Forman-Ricci Curvature for Complex Networks

Melanie Weber[1,2], Emil Saucan[2] and Jürgen Jost[2]

[1] Princeton University, Program in Applied and Computational Mathematics

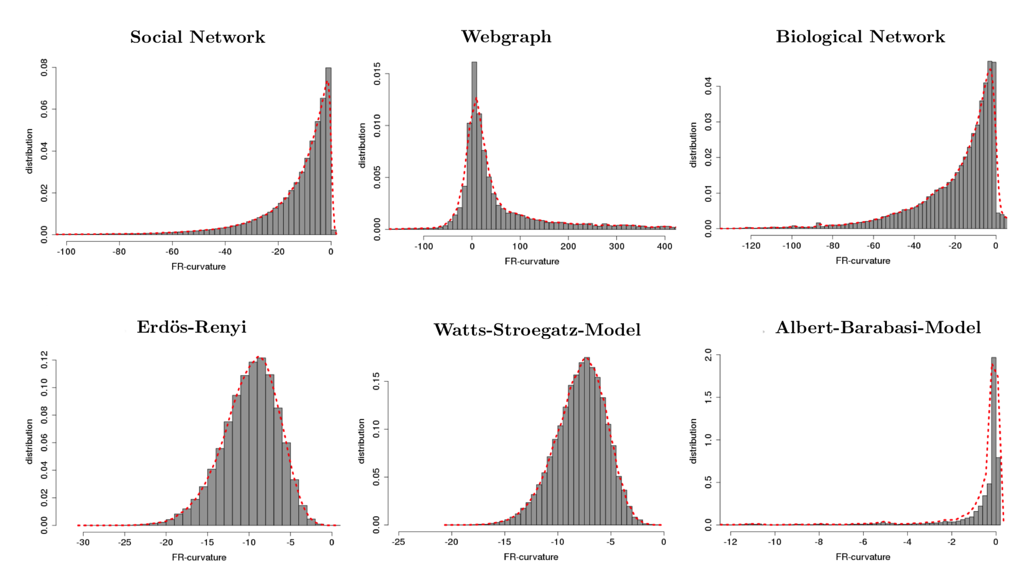
[2] Max Planck Institute for Mathematics in the Sciences, Leipzig/ Germany

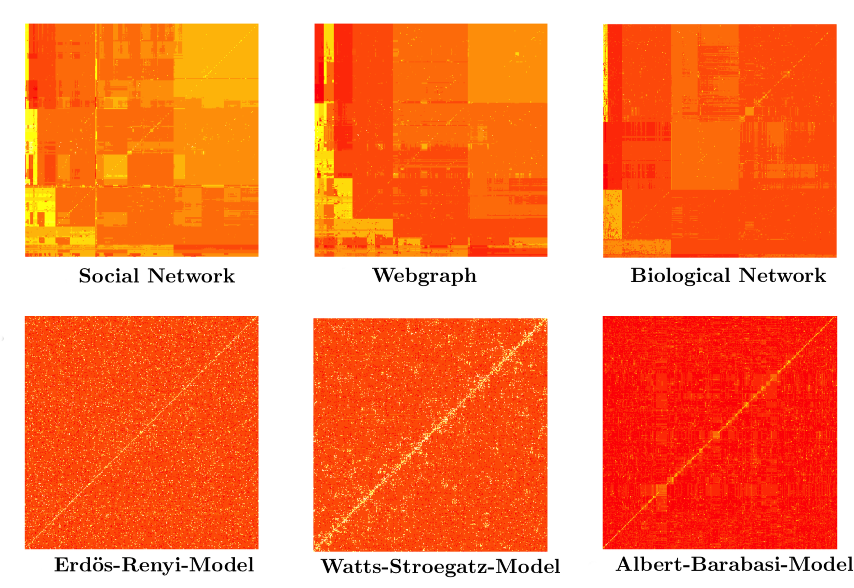
Abstract

With the rapid rise of data science since the late 1990s, networks have emerged as a powerful tool to represent complex systems. We present geometric tools for characterizing such complex networks through the analysis of so far widely neglected network properties to provide novel insights into their structure and evolution.

1. We introduce a discrete Forman-Ricci curvature and its corresponding geometric flows as characteristics for static and dynamics networks. Our work suggests a number of data mining applications including denoising and the extrapolation of network evolution.

2. We extend the introduced formalism to higher dimensions in an attempt to study the global shape of networks. Our setting allows for a network-theoretic formulation of a Gauß-Bonnet style theorem and the computation of Euler characteristics. Attempting to describe the long-term behavior of networks qualitatively, we introduce prototype networks that give rise to a global classification scheme based on Ricci-curvature.





Applications

1. Network Classification with curvature-based distance
2. Detection of interesting regions in dynamic networks
3. Network extrapolation, prediction of network evolution
4. Clustering
5. Denoising

Theory

In [1], we introduce a discrete Ricci curvature building on an earlier notion by R. Forman

$$\mathrm{Ric}_{\mathbf{F}}(e) = \omega(e)\left(\frac{\omega(v_1)}{\omega(e)} + \frac{\omega(v_2)}{\omega(e)} - \sum_{\omega(e_{v_1})\sim e,\, e_{v_2}\sim e}\left[\frac{\omega(v_1)}{\sqrt{\omega(e)\omega(e_{v_1})}} + \frac{\omega(v_2)}{\sqrt{\omega(e)\omega(e_{v_2})}}\right]\right)$$

that proved to be a powerful edge-based characteristic. The corresponding Ricci flow provides a counterpart for dynamic networks

$$\tilde{\gamma}(e) - \gamma(e) = -\mathrm{Ric}_{\mathbf{F}}(\gamma(e)) \cdot \gamma(e)\,,$$

that gives rise to a number of data mining tools as proposed in [1,3]. A more theoretical work [2] extends the 1-dimensional notion for graphs to a higher-dimensional formulation for polyhedral complexes. We define Euler characteristics for networks as

$$\chi(X) = \sum_{k=0}^{d}(-1)^k F^k$$

or

$$\chi(X) = \sum_{v\in F_0} R_0(v) - \sum_{e\in F_1} R_1(e) + \sum_{f\in F_2} R_2(f)$$

in terms of the k-dimensional Ricci curvature R_k for nodes (k=0), edges (k=1) and 2-dimensional faces (k=2). For the simplified case of unweighted networks, this gives

$$\begin{aligned}\chi(X) = &\sum_{v\in V(X)}\left(1 + \frac{3}{2}\deg(v) - \deg^2(v)\right)\\ &- \sum_{e\in E(X)}\left(4 + 6\cdot\#\{f_n^2 > e\} + \sum_{f_n^2>e} n + \sum_{v<e}\deg(v)\right)\\ &+ \sum_{f_n^2\in F_n(X)} 1 + 6n + n^2\end{aligned}$$

which can be calculated directly from a given network. For simplicity, we only consider triangular, 2-dimensional faces and neglect higher order faces. In this notion, one can prove the following Gauss-Bonnet style argument:

Lemma:
Let X be a 2-dimensional cell complex, satisfying certain geometric constraints (see [2]). Then, if $\overline{R}_1 > 0$, then $\chi(X) > 0$.

i.e. we can predict the evolution of a network (characterized by the sign of its curvature) by the Euler characteristic. This allows for defining the prototype networks as follows:

Definition: *Let X be a 2-dimensional polyhedral complex with Euler characteristic χ as given by the Gauß-Bonnet formula. Then we define X to be*

1. *Spherical, if $\chi > 0$,*
2. *Euclidean, if $\chi = 0$,*
3. *Hyperbolic, if $\chi < 0$.*

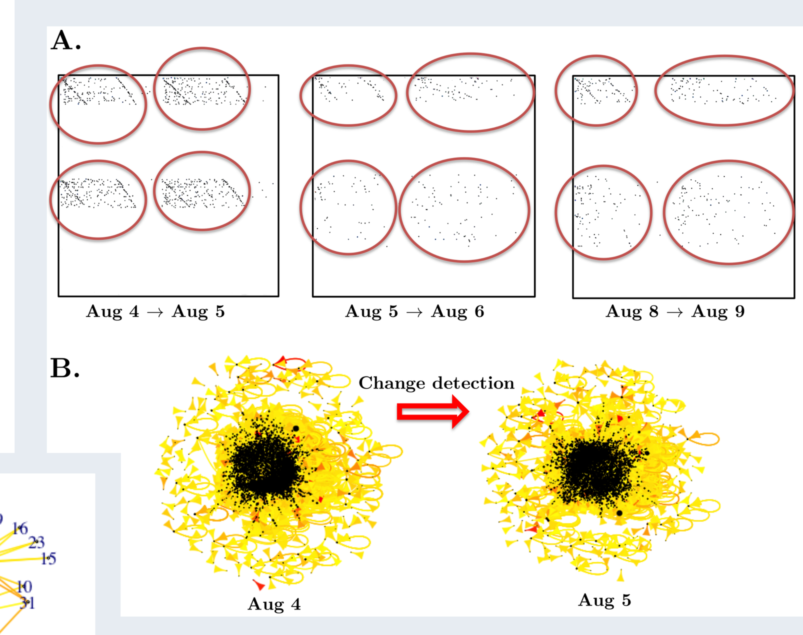
Computational Results: Curvature-based Analysis

We demonstrate how Forman-Ricci curvature captures essential features of real-world networks, including community structure and directionality. In comparison to established network characteristics that are node-based and highly dependent on node degrees, we show that edge weights encode important information that node-based characteristics fail to capture.

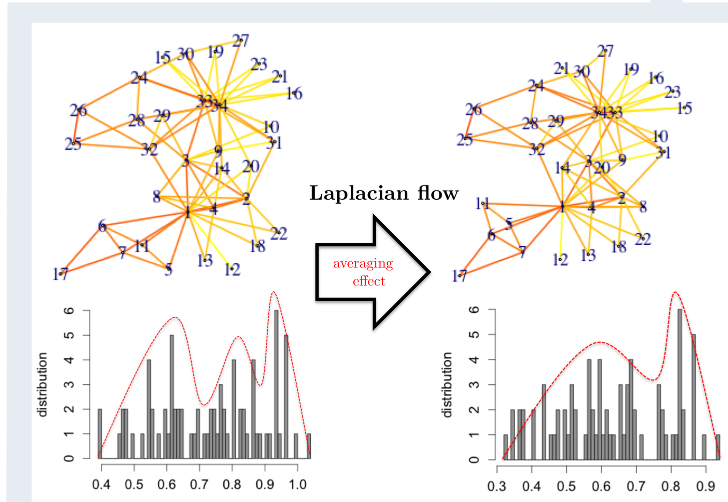
The evaluation of a set of real-world networks – drawn from fields of current major interest in Data Science – and comparison with three established model networks (Erdös-Rényi, Watts-Strogatz, Albert-Barabási) showed close similarity between the real-world networks and the Albert-Barabási model. By measuring the (dis-)similarity between graphs with a curvature-based distance we introduce a classification scheme for networks. While common distance measures would require an alignment of nodes (NP-hard), we perform curvature-based distance estimates in $O(n^3)$.

The curvature maps visualize and statistically evaluate the results. By adding a spatial dimension to commonly used histograms, one gains insight into the community structure and directionality of networks.

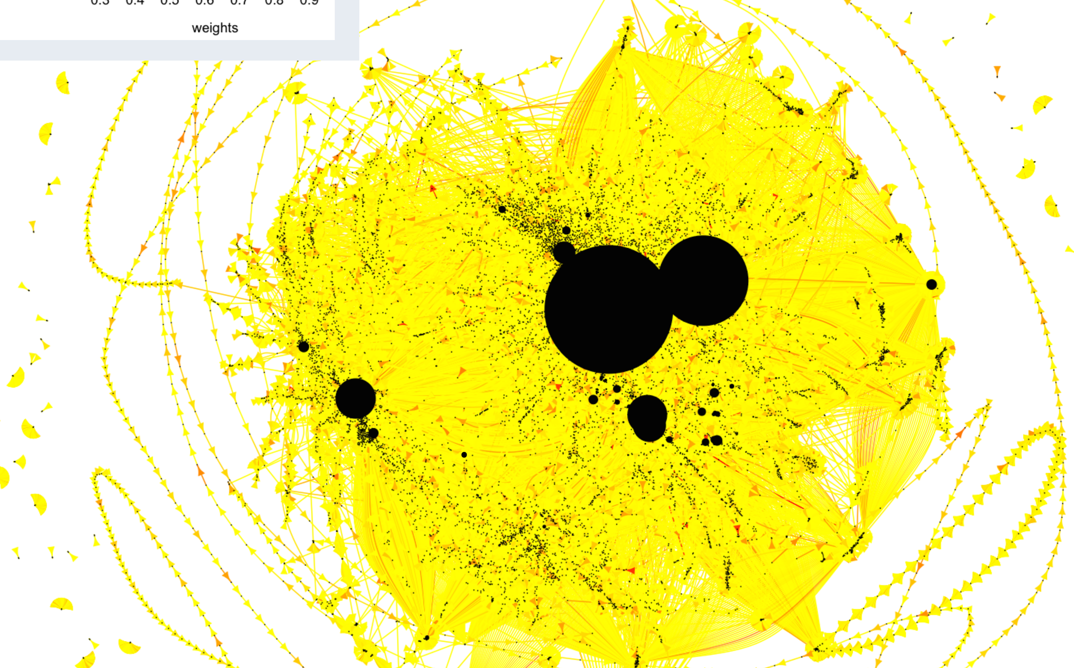
2. Detecting interesting regions (Gnutella file exchanges)



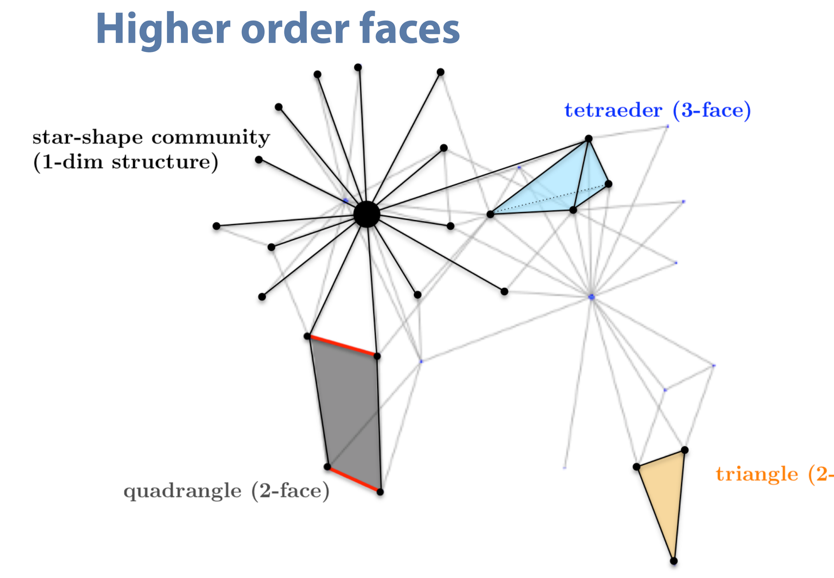
5. Denoising



Google Webgraph (curvature-colored plot)



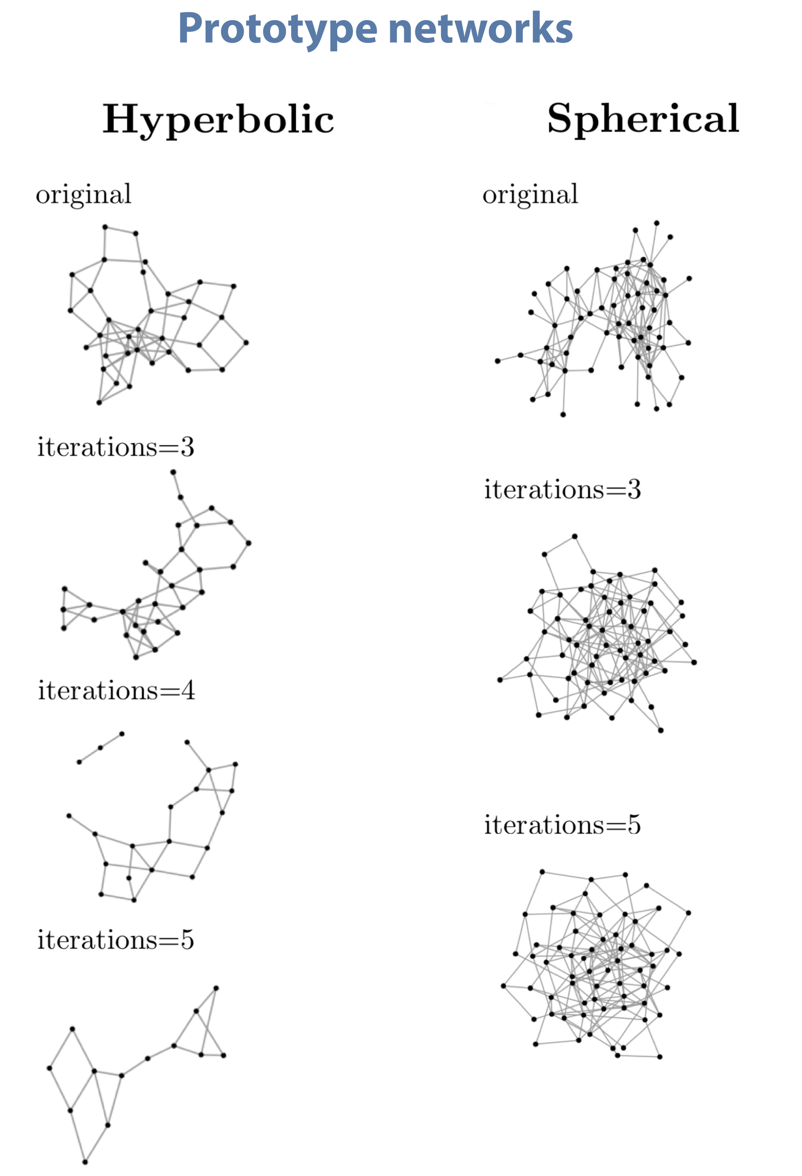
Higher order faces



Can one see the shape of a network?

We introduce a network-theoretic Gauß-Bonnet argument that allows for computation of Euler characteristics for networks via combinatorial curvature functions. With this, we attempt to define a "prototype" networks that gives rise to a classification scheme based on the network's shape. Using small examples, we show the evolution of such prototypic limit cases: By evaluating Ricci curvature, one can see the structure of a network.

Prototype networks



References

[1] M. Weber, E. Saucan and J. Jost (2016): Characterizing complex networks with Forman-Ricci curvature and associated geometric flows. arXiv:1607.08654 to appear: Journal of Complex Networks.

[2] M. Weber, E. Saucan and J. Jost (2016): Can one see the shape of a network? arXiv: 1608.07838

[3] M. Weber, J. Jost and E. Saucan (2016): Forman-Ricci flow for change detection in large dynamic data sets. arXiv:1604.0663 Proceedings of ICICS '16.

Contact
Melanie Weber
Fine Hall 221
mw25@math.princeton.edu