(n/\epsilon^2)$	$O\left(\frac{1}{\epsilon^4}\right)$	$O\left(n + \frac{n^{2/3}}{\epsilon^2}\right)$	$O\left(\frac{1}{\epsilon^3}\right)$
LO	$O(1/\epsilon^2)$	$O\left(\frac{1}{\epsilon^2}\right)$	$O\left(\frac{1}{\epsilon^2}\right)$	$O\left(\frac{1}{\epsilon^2}\right)$

Table 1: Oracle complexities of our stochastic Frank-Wolfe methods versus RFW (Weber and Sra, 2017).

and Sra (2016) present a systematic study of first-order methods for geodesically convex problems, followed by a variance-reduced Riemannian SVRG (Zhang et al., 2016) (see also (Sato et al., 2017)) that also applies to geodesically nonconvex functions. Kasai et al. (2018b) study gradient decent variants, as well as a Riemannian ADAM (Kasai et al., 2019). A caveat of these methods is that to ensure convergence they need to perform a potentially costly projection, or otherwise make the strong (and often unrealistic) assumption that their iterates remain in a compact set. In contrast, Rfw generates feasible iterates and avoids projection, leading to a cleaner analysis and more practical algorithm in cases where the “linear” oracle is efficiently implementable (Weber and Sra, 2017).

Riemannian optimization has also been applied in the ML literature, including for the computation of hyperbolic embeddings (Sala et al., 2018), low-rank matrix and tensor factorization (Vandereycken, 2013) and eigenvector based methods (Journée et al., 2010; Zhang et al., 2016; Tripuraneni et al., 2018).

## 2 Background and Notation

We recall some basic background from Riemannian geometry; see (Jost, 2011) for a thorough introduction.

**Riemannian manifolds.** A *manifold*  $\mathcal{M}$  is a locally Euclidean space equipped with a differential structure. Its corresponding *tangent spaces*  $T_x\mathcal{M}$  consist of tangent vectors at points  $x \in \mathcal{M}$ . We define an *exponential map*  $\text{Exp} : T_x\mathcal{M} \rightarrow \mathcal{M}$  as follows. Say  $g_x \in T_x\mathcal{M}$ ; then  $y = \text{Exp}_x(g_x) \in \mathcal{M}$  with respect to a geodesic  $\gamma : [0, 1] \mapsto \mathcal{M}$  with  $\gamma(0) = x$ ,  $\gamma(1) = y$  and  $\dot{\gamma}(0) = g_x$ . We will also use the *inverse exponential map*  $\text{Exp}^{-1} : \mathcal{M} \rightarrow T_x\mathcal{M}$  that defines a diffeomorphism from the neighborhood of  $x \in \mathcal{M}$  onto the neighborhood of  $0 \in T_x\mathcal{M}$  with  $\text{Exp}_x^{-1}(x) = 0$ .

*Riemannian manifolds* are smooth manifolds with an inner product  $\mathfrak{g}_x(u, v) = \langle u, v \rangle_x$  defined on  $T_x\mathcal{M}$  for each  $x \in \mathcal{M}$ . The inner product gives rise to a norm  $\|v\|_x := \sqrt{\mathfrak{g}_x(v, v)}$  for  $v \in T_x\mathcal{M}$ . We will further denote the geodesic distance of  $x, y \in \mathcal{M}$  as  $d(x, y)$ . For comparing vectors of different tangent spaces, we use the following notion of *parallel transport*: Let  $x, y \in \mathcal{M}$ ,  $x \neq y$ . Then, the operator  $\Gamma_x^y g_x$  maps  $g_x \in T_x\mathcal{M}$  to the tangent space  $T_y\mathcal{M}$  along a geodesic  $\gamma$  with  $\gamma(0) = x$  and  $\gamma(1) = y$ . Note that the inner product on the tangent spaces is preserved under this mapping.

**Functions on Riemannian manifolds.** The *Riemannian gradient*  $\text{grad } \phi(x)$  of a differentiable function  $\phi : \mathcal{M} \rightarrow \mathbb{R}$  is defined as the unique vector  $v \in T_x\mathcal{M}$  with  $D\phi(x)[v] = \langle \text{grad } \phi(x), v \rangle_x$ . For our algorithms we further need a notion of smoothness: Let  $\phi : \mathcal{M} \rightarrow \mathbb{R}$

be differentiable. We say that  $\phi$  is  $L$ -smooth, if

$$\|\text{grad } \phi(y) - \Gamma_x^y \text{grad } \phi(x)\| \leq Ld(x, y), \forall x, y \in \mathcal{M}, \quad (2.1)$$

or equivalently, if for all  $x, y \in \mathcal{M}$ ,  $\phi$  satisfies

$$\phi(y) \leq \phi(x) + \langle g_x, \text{Exp}_x^{-1}(y) \rangle_x + \frac{L}{2}d^2(x, y). \quad (2.2)$$

Another important property is *geodesic convexity* (g-convexity), which is defined as follows: assume  $g_x$  is a (sub)gradient of  $\phi$ , then

$$\phi(y) \geq \phi(x) + \langle g_x, \text{Exp}_x^{-1}(y) \rangle_x \forall x, y \in \mathcal{M}. \quad (2.3)$$

**Projection-free vs. Projection-based methods.** Classic Riemannian optimization has focused mostly on projection-based methods, such as *Riemannian Gradient Decent* (RGD) or *Riemannian Steepest Decent* (RSD) (Absil et al., 2008). A convergence analysis of such methods typically assumes the gradient to be Lipschitz. However, the objectives typically considered in most optimization and machine learning tasks are not Lipschitz on the whole manifold. Hence, a compactness condition is required. Crucially, in projection-based methods, the retraction back onto the manifold is typically not guaranteed to land in this compact set. Therefore, additional work (e.g., a projection step) is needed to ensure that the update remains in the compact region where the gradient is Lipschitz. On the other hand, Fw methods bypass this issue, because their update is guaranteed to stay within the compact feasible region. Further, for descent based methods it can suffice to ensure boundedness of the initial level set, but crucially, stochastic methods are *not* descent methods, and this argument does not apply. Finally, in some problems, the Riemannian linear oracle can be much less expensive than computing a projection back onto the compact set. This is especially significant for the applications highlighted in this paper, where the linear oracle even admits a closed form solution.

### 3 Algorithms

In this section, we introduce three stochastic variants of RFW and analyze their convergence.

#### 3.1 Stochastic Riemannian Frank-Wolfe

Our first method, SRFW (Alg. 1), is a direct analog of stochastic Euclidean FW. It has two key computational components : a stochastic gradient and a “linear” oracle. Specifically, it needs the *stochastic “linear” oracle*

$$y_k \leftarrow \underset{y \in \mathcal{X}}{\text{argmin}} \langle G(\xi^k, x_k), \text{Exp}_{x_k}^{-1}(y) \rangle, \quad (3.1)$$

where  $G(\cdot, \cdot)$  is an unbiased estimator of the Riemannian gradient ( $\mathbb{E}_\xi G(\xi, x) = \text{grad } \Phi(x)$ ). In contrast to Euclidean FW, the oracle (3.1) actually involves solving a nonlinear, nonconvex optimization problem. Whenever this problem is efficiently solvable, we can benefit from the FW strategy. Our experiments provide two concrete examples.

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**Algorithm 1** Stochastic Riemannian Frank-Wolfe (SRFW)

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- 1: Initialize  $x_0 \in \mathcal{X}$ , assume access to the geodesic map  $\gamma : [0, 1] \rightarrow \mathcal{M}$ .
  - 2: Set number of iterations  $K$  and minibatch sizes  $\{b_k\}_{k=0}^{K-1}$ .
  - 3: **for**  $k = 0, 1, \dots, K-1$  **do**
  - 4:     Sample i.i.d.  $\{\xi_1^k, \dots, \xi_{b_k}^k\}$  uniformly at random according  $\mathcal{P}$ .
  - 5:     **for**  $\tilde{k} = 0, 1, \dots, K-1$  **do**
  - 6:          $y_{\tilde{k}} \leftarrow \operatorname{argmin}_{y \in \mathcal{X}} \langle \frac{1}{b_k} \sum_{i=1}^{b_k} \operatorname{grad} \phi(x_{\tilde{k}}, \xi_i^k), \operatorname{Exp}_{x_{\tilde{k}}}^{-1}(y) \rangle$
  - 7:         Compute step size  $\eta_k$  and set  $x_{\tilde{k}+1} \leftarrow \gamma(s_{\tilde{k}})$ , where  $\gamma(0) = x_{\tilde{k}}$  and  $\gamma(1) = y_{\tilde{k}}$ .
  - 8:     **end for**
  - 9:      $x^k \leftarrow x_{\tilde{k}}$
  - 10: **end for**
  - 11: Output  $\hat{x}$  chosen uniformly at random from  $\{x^k\}_{k=0}^{K-1}$ .
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Following (Reddi et al., 2016), we consider a mini-batch variant of the oracle (3.1), namely

$$y_k \leftarrow \operatorname{argmin}_{y \in \mathcal{X}} \left\langle \frac{1}{b_k} \sum_{i=1}^{b_k} \operatorname{grad} \phi(x_k, \xi_i), \operatorname{Exp}_{x_k}^{-1}(y) \right\rangle,$$

where  $\xi_i \sim \mathcal{P}$  are drawn i.i.d., and thus the mini-batch gradient is also unbiased. We evaluate the goodness of this mini-batch gradient approximation with the following lemma:

**Lemma 3.1.** *Let  $\Phi(x) = \mathbb{E}_{\xi} [\phi(x, \xi_i)]$  with random variables  $\{\xi_i\}_{i=1}^b = \xi \sim \mathcal{P}$ . Furthermore, let  $g(x) := \frac{1}{b} \sum_{i=1}^b \operatorname{grad} \phi(x, \xi_i)$  denote the gradient estimate from a batch  $\xi$  and  $\max \|\operatorname{grad} \phi(x, \xi)\| \leq G$  an upper bound on the estimator. Then,  $\mathbb{E} [\|g(x) - \operatorname{grad} \Phi(x)\|] \leq \frac{G}{\sqrt{b}}$ .*

With this characterization of the approximation error, we can analyze convergence for both nonconvex and g-convex objectives. In the nonconvex case, we measure convergence in terms of the *Frank-Wolfe gap*:

$$\mathcal{G}(x) = \max_{y \in \mathcal{X}} \langle \operatorname{Exp}_x^{-1}(y), -\operatorname{grad} \Phi(x) \rangle. \quad (3.2)$$

In particular, we show that SRFW converges at a sublinear rate to first-order optimal points:

**Theorem 3.2.** *With constant step size  $\eta_k = \frac{1}{\sqrt{K}}$  and constant batch sizes  $b_k = K$ , Algorithm 1 converges globally in expectation as  $\mathbb{E} [\mathcal{G}(\hat{x})] = O(1/\sqrt{K})$ .*

For g-convex objectives, we can obtain a convergence result in terms of the optimality gap  $\Delta_k := \Phi(x_k) - \Phi(x^*)$ . Here, SRFW converges at a sublinear rate to the global optimum.

**Corollary 3.3.** *Assuming that  $\Phi$  is g-convex, under the settings of Theorem 3.2 the optimality gap converges as  $\mathbb{E} [\Delta_k] = O(1/\sqrt{K})$ .*

We defer the proof of Theorem 3.2, Corollary 3.6, Lemma 3.1, and their accompanying auxiliary lemmas to Appendix A.

A shortcoming of SRFW is its large batch sizes. However, we can significantly reduce the batch sizes by choosing non-constant decreasing step sizes. This batch-size reduced version (R-SRFW) comes at the cost of weaker theoretical guarantees (see Theorem A.8 in the appendix), but yields significant performance gains in practice as we demonstrate in Appendix A.

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**Algorithm 2** Semi-stochastic variance-reduced Riemannian Frank-Wolfe (SVR-RfW)

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```
1: Initialize  $x_0 \in \mathcal{X}$ ; assume access to the geodesic map  $\gamma : [0, 1] \rightarrow \mathcal{M}$ .
2: Choose number of iterations  $S$  and size of epochs  $m$  and set mini batch sizes  $\{b_k\}_{k=0}^{K-1}$ .
3: for  $s = 0, 1, \dots, S - 1$  do
4:    $\tilde{x}^s = x_m^s$ .
5:   Compute gradient at  $\tilde{x}^s$ :  $\text{grad } \Phi(\tilde{x}^s) = \frac{1}{N} \sum_{i=1}^N \text{grad } \phi_i(\tilde{x}^s)$ .
6:   for  $\tilde{k} = 0, 1, \dots, m$  do
7:     Sample i.i.d.  $I_k := \{i_1, \dots, i_{b_k}\} \subseteq [n]$  (minibatches).
8:      $z_{k+1}^{s+1} \leftarrow \text{argmin}_{z \in \mathcal{X}} \langle \frac{1}{b_k} \sum_{j=i_1, \dots, i_{b_k}} \text{grad } \phi_j(x_t^{s+1}) - \Gamma_{\tilde{x}^s}^{x_t^{s+1}} (\text{grad } \phi_j(\tilde{x}^s)) + \text{grad } \Phi(\tilde{x}^s) \rangle, \text{Exp}_{\tilde{x}^s}^{-1}(z)$ 
9:     Compute step size  $\eta_k$  and set  $x_{k+1}^{s+1} \leftarrow \gamma(s_{\tilde{k}})$ , where  $\gamma(0) = x_k^{s+1}$  and  $\gamma(1) = z_{k+1}^{s+1}$ .
10:  end for
11: end for
12: Output  $\hat{x} = \tilde{x}_K^S$ .
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### 3.2 Stochastic variance-reduced Frank-Wolfe

In addition to the purely stochastic SRfW method, for the finite-sum problem (1.2) we can obtain a stochastic FW algorithm via a (semi-stochastic) *variance-reduced* approach. By exploiting the finite sum structure, we can obtain provably faster FW algorithms.

#### 3.2.1 Svr-Rfw

We first propose SVR-RfW (Algorithm 2) that combines a classic variance-reduced estimate of the gradient with RfW. This approach computes the full gradient at the beginning of each epoch and uses batch estimates within epochs. The variance-reduced gradient estimate guarantees the following bound on the approximation error:

**Lemma 3.4** (Goodness of variance-reduced gradient estimate). *Consider the  $k^{\text{th}}$  iteration in the  $s^{\text{th}}$  epoch and the stochastic variance-reduced gradient estimate*

$$g_k(x_k^{s+1}) = \frac{1}{b_k} \sum_j \text{grad } \phi_j(x_k^{s+1}) - \Gamma_{\tilde{x}^s}^{x_k^{s+1}} (\text{grad } \phi_j(\tilde{x}^s) + \text{grad } \Phi(\tilde{x}^s)) ,$$

with the  $\{\phi_i\}$  being  $L$ -Lipschitz. Then

$$\mathbb{E} \left[ \|\text{grad } \Phi(x_k^{s+1}) - g_k\| \right] \leq \frac{L}{\sqrt{b_k}} \|x_k^{s+1} - \tilde{x}^s\| .$$

Using Lemma 3.4 we can recover the following sublinear convergence rate:

**Theorem 3.5.** *With steps size  $\eta_k = \frac{1}{\sqrt{mS}}$  and constant batch sizes  $b_k = m$ , Alg. 2 converges in expectation with  $\mathbb{E} [\mathcal{G}(\hat{x})] = O\left(\frac{1}{\sqrt{mS}}\right)$ .*

Analogously to SRfW, SVR-RfW converges sublinearly to the global optimum, if the objective is  $g$ -convex. As before, we use  $\Delta_k = \Phi(x_k) - \Phi(x^*)$ .

**Corollary 3.6.** *Assuming that  $\Phi$  is  $g$ -convex, one can show an analogous convergence rate for the optimality gap, i.e.,  $\mathbb{E} [\Delta_k] = O(1/\sqrt{K})$ .*

The proofs are very similar to that of Theorem 3.2. They can be found together with the proof of Lemma 3.4 in Appendix B.

A significant shortcoming of the semi-stochastic approach is the need for repeated computation of the full gradient which limits its scalability. In the following section, we introduce an improved version that circumvents these costly computations.

### 3.2.2 Improved gradient estimation

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#### Algorithm 3 SPIDER-RFW

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- 1: Initialize  $x_0 \in \mathcal{X}$ , number of iterations  $K$ , size of epochs  $m$ . Assume access to  $\gamma : [0, 1] \rightarrow \mathcal{M}$ .
  - 2: **for**  $k = 0, 1, \dots, K$  **do**
  - 3:     **if**  $\text{mod}(k, m) = 0$  **then**
  - 4:         Sample i.i.d.  $S_1 = \{\zeta_1, \dots, \zeta_{|S_1|}\}$  with predefined  $|S_1|$ .
  - 5:         Compute gradient  $g_k \leftarrow \text{grad } \Phi_{S_1}(x_k)$ .
  - 6:     **else**
  - 7:         Sample i.i.d.  $S_2 = \{\zeta_1, \dots, \zeta_{|S_2|}\}$  with  $|S_2| = \lceil \min\{n, \frac{2mL^2 \|\text{Exp}_{x_{k-1}}^{-1}(x_k)\|}{\epsilon^2}\} \rceil$ .
  - 8:         Compute gradient  $g_k \leftarrow \text{grad } \Phi_{S_2}(x_k) - \Gamma_{x_{k-1}}^{x_k} (\text{grad } \Phi_{S_2}(x_{k-1}) - g_{k-1})$ .
  - 9:     **end if**
  - 10:      $z_{k+1} \leftarrow \text{argmin}_{z \in \mathcal{X}} \langle g_k, \text{Exp}_{x_k}^{-1}(z) \rangle$ .
  - 11:      $x_{k+1} \leftarrow \gamma(\eta_k)$ , where  $\gamma(0) = x_k$  and  $\gamma(1) = z_{k+1}$ .
  - 12: **end for**
  - 13: Output  $\hat{x}$  chosen uniformly at random from  $\{x^k\}_{k=0}^{K-1}$ .
- 

Recently, Fang et al. (2018) introduced SPIDER as an efficient way of estimating the (Euclidean) gradient. Based on the idea of variance-reduction, the algorithm iterates between gradient estimates with different sample size. In particular, it recomputes the gradient at the beginning of each epoch with a larger (constant) batch size; the smaller batch sizes within epochs decrease as we move closer to the optimum. In (Zhang et al., 2018) and (Zhou et al., 2018) the authors explored this technique for Riemannian gradient decent. In the following, we will introduce an improved variance-reduced SRFW using SPIDER. The Riemannian extension of these ideas to the constrained RFW setting requires overcoming some technical challenges; but this work pays off, because the resulting variance reduced version of Rfw displays strong empirical performance.

We write  $\text{grad } \Phi_S(x_k) = \frac{1}{|S|} \sum_{i=1}^{|S|} \text{grad } \phi(x_k, \zeta_i)$  for the approximation of the gradient with respect to a sample  $S = (\zeta_1, \dots, \zeta_{|S|})$ . Furthermore, consider the following parameter choice ( $K = \#$  iterations):

$$\eta = \frac{1}{\sqrt{K}} \quad (\text{step size}) \quad (3.3)$$

$$m = \sqrt{K} = \frac{1}{\epsilon} \quad (\# \text{ epochs}) \quad (3.4)$$

$$S_1 = \frac{2G^2}{\epsilon^2}. \quad (3.5)$$

Here,  $\epsilon$  characterizes the goodness of the gradient estimate.  $S_2$  is recomputed in each iteration as given in Algorithm 3. Note that here  $n$  is determined by the number of terms in the finite-sum approximation or  $n = \infty$  in the general (stochastic) case.

We start by analyzing the SPIDER gradient estimate; this result is central to our convergence analysis. First, we show that  $(\|g_k - \text{grad } \Phi(x_k)\|)_k$  is a martingale (Lemma C.3 and Corollary C.4). Then, using martingale properties, we can prove the following bound on the approximation error.

**Lemma 3.7** (Goodness of SPIDER-approximation). The expected deviation of the estimate  $g_k$  from the true gradient is bounded as  $\mathbb{E} [\|g_k - \text{grad } \Phi(x_k)\| | \mathcal{F}_k] \leq \epsilon$ .

With this result, we arrive at our main result for SPIDER-RFW which shows that for non-convex objectives it attains a global sublinear convergence rate.

**Theorem 3.8** (Convergence SPIDER-RFW). With the parameter choices (3.3), Algorithm 3 converges in expectation with rate  $\mathbb{E} [\mathcal{G}(x_K)] = O\left(\frac{1}{\sqrt{K}}\right)$ .

In the special case of  $g$ -convex objectives, we obtain a result on function suboptimality.

**Corollary 3.9.** Assuming that  $\Phi$  is  $g$ -convex, one can show a similar convergence rate for the optimality gap, i.e.,  $\mathbb{E} [\Delta_k] = O(1/\sqrt{K})$ .

The proofs of all three result and the accompanying auxiliary lemmas are deferred to Appendix C.

## 4 Experiments

In this section we validate the proposed algorithms by comparing them against deterministic RFW (Weber and Sra, 2017) and the state-of-the-art Riemannian stochastic optimization methods. All experiments were performed in MATLAB.

### 4.1 Riemannian centroid

The computation of the *Riemannian centroid* (also known as the *Karcher mean*) is a canonical benchmark task for testing Riemannian optimization (Zhang et al., 2016; Kasai et al., 2018b,a). It asks to find the mean of the set  $M = \{M_i\}$  of  $N \times N$  positive definite matrices (we write  $|M| = m$ ) with respect to the Riemannian metric. This task requires solving

$$\min_{H \preceq X \preceq A} \sum_{i=1}^m w_i \delta_R^2(X, M_i) = \sum_{i=1}^m w_i \|\log(X^{-1/2} M_i X^{-1/2})\|_F^2,$$

where  $\|\cdot\|_F$  denotes the Frobenius norm. The constraints are given by the well-known matrix means inequality: The *harmonic mean*  $H := \left(\sum_i w_i M_i^{-1}\right)^{-1}$  gives a lower bound on the geometric matrix mean, while the arithmetic mean  $A := \sum_i w_i M_i$  provides an upper bound (Bhatia, 2007).

Writing  $\phi_i(X) = w_i \delta_R^2(X, M_i)$ , we note that the gradient of the objective is given by  $\nabla \phi_i(X) = w_i X^{-1} \log(X M_i^{-1})$ , whereby the corresponding ‘‘linear’’ oracle reduces to solving

$$Z_k \leftarrow \underset{H \preceq Z \preceq A}{\text{argmin}} \langle X_k^{1/2} \nabla \phi_i(X_k) X_k^{1/2}, \log(X_k^{-1/2} Z X_k^{-1/2}) \rangle. \quad (4.1)$$



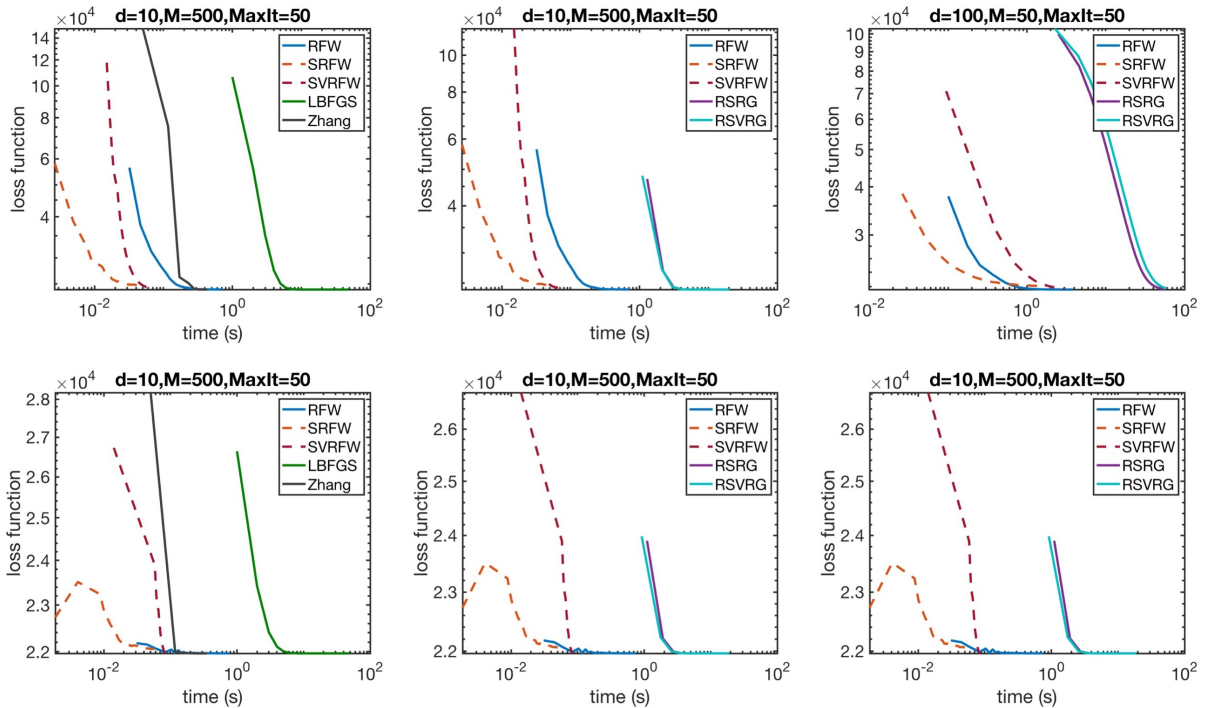


Figure 1: **Riemannian centroid.** Rfw and stochastic variants versus state-of-the-art Riemannian optimization methods (*parameters*:  $d$  - size of matrices,  $M$  - number of matrices,  $MaxIt$  - number of iterations; *initialization*:  $X_0 = H(M)$ ). Hereby, we compare against deterministic algorithms (LBFGS and ZHANG, left) as well as recent state-of-the-art stochastic Riemannian algorithms R-SRG and RSVRG (middle and right). The results in the top row are for well-conditioned, the results in the bottom row for ill-conditioned matrices.

Remarkably, (4.1) can be solved in closed form (Weber and Sra, 2017, Theorem 7), which allows us to exploit it for use in our stochastic Frank-Wolfe methods.

Besides its importance as a benchmark, the Karcher mean is a fundamental subroutine in performing statistical data analysis, for instance, the computation of hyperbolic embeddings (Sala et al., 2018). Although the Karcher mean problem is nonconvex in Euclidean space, it is  $g$ -convex in the Riemannian setting. This allows one to apply RFW, in addition to all the stochastic methods discussed above (which do not necessarily require  $g$ -convexity). RFW requires the computation of the full gradient in each iteration step, whereas our stochastic methods implement gradient estimates at a significantly reduced computational cost. This results in observable performance gains as shown in our experiments (Fig. 1).

To evaluate the efficiency of our methods, we compare against state-of-the-art algorithms. First, *Riemannian BFGS*, a quasi-Newton method (Yuan et al., 2016), for which we use an improved limited-memory version of the method available in *Manopt* (Boumal et al., 2014a). Secondly *Zhang's method* (Zhang, 2017), a recently published majorization-minimization method for computing the geometric matrix mean. Against both (deterministic) algorithms we observe strong performance gains (Fig. 1). In (Weber and Sra, 2017), RFW's is compared with

a wide range of Riemannian optimization methods and varying choices of hyperparameters; of those LBFSG and Zhang’s method were reported to be especially competitive, which motivates our choice. We further present two instances of comparing stochastic RFW against stochastic gradient-based methods (R-SRG and RsvRG (Kasai et al., 2018a)), both of which are outperformed by our FW approach.

We note that the comparison experiments are not quite fair to our methods, as neither R-SRG nor RsvRG implement the noted projection operation (see discussion on page 3) required to align their implementation with their theory.

## 4.2 Wasserstein Barycenters

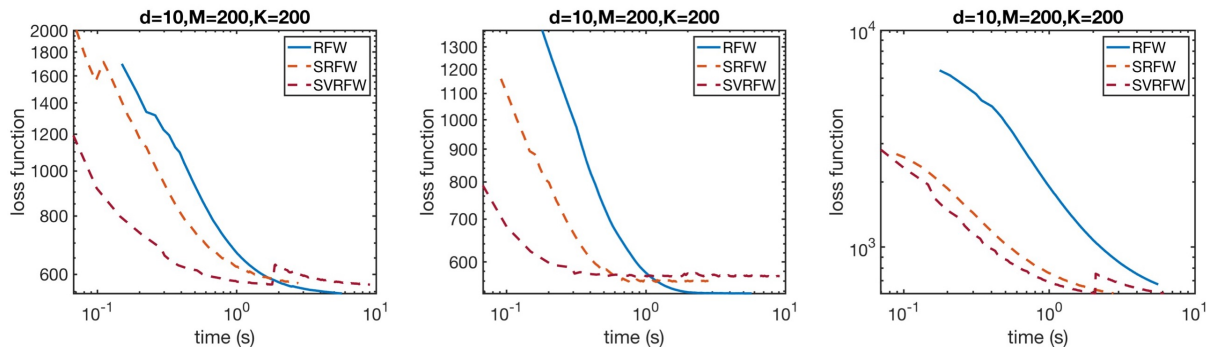


Figure 2: **Wasserstein barycenters.** Performance of RFW and stochastic variants for well-conditioned inputs of fixed size ( $d$ : size of matrices,  $M$ : number of matrices,  $K$ : number of iterations) with different initializations:  $X_0 \sim \mathcal{C}$  (left),  $X_0 = \frac{1}{2}(\alpha I + A)$  (middle) and  $X_0 = A$  (right).

A basic task in statistics is the computation of means of empirical probability measures with respect to the optimal transport metric (or *Wasserstein distance*). Here, we consider the problem of computing such *Wasserstein barycenters* of multivariate (centered) Gaussians. This corresponds to the following minimization task on the PSD manifold:<sup>1</sup>

$$\min_{\alpha I \preceq X \preceq A} \sum_{i=1}^M d_W^2(X, \mathcal{C}) = \sum_i w_i [\text{tr}(C_i + X) - 2 \text{tr}(C_i^{1/2} X C_i^{1/2})^{1/2}], \quad (4.2)$$

where  $\mathcal{C} = \{C_i\} \subseteq \mathbb{P}(n)$ ,  $|\mathcal{C}| = m$  are the covariance matrices of the Gaussians and  $\alpha$  denotes their minimal eigenvalue over  $\mathcal{C}$ .

A closely related problem is the task of computing Wasserstein barycenters of *matrix-variate Gaussians*, i.e., multivariate Gaussians whose covariance matrices are expressed as suitable Kronecker products. Such models are of interest in several inference problems, see for instance (Stegle et al., 2011). By plugging in Kronecker structured covariances into (4.2), the

<sup>1</sup>Interestingly, this problem turns out to be Euclidean convex (more precisely, it is a nonlinear semidefinite program); but is better suited for a Riemannian approach given its structure.

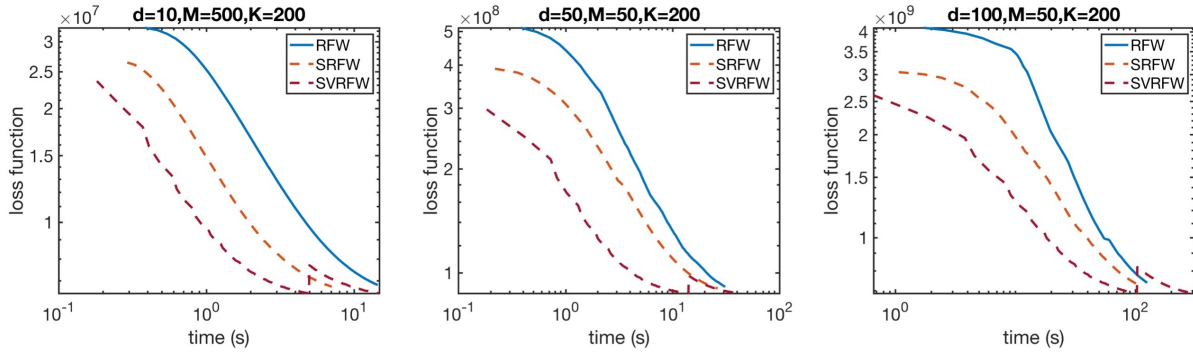


Figure 3: **Wasserstein barycenters for MVNs.** Performance of RFW and stochastic variants for well-conditioned inputs of different sizes ( $d$ : size of matrices,  $M$ : number of matrices,  $K$ : number of iterations); initialized at  $X_0 = A$ .

corresponding barycenter problem takes the form

$$\min_{X \succ 0} \sum_{i=1}^n \text{tr}(A_i \otimes A_i) + \text{tr}(X \otimes X) - 2 \text{tr}[(A_i \otimes A_i)^{1/2}(X \otimes X)(A_i \otimes A_i)^{1/2}]^{1/2}. \quad (4.3)$$

Remarkably, despite the product terms, problem (4.3) turns out to be (Euclidean) convex (Lemma 4.1). This allows one to apply (g-) convex optimization tools, and use convexity to conclude global optimality. This result should be of independent interest.

**Lemma 4.1.** *The minimization problem 4.3 is convex.*

The proof uses properties of the Kronecker product and invokes the Ando-Lieb concavity theorem (Ando, 1979). It can be found in Appendix D.

One can show that the Wasserstein mean is upper bounded by the arithmetic mean  $A$  and lower bounded by  $\alpha I$ , rendering this a constrained optimization task. For computing the gradient, note that the Riemannian gradient  $\text{grad } \phi(X)$  can be written as  $\text{grad } \phi(X) = X \nabla \phi(X) - \nabla \phi(X) X$ , where  $\nabla \phi$  is the Euclidean gradient (where  $\phi$  denotes the objective in (4.2)). It is easy to show, that

$$\nabla \phi(X) = \sum_i w_i \left( I - (C_i X)^{-1/2} C_i \right),$$

which directly gives the gradient of the objective.

We evaluate the performance of our stochastic Rfw methods against the deterministic Rfw method for different initializations (Fig. 3). Our results indicate that all three initializations are suitable. This suggests, that (stochastic) Rfw is not sensitive to initialization and performs well even if not initialized close to the optimum. In a second experiment, we compute Wasserstein barycenters of MVNs for different input sizes. Both experiments indicates that especially the purely stochastic SRFW improves on RFW with comparable accuracy and stability. We did not compare against projection-based methods in the case of Wasserstein barycenters, since to our knowledge there are no implementations with the appropriate projections available.

## 5 Discussion

We introduced three stochastic Riemannian Frank-Wolfe methods, which go well-beyond the deterministic RFW recently proposed by (Weber and Sra, 2017). In particular, we (i) eliminate the restriction to  $g$ -convex objectives, allowing for an application to nonconvex problems; and (ii) improve the oracle complexities by replacing the computation of full gradients with stochastic gradient estimates. For the latter task, we analyze both fully stochastic and semi-stochastic variance-reduced estimators. Moreover, we implement the recently proposed SPIDER technique that significantly improves the classic variance-reduced approach by circumventing the need to compute full gradients.

We discuss applications of our methods to the computation of the Riemannian centroid and Wasserstein barycenters, both fundamental subroutines of potential value in several applications, including in machine learning. In validation experiments, we observe strong performance gains compared to the deterministic RFW as well as state-of-the-art deterministic and stochastic Riemannian methods.

This paper focused on developing a non-asymptotic convergence analysis and on establishing theoretical guarantees for our methods. Future work includes implementation of our algorithms for other manifolds and other classic Riemannian optimization tasks (see, e.g., (Absil and Hosseini, 2017)). This includes tasks with constraints on determinants or condition numbers. An important example for the latter is the task of learning a DPP kernel (see, e.g., (Mariet and Sra, 2015)), which can be formulated as a stochastic,  $g$ -convex problem. We hope to explore practical applications of our approach to large-scale constrained problems in ML and statistics.

Furthermore, instead of using exponential maps, one can reformulate our proposed methods using retractions. For projected gradient methods, the practicality of retraction-based approaches has been established (Absil et al., 2008), rendering this a promising extension for future research.

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## A Stochastic Riemannian Frank-Wolfe (SRFW)

### A.1 Convergence Analysis

Here and in the following  $x_k, x_{k+1}$  and  $y$  are as specified in Alg. 1.

**Assumption 1.** We make the following assumptions on  $\Phi$ :

1.  $\Phi$  is  $L$ -smooth.
2.  $\max_{\xi \in \text{supp}(\mathcal{P})} \|\text{grad } \phi(x, \xi)\| \leq G$  for some constant  $G \geq 0$ .

**Definition A.1** (Curvature constant). Let  $x, y, z \in \mathcal{X}$  and  $\gamma : [0, 1] \rightarrow \mathcal{M}$  a geodesic map with  $\gamma(0) = x$ ,  $\gamma(1) = z$  and  $y = \gamma(\eta)$  with  $\eta \in [0, 1]$ . Define

$$M_\Phi := \sup_{\substack{x, y, z \in \mathcal{X} \\ y = \gamma(\eta)}} \frac{2}{\eta^2} \left[ \Phi(y) - \Phi(x) - \langle \text{grad } \Phi(x), \text{Exp}_x^{-1}(y) \rangle \right]. \quad (\text{A.1})$$

**Lemma A.2** (Weber and Sra (2017)). Let  $\Phi : \mathcal{M} \rightarrow \mathbb{R}$  be  $L$ -smooth on  $\mathcal{X}$ ; let  $\text{diam}(\mathcal{X}) := \sup_{x, y \in \mathcal{X}} d(x, y)$ . Then, the curvature constant  $M_\Phi$  satisfies the bound

$$M_\Phi \leq L \text{diam}(\mathcal{X})^2.$$

*Proof.* Let  $x, z \in \mathcal{X}$  and  $y = \gamma(\eta)$ . This implies  $\frac{1}{\eta^2} d(x, y)^2 \leq d(x, z)^2$ . With Eq. 2.1 we have

$$\|\Phi(z) - \Phi(x) - \langle \text{grad } \Phi(x), \text{Exp}_x^{-1}(z) \rangle\|^2 \leq \frac{L}{2} d(x, z)^2. \quad (\text{A.2})$$

From this and the definition of the curvature constant we see

$$M_\Phi \leq \sup \frac{2}{\eta^2} \frac{L}{2} d(x, y)^2 \leq \sup L d(x, z)^2 \leq L \cdot \text{diam}^2(\mathcal{X}). \quad (\text{A.3})$$

□

**Lemma A.3** (Weber and Sra (2017)). Let  $\mathcal{X}$  be a constrained set. There exists a constant  $M_\Phi \geq 0$  such that for  $x_k, x_{k+1}, y_k \in \mathcal{X}$  as specified in Alg. 1, and for  $\eta_k \in (0, 1)$

$$\Phi(x_{k+1}) \leq \Phi(x_k) + \eta_k \langle \text{grad } \Phi(x_k), \text{Exp}_{x_k}^{-1}(y_k) \rangle + \frac{1}{2} M_\Phi \eta_k^2.$$

*Proof.* Let  $M_\Phi$  be the above defined curvature constant. Then, from its definition we get

$$M_\Phi \geq \frac{2}{\eta^2} \left( \Phi(x_{k+1}) - \Phi(x_k) - \langle \text{grad } \Phi(x_k), \text{Exp}_{x_k}^{-1}(x_{k+1}) \rangle \right),$$

which we can rewrite as

$$\Phi(x_{k+1}) \leq \Phi(x_k) + \langle \text{grad } \Phi(x_k), \text{Exp}_{x_k}^{-1}(x_{k+1}) \rangle + \frac{1}{2} \eta^2 M_\Phi.$$

For  $x_k, x_{k+1}, y_k \in \mathcal{X}$  as specified in Alg. 1 the update is given by

$$x_{k+1} = \gamma(\eta_k) = \text{Exp}_{x_k}(-\eta_k g_{x_k}),$$



which gives

$$\begin{aligned} \text{Exp}_{x_k}^{-1}(x_{k+1}) &= -\eta_k \underbrace{g_{x_k}}_{=-\text{Exp}_{x_k}^{-1}(y_k)} = \eta_k \text{Exp}_{x_k}^{-1}(y_k). \end{aligned}$$

With this, we can rewrite the second term above as

$$\begin{aligned} \langle g_{x_k}, \text{Exp}_{x_k}^{-1}(x_{k+1}) \rangle &= \langle g_{x_k}, \eta_k \text{Exp}_{x_k}^{-1}(y_k) \rangle \\ &= \eta_k \langle g_{x_k}, \text{Exp}_{x_k}^{-1}(y_k) \rangle. \end{aligned}$$

Putting everything together we get the desired bound:

$$\Phi(x_{k+1}) \leq \Phi(x_k) + \eta_k \langle g_{x_k}, \text{Exp}_{x_k}^{-1}(y_k) \rangle + \frac{1}{2} M_\Phi \eta_k^2.$$

□

**Lemma A.4** (Goodness of RM-approximation). *Let  $\Phi(x) = \mathbb{E}_\xi [\phi(x, \xi_i)]$  with r.v.  $\{\xi_i\}_{i=1}^b = \xi \sim \mathcal{P}$ . Furthermore, let*

$$g(x) := \frac{1}{b} \sum_{i=1}^b \text{grad } \phi(x, \xi_i)$$

denote the gradient estimate from a batch  $\xi$ . Then,

$$\mathbb{E} [\|g(x) - \text{grad } \Phi(x)\|] \leq \frac{G}{\sqrt{b}}.$$

*Proof.* We have

$$\begin{aligned} \mathbb{E} \left[ \underbrace{\|g(x) - \text{grad } \Phi(x)\|}_{= \mathbb{E}[g(x)]}^2 \right] &= \dagger \mathbb{E} \left[ \underbrace{\|g(x) - \mathbb{E}[g(x)]\|}_{\geq 0}^2 \right] \leq \mathbb{E} [\|g(x)\|^2] \\ &= \mathbb{E} \left[ \left\| \frac{1}{b} \sum_{i=1}^b \text{grad } \phi(x, \xi_i) \right\|^2 \right] \leq \ddagger \frac{1}{b^2} \mathbb{E} \left[ \underbrace{\sum_{i=1}^b \|\text{grad } \phi(x, \xi_i)\|^2}_{\leq G^2} \right] \leq \dagger\dagger \frac{G^2}{b}, \end{aligned}$$

where (†) follows from  $g_k$  being an unbiased estimate of  $\text{grad } \Phi(x)$ , (‡) from the triangle inequality and (††) from Ass. 1. Furthermore, with Jensen's inequality:

$$\mathbb{E} [\|g(x) - \text{grad } \Phi(x)\|^2] \geq [\mathbb{E} (\|g(x) - \text{grad } \Phi(x)\|)]^2.$$

Putting together both and taking the square root on both sides gives the desired claim:

$$\mathbb{E} [\|g(x) - \text{grad } \Phi(x)\|] \leq \frac{G}{\sqrt{b}}.$$

□

To evaluate convergence rates, consider the following convergence criterion:

**Definition A.5** (Frank-Wolfe gap).

$$\mathcal{G}(x) = \max_{y \in \mathcal{X}} \langle \text{Exp}_x^{-1}(y), -\text{grad } \Phi(x) \rangle$$

Assuming that the RM-approximation  $g(x)$  gives an unbiased estimate of the gradient  $\text{grad } \Phi(x)$ , we have  $\mathbb{E} [\mathcal{G}(x)] = \max_{y \in \mathcal{X}} \langle \text{Exp}_x^{-1}(y), -g(x) \rangle$ .

**Theorem A.6.** *With constant steps size  $\eta_k = \frac{1}{\sqrt{K}}$  and constant batch sizes  $b_k = K$ , Alg. 1 converges globally in expectation with a sublinear rate, i.e.*

$$\mathbb{E} [\mathcal{G}(\hat{x})] = O(1/\sqrt{K}) .$$

*Proof.* (Thm. 3.2) Let again

$$g_k(x_k) := \frac{1}{b_k} \sum_{i=1}^{b_k} \text{grad } \phi(x_k, \xi_i^k) \quad (\text{A.4})$$

denote the gradient estimate from the  $k^{\text{th}}$  batch. Then

$$\Phi(x_{k+1}) \leq^{\dagger} \Phi(x_k) + \eta_k \langle \text{grad } \Phi(x_k), \text{Exp}_{x_k}^{-1}(y_k) \rangle + \frac{1}{2} M_{\Phi} \eta_k^2 \quad (\text{A.5})$$

$$\leq^{\ddagger} \Phi(x_k) + \eta_k \langle g_k(x_k), \text{Exp}_{x_k}^{-1}(y_k) \rangle + \eta_k \langle \text{grad } \Phi(x_k) - g_k(x_k), \text{Exp}_{x_k}^{-1}(y_k) \rangle + \frac{1}{2} M_{\Phi} \eta_k^2 \quad (\text{A.6})$$

Here,  $(\dagger)$  follows from Lem. A.3 and  $(\ddagger)$  from ‘adding a zero’ with respect to  $g_k$ . We then apply Cauchy-Schwartz to the inner product and make use of the fact that the geodesic distance between points in  $\mathcal{X}$  is bounded by its diameter:

$$\langle \text{grad } \Phi(x_k) - g_k(x_k), \text{Exp}_{x_k}^{-1}(y_k) \rangle \leq \|\text{grad } \Phi(x_k) - g_k(x_k)\| \cdot \underbrace{\|\text{Exp}_{x_k}^{-1}(y_k)\|}_{\leq \text{diam}(\mathcal{X})} . \quad (\text{A.7})$$

This gives (with  $D := \text{diam}(\mathcal{X})$ )

$$\Phi(x_{k+1}) \leq \Phi(x_k) + \eta_k \langle g_k(x_k), \text{Exp}_{x_k}^{-1}(y_k) \rangle + \eta_k D \|\text{grad } \Phi(x_k) - g_k(x_k)\| + \frac{1}{2} M_{\Phi} \eta_k^2 .$$

Let  $\Delta_k := \Phi(x_k) - \Phi(x^*)$  denote the optimality gap. Subtracting  $\Phi(x^*)$  from both sides, we can rewrite the above inequality as

$$\Delta_{k+1} \leq \Delta_k + \eta_k \langle g_k(x_k), \text{Exp}_{x_k}^{-1}(y_k) \rangle + \eta_k D \|\text{grad } \Phi(x_k) - g_k(x_k)\| + \frac{1}{2} M_{\Phi} \eta_k^2 .$$

Taking expectations and applying Lem. 3.1 to the third term on the right-hand-side, we get

$$\mathbb{E} [\Delta_{k+1}] \leq \mathbb{E} [\Delta_k] - \eta_k \mathbb{E} [\mathcal{G}(x_k)] + \eta_k D \frac{G}{\sqrt{b_k}} + \frac{1}{2} M_{\Phi} \eta_k^2 ,$$

where we have rewritten the second term in terms of the Frank-Wolfe gap

$$\mathbb{E} [\mathcal{G}(x_k)] = -\mathbb{E} \left[ \langle g_k(x_k), \text{Exp}_{x_k}^{-1}(y_k) \rangle \right] .$$

The equality follows from  $y_k$  being optimal w.r.t. the oracle as defined in Alg. 1. Summing over all  $k$  batches and telescoping gives

$$\sum_k \eta_k \mathbb{E} [\mathcal{G}(x_k)] \leq \mathbb{E} [\Delta_0] - \underbrace{\mathbb{E} [\Delta_K]}_{\geq 0} + \sum_k \eta_k D \frac{G}{\sqrt{b_k}} + \sum_k \frac{1}{2} M_\Phi \eta_k^2 \quad (\text{A.8})$$

$$\leq \mathbb{E} [\Delta_0] + \sum_k \eta_k D \frac{G}{\sqrt{b_k}} + \sum_k \frac{1}{2} M_\Phi \eta_k^2. \quad (\text{A.9})$$

From Alg. 1 we see that the output  $\hat{x}$  is chosen uniformly at random from  $\{x_1, \dots, x_K\}$ , i.e.  $\mathbb{E} [\mathcal{G}(x_k)] = \mathbb{E} [\mathcal{G}(\hat{x})]$ . Now, with constant step sizes  $\eta_k = \eta$  and batch sizes  $b_k = b$ , we have

$$K\eta \mathbb{E} [\mathcal{G}(\hat{x})] \leq \mathbb{E} [\Delta_0] + K\eta D \frac{G}{\sqrt{b}} + K \frac{1}{2} M_\Phi \eta^2.$$

From  $\eta = \frac{1}{\sqrt{K}}$  and  $b = K$  we see that

$$\mathbb{E} [\mathcal{G}(\hat{x})] \leq \frac{1}{\sqrt{K}} \left( \mathbb{E} [\Delta_0] + DG + \frac{1}{2} M_\Phi \right),$$

which shows the desired sublinear convergence rate.  $\square$

In addition to the nonconvex case, we also present a result for g-convex objectives:

**Corollary A.7.** *Assuming that  $\Phi$  is g-convex, one can show an analogous convergence rate for the optimality gap, i.e.*

$$\mathbb{E} [\Delta_{\hat{k}}] = O(1/\sqrt{K}). \quad (\text{A.10})$$

*Proof.* In the proof of Prop. 3.2, Eq. B.2 note that

$$\begin{aligned} \Phi(x_{k+1}) &\leq \Phi(x_k) + \eta_k \langle g_k(x_k), \text{Exp}_{x_k}^{-1}(y_k) \rangle + \eta_k \langle \text{grad } \Phi(x_k) - g_k(x_k), \text{Exp}_{x_k}^{-1}(y_k) \rangle + \frac{1}{2} M_\Phi \eta_k^2 \\ &\leq^\dagger \Phi(x_k) + \eta_k \langle g_k(x_k), \text{Exp}_{x_k}^{-1}(x^*) \rangle + \eta_k \langle \text{grad } \Phi(x_k) - g_k(x_k), \text{Exp}_{x_k}^{-1}(y_k) \rangle + \frac{1}{2} M_\Phi \eta_k^2 \\ &\leq^\ddagger \Phi(x_k) - \eta_k (\Phi(x_k) - \Phi(x^*)) + \eta_k \langle \text{grad } \Phi(x_k) - g_k(x_k), \text{Exp}_{x_k}^{-1}(y_k) \rangle + \frac{1}{2} M_\Phi \eta_k^2 \end{aligned}$$

where  $\dagger$ ) follows from  $y_k$  being the argmin as defined in Alg. 1 and the fourth inequality ( $\ddagger$ ) from the g-convexity of  $\Phi$ . Note, that the second term can be written as

$$\eta_k (\Phi(x_k) - \Phi(x^*)) = \eta_k \Delta_k.$$

Then, following the proof above, we get the desired convergence rate.  $\square$

## A.2 Decreasing batch sizes

The current hyperparameters assume a large batch size that is probably not practical. Can we maintain sublinear convergence rates for a decreasing step size  $\eta_k$  and smaller batch sizes  $b_k$ ?

**Theorem A.8.** *With decreasing steps sizes  $\eta_k = \frac{1}{k}$  and constant batch sizes  $b_k = \log_2^2 K$ , Alg. 1 converges in expectation as  $O\left(\frac{1}{\log_2 K}\right)$ , i.e.*

$$\mathbb{E} [\mathcal{G}(\hat{x})] = O\left(\frac{1}{\log_2 K}\right).$$

*Proof.* We follow the proof of Prop. 3.2 to inequality B.6 and note that again, due to the choice of the output, we have  $\mathbb{E} [\mathcal{G}(x_k)] = \mathbb{E} [\mathcal{G}(\hat{x})]$ . This gives:

$$\mathbb{E} [\mathcal{G}(\hat{x})] \sum_k \eta_k \leq \mathbb{E}(\Delta_0) + \sum_k \eta_k D \frac{G}{\sqrt{b_k}} + \frac{1}{2} M_\Phi \sum_k \eta_k^2.$$

Let  $K = 2^n$ . We note that

$$\frac{n}{2} + 1 \leq \sum_{k=1}^{2^n} \frac{1}{k} \leq \ln(2^n) + 1,$$

and  $\sum_{k=1}^{2^n} \frac{1}{k^2} \leq 2$ . With that and  $b = b_k = n^2$ , we have

$$\left(\frac{n}{2} + 1\right) \mathbb{E} [\mathcal{G}(\hat{x})] \leq \mathbb{E}(\Delta_0) + \frac{DG}{n} \underbrace{(\ln(2^n) + 1)}_{=n \ln(2)} + M_\Phi,$$

and by rearranging terms

$$\mathbb{E} [\mathcal{G}(\hat{x})] \leq \frac{2}{n+2} [\mathbb{E}(\Delta_0) + DG \ln(2) + M_\Phi].$$

The claim follows from  $n = \log_2 K$  and the term in brackets being constant.  $\square$

**Remark A.9.** The above result allows for a smaller batch size, albeit with weaker convergence guarantees, i.e. a slower  $O(1/\log(K))$  convergence rate.

## B Stochastic variance-reduced Riemannian Frank-Wolfe

We restate the central convergence result in the main text:

**Theorem B.1** (Convergence SRFW). *With steps size  $\eta_k = \frac{1}{\sqrt{mS}}$  and constant batch sizes  $b_k = m$ , Alg. 2 converges in expectation with  $\mathbb{E} [\mathcal{G}(\hat{x})] = O\left(\frac{1}{\sqrt{mS}}\right)$ .*

Before we can proof the theorem, we have to analyze the goodness of the variance-reduced gradient estimate:

**Lemma B.2** (Goodness of variance-reduced gradient estimate). *Consider the  $k^{\text{th}}$  iteration in the  $s^{\text{th}}$  epoch and the stochastic variance-reduced gradient estimate*

$$g_k(x_k^{s+1}) = \frac{1}{b_k} \sum_j \text{grad } \phi_j(x_k^{s+1}) - \Gamma_{\tilde{x}^s}^{x_k^{s+1}} (\text{grad } \phi_j(\tilde{x}^s) + \text{grad } \Phi(\tilde{x}^s)).$$

*Then, the following inequality holds:*

$$\mathbb{E} \left[ \|\text{grad } \Phi(x_k^{s+1}) - g_k\| \right] \leq \frac{L}{\sqrt{b_k}} \|x_k^{s+1} - \tilde{x}^s\|.$$

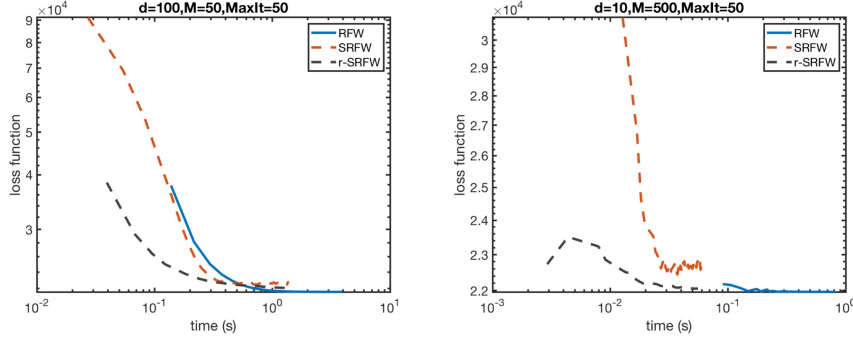


Figure 4: **Comparison of Srfw and r-Srfw.** Performance SRFW and R-SRFW in comparison with RFW for well-conditioned inputs of different sizes ( $d$ : size of matrices,  $M$ : number of matrices,  $K$ : number of iterations); initialized at  $X_0 = H$ .

*Proof.* Following Alg. 2, let  $I_k = \{i_1, \dots, i_{b_k}\}$  denote the sample on the  $k$ th iteration of the  $s$ th epoch. We introduce the shorthands

$$\zeta_k^{s+1} = \frac{1}{b_k} \sum_{l=1}^{b_k} \text{grad } \phi_{i_l}(x_k^{s+1}) - \Gamma_{\tilde{x}^s}^{x_k^{s+1}} \text{grad } \phi_{i_l}(\tilde{x}^s)$$

$$\zeta_{k,i}^{s+1} \text{grad } \phi_i(x_k^{s+1}) - \Gamma_{\tilde{x}^s}^{x_k^{s+1}} \text{grad } \phi_i(\tilde{x}^s),$$

i.e.,  $\zeta_k^{s+1} = \frac{1}{b_k} \sum_{i=1}^{b_k} \zeta_{k,i}^{s+1}$ . Then we have

$$\begin{aligned} \mathbb{E}_{I_k} \left[ \|\text{grad } \Phi(x_k^{s+1}) - g_k(x_k^{s+1})\|^2 \right] &= \mathbb{E}_{I_k} \left[ \|\zeta_k^{s+1} - \text{grad } \Phi(x_k^{s+1}) + \Gamma_{\tilde{x}^s}^{x_k^{s+1}} \text{grad } \Phi(\tilde{x}^s)\|^2 \right] \\ &=^+ \mathbb{E}_{I_k} \left[ \|\zeta_k^{s+1} - \mathbb{E}(\zeta_k^{s+1})\|^2 \right]. \end{aligned}$$

Here, (+) follows from the following argument:

$$\begin{aligned} \mathbb{E}_{I_k}(\zeta_k^{s+1}) &= \mathbb{E}_{I_k} \left( \frac{1}{b_k} \sum_i \zeta_{k,i}^{s+1} \right) = \frac{1}{b_k} \sum_i \mathbb{E}(\zeta_{k,i}^{s+1}) = \frac{1}{b_k} \sum_i \text{grad } \Phi(x + k^{s+1}) - \Gamma_{\tilde{x}^s}^{x_k^{s+1}} \text{grad } \Phi(\tilde{x}^s) \\ &= \text{grad } \Phi(x + k^{s+1}) - \Gamma_{\tilde{x}^s}^{x_k^{s+1}} \text{grad } \Phi(\tilde{x}^s). \end{aligned}$$

We further have

$$\begin{aligned}
\mathbb{E}_{I_k} \left[ \|\zeta_k^{s+1} - \mathbb{E}(\zeta_k^{s+1})\|^2 \right] &= \mathbb{E}_{I_k} \left[ \left\| \frac{1}{b_k} \sum_i \zeta_{k,i}^{s+1} - \mathbb{E}_{I_k} \left( \frac{1}{b_k} \sum_i \zeta_{k,i}^{s+1} \right) \right\|^2 \right] \\
&= \mathbb{E}_{I_k} \left[ \left\| \frac{1}{b_k} \sum_i (\zeta_{k,i}^{s+1} - \mathbb{E}_{I_k}(\zeta_{k,i}^{s+1})) \right\|^2 \right] \\
&\leq \frac{1}{b_k^2} \mathbb{E}_{I_k} \left[ \left( \sum_i \|\zeta_{k,i}^{s+1} - \mathbb{E}_{I_k}(\zeta_{k,i}^{s+1})\| \right)^2 \right] \\
&= \frac{1}{b_k^2} \mathbb{E}_{I_k} \left[ \underbrace{\|\zeta_{k,i}^{s+1} - \mathbb{E}_{I_k}(\zeta_{k,i}^{s+1})\|^2}_{\geq 0^*} \right] \\
&\leq^\ddagger \frac{1}{b_k^2} \sum_i \mathbb{E}_{I_k} \left[ \|\text{grad } \phi_i(x_k^{s+1}) - \Gamma_{\tilde{x}^s}^{x_k^{s+1}} \text{grad } \Phi(\tilde{x}^s)\|^2 \right] \\
&\leq^{++} \frac{1}{b_k^2} \sum_i L^2 \|x_k^{s+1} - \tilde{x}^s\|^2 \\
&= \frac{L^2}{b_k} \|x_k^{s+1} - \tilde{x}^s\|,
\end{aligned}$$

where  $(\ddagger)$  follows from  $(*)$  and the definition of  $\zeta_{k,i}^{s+1}$ ,  $(++)$  follows from the  $\phi_i$  being  $L$ -Lipschitz smooth. This shows

$$\mathbb{E} \left[ \|\text{grad } \Phi(x_k^{s+1}) - g_k\|^2 \right] \leq \frac{L^2}{b_k} \|x_k^{s+1} - \tilde{x}^s\|^2.$$

Jensen's inequality gives

$$\mathbb{E} \left[ \|\text{grad } \Phi(x_k^{s+1}) - g_k\|^2 \right] \geq \mathbb{E} \left[ \|\text{grad } \Phi(x_k^{s+1}) - g_k\| \right]^2,$$

and, putting everything together and taking the square root on both sides, the claim follows:

$$\mathbb{E} \left[ \|\text{grad } \Phi(x_k^{s+1}) - g_k\| \right] \leq \frac{L}{\sqrt{b_k}} \|x_k^{s+1} - \tilde{x}^s\|.$$

□

We can now prove the convergence result.

*Proof.* (Thm. 3.5) Let again

$$g_k(x_k^{s+1}) = \frac{1}{b_k} \sum_j \text{grad } \phi_j(x_k^{s+1}) - \Gamma_{\tilde{x}^s}^{x_k^{s+1}} (\text{grad } \phi_j(\tilde{x}^s) + \text{grad } \Phi(\tilde{x}^s)) \quad (\text{B.1})$$

denote the variance-reduced gradient estimate from the  $k^{\text{th}}$  iteration in the  $s^{\text{th}}$  epoch. Then

$$\Phi(x_{k+1}^{s+1}) \leq^\dagger \Phi(x_k^{s+1}) + \eta_k \langle \text{grad } \Phi(x_k^{s+1}), \text{Exp}_{x_k^{s+1}}^{-1}(y_k) \rangle + \frac{1}{2} M_\Phi \eta_k^2 \quad (\text{B.2})$$

$$\leq^\ddagger \Phi(x_k^{s+1}) + \eta_k \langle g_k(x_k^{s+1}), \text{Exp}_{x_k^{s+1}}^{-1}(y_k) \rangle \quad (\text{B.3})$$

$$+ \eta_k \langle \text{grad } \Phi(x_k^{s+1}) - g_k(x_k^{s+1}), \text{Exp}_{x_k^{s+1}}^{-1}(y_k) \rangle + \frac{1}{2} M_\Phi \eta_k^2 \quad (\text{B.4})$$

Here, (†) follows from Lem. A.3 and (‡) from ‘adding a zero’ with respect to  $g_k$ . We then apply Cauchy-Schwartz to the inner product and make use of the fact that the geodesic distance between points in  $\mathcal{X}$  is bounded by its diameter:

$$\langle \text{grad } \Phi(x_k^{s+1}) - g_k(x_k^{s+1}), \text{Exp}_{x_k^{s+1}}^{-1}(y_k) \rangle \leq \|\text{grad } \Phi(x_k^{s+1}) - g_k(x_k^{s+1})\| \cdot \underbrace{\|\text{Exp}_{x_k^{s+1}}^{-1}(y_k)\|}_{\leq \text{diam}(\mathcal{X})}. \quad (\text{B.5})$$

This gives (with  $D := \text{diam}(\mathcal{X})$ )

$$\Phi(x_{k+1}^{s+1}) \leq \Phi(x_k^{s+1}) + \eta_k \langle g_k(x_k^{s+1}), \text{Exp}_{x_k^{s+1}}^{-1}(y_k) \rangle + \eta_k D \|\text{grad } \Phi(x_k^{s+1}) - g_k(x_k^{s+1})\| + \frac{1}{2} M_\Phi \eta_k^2.$$

Let  $\Delta_{k,s+1} := \Phi(x_k^{s+1}) - \Phi(x^*)$  denote the optimality gap. Subtracting  $\Phi(x^*)$  from both sides, we can rewrite the above inequality as

$$\Delta_{k+1,s+1} \leq \Delta_{k,s+1} + \eta_k \langle g_k(x_k^{s+1}), \text{Exp}_{x_k^{s+1}}^{-1}(y_k) \rangle + \eta_k D \|\text{grad } \Phi(x_k^{s+1}) - g_k(x_k^{s+1})\| + \frac{1}{2} M_\Phi \eta_k^2.$$

Taking expectations and applying Lem. 3.1 to the third term on the right-hand-side, we get

$$\mathbb{E} [\Delta_{k+1,s+1}] \leq \mathbb{E} [\Delta_{k,s+1}] - \eta_k \mathbb{E} [\mathcal{G}(x_k^{s+1})] + \eta_k D \frac{G}{\sqrt{b_k}} + \frac{1}{2} M_\Phi \eta_k^2,$$

where we have rewritten the second term in terms of the Frank-Wolfe gap

$$\mathbb{E} [\mathcal{G}(x_k^{s+1})] = -\mathbb{E} \left[ \langle g_k(x_k^{s+1}), \text{Exp}_{x_k^{s+1}}^{-1}(y_k) \rangle \right].$$

The equality follows from  $y_k$  being optimal w.r.t. the oracle as defined in Alg. 1. Summing over all  $s$  epochs  $m$  iterations within each epoch and telescoping gives

$$\sum_s \sum_k \eta_k \mathbb{E} [\mathcal{G}(x_k^{s+1})] \leq \mathbb{E} [\Delta_{0,0}] - \underbrace{\mathbb{E} [\Delta_{m,S}]_{\geq 0}} + \sum_k \eta_k D \frac{G}{\sqrt{b_k}} + \sum_k \frac{1}{2} M_\Phi \eta_k^2 \quad (\text{B.6})$$

$$\leq \mathbb{E} [\Delta_{0,0}] + \sum_k \eta_k D \frac{G}{\sqrt{b_k}} + \sum_k \frac{1}{2} M_\Phi \eta_k^2. \quad (\text{B.7})$$

Note, that here  $\Delta_{0,0}$  denotes the initial optimality gap. Now, with constant step sizes  $\eta_k = \eta = \frac{1}{\sqrt{mS}}$  and batch sizes  $b_k = b = m$ , we have

$$\begin{aligned} \sqrt{mS} \mathbb{E} [\mathcal{G}(\hat{x})] &\leq \mathbb{E} [\Delta_{0,0}] + \frac{\sqrt{mS}}{\sqrt{m}} LD + \frac{1}{2} M_\Phi \\ \mathbb{E} [\mathcal{G}(\hat{x})] &\leq \frac{1}{\sqrt{mS}} \mathbb{E} [\Delta_{0,0}] + \frac{1}{\sqrt{m}} LD + \frac{1}{2\sqrt{mS}} M_\Phi. \end{aligned}$$

Since  $m > 0$ , the claim follows as

$$\mathbb{E} [\mathcal{G}(\hat{x})] \leq \frac{1}{\sqrt{mS}} \left( \mathbb{E} [\Delta_{0,0}] + LD + \frac{1}{2} M_\Phi \right), \quad (\text{B.8})$$

i.e.,  $\mathbb{E} [\mathcal{G}(\hat{x})] = O(\frac{1}{\sqrt{mS}})$ . □

## C SPIDER Frank-Wolfe

**Proposition C.1** (Convergence SPIDER-RFW). *With the parameter choices 3.3, Alg. 3 converges in expectation with the following rate:*

$$\mathbb{E} [\mathcal{G}(x_K)] = O\left(\frac{1}{\sqrt{K}}\right). \quad (\text{C.1})$$

Before proving this result, we have to analyze the goodness of the gradient estimate. We denote with  $\mathcal{F}_k$  the  $\sigma$ -field generated by  $x_k$ , the family  $\{\mathcal{F}_k\}_{k \geq 0}$  forms a filtration. The SPIDER estimate at step  $k$  is computed as

$$g_k = \begin{cases} \text{grad } \Phi_{S_1}(x_k), & \text{mod}(k, m) = 0 \\ \text{grad } \Phi_{S_2}(x_k) - \Gamma_{x_{k-1}}^{x_k} (\text{grad } \Phi_{S_2}(x_{k-1}) - g_{k-1}), & \text{else} \end{cases}. \quad (\text{C.2})$$

The deviation from the true gradient can be bounded as follows:

**Lemma C.2** (Goodness of SPIDER-approximation). *The expected deviation of the estimate  $g_k$  (Eq. C.2) from the true gradient is bounded as follows:*

$$\mathbb{E} [\|g_k - \text{grad } \Phi(x_k)\| | \mathcal{F}_k] \leq \epsilon. \quad (\text{C.3})$$

**Lemma C.3.**  *$(g_k - \text{grad } \Phi(x_k))_k$  forms a martingale with respect to the filtration  $\{\mathcal{F}_k\}$ .*

*Proof.*

$$\begin{aligned} \mathbb{E} [g_k - \text{grad } \Phi(x_k) | \mathcal{F}_k] &= \mathbb{E} [\text{grad } \Phi_{S_2}(x_k) - \Gamma_{x_{k-1}}^{x_k} (\text{grad } \Phi_{S_2}(x_{k-1}) - g_{k-1}) - \text{grad } \Phi(x_k) | \mathcal{F}_k] \\ &= \underbrace{\mathbb{E} [\text{grad } \Phi_{S_2}(x_k) - \text{grad } \Phi(x_k) | \mathcal{F}_k]}_{=0, \text{ unbiased estimate}} + \mathbb{E} [\Gamma_{x_{k-1}}^{x_k} g_{k-1} - \text{grad } \Phi_{S_2}(x_{k-1}) | \mathcal{F}_k] \\ &= \Gamma_{x_{k-1}}^{x_k} g_{k-1} - \text{grad } \Phi_{S_2}(x_{k-1}) \end{aligned}$$

□

**Corollary C.4.**  *$(\|g_k - \text{grad } \Phi(x_k)\|)_k$  is a martingale too.*

*Proof.* By Jensen's inequality and the convexity of the norm,  $(\|g_k - \text{grad } \Phi(x_k)\|)_k$  is a sub-martingale. Since it has constant expectation (i.e., by assumption  $\mathbb{E} (\|g_k - \text{grad } \Phi(x_k)\|) = 0$ ), it is also a martingale. □

**Remark C.5.** Let  $M = (M_k)_k$  denote the martingale with  $M_k = \|g_k - \text{grad } \Phi(x_k)\|$ . Note that  $M_k \in \mathcal{L}^2$ ; therefore we have orthogonality of increments, i.e.,

$$\langle M_t - M_s, M_v - M_u \rangle = 0 \quad (v \geq u \geq t \geq s).$$

This implies

$$\mathbb{E} [M_k^2] = \mathbb{E} [M_{k-1}^2] + \mathbb{E} [(M_k - M_{k-1})^2].$$

and recursively

$$\mathbb{E} [M_k^2] = \mathbb{E} [M_0^2] + \sum_{i=1}^k \mathbb{E} [(M_i - M_{i-1})^2].$$



We can now proof Lemma C.2:

*Proof.* (Lem. C.2) First, consider the case  $\mathbf{mod}(m, k) = 0$ :

$$\mathbb{E} [\|g_k - \text{grad } \Phi(x_k)\|^2 | \mathcal{F}_k] = \mathbb{E} [\|\text{grad } \Phi_{S_1}(x_k) - \text{grad } \Phi(x_k)\|^2 | \mathcal{F}_k] \stackrel{\dagger}{\leq} \frac{G^2}{|S_1|} = \frac{G^2 \epsilon^2}{2G^2} = \frac{\epsilon^2}{2}, \quad (\text{C.4})$$

where (†) follows from Lemma 3.1. Now, consider  $\mathbf{mod}(m, k) \neq 0$ . We have:

$$\begin{aligned} & \mathbb{E} [\|g_k - \text{grad } \Phi(x_k)\|^2 | \mathcal{F}_k] \\ &= \stackrel{++}{=} \mathbb{E} [\Gamma_{x_{k-1}}^{x_k} \|g_{k-1} - \text{grad } \Phi(x_{k-1})\|^2 | \mathcal{F}_k] + \mathbb{E} [\|g_k - \text{grad } \Phi(x_k) - \Gamma_{x_{k-1}}^{x_k} (\text{grad } \Phi(x_{k-1}) - g_{k-1})\|^2 | \mathcal{F}_k] \\ &= \stackrel{\ddagger}{=} \mathbb{E} [\Gamma_{x_{k-1}}^{x_k} \|g_{k-1} - \text{grad } \Phi(x_{k-1})\|^2 | \mathcal{F}_k] \\ &\quad + \mathbb{E} [\|\text{grad } \Phi_{S_2}(x_k) - \Gamma_{x_{k-1}}^{x_k} (\text{grad } \Phi_{S_2}(x_{k-1}) - g_{k-1}) - \text{grad } \Phi(x_k) - \Gamma_{x_{k-1}}^{x_k} (g_{k-1} - \text{grad } \Phi(x_{k-1}))\|^2 | \mathcal{F}_k] \\ &= \stackrel{++}{=} \mathbb{E} [\Gamma_{x_{k-1}}^{x_k} \|g_{k-1} - \text{grad } \Phi(x_{k-1})\|^2 | \mathcal{F}_k] \\ &\quad + \mathbb{E} [\|\text{grad } \Phi_{S_2}(x_k) - \text{grad } \Phi(x_k) - \Gamma_{x_{k-1}}^{x_k} (\text{grad } \Phi_{S_2}(x_{k-1}) - \text{grad } \Phi(x_{k-1}))\|^2 | \mathcal{F}_k], \end{aligned}$$

where (++) follows from Remark C.5, (‡) from substituting  $g_k$  according to Alg. 3 and (††) from summarizing the  $(x_{k_1})$ -terms. We introduce the shorthand

$$\zeta_i = \text{grad } \phi(x_k, \xi) - \text{grad } \phi(x_k) - \Gamma_{x_{k-1}}^{x_k} (\text{grad } \phi(x_{k-1}, \xi) - \text{grad } \phi(x_{k-1})).$$

Then, we get

$$\begin{aligned} & \mathbb{E} [\|\text{grad } \Phi_{S_2}(x_k) - \text{grad } \Phi(x_k) - \Gamma_{x_{k-1}}^{x_k} (\text{grad } \Phi_{S_2}(x_{k-1}) - \text{grad } \Phi(x_{k-1}))\|^2 | \mathcal{F}_k] \\ &= \mathbb{E} \left[ \left\| \frac{1}{|S_2|} \sum_{i=1}^{|S_2|} \zeta_i \right\|^2 | \mathcal{F}_k \right] = \frac{1}{|S_2|^2} \mathbb{E} \left[ \left\| \sum_{i=1}^{|S_2|} \zeta_i \right\|^2 | \mathcal{F}_k \right] \\ &\stackrel{\dagger}{\leq} \frac{1}{|S_2|^2} \mathbb{E} \left[ \left( \sum_{i=1}^{|S_2|} \|\zeta_i\| \right)^2 | \mathcal{F}_k \right] = \stackrel{\ddagger}{=} \frac{1}{|S_2|^2} \mathbb{E} \left[ \sum_{i=1}^{|S_2|} \|\zeta_i\|^2 | \mathcal{F}_k \right] \\ &= \frac{1}{|S_2|^2} \sum_{i=1}^{|S_2|} \mathbb{E} [\|\zeta_i\|^2 | \mathcal{F}_k] = \stackrel{++}{=} \frac{1}{|S_2|} \mathbb{E} [\|\zeta_i\|^2 | \mathcal{F}_k] \\ &= \frac{1}{|S_2|} \mathbb{E} [\|\text{grad } \phi(x_k, \xi) - \text{grad } \phi(x_k) - \Gamma_{x_{k-1}}^{x_k} (\text{grad } \phi(x_{k-1}, \xi) - \text{grad } \phi(x_{k-1}))\|^2 | \mathcal{F}_k]. \end{aligned}$$

where (†) follows from the triangle-inequality, (‡) from  $\mathbb{E} [\zeta_i] = 0$ , i.e., all mixed terms vanish, and (++) from the  $\zeta_i$  being i.i.d. Note, that

$$\begin{aligned} \mathbb{E} [\text{grad } \phi(x_k, \xi) | \mathcal{F}_k] &= \text{grad } \phi(x_k) \\ \mathbb{E} [\Gamma_{x_{k-1}}^{x_k} \text{grad } \phi(x_{k-1}, \xi) | \mathcal{F}_k] &= \text{grad } \phi(x_{k-1}). \end{aligned}$$

With this, we have

$$\begin{aligned}
& \mathbb{E} [\|\text{grad } \phi(x_k, \zeta) - \text{grad } \phi(x_k) - \Gamma_{x_{k-1}}^{x_k} (\text{grad } \phi(x_{k-1}, \zeta) - \text{grad } \phi(x_{k-1}))\|^2 | \mathcal{F}_k] \\
&= \mathbb{E} [\|\text{grad } \phi(x_k, \zeta) - \Gamma_{x_{k-1}}^{x_k} \text{grad } \phi(x_{k-1}, \zeta) - (\text{grad } \phi(x_k) - \Gamma_{x_{k-1}}^{x_k} \text{grad } \phi(x_{k-1}))\|^2 | \mathcal{F}_k] \\
&= \mathbb{E} \left[ \underbrace{\|\text{grad } \phi(x_k, \zeta) - \Gamma_{x_{k-1}}^{x_k} \text{grad } \phi(x_{k-1}, \zeta) - \mathbb{E} [\|\text{grad } \phi(x_k, \zeta) - \Gamma_{x_{k-1}}^{x_k} \text{grad } \phi(x_{k-1}, \zeta)\|^2 | \mathcal{F}_k]\|}_{\geq 0}^2 | \mathcal{F}_k \right] \\
&\leq \mathbb{E} [\|\text{grad } \phi(x_k, \zeta) - \Gamma_{x_{k-1}}^{x_k} \text{grad } \phi(x_{k-1}, \zeta)\|^2 | \mathcal{F}_k]
\end{aligned}$$

Inserting this above, we get

$$\begin{aligned}
& \mathbb{E} [\|g_k - \text{grad } \Phi(x_k)\|^2 | \mathcal{F}_k] \\
&\leq \mathbb{E} [\Gamma_{x_{k-1}}^{x_k} \|g_{k-1} - \text{grad } \Phi(x_{k-1})\|^2 | \mathcal{F}_k] + \frac{1}{|S_2|} \mathbb{E} [\|\text{grad } \phi(x_k, \zeta) - \Gamma_{x_{k-1}}^{x_k} (\text{grad } \phi(x_{k-1}, \zeta))\|^2 | \mathcal{F}_k] \\
&\leq^{\dagger} \mathbb{E} [\Gamma_{x_{k-1}}^{x_k} \|g_{k-1} - \text{grad } \Phi(x_{k-1})\|^2 | \mathcal{F}_k] + \frac{1}{|S_2|} L^2 \mathbb{E} [\|\text{Exp}_{x_{k-1}}^{-1}(x_k)\|^2 | \mathcal{F}_k] \\
&=^{\ddagger} \mathbb{E} [\Gamma_{x_{k-1}}^{x_k} \|g_{k-1} - \text{grad } \Phi(x_{k-1})\|^2 | \mathcal{F}_k] + \frac{\epsilon^2}{2mL^2 \|\text{Exp}_{x_{k-1}}^{-1}(x_k)\|} L^2 \|\text{Exp}_{x_{k-1}}^{-1}(x_k)\| \\
&= \mathbb{E} [\Gamma_{x_{k-1}}^{x_k} \|g_{k-1} - \text{grad } \Phi(x_{k-1})\|^2 | \mathcal{F}_k] + \frac{\epsilon^2}{2m}
\end{aligned}$$

where  $(\dagger)$  follows from  $\phi$  being  $L$ -Lipschitz and  $(\ddagger)$  from the choice of  $|S_2|$  in Alg. 3. Recursively going back to the beginning of the epoch (see Remark C.5), we get (with  $k_0 = \lfloor \frac{k}{m} \rfloor m$ ):

$$\mathbb{E} [\|g_k - \text{grad } \Phi(x_k)\|^2 | \mathcal{F}_k] \leq \underbrace{\mathbb{E} [\|g_{k_0} - \text{grad } \Phi(x_{k_0})\|^2 | \mathcal{F}_{k_0}]}_{\leq \frac{\epsilon^2}{2} \text{ Eq. C.4}} + m \frac{\epsilon^2}{2m} \leq \epsilon^2.$$

With Jensen's inequality, we have

$$(\mathbb{E} [\|g_k - \text{grad } \Phi(x_k)\|^2 | \mathcal{F}_k])^2 \leq \mathbb{E} [\|g_k - \text{grad } \Phi(x_k)\|^4 | \mathcal{F}_k] \leq \epsilon^2,$$

which gives

$$\mathbb{E} [\|g_k - \text{grad } \Phi(x_k)\| | \mathcal{F}_k] \leq \epsilon.$$

□

Now, we can prove Proposition C.1:

*Proof.* (Prop. C.1) We again have

$$\begin{aligned}
\Phi(x_{k+1}) &\leq^{\dagger} \Phi(x_k) + \eta_k \langle \text{grad } \Phi(x_k), \text{Exp}_{x_k}^{-1}(y_k) \rangle + \frac{1}{2} M_{\Phi} \eta_k^2 \\
&\leq^{\ddagger} \Phi(x_k) + \eta_k \langle g_k(x_k), \text{Exp}_{x_k}^{-1}(y_k) \rangle + \eta_k \langle \text{grad } \Phi(x_k) - g_k(x_k), \text{Exp}_{x_k}^{-1}(y_k) \rangle + \frac{1}{2} M_{\Phi} \eta_k^2,
\end{aligned}$$

where, (†) follows from Lem. A.3 and (‡) from ‘adding a zero’ with respect to  $g_k$ . We again apply Cauchy-Schwartz to the inner product and make use of the fact that the geodesic distance between points in  $\mathcal{X}$  is bounded by its diameter:

$$\langle \text{grad } \Phi(x_k) - g_k(x_k), \text{Exp}_{x_k}^{-1}(y_k) \rangle \leq \|\text{grad } \Phi(x_k) - g_k(x_k)\| \cdot \underbrace{\|\text{Exp}_{x_k}^{-1}(y_k)\|}_{\leq \text{diam}(\mathcal{X})}. \quad (\text{C.5})$$

This gives (with  $D := \text{diam}(\mathcal{X})$ )

$$\Phi(x_{k+1}) \leq \Phi(x_k) + \eta_k \langle g_k(x_k), \text{Exp}_{x_k}^{-1}(y_k) \rangle + \eta_k D \|\text{grad } \Phi(x_k) - g_k(x_k)\| + \frac{1}{2} M_\Phi \eta_k^2.$$

We know subtract  $\Phi(x^*)$  from both sides and introduce the shorthand  $\Delta_k = \Phi(x_k) - \Phi(x^*)$  for the optimality gap. Taking expectations, we get

$$\mathbb{E} [\Delta_{k+1}] \leq \mathbb{E} [\Delta_k] + \eta_k \underbrace{\mathbb{E} [\langle g_k(x_k), \text{Exp}_{x_k}^{-1}(y_k) \rangle]}_{=-\mathbb{E}[\mathcal{G}(x_k)]} + \eta_k D \underbrace{\mathbb{E} [\|\text{grad } \Phi(x_k) - g_k(x_k)\|]}_{\leq \epsilon} + \frac{1}{2} M_\Phi \eta_k^2.$$

With Lemma C.2 this can be rewritten in terms of the optimality gap as:

$$\mathbb{E} [\Delta_{k+1}] \leq \mathbb{E} [\Delta_k] - \eta_k \mathbb{E} [\mathcal{G}(x_k)] + \eta_k D \epsilon + \frac{1}{2} M_\Phi \eta_k^2.$$

Summing and telescoping gives (with  $\mathbb{E} [\mathcal{G}(x_k)] = \mathbb{E} [\mathcal{G}(\hat{x})] \forall k$ ):

$$\begin{aligned} \mathbb{E} [\mathcal{G}(\hat{x})] \sum_k \eta_k &\leq \mathbb{E} [\Delta_0] - \underbrace{\mathbb{E} [\Delta_K]}_{\geq 0} + D \epsilon \sum_k \eta_k + \frac{1}{2} M_\Phi \sum_k \eta_k^2 \\ &\leq \mathbb{E} [\Delta_0] + D \epsilon \sum_k \eta_k + \frac{1}{2} M_\Phi \sum_k \eta_k^2. \end{aligned}$$

With  $\eta_k = \eta = \frac{1}{\sqrt{K}}$ , this becomes

$$\underbrace{K\eta}_{=\sqrt{K}} \mathbb{E} [\mathcal{G}(\hat{x})] \leq \mathbb{E} [\Delta_0] + D \epsilon \underbrace{K\eta}_{=\sqrt{K}} + \frac{1}{2} M_\Phi \underbrace{K\eta^2}_{=1}.$$

Note, that  $\epsilon = \frac{1}{m} = \frac{1}{\sqrt{K}}$ . Dividing by  $\sqrt{K}$  then gives the claim:

$$\mathbb{E} [\mathcal{G}(\hat{x})] \leq \frac{1}{\sqrt{K}} \left( \mathbb{E} [\Delta_0] + D \underbrace{\epsilon \sqrt{K}}_{=1} + \frac{1}{2} M_\Phi \right). \quad (\text{C.6})$$

□

**Corollary C.6.** *Assuming that  $\Phi$  is  $g$ -convex, one can show a similar convergence rate for the optimality gap, i.e.*

$$\mathbb{E} [\Delta_{\hat{k}}] = O(1/\sqrt{K}). \quad (\text{C.7})$$

*Proof.* Analogous to the proof of Cor. 3.6. □

## D Wasserstein barycenters of matrix-variate Gaussians

We restate the result from the main text:

**Lemma D.1.** *The minimization problem*

$$\min_{X>0} \left[ \sum_{i=1}^n \text{tr}(A_i \otimes A_i) + \text{tr}(X \otimes X) - 2 \text{tr} \left( (A_i \otimes A_i)^{1/2} (X \otimes X) (A_i \otimes A_i)^{1/2} \right)^{1/2} \right]$$

is convex.

For the proof, recall the following well-known properties of Kronecker products:

**Lemma D.2** (Properties of Kronecker products). *Let  $A, B, C, D \in \mathbb{P}^d$ .*

1.  $(A \otimes A)^{1/2} = A^{1/2} \otimes A^{1/2}$ ;
2.  $AC \otimes BD = (A \otimes B)(C \otimes D)$ .

Furthermore, recall the Ando-Lieb theorem ([Ando, 1979](#)):

**Theorem D.3** (Ando-Lieb). *Let  $A, B \in \mathbb{P}^d$ . Then the map  $(A, B) \mapsto A^\gamma \otimes B^{1-\gamma}$  is jointly concave for  $0 < \gamma < 1$ .*

Equipped with those two arguments, we can prove the lemma.

*Proof.* First, note that

$$\begin{aligned} \text{tr}(A_i \otimes A_i) &= (\text{tr } A_i)(\text{tr } A_i) = (\text{tr } A_i)^2 \quad \forall i = 1, \dots, n \\ \text{tr}(X \otimes X) &= (\text{tr } X)(\text{tr } X) = (\text{tr } X)^2. \end{aligned}$$

Next, consider the third term. We have

$$\begin{aligned} \text{tr} \left[ \left( (A_i \otimes A_i)^{1/2} (X \otimes X) (A_i \otimes A_i)^{1/2} \right)^{1/2} \right] &=^\dagger \text{tr} \left[ \left( (A_i^{1/2} X \otimes A_i^{1/2} X) (A_i \otimes A_i)^{1/2} \right)^{1/2} \right] \\ &=^\dagger \text{tr} \left[ \left( (A_i^{1/2} X A_i^{1/2}) \otimes (A_i^{1/2} X A_i^{1/2}) \right)^{1/2} \right] \\ &=^\ddagger \text{tr} \left[ \left( A_i^{1/2} X A_i^{1/2} \right)^{1/2} \otimes \left( A_i^{1/2} X A_i^{1/2} \right)^{1/2} \right], \end{aligned}$$

where  $(\dagger)$  follows from Lem. D.2(ii) and  $(\ddagger)$  from Lem. D.2(i). Note that  $X \mapsto A^{1/2} X A^{1/2}$  is a linear map. Therefore, we can now apply Ando's theorem with  $\gamma = \frac{1}{2}$ , which establishes the concavity of the trace term. Its negative is convex and consequently, the objective is a sum of convex functions. The claim follows from the convexity of sums of convex functions.  $\square$