A Short Proof of the Wonderful Lemma

Maria Chudnovsky *
Princeton University, Princeton, NJ 08544, USA

April 1, 2017

Abstract

The Wonderful Lemma, that was first proved by Roussel and Rubio, is one of the most important tools in the proof of the Strong Perfect Graph Theorem. Here we give a short proof of this lemma.

1 Introduction

All graphs in this paper are finite and simple. We denote by G^c the *complement* of the graph G. For two graphs H and G, H is an *induced subgraph* of G if $V(H) \subseteq V(G)$, and a pair of vertices $u, v \in V(H)$ is adjacent if and only if it is adjacent in G. A *hole* in a graph is an induced subgraph that is isomorphic to the cycle C_k with $k \geq 4$, and k is the *length* of the hole. A hole is *odd* if k is odd, and *even* otherwise. An *antihole* in a graph is an induced subgraph that is isomorphic to C_k^c with $k \geq 4$, and again k is the *length* of the antihole. Similarly, an antihole is *odd* if k is odd, and *even* otherwise.

A graph G is called *perfect* if for every induced subgraph H of G, $\chi(H) = \omega(H)$; and *Berge* if it has no odd holes and no odd antiholes. In 2006 a celebrated conjecture of Claude Berge [1] was proved [2]:

Theorem 1.1 *A graph is Berge if and only if it is perfect.*

The original proof was 150 pages long (it has since been shortened somewhat [3]). A key tool that is used many many times throughout the proof is what the authors of [2] lovingly call The Wonderful Lemma, that was first proved in [5] (it was also proved independently by the authors of [2] in joint work with Carsten Thomassen). Roughly, this lemma states that certain subsets of vertices in Berge graphs behave similarly to singletons.

Let us explain this more precisely. Let G be a graph. A set $X \subseteq V(G)$ is *anticonnected* if $X \neq \emptyset$ and the graph $G^c|X$ is connected. A vertex $v \in V(G) \setminus X$ is X*-complete* if v is adjacent to every member of X, and an edge is X*-complete* if both its ends are X-complete. A *path* P in a graph is a sequence of distinct vertices v_0-$\dots$-v_k where v_i is adjacent to v_j if and only if $|i-j| = 1$, and we call k the *length* of the path. The path is *odd* if k is odd. We define $V(P) = \{v_0, \dots, v_k\}$. We call the set $\{v_1, \dots, v_{k-1}\}$ the interior of P, and denote it by P^*. For $v_i, v_j \in V(P)$ with $i \leq j$, we write v_i-P-v_j

*Partially supported by NSF grants DMS-1550991.

to be the path v_i-v_{i+1}-...-v_j. A *leap* for the path P is a pair of non-adjacent vertices x, y such that x is adjacent to v_0, v_1, v_k and has no other neighbor in $V(P)$, and y is adjacent to v_0, v_{k-1}, v_k, and has no other neighbor in $V(P)$.

An *antipath* Q in G is a sequence of distinct vertices v_0-...-v_k where v_i is non-adjacent to v_j if and only if $|i-j|=1$, and we call k the *length* of the antipath. The antipath is *odd* if k is odd. We define $V(Q)=\{v_0,\dots,v_k\}$. We call the set $\{v_1,\dots,v_{k-1}\}$ the interior of Q, and denote it by Q^*. Using similar notation, we often describe a hole or an antihole in a graph is a sequence of vertices v_1-...-v_k-v_1, where $v_iv_j \in E(G)$ if and only if $|i-j| \in \{1, k-1\}$. We can now state the Wonderful Lemma.

Theorem 1.2 *Let G be a Berge graph, let $X \subseteq V(G)$ be an anticonnected set, and let $P = v_0$-...-v_k be an odd path whose ends are X-complete. Then one of the following holds:*

1. *Some edge of P is X-complete, or*
2. *$k \geq 5$ and X contains a leap for P, or*
3. *$k = 3$ and there is an odd antipath Q with ends v_1, v_2 and $Q^* \subseteq X$.*

The main result of this paper is a new shorter proof of 1.2, that is also quite different from a more recent proof of [6]. The current proof uses ideas from [4].

2 Corollaries

In [2] three corollaries of 1.2 were proved. It turns out that they can be strengthened a little, as follows. Let G be a Berge graph, and let $X \subseteq V(G)$ be anticonnected. If we know that one of the outcomes of 1.2 holds for certain odd paths with X-complete ends, then the conclusions of each of the corollaries also hold for X. This observation will be very useful in our proof of 1.2, and we explain the details in this section.

The first (strengthened) corollary is:

Theorem 2.1 *Let G be a Berge graph, and let $X \subseteq V(G)$ be anticonnected. Let P be an odd path with X-complete ends and such that no edge of P is X-complete (in particular, P has length at least three). Suppose that one of the outcomes of 1.2 holds for X and P in G. Then every X-complete vertex of G has a neighbor in P^*.*

Proof. Suppose $v \in V(G)$ is X-complete and has no neighbor in P^*. Then $v \notin X \cup V(P)$. Since no edge of P is X-complete, one of the last two outcomes of 1.2 holds for X and P in G. If the second outcome holds and X contains a leap x, y for P, then v-x-v_1-...-v_{k-1}-y-v is an odd hole in G, and if the third outcome holds, then v-v_1-Q-v_2-v is an odd antihole in G, in both cases a contradiction. This proves 2.1. ∎

For a path or a hole D and an anticonnected set $X \subseteq V(G) \setminus V(D)$, an *$X$-segment* of D is a path P of D such that $|V(P)| > 2$, the ends of P are X-complete, and no vertex of P^* is X-complete. The second (strengthened) corollary is:

Theorem 2.2 *Let G be a Berge graph, and let X be an anticonnected set in G. Let $C = v_1\text{-}\dots\text{-}v_s\text{-}v_1$ be a hole in $G \setminus X$. Suppose that one of the outcomes of 1.2 holds for every odd path with X-complete ends in G. Then either*

1. *An even number of edges of C is X-complete, or*
2. *C has exactly one X-complete edge, and exactly two X-complete vertices.*

Proof. We may assume that C contains an odd number of X-complete edges, and at least three X-complete vertices. Thus every edge of C is either X-complete, or belongs to exactly one X-segment. Since G is Berge, it follows that C has an even number of edges, and so some X-segment P of C is odd. Let e be an X-complete edge of C. Since P is a path, and since no vertex of P^* is X-complete, it follows that at least one end of e has no neighbor in P^*, contrary to 2.1. This proves 2.2. ∎

Finally, the (strengthened) third corollary is:

Theorem 2.3 *Let G be a Berge graph, and let X be an anticonnected set in G. Let P be an odd path with X-complete ends in G, and suppose some vertex of P^* is X-complete. Suppose that one of the outcomes of 1.2 holds for every odd path with X-complete ends in G. Then an odd number of edges of P is X-complete.*

Proof. Since v_0, v_k are X-complete, it follows that every edge of P is either X-complete, or belongs to exactly one X-segment of P. We may assume that the number of X-complete edges in P is even, and therefore, since the total number of edges of P is odd, some X-segment of P, say R, is odd. But at least one of v_0, v_k has no neighbor in R^*, contrary to 2.1. This proves 2.3. ∎

3 The proof of 1.2

In this section we prove 1.2.

Proof. We proceed by induction on $|X|$, and for fixed $|X|$ by induction on k. If $|X| = 1$, the first outcome of 1.2 holds since G has no odd hole, so we may assume that $|X| \geq 2$. If $k = 1$, then the first outcome of 1.2 holds, so we may assume that $k \geq 3$.

Assume first that $k = 3$. We may assume that neither of v_1, v_2 is X-complete (for otherwise the first outcome hold), and since X is anticonnected, it follows that there is an antipath Q with ends v_1, v_2 and with $Q^* \subseteq X$. Since $v_1\text{-}Q\text{-}v_2\text{-}v_0\text{-}v_3\text{-}p_1$ is not an odd antihole in G, it follows that Q is odd, and the third outcome of 1.2 holds. Thus we may assume that $k \geq 5$.

(1) We may assume that no vertex of P^* is X-complete.

Suppose for some $i \in \{1, \dots, k-1\}$ the vertex v_i is X-complete. Then one of $v_0\text{-}P\text{-}v_i$ and $v_i\text{-}P\text{-}v_k$ is an odd path whose ends are X-complete, say R. We may assume that no edge of R is X-complete, for otherwise the first outcome of 1.2 holds. Since $|V(R)| < k$, inductively one of the outcomes of 1.2 holds for X and R. But one of v_0, v_k is an X-complete vertex with no neighbor in R^*, contrary to 2.1. This proves (1).

Let T be a spanning tree of $G^c|X$, and let a be a leaf of T. Define $A = X \setminus \{a\}$. Since $T \setminus a$ is a spanning tree of $G^c|A$, it follows that A is anticonnected. Since $|A| < |X|$, it follows inductively that one of the outcomes of 1.2 holds for every odd path Q with A-complete ends, and so we can apply 2.1, 2.2 and 2.3 to A.

(2) We may assume that exactly one of v_1, v_{k-1} is A-complete, and no other vertex of P^* is A-complete.

If no vertex of P^* is A-complete, then the second outcome of 1.2 holds for P and A, and so X contains a leap for P, and 1.2 holds. Thus we may assume that some vertex of P^* is A-complete. We write "a-segment of P" to mean "{a}-segment of P". Since v_0, v_k are adjacent to a, it follows that every edge of P is in an a-segment or is $\{a\}$-complete. By 2.3, we deduce that an odd number of edges of P is A-complete, and so, by (1), some a-segment R of P contains an odd number of A-complete edges. Let $R = v_i\text{-}\dots\text{-}v_j$. Since by (1) no vertex of P^* is X-complete, $j - i \geq 2$. Since X is anticonnected, a is not A-complete. Thus $a\text{-}v_i\text{-}R\text{-}v_j\text{-}a$ is a hole, say C, and an odd number of edges of C is A-complete. By 2.2, we deduce that C has exactly one A-complete edge, say $v_s v_{s+1}$, and exactly two A-complete vertices, namely v_s and v_{s+1}. By symmetry, we may assume that $s - i$ is odd. Then $i \neq s$, and hence v_i is not X-complete and $i \neq 0$. Suppose first that $s \neq k - 1$. Since by (1) no vertex of P^* is X-complete, it follows that a is non-adjacent to v_s, v_{s+1}. Now $v_k\text{-}a\text{-}v_i\text{-}R\text{-}v_s$ is an odd path whose ends are A-complete, and none of whose internal vertices are A-complete, but v_{s+1} has no neighbors in the interior of this path, contrary to 2.1. This proves that $s = k - 1$, and hence $s + 1 = k = j$. By (1) a is non-adjacent to v_{i-1}.

Suppose that some other vertex $v_t \in P^*$ is A-complete. Then $t < i$, and in particular $i > 1$. Now $T = v_0\text{-}a\text{-}v_i\text{-}R\text{-}v_s$ is an odd path with A-complete ends and no A-complete vertex in T^*, and so by 2.1 v_t has a neighbor in T^*. Since $t < i$ and by (1) v_t is not X-complete (and therefore not adjacent to a), it follows that $t = i - 1$ and no vertex of $P^* \setminus \{v_t, v_{k-1}\}$ is A-complete. Let $q < i$ be maximum such that v_q is adjacent to a. Then $q < i - 1$. Since $a - v_i - P - v_k - a$ is not an odd hole, it follows that i is odd. Since $a - v_q - P - v_i - a$ is not an odd hole, it follows that q is odd, and in particular $q > 0$. But now $v_{i-1} - P - v_q - a - v_k$ is an odd path with A-complete ends and no A-complete edge, and v_{k-1} has no neighbor in its interior, contrary to 2.1. This proves (2).

By (2), if there are three distinct vertices of X each of which is a leaf of a spanning tree of $G^c|X$, then one of v_1, v_{k-1} is X-complete, contrary to (1). Since $|X| \geq 2$, this implies that there exist distinct $a, b \in X$ such that a and b are the only leaves of every spanning tree of $G^c|T$. Consequently $G^c|X$ is a path from a to b, and so $G|X$ is an antipath from a to b. Write $Q = a\text{-}u_1\text{-}\dots\text{-}u_s\text{-}b$, $A = X \setminus \{a\}$, $B = X \setminus \{b\}$, $C = X \setminus \{a, b\}$. By (2) and since by (1) no vertex of P^* is X-complete, we may assume that v_1 is A-complete and non-adjacent to a, and v_{k-1} is B-complete and non-adjacent to b. Since $k > 3$ and $v_1\text{-}a\text{-}Q\text{-}b\text{-}v_{k-1}$ is not an odd antihole, it follows that s is even. Let $R = v_1\text{-}P\text{-}v_{k-1}$. If $X = \{a, b\}$, then $A = \{b\}$ and $B = \{a\}$, and by (2) a, b is a leap for P, so the second outcome of 1.2 holds. So we may assume that $X \neq \{a, b\}$. Then $C \neq \emptyset$, and R is an odd path with C-complete ends. Since v_0 has no neighbor in R^*, and since $|C| < |X|$, inductively 2.1 implies that some vertex r of R^* is C-complete. Now by (2), r is non-adjacent to a, b, and so $r\text{-}a\text{-}Q\text{-}b\text{-}r$ is an odd antihole, a contradiction. This completes the proof. ∎

References

[1] C. Berge, Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind, *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe* **10** (1961), 114.

[2] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, The strong perfect graph theorem, *Annals of Math* **164** (2006), 51-229.

[3] M. Chudnovsky and P. Seymour, Even pairs in Berge graphs, *J. Combinatorial Theory, Ser. B* **99** (2009), 370-377.

[4] F. Maffray and N. Trotignon, A class of perfectly contractile graphs, *J. Combinatorial Theory, Ser. B* **96** (2006), 1-19.

[5] F. Roussel and P. Rubio, About skew partitions in minimal imperfect graphs, *J. Combinatorial Theory, Ser. B* **83** (2001), 171-190.

[6] N. Trotignon and K.Vušković, On Roussel-Rubio-type lemmas and their consequences, *Discrete Math* **311** (2011), 684-687.