An approximate version of Hadwiger's conjecture for claw-free graphs

Maria Chudnovsky*
Columbia University,
New York, NY 10027, USA

and

Alexandra Ovetsky Fradkin
Princeton University,
Princeton, NJ 08544, USA

March 24, 2009

Abstract

Hadwiger's conjecture states that every graph with chromatic number χ has a clique minor of size χ. In this paper we prove a weakened version of this conjecture for the class of claw-free graphs (graphs that do not have a vertex with three pairwise nonadjacent neighbors). Our main result is that a claw-free graph with chromatic number χ has a clique minor of size $\lceil \frac{2}{3}\chi \rceil$.

1 Introduction

In 1943, Hadwiger [9] conjectured that for every loopless graph G and every integer $t \geq 0$, either G is t-colorable, or G has a K_{t+1}-minor (we define colorability and minors later in this section). Since then, Hadwiger's conjecture has received a lot of attention and is now considered by many to be one of the most interesting problems in graph theory. Currently, Hadwiger's conjecture has been proved for $t \leq 5$ and remains open for $t > 5$. The cases where $t \leq 3$ were proved by Hadwiger in [9] and the case $t = 4$ was shown by Wagner [14] to be

*Part of this research was conducted while the author served as a Clay Mathematics Institute Research Fellow. Partially supportted by NSF grant DMS-0758364.

equivalent to the *four color theorem* [1], [2]. Finally, the case $t = 5$ was proved in 1993 by Robertson, Seymour, and Thomas [13] using the four color theorem.

Hadwiger's conjecture has also been proved for some special classes of graphs. The *line graph* of a graph G, denoted by $L(G)$, is a graph whose vertices are the edges of G, and if $u, v \in E(G)$ then $uv \in E(L(G))$ if and only if u and v share a vertex in G. Please note that in this definition the graph G may have parallel edges. In 2004, Reed and Seymour [12] proved Hadwiger's conjecture for line graphs. A graph G is a *quasi-line graph* if for every vertex v, the set of neighbors of v can be expressed as the union of two cliques. Note that this is a partition of the vertex set of the neighborhood of v. It is easy to verify that the class of line graphs is a proper subset of the class of quasi-line graphs. In a recent work [5], the authors have shown that Hadwiger's conjecture holds for quasi-line graphs.

In this paper we prove a weakened version of Hadwiger's conjecture for a class of graphs known as *claw-free graphs*, a proper superset of the class of quasi-line graphs. A graph is claw-free if it does not contain a *claw*, that is a $K_{1,3}$, as an induced subgraph. The main result of this paper is the following:

1.1 *Let G be a claw-free graph with chromatic number χ. Then G has a clique minor of size $\lceil \frac{2}{3}\chi \rceil$.*

Our proof of 1.1 uses a structure theorem for claw-free graphs that appears in [7]. We describe this theorem in the next section. However, before we do that we must set up some notation that will be useful in the rest of the paper.

Let G be a finite loopless graph. Denote the set of vertices of G by $V(G)$ and the set of edges of G by $E(G)$. A *k-coloring* of G is a map $c : V(G) \to \{1, \ldots, k\}$ such that for every pair of adjacent vertices $v, w \in V(G)$, $c(v) \neq c(w)$. We may also refer to a k-coloring simply as a "coloring". The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer such that there is a $\chi(G)$-coloring of G.

For $v \in V(G)$, we denote the set of neighbors of v in G by $N_G(v)$ (so $v \notin N_G(v)$) and for $X \subseteq V(G)$, we denote the set $(\bigcup_{x \in X} N_G(x)) \setminus X$ by $N_G(X)$. For $X, Y \subseteq V(G)$, we say that X *dominates* Y if $Y \subseteq N_G(X) \cup X$. For $X \subset V(G)$, let $G|X$ denote the subgraph of G induced on X and let $G \setminus X$ denote the subgraph of G induced on $V(G) \setminus X$. We define a *path* P in G to be an induced connected subgraph of G such that either P is a one-vertex graph, or two vertices of P have degree one and all the others have degree two. The *length* of P is the number of edges in P. The complement of G is the graph $\overline{G}$, on the same vertex set as G, and such that two vertices are adjacent in $\overline{G}$ if and only if they are nonadjacent in G. A *clique* in G is a set of vertices of G that are all pairwise adjacent. A *stable set* in G is a clique in $\overline{G}$. A *triad* is a stable set of size 3. The *clique number* of G, denoted by $\omega(G)$, is the size of a maximum clique in G. The *stability number* of G, denoted by $\alpha(G)$, is the size of a maximum stable set in G. The *complete graph* on t vertices, denoted by K_t, is a graph such that $|V(K_t)| = t$ and $V(K_t)$ is a clique. A *component* is a maximal connected subgraph of G; an *anticomponent* is a maximal connected subgraph of $\overline{G}$. A set $S \subset V(G)$ is a *cutset* if $G \setminus S$ has more components than G. We say that S is a *clique cutset* if it is both a clique and a cutset.

We say that two subgraphs S_1, S_2 of G are *adjacent* if there is an edge between $V(S_1)$ and $V(S_2)$. A graph H is said to be a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. Let H be a graph with $V(H) = \{v_1, \ldots, v_n\}$. Then H is a minor of G if and only if there are $|V(H)|$ non-null connected subgraphs $A_1, \ldots, A_n$ of G, such that $V(A_i) \cap V(A_j) = \emptyset$, and A_i and A_j are adjacent if v_i is adjacent to v_j. We say that a graph G has a clique minor of size t if K_t is a minor of G.

This paper is organized as follows. In the next section we state (a corollary of) the structure theorem for claw-free graphs that appears in [7]. Section 3 contains some lemmas about claw-free graphs that are used in later proofs. In Section 4 we deal with a certain structure that may be present in a claw-free graph (called a "non-reduced W-join"), and conclude that a minimal counterexample to 1.1 admits no such structure. Sections 5—8 are devoted to dealing with the different outcomes of the structure theorem of [7]; in each of the sections we prove that a minimal counterexample to 1.1 does not fall into the particular class of graphs that section is concerned with. Finally, in the end of Section 8, we collect all these results to prove 1.1. Section 9 describes what we know about proving Hadwiger's conjecture itself for claw-free graphs (and not just the weakening 1.1); it also points out the difficult cases that we were unable to deal with.

2 Structure theorem for claw-free graphs

The goal of this section is to state and describe the structure theorem for claw-free graphs appearing in [7] (or, more precisely, its corollary). We begin with some definitions which are modified from [7].

Let X, Y be two subsets of $V(G)$ with $X \cap Y = \emptyset$. We say that X and Y are *complete* to each other if every vertex of X is adjacent to every vertex of Y, and we say that they are *anticomplete* to each other if no vertex of X is adjacent to a member of Y. Similarly, if $A \subseteq V(G)$ and $v \in V(G) \setminus A$, then v is A-*complete* if v is adjacent to every vertex in A, and A-*anticomplete* if v has no neighbor in A.

Let $F \subseteq V(G)^2$ be a set of unordered pairs of distinct vertices of G such that every vertex appears in at most one pair. Then H is a *thickening* of (G, F) if for every $v \in V(G)$ there is a nonempty subset $X_v \subseteq V(H)$, all pairwise disjoint and with union $V(H)$ satisfying the following:

- for each $v \in V(G)$, X_v is a clique of H
- if $u, v \in V(G)$ are adjacent in G and $\{u, v\} \notin F$, then X_u is complete to X_v in H
- if $u, v \in V(G)$ are nonadjacent in G and $\{u, v\} \notin F$, then X_u is anticomplete to X_v in H
- if $\{u, v\} \in F$ then X_u is neither complete nor anticomplete to X_v in H.

Here are some classes of claw-free graphs that come up in the structure theorem.

- **Graphs from the icosahedron.** The *icosahedron* is the unique planar graph with twelve vertices all of degree five. Let it have vertices $v_0, v_1, \ldots, v_{11}$, where for $1 \leq i \leq 10$, v_i is adjacent to v_{i+1}, v_{i+2} (reading subscripts modulo 10), and v_0 is adjacent to v_1, v_3, v_5, v_7, v_9, and v_{11} is adjacent to $v_2, v_4, v_6, v_8, v_{10}$. Let this graph be G_0. Let G_1 be obtained from G_0 by deleting v_{11} and let G_2 be obtained from G_1 by deleting v_{10}. Furthermore, let $F' = \{\{v_1, v_4\}, \{v_6, v_9\}\}$ and let $F \subseteq F'$.

  Let $G \in \mathcal{T}_1$ if G is a thickening of $(G_0, \emptyset)$, $(G_1, \emptyset)$, or (G_2, F) for some F.

- **Fuzzy long circular interval graphs.** Let Σ be a circle, and let $F_1, \ldots, F_k \subseteq \Sigma$ be homeomorphic to the interval $[0, 1]$, such that no two of $F_1, \ldots, F_k$ share an endpoint, and no three of them have union Σ. Now let $V \subseteq \Sigma$ be finite, and let H be a graph with vertex set V in which distinct $u, v \in V$ are adjacent precisely if $u, v \in F_i$ for some i.

  Let $F' \subseteq V(H)^2$ be the set of pairs $\{u, v\}$ such that u, v are distinct endpoints of F_i for some i. Let $F \subseteq F'$ such that every vertex of G appears in at most one member of F. Then G is a *fuzzy long circular interval graph* if for some such H and F, G is a thickening of (H, F).

  Let $G \in \mathcal{T}_2$ if G is a fuzzy long circular interval graph.

- **Fuzzy antiprismatic graphs.** A graph K is *antiprismatic* if for every $X \subseteq V(K)$ with $|X| = 4$, X is not a claw and there are at least two pairs of vertices in X that are adjacent. Let H be a graph and let $F \subseteq V(H)^2$ be a set of pairs $\{u, v\}$ such that every vertex of H is in at most one member of F and

  - no triad of H contains u and no triad of H contains v, or
  - there is a triad of H containing both u and v and no other triad of H contains u or v.

  Thus F is the set of "changeable edges" discussed in [6]. The pair (H, F) is *antiprismatic* if for every $F' \subseteq F$, the graph obtained from H by changing the adjacency of all the vertex pairs in F' is antiprismatic. We say that a graph G is a *fuzzy antiprismatic graph* if G is a thickening of (H, F) for some antiprismatic pair (H, F).

  Let $G \in \mathcal{T}_3$ if G is a fuzzy antiprismatic graph.

Next, we define what it means for a claw-free graph to admit a "strip-structure".

A *hypergraph* H consists of a finite set $V(H)$, a finite set $E(H)$, and an incidence relation between $V(H)$ and $E(H)$ (that is, a subset of $V(H) \times E(H)$). For the statement of the structure theorem, we only need hypergraphs such that every member of $E(H)$ is incident with either one or two members of $V(H)$ (thus, these hypergraphs are graphs if we allow "graphs" to have loops and parallel edges). For $F \in E(H)$, $\overline{F}$ denotes the set of all $h \in V(H)$ incident with F.

Let G be a graph. A *strip-structure* (H, η) of G consists of a hypergraph H with $E(H) \neq \emptyset$, and a function η mapping each $F \in E(H)$ to a subset $\eta(F)$ of $V(G)$, and mapping each pair

(F,h) with $F \in E(H)$ and $h \in \overline{F}$ to a subset $\eta(F,h)$ of $\eta(F)$, satisfying the following conditions.

(SD1) The sets $\eta(F)$ $(F \in E(H))$ are nonempty and pairwise disjoint and have union $V(G)$.

(SD2) For each $h \in V(H)$, the union of the sets $\eta(F,h)$ for all $F \in E(H)$ with $h \in \overline{F}$ is a clique of G.

(SD3) For all distinct $F_1, F_2 \in E(H)$, if $v_1 \in \eta(F_1)$ and $v_2 \in \eta(F_2)$ are adjacent in G, then there exists $h \in \overline{F_1} \cap \overline{F_2}$ such that $v_1 \in \eta(F_1,h)$ and $v_2 \in \eta(F_2,h)$.

There is also a fourth condition, but it is technical and we will not need it in this paper.

Let (H,η) be a strip-structure of a graph G, and let $F \in E(H)$, where $\overline{F} = \{h_1,\dots,h_k\}$. Let $v_1,\dots,v_k$ be new vertices, and let J be the graph obtained from $G|\eta(F)$ by adding $v_1,\dots,v_k$, where v_i is complete to $\eta(F,h_i)$ and anticomplete to all other vertices of J. Then $(J,\{v_1,\dots,v_k\})$ is called the *strip of* (H,η) *at* F. A strip-structure (H,η) is *nontrivial* if $|E(H)| \geq 2$.

Next, we list some strips (J,Z) that we will need for the structure theorem.

- $\mathcal{Z}_1$: Let H be a graph with vertex set $\{v_1,\dots,v_n\}$, such that for $1 \leq i < j < k \leq n$, if v_i, v_k are adjacent then v_j is adjacent to both v_i, v_k. Let $n \geq 2$, let v_1, v_n be nonadjacent, and let there be no vertex adjacent to both v_1 and v_n. Let $F' \subseteq V(H)^2$ be the set of pairs $\{v_i, v_j\}$ such that $i < j$, $v_i \neq v_1$ and $v_j \neq v_n$, v_i is nonadjacent to v_{j+1}, and v_j is nonadjacent to v_{i-1}. Furthermore, let $F \subseteq F'$ such that every vertex of H appears in at most one member of F. Then G is a *fuzzy linear interval graph* if for some H and F, G is a thickening of (H,F) with $|X_{v_1}| = |X_{v_n}| = 1$. Let $X_{v_1} = \{u_1\}$, $X_{v_n} = \{u_n\}$, and $Z = \{u_1, u_n\}$.

- $\mathcal{Z}_2$: Let $n \geq 2$. Construct a graph H as follows. Its vertex set is the disjoint union of three sets A, B, C, where $|A| = |B| = n+1$ and $|C| = n$, say $A = \{a_0, a_1, \dots, a_n\}, B = \{b_0, b_1, \dots, b_n\}$, and $C = \{c_1,\dots,c_n\}$. Adjacency is as follows. A, B, C are cliques. For $0 \leq i,j \leq n$ with $(i,j) \neq (0,0)$, let a_i, b_j be adjacent if and only if $i = j$, and for $1 \leq i \leq n$ and $0 \leq j \leq n$, let c_i be adjacent to a_j, b_j if and only if $i \neq j \neq 0$. All other pairs not specified so far are nonadjacent. Now let $X \subseteq A \cup B \cup C \setminus \{a_0, b_0\}$ with $|C \setminus X| \geq 2$. Let $H' = H \setminus X$ and let G be a thickening of (H', F) with $|X_{a_0}| = |X_{b_0}| = 1$ and $F \subseteq V(H')^2$ (we will not specify the possibilities for the set F because they are technical and we will not need them in our proof). Let $X_{a_0} = \{a'_0\}$, $X_{b_0} = \{b'_0\}$, and $Z = \{a'_0, b'_0\}$.

- $\mathcal{Z}_3$: Let H be a graph, and let h_1-h_2-h_3-h_4-h_5 be the vertices of a path of H in order, such that h_1, h_5 both have degree one in H, and every edge of H is incident with one of h_2, h_3, h_4. Let H' be obtained from the line graph of H by making the edges

h_2h_3 and h_3h_4 of H (vertices of H') nonadjacent. Let $F \subseteq \{\{h_2h_3, h_3h_4\}\}$ and let G be a thickening of (H', F) with $|X_{h_1h_2}| = |X_{h_4h_5}| = 1$. Let $X_{h_1h_2} = \{u\}$, $X_{h_4h_5} = \{v\}$, and $Z = \{u, v\}$.

$\mathcal{Z}_4$: Let H be the graph with vertex set $\{a_0, a_1, a_2, b_0, b_1, b_2, b_3, c_1, c_2\}$ and adjacency as follows: $\{a_0, a_1, a_2\}, \{b_0, b_1, b_2, b_3\}, \{a_2, c_1, c_2\}$, and $\{a_1, b_1, c_2\}$ are cliques; b_2, c_1 are adjacent; and all other pairs are nonadjacent. Let $F = \{\{b_2, c_2\}, \{b_3, c_1\}\}$ and let G be a thickening of (H, F) with $|X_{a_0}| = |X_{b_0}| = 1$. Let $X_{a_0} = \{a_0'\}$, $X_{b_0} = \{b_0'\}$, and $Z = \{a_0', b_0'\}$.

$\mathcal{Z}_5$: Let H be the graph with vertex set $\{v_1, \ldots, v_{12}\}$, and with adjacency as follows. v_1-$\cdots$-v_6-v_1 is an induced cycle in G of length 6. Next, v_7 is adjacent to v_1, v_2; v_8 is adjacent to v_4, v_5; v_9 is adjacent to v_6, v_1, v_2, v_3; v_{10} is adjacent to v_3, v_4, v_5, v_6, v_9; v_{11} is adjacent to $v_3, v_4, v_6, v_1, v_9, v_{10}$; and v_{12} is adjacent to $v_2, v_3, v_5, v_6, v_9, v_{10}$. No other pairs are adjacent. Let H' be a graph isomorphic to $H \setminus X$ for some $X \subseteq \{v_{11}, v_{12}\}$ and let $F \subseteq \{\{v_9, v_{10}\}\}$. Let G be a thickening of (H', F) with $|X_{a_0}| = |X_{b_0}| = 1$. Let $X_{v_7} = \{v_7'\}$, $X_{v_8} = \{v_8'\}$, and $Z = \{v_7', v_8'\}$.

We are now ready to state a structure theorem for claw-free graphs that is an easy corollary of the main result of [7].

2.1 *Let G be a connected claw-free graph. Then either*

- *G is a member of $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, or*
- *$V(G)$ is the union of three cliques, or*
- *G admits a nontrivial strip-structure such that for each strip (J, Z), $1 \leq |Z| \leq 2$, and if $|Z| = 2$, then either*
  - *$|V(J)| = 3$ and Z is complete to $V(J) \setminus Z$, or*
  - *(J, Z) is a member of $\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4 \cup \mathcal{Z}_5$.*

3 Tools

In this section we prove a few preliminary results about minimal counterexamples to our main theorem. In particular, we prove that a minimal counterexample to 1.1 does not admit a clique cutset and has $\chi(G) \leq \lceil \frac{n}{2} \rceil$. We also prove that if G is a minimal counterexample to 1.1 and K_1, K_2 are two cliques in G, then there exist $\min(|K_1|, |K_2|)$ vertex disjoint paths between K_1 and K_2 in G. We found these results to be useful tools in the proof of 1.1.

3.1 *Let G be a graph which does not have a clique minor of size $\lceil \frac{2}{3}\chi(G) \rceil$, and assume that every proper induced subgraph G' of G has a clique minor of size $\lceil \frac{2}{3}\chi(G') \rceil$. Then G does not admit a clique cutset.*

Proof. Suppose that G admits a clique cutset S. Then there exists a partition (X_1, X_2) of $V(G) \setminus V(S)$ such that there are no edges between X_1 and X_2. For $i = 1, 2$, let $G_i = G|(X_i \cup S)$. We claim that $\max(\chi(G_1), \chi(G_2)) \geq \chi(G)$. For suppose not. Then there exist colorings of G_1, G_2 with fewer than $\chi(G)$ colors. We can permute the colors of these colorings so that they agree on S and from this obtain a coloring of G with fewer than $\chi(G)$ colors, which is a contradiction. This proves the claim.

Without loss of generality, suppose that $\chi(G_1) \geq \chi(G_2)$. Since G_1 is a proper induced subgraph of G, it follows that G_1 has a clique minor of size $\lceil \frac{2}{3}\chi(G_1) \rceil \geq \lceil \frac{2}{3}\chi(G) \rceil$. Hence, G has a clique minor of size $\lceil \frac{2}{3}\chi(G) \rceil$, a contradiction. This proves 3.1. ■

3.2 *Let G be a graph which does not have a clique minor of size $\lceil \frac{2}{3}\chi(G) \rceil$, and assume that every proper induced subgraph G' of G has a clique minor of size $\lceil \frac{2}{3}\chi(G') \rceil$. Then $\overline{G}$ is connected.*

Proof. Suppose that $\overline{G}$ is not connected. Then there exists a partition (X_1, X_2) of $V(G)$ such that X_1, X_2 are nonempty and every member of X_1 is adjacent to every member of X_2 in G. For $i = 1, 2$, let $G_i = G|X_i$. Then $\chi(G) = \chi(G_1) + \chi(G_2)$, and G_i has a clique minor of size $\lceil \frac{2}{3}\chi(G_i) \rceil$ for $i = 1, 2$. But then G has a clique minor of size $\lceil \frac{2}{3}\chi(G_1) \rceil + \lceil \frac{2}{3}\chi(G_2) \rceil \geq \lceil \frac{2}{3}\chi(G) \rceil$, a contradiction. This proves 3.2. ■

3.3 *Let G be a claw-free graph with $|V(G)| = n$ which does not have a clique minor of size $\lceil \frac{2}{3}\chi(G) \rceil$, and assume that every proper induced subgraph G' of G has a clique minor of size $\lceil \frac{2}{3}\chi(G') \rceil$. Then $\chi(G) \leq \lceil \frac{n}{2} \rceil$.*

Proof. Suppose that $\chi(G) > \lceil \frac{n}{2} \rceil$. It follows that $\overline{G}$ has no matching of size $\lfloor \frac{n}{2} \rfloor$. For a set X, let $o(X)$ be the number of odd components of $\overline{G} \setminus X$. Let μ be the size of a largest matching in $\overline{G}$. Then by the Tutte-Berge formula [3], there exists a set $X \subseteq V(G)$ such that $o(X) = |X| + n - 2\mu$. Since there is no matching of size $\lfloor \frac{n}{2} \rfloor$ in $\overline{G}$, it follows that $n - 2\mu > 1$ and so $o(X) > 1$. Therefore, $|X| > 0$, since by 3.2 $\overline{G}$ is connected. We claim that $\alpha(G \setminus X) = 2$. For suppose otherwise. Then some anticomponent, say C, of G contains a triad $\{v_1, v_2, v_3\}$ (since anticomponents are complete to each other by definition). Let $v_4 \in V(G) \setminus (X \cup V(C))$. Then $G|\{v_1, v_2, v_3, v_4\}$ is a claw, a contradiction. This proves the claim.

Let $C_1, \ldots, C_k$ be the anticomponents of $G \setminus X$. Then

$$\chi(G \setminus X) \geq \sum_{i=1}^{k} \lceil \frac{C_i}{2} \rceil = \frac{n - |X| + o(X)}{2} = \frac{n - |X| + |X| + n - 2\mu}{2} = n - \mu \geq \chi(G).$$

But $G \setminus X$ is a proper induced subgraph of G, and so $G \setminus X$ has a clique minor of size $\lceil \frac{2}{3}\chi(G \setminus X) \rceil = \lceil \frac{2}{3}\chi(G) \rceil$ and consequently so does G. This proves 3.3. ■

3.4 *Let G be a claw-free graph which does not have a clique minor of size $\lceil \frac{2}{3}\chi(G) \rceil$, and subject to that with $|V(G)|$ minimum. Let K_1, K_2 be two cliques in G. Then there exist min$(|K_1|, |K_2|)$ vertex disjoint paths between K_1 and K_2 in G.*

Proof. Suppose not. Let S be a smallest cutset separating K_1 and K_2. Then Menger's Theorem [11] implies that $|S| < \min(|K_1|, |K_2|)$. It follows that there exists a partition (X_1, X_2) of $V(G) \setminus V(S)$ such that $K_i \subset X_i \cup S$ and there are no edges between X_1 and X_2. Let G_i be the graph obtained from $G|(X_i \cup S)$ by adding an edge s_1s_2 for every pair of nonadjacent vertices $s_1, s_2 \in S$.

(1) max$(\chi(G_1), \chi(G_2)) \geq \chi(G)$.

Suppose not. Then there exist colorings of G_1, G_2 with fewer than $\chi(G)$ colors. We can permute the colors of these colorings so that they agree on S and from this obtain a coloring of G with fewer than $\chi(G)$ colors, which is a contradiction. This proves (1).

(2) For all $v \in S$, v has a neighbor in X_1 and in X_2.

Without loss of generality, suppose there exists $v \in S$ with no neighbor in X_1. Then if $v \notin K_1$ we can add v to X_2 and obtain a smaller cutset, $S \setminus \{v\}$, separating K_1 and K_2, contradicting the minimality of S. So $v \in S \cap K_1$, and since v is anticomplete to X_1 and $K_1 \subseteq X_1 \cup S$, it follows that $K_1 \subseteq S$. But $|S| < \min(|K_1|, |K_2|)$, which is a contradiction. This proves (2).

(3) G_i is a claw-free graph for $i = 1, 2$.

For $v \in X_i$, v has the same neighbors in G_i as in G and the edges between the neighbors in G_i are a superset of those in G. Hence, the neighbors of v in G_i still do not contain a triad. For $v \in S$, we claim that the set of neighbors of v in X_i is a clique. For suppose v has two neighbors $x_1, x_1' \in X_1$ that are nonadjacent to each other. By (2), v has a neighbor $x_2 \in X_2$. But now x_1, x_1', x_2 are three pairwise nonadjacent vertices in the neighborhood of v in G, contrary to the fact that G is a claw-free graph. This proves the claim. Since $N_{G_i}(v) = (N_G(v) \cap X_i) \cup S$, it follows that G_i is claw-free. This proves (3).

Without loss of generality, let $\chi(G_1) \geq \chi(G_2)$. Let $S = \{s_1, \ldots, s_n\}$ and let $\mathbb{P} = \{P_1, \ldots, P_n\}$ be $|S|$ vertex disjoint paths between S and K_2 in G_2 such that $s_i \in P_i$. Such paths exist by Menger's Theorem [11] and the minimality of S. Let $\phi : S \to \mathbb{P}$ be a bijection defined by $\phi(s_i) = P_i$.

By the minimality of $|V(G)|$, there exists a set $\mathbb{S}$ of $\lceil \frac{2}{3}\chi(G_1) \rceil$ connected disjoint subgraphs of G_1 that are pairwise adjacent in G_1. For $H \in \mathbb{S}$ define $\psi(H)$ by

$$\psi(H) = (H \setminus S) \cup \bigcup_{s \in V(H) \cap S} \phi(s).$$

Then $\psi(H)$ is a subgraph of G. Define $\mathbb{Q} = \{\psi(H) : H \in \mathbb{S}\}$. Then $\mathbb{Q}$ is a set of $\lceil \frac{2}{3}\chi(G_1)\rceil \geq \lceil \frac{2}{3}\chi(G)\rceil$ connected disjoint subgraphs of G. We claim that the members of $\mathbb{Q}$ are pairwise adjacent. Suppose not. Choose $Q_1, Q_2 \in \mathbb{Q}$ that are not adjacent. For $i = 1, 2$, let H_i be the member of $\mathbb{S}$ such that $Q_i = \psi(H_i)$. Since K_2 is a clique in G, it follows that not both $V(Q_1)$ and $V(Q_2)$ contain a vertex of K_2, and therefore, not both $V(H_1)$ and $V(H_2)$ contain a vertex of S. Since H_1 and H_2 are adjacent, we deduce that there exist $h_1 \in V(H_1)$ and $h_2 \in V(H_2)$ such that not both h_1, h_2 are in S and h_1h_2 is an edge of G_1. But now by the definition of ψ and G_1, $h_1 \in V(Q_1)$, $h_2 \in V(Q_2)$ and h_1h_2 is an edge of G, contrary to the fact that Q_1 and Q_2 are nonadjacent. This proves the claim. Hence G has a clique minor of size $\lceil \frac{2}{3}\chi(G)\rceil$, a contradiction. This completes the proof of 3.4. ■

4 W-joins

Let (A, B) be disjoint subsets of $V(G)$. The pair (A, B) is called a *homogeneous pair* in G if A, B are cliques, and for every vertex $v \in V(G) \setminus (A \cup B)$, v is either A-complete or A-anticomplete and either B-complete or B-anticomplete. A *W-join* (A, B) is a homogeneous pair in which A is neither complete nor anticomplete to B. We say that a W-join (A, B) is *reduced* if we can partition A into two sets A_1 and A_2 and we can partition B into B_1, B_2 such that A_1 is complete to B_1, A_2 is anticomplete to B, and B_2 is anticomplete to A. Note that since A is neither complete nor anticomplete to B, it follows that both A_1 and B_1 are non-empty and at least one of A_2, B_2 is non-empty. We call a W-join that is not reduced a *non-reduced* W-join.

Let H be a thickening of (G, F) and let $\{u, v\} \in F$. Then we notice that (X_u, X_v) is a W-join in H. If for every $\{u, v\} \in F$ we have that (X_u, X_v) is a reduced W-join then we say that H is a *reduced thickening* of G.

In this section we prove that a minimal counterexample G to 1.1 does not admit a non-reduced W-join (and hence if G is a thickening of (H, F) for some claw-free graph H and $F \subseteq V(H)^2$, then it is a reduced thickening). We start with a preliminary result, that appears in [7], but we include its proof here, for completeness.

4.1 *Let G be a claw-free graph and let (A, B) be a W-join. Let H be a graph obtained from G by arbitrarily changing the adjacency between some vertices of A and some vertices of B (all the other adjacencies remain unchanged). Then H is claw-free.*

Proof. Let C be the set of vertices of G that are A-complete and B-complete, D be the set of vertices of G that are A-complete and B-anticomplete, E the set of vertices of G that are A-anticomplete and B-complete, and F the set of vertices of G that are A-anticomplete and B-anticomplete. Observe that since G is claw-free, both D and E are cliques. Let $v \in V(H)$. We need to show that the set $N_H(v)$ does not contain a stable set of size three. We do so by considering the following cases:

1. $v \in D \cup E \cup F$. In this case $H|(N_H(v)) = G|(N_G(v))$ and hence $H|(N_H(v))$ does not contain a stable set of size three since G is a claw-free graph.

2. $v \in A \cup B$. From the symmetry, we may assume that $v \in A$. Let $B(v) = N_H(v) \cap B$. Suppose there is a triad T in $N_H(v)$. Since T is not a triad in $N_G(v)$ it follows that $T \cap B(v) \neq \emptyset$. Since $B(v)$ is a clique we deduce that $|T \cap B(v)| = 1$, let t be the unique vertex of $T \cap B(v)$. Since T is a triad, it follows that $T \setminus \{t\} \subseteq A \cup D$. But $A \cup D$ is a clique, contrary to the fact that $|T \setminus \{t\}| = 2$.

3. $v \in C$. First, we note that v has no neighbors in F. Suppose v has a neighbor $f \in F$. Since A is not complete to B, there exist $a \in A$ and $b \in B$ that are nonadjacent. But then f, a, b are three pairwise nonadjacent vertices in $N_G(v)$, contrary to the fact that G is claw-free. This implies that $N_H(v) \subseteq A \cup B \cup C \cup D \cup E$. Suppose that $N_H(v)$ contains three pairwise nonadjacent vertices, say v_1, v_2, v_3. We claim that $\{v_1, v_2, v_3\} \subseteq C \cup D \cup E$. For suppose $v_1 \in A \cup B$, say $v_1 \in A$. Since $\{v_1, v_2, v_3\}$ is a triad, it follows that $\{v_2, v_3\} \subseteq B \cup E$, contrary to the fact that $B \cup E$ is a clique. This proves the claim. But $H|(C \cup D \cup E) = G|(C \cup D \cup E)$ and hence v_1, v_2, v_3 are three pairwise nonadjacent vertices in the neighborhood of v in G, contrary to G being claw-free.

This proves 4.1. ■

Next we prove a result that allows us to handle non-reduced W-joins. We remark that this is a slight strengthening of a lemma from [5] and the proof is basically the same.

4.2 *Let G be a claw-free graph and suppose that G admits a non-reduced W-join. Then there exists a subgraph H of G with the following properties:*

1. *H is a claw-free graph, $|V(H)| = |V(G)|$ and $|E(H)| < |E(G)|$.*
2. *$\chi(H) = \chi(G)$.*

Proof. Let (A, B) be a non-reduced W-join of G. Let C be the set of vertices of G that are A-complete and B-complete, D be the set of vertices of G that are A-complete and B-anticomplete, E the set of vertices of G that are A-anticomplete and B-complete, and F the set of vertices of G that are A-anticomplete and B-anticomplete. We note that both D and E are cliques. Let $J = G|(A \cup B)$. Then $\overline{J}$ is bipartite. Let M be a maximum matching in $\overline{J}$ and let $|M| = m$. We claim that we can color J with $|A| + |B| - m$ colors. This follows from the fact that we can color the vertices of M with m colors and $|(A \cup B) \setminus V(M)| = |A| + |B| - 2m$.

By König's Theorem [10], $|M|$ equals the minimum size of a vertex cover of $\overline{J}$, that is, the minimum number of vertices hitting all edges of $\overline{J}$. Let X be a minimum vertex cover of $\overline{J}$. Then $A \setminus X$ is complete to $B \setminus X$ in G.

Let $A' = A \cap X$ and $B' = B \cap X$. Let H be the graph obtained from G by deleting the edges between the members of A' and the members of B and the edges between the members of B' and the members of A. Then since (A, B) is a non-reduced W-join in G, $|E(H)| < |E(G)|$ and by 4.1, H is a claw-free graph, and thus the first assertion of the theorem holds.

To prove the second assertion of the theorem, it is enough to show that every coloring of H can be used to obtain a coloring of G using the same number of colors. Let c_H be a coloring of H. Now since $(A \setminus A') \cup (B \setminus B')$ is a clique in H and $|A' \cup B'| = m$, it follows that every coloring of $H|(A \cup B)$, and in particular c_H, uses at least $|A| + |B| - m$ colors. Hence, at most m colors appear on both A and B. We construct a coloring of G as follows. We use each of the colors of c_H that appears on both A and B to color the vertices of $V(M)$ (using each color for two vertices of M that are matched to each other in M). We use the rest of the colors of c_H which appear on A for the remaining vertices of A, and the rest of the colors of C_H that appear on B for the rest of the vertices of B. This yields a coloring of J. We keep the colors of the vertices of $V(G) \setminus (A \cup B)$ unchanged. The coloring just defined is a proper coloring of G, and it uses the same number of colors as c_H. This proves the second assertion of the theorem and completes the proof of 4.2. ∎

4.2 implies the following:

4.3 *Let G be a claw-free graph that does not have a clique minor of size $\lceil \frac{2}{3}\chi(G) \rceil$. Assume that for every claw-free graph G' with $|V(G')| = |V(G)|$ and $|E(G')| < |E(G)|$, G' has a clique minor of size $\lceil \frac{2}{3}\chi(G') \rceil$. Assume also that G is a thickening of (H, F) for some claw-free graph H and $F \subseteq V(H)^2$. Then G is a reduced thickening of (H, F).*

5 The icosahedron

5.1 *Let $G \in \mathcal{T}_1$. Suppose that every claw-free graph G' such that either $|V(G')| < |V(G)|$, or $|V(G')| = |V(G)|$ and $|E(G')| < |E(G)|$ has a clique minor of size $\lceil \frac{2}{3}\chi(G') \rceil$. Then G has a clique minor of size $\lceil \frac{2}{3}\chi(G) \rceil$.*

Proof. We begin with an observation.

(1) Suppose there exists an induced 5-edge path $P = w_1 \text{-} \cdots \text{-} w_6$ of G such that for $1 \le i \le 6$ there exist pairwise disjoint $Y_{w_i} \subseteq V(G)$ with $w_i \in Y_{w_i}$ satisfying the following properties:

1. *For $1 \le i \le 6$, Y_{w_i} is a clique*
2. *For $1 \le i \le 6$, $N_G(Y_{w_i}) \subseteq N_G(w_i)$*
3. *$Y_{w_1} \cup Y_{w_2}$ is anticomplete to $Y_{w_5} \cup Y_{w_6}$*
4. *$S_1 = \{w_1, w_2, w_3\}$ dominates $V(G) \setminus (Y_{w_5} \cup Y_{w_6})$ and $S_2 = \{w_4, w_5, w_6\}$ dominates $V(G) \setminus (Y_{w_1} \cup Y_{w_2})$*
5. *For $i = 2, 3$, w_i is complete to $Y_{w_{i-1}}$ and for $i = 4, 5$, w_i is complete to $Y_{w_{i+1}}$.*

Then the theorem holds.

Let $G' = G \setminus V(P)$. Since P is 2-colorable, $\chi(G') \geq \chi(G) - 2$. By assumption, G' has a clique minor of size $\lceil \frac{2}{3}\chi(G') \rceil \geq \lceil \frac{2}{3}(\chi(G)-2) \rceil \geq \lceil \frac{2}{3}\chi(G) \rceil - 2$. This means that there exists a set $\mathcal{S}$ of $\lceil \frac{2}{3}\chi(G) \rceil - 2$ connected, disjoint subgraphs of G' that are pairwise adjacent. Suppose that no member of $\mathcal{S}$ is a subgraph of $G|(Y_{w_1} \cup Y_{w_2})$ or $G|(Y_{w_5} \cup Y_{w_6})$. Then, since S_1 dominates $V(G) \setminus (Y_{w_5} \cup Y_{w_6})$, S_2 dominates $V(G) \setminus (Y_{w_1} \cup Y_{w_2})$, and $G|S_1$ is adjacent to $G|S_2$, it follows that $\mathcal{S} \cup \{G|S_1, G|S_2\}$ is a set of $\lceil \frac{2}{3}\chi(G) \rceil$ connected, disjoint subgraphs of G that are pairwise adjacent.

Hence, we may assume that some member of $\mathcal{S}$ is a subgraph of $G|(Y_{w_1} \cup Y_{w_2})$ or $G|(Y_{w_5} \cup Y_{w_6})$. From symmetry, we may assume that there exists $T \in \mathcal{S}$ such that $V(T) \subseteq Y_{w_1} \cup Y_{w_2}$. Note that this implies that no member of $\mathcal{S}$ is a subgraph of $G|(Y_{w_5} \cup Y_{w_6})$ since $Y_{w_1} \cup Y_{w_2}$ is anticomplete to $Y_{w_5} \cup Y_{w_6}$. Suppose that no member of $\mathcal{S}$ is a subgraph of $G|Y_{w_1}$. Let $S_1' = \{w_1, w_2\}$ and $S_2' = \{w_3, w_4, w_5, w_6\}$. Then, since S_2' dominates $V(G) \setminus Y_{w_1}$, and $N_G(T) \setminus S_1' \subseteq N_G(S_1')$, and $G|S_1'$ is adjacent to $G|S_2'$, it follows that $\mathcal{S} \cup \{G|S_1', G|S_2'\}$ is a set of $\lceil \frac{2}{3}\chi(G) \rceil$ connected, disjoint subgraphs of G that are pairwise adjacent.

Hence, we may assume that some member of $\mathcal{S}$ is a subgraph of $G|Y_{w_1}$. So there exists $T \in \mathcal{S}$ such that $V(T) \subseteq Y_{w_1}$. Let $S_1'' = \{w_1\}$ and $S_2'' = \{w_2, w_3, w_4, w_5, w_6\}$. Then, since S_2'' dominates $V(G)$ and $N_G(T) \setminus S_1'' \subseteq N_G(S_1'')$, and $G|S_1''$ is adjacent to $G|S_2''$, it follows that $\mathcal{S} \cup \{G|S_1'', G|S_2''\}$ is a set of $\lceil \frac{2}{3}\chi(G) \rceil$ connected, disjoint subgraphs of G that are pairwise adjacent. This proves (1).

Let $v_0, v_1, \ldots, v_{11}$ be as in the definition of the icosahedron. Further let G_0, G_1, G_2, and F be as in the definition of $\mathcal{T}_1$. Then G is a thickening of either $(G_0, \emptyset)$, $(G_1, \emptyset)$, or (G_2, F) for $F \subseteq \{(v_1, v_4), (v_6, v_9)\}$. By 4.3, G is a reduced thickening. For $0 \leq i \leq 11$, let X_{v_i} be as in the definition of thickening (where $X_{v_{11}}$ is empty when G is a thickening of $(G_1, \emptyset)$ or (G_2, F), and $X_{v_{10}}$ is empty when G is a thickening of (G_2, F)). If G is a thickening of $(G_0, \emptyset)$ or $(G_1, \emptyset)$, then for $i = 0, 2, 4, 6, 8, 9$, let $u_i \in X_{v_i}$. If G is a thickening of (G_2, F), then for $i = 0, 2, 4, 6, 8, 9$ choose $u_i \in X_{v_i}$ such that u_4 has a neighbor in X_{v_1}, u_6 and u_9 are nonadjacent, and subject to that $N_G(u_9)$ is maximal.

Let $P = G|\{u_0, u_2, u_4, u_6, u_8, u_9\}$. Note that P is a 5-edge path in G. For $i = 0, 2, 4, 8$, let $Y_{u_i} = X_{v_i}$. Let Y_{u_6} consist of those members of X_{v_6} that are nonadjacent to u_9, and let Y_{u_9} consist of those members of X_{v_9} that are nonadjacent to u_6. Notice that because the thickening is reduced, u_4 has a neighbor in X_{v_1}, and u_9 was chosen with maximal neighbors, it follows that $N_G(Y_{u_4}) \subseteq N_G(u_4)$, $N_G(Y_{u_6}) \subseteq N_G(u_6)$ and $N_G(Y_{u_9}) \subseteq N_G(u_9)$. Then it is easy to check that P and $\{Y_{u_i}\}$ satisfy the five conditions of (1) and so this proves 5.1. ■

6 Three cliques

In this section we prove 1.1 for graphs G such that $V(G)$ is the union of three cliques.

6.1 *Let G be a claw-free graph such that $V(G)$ is the union of three cliques. Then G has a clique minor of size $\lceil \frac{2}{3}\chi(G) \rceil$.*

Proof. Let $n = |V(G)|$. We may assume that every proper induced subgraph G' of G has a clique minor of size $\lceil \frac{2}{3}\chi(G') \rceil$. By 3.3, $\chi(G) \leq \lceil \frac{n}{2} \rceil$. It follows that $\lceil \frac{2}{3}\chi(G) \rceil \leq \lceil \frac{n}{3} \rceil + 1$. Let $V(G) = A \cup B \cup C$ where A, B, C are cliques. Without loss of generality, we may assume that $|A| \geq |B| \geq |C|$. Then $|A| \geq \lceil \frac{n}{3} \rceil$. If $|A| > \lceil \frac{n}{3} \rceil$ then we are done. Otherwise, let

$$\mathcal{S} = (\bigcup_{a \in A} \{G|\{a\}\}) \cup \{G|(B \cup C)\}.$$

Then $|\mathcal{S}| = \lceil \frac{n}{3} \rceil + 1$ and we claim that the members of $\mathcal{S}$ are connected, disjoint, and pairwise adjacent. It suffices to check that $G|(B \cup C)$ is connected and every member of A has a neighbor in $B \cup C$. Since by 3.1 A is not a clique cutset in G, it follows that $G|(B \cup C)$ is connected; and since $A \setminus \{a\}$ is not a clique cutset for every $a \in A$, it follows that every $a \in A$ has a neighbor in $B \cup C$. This proves the claim and therefore G has a clique minor of size $\lceil \frac{n}{3} \rceil + 1 \geq \lceil \frac{2}{3}\chi(G) \rceil$. This proves 6.1. ■

7 Antiprismatic graphs

7.1 *Let G be a claw-free graph which does not have a clique minor of size $\lceil \frac{2}{3}\chi(G) \rceil$, and assume that every proper induced subgraph G' of G has a clique minor of size $\lceil \frac{2}{3}\chi(G') \rceil$. Then $\alpha(G) > 2$.*

Proof. Clearly, we may assume that $\alpha(G) \geq 2$. Let $n = |V(G)|$. Suppose that $\alpha(G) = 2$. By 3.3, $\chi(G) \leq \lceil \frac{n}{2} \rceil$. Since $\chi(G) \geq \lceil \frac{n}{\alpha(G)} \rceil$, it follows that $\chi(G) = \lceil \frac{n}{2} \rceil$. Then $\lceil \frac{2}{3}\chi(G) \rceil \leq \lceil \frac{n+1}{3} \rceil$. We prove by induction on n that every graph with $\alpha = 2$ has a clique minor of size at least $\lceil \frac{n+1}{3} \rceil$. This clearly holds when $n = 1$. If G does not have a path of length 2, then G is the disjoint union of at most two cliques and so it follows that G has a clique minor of size at least $\lceil \frac{n}{2} \rceil \geq \lceil \frac{n+1}{3} \rceil$. So we may assume that G has a path P of length 2. Then $G \setminus V(P)$ has a clique minor of size at least $\lceil \frac{n-2}{3} \rceil$. Since P dominates $V(G) \setminus V(P)$, it follows that we can add P to the list of connected subgraphs of $G \setminus V(P)$ forming a clique minor of size $\lceil \frac{n-2}{3} \rceil$, and so G has a clique minor of size at least $\lceil \frac{n-2}{3} \rceil + 1 = \lceil \frac{n+1}{3} \rceil$. This proves the inductive step and completes the proof of 7.1.

■

Next, we prove a few basic facts about antiprismatic graphs.

7.2 *Let G be an antiprismatic graph such that G does not have two disjoint triads. Then there exists a vertex $v \in V(G)$ meeting all triads of G.*

Proof. Suppose not. If $\alpha(G) \leq 2$, then 7.2 holds vacuously for any vertex $v \in V(G)$. So we may assume that $\alpha(G) = 3$. Let $\{v_1, v_2, v_3\}$ be a triad of G. Since v_1 does not meet all triads, it follows that there exists a triad T disjoint from v_1. Since there are no two disjoint triads, T contains at least one of v_2, v_3, and because the graph is antiprismatic it does not contain both. Without loss of generality, we may assume that $v_2 \in T$. Let $T = \{v_2, v_4, v_5\}$. Since v_2 does not meet all triads, if follows that there exists a triad T' disjoint from v_2. Because there are no two disjoint triads, T' contains one of v_1, v_3 and one of v_4, v_5. However, because G is antiprismatic, $\{v_1, v_3\}$ is complete to $\{v_4, v_5\}$, a contradiction. Hence, we conclude that some vertex of G meets all triads and this proves 7.2. ■

7.3 *Let G be an antiprismatic graph and let $P = v_1\text{-}v_2\text{-}v_3$ be a two-edge path in G. Let $X \subseteq V(G)$ be the set of vertices not dominated by $V(P)$. Then $|X| \leq 1$, and if no triad of G contains both v_1 and v_3 then $|X| = 0$.*

Proof. Suppose first that no triad of G contains both v_1 and v_3. Then $V(P)$ dominates $V(G)$ and so $|X| = 0$. Hence, we may assume that there exists a triad $T = \{v_1, v_3, x\}$ in G. Then, since G is antiprismatic, it follows that $V(P)$ does not dominate x. Let $y \in V(G) \setminus \{v_1, v_2, v_3, x\}$. Then since G is antiprismatic, y has two neighbors in T. In particular, y is adjacent to at least one of v_1, v_3 and hence y is dominated by $V(P)$. It follows that $|X| = 1$ and this proves 7.3. ■

7.4 *Let G be a fuzzy antiprismatic graph such that every claw-free graph G' with either $|V(G')| < |V(G)|$ or $|V(G')| = |V(G)|$ and $|E(G')| < |E(G)|$ has a clique minor of size $\lceil \frac{2}{3}\chi(G') \rceil$. Then G has a clique minor of size $\lceil \frac{2}{3}\chi(G) \rceil$.*

Proof. If $\alpha(G) \leq 2$ then the result follows from 7.1. So we may assume that $\alpha(G) \geq 3$. Let H be a graph and let $F \subseteq V(H)^2$ such that the pair (H, F) is antiprismatic, and G is a thickening of (H, F). By 4.3, G is a reduced thickening of (H, F). For $v \in V(H)$ let X_v be as in the definition of a thickening.

(1) Let $\{x, y\} \in F$, such that x is adjacent to y in H, and let H' be obtained from H by deleting the edge xy. Then for every $\{u, v\} \in F$, either

- *no triad of H' contains u and no triad of H' contains v, or*
- *there is a triad of H' containing both u and v and no other triad of H' contains u or v;*

and the pair (H', F) is antiprismatic.

Since every vertex of H belongs to at most one pair of F, to prove the first assertion of (1), it is enough to check that either

- no triad of H' contains x and no triad of H' contains y, or

- there is a triad of H' containing both x and y and no other triad of H' contains x or y.

Let T be a triad of H' such that $x \in T$. First we observe that since x is adjacent to y in H, and since the pair (H, F) is antiprismatic, it follows that T is not a triad of H, and therefore $y \in T$. Let $T = \{x, y, w\}$. We need to show that no other triad of H' contains x or y. Suppose T' is another triad of H' containing x. Applying the observation above to T', we deduce that $y \in T'$. Let $T' = \{x, y, w'\}$. But now only one pair of vertices of the set $\{x, y, w, w'\}$ is adjacent in H', which is a contradiction since (H, F) is antiprismatic and H' is obtained from H by changing the adjacency of a vertex pair in F. This proves the first assertion of (1). The second assertion follows immediately from the fact that the pair (H, F) is antiprismatic. This proves (1).

In view of (1), we may assume that if $\{u, v\} \in F$, then u is nonadjacent to v in H.

(2) If H has two disjoint triads, then 7.4 holds.

Let $\{v_1, v_2, v_3\}$ and $\{u_1, u_2, u_3\}$ be two disjoint triads of H. Then, since H is antiprismatic, $H|\{v_1, v_2, v_3, u_1, u_2, u_3\}$ is an induced cycle of length 6, and without loss of generality we may assume that for $i = 1, 2, 3$, v_i is adjacent to u_i and u_{i+1} (where the subscripts are read modulo 3). It follows from the definition of a fuzzy antiprismatic graph that for $i = 1, 2, 3$, X_{v_i} is complete to X_{u_i} and $X_{u_{i+1}}$ (where the subscripts are read modulo 3), and that at most one of the pairs $\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}$ and at most one of the pairs $\{u_1, u_2\}, \{u_2, u_3\}, \{u_1, u_3\}$ belong to F. In view of that, for $i = 1, 2, 3$, we can choose $v'_i \in X_{v_i}$ such that $\{v'_1, v'_2, v'_3\}$ is a triad and subject to that $N_G(v'_1)$ is maximal, and $u'_i \in X_{u_i}$ such that $\{u'_1, u'_2, u'_3\}$ is a triad and subject to that $N_G(u'_3)$ is maximal. Let $Y_{v_1} \subseteq X_{v_1}$ be the set of those elements that are nonadjacent to v'_2 and v'_3, and let $Y_{u_3} \subseteq X_{u_3}$ be the set of those elements that are nonadjacent to u'_1 and u'_2.

Let $S_1 = \{v'_1, u'_1, u'_2\}$, $S_2 = \{v'_2, v'_3, u'_3\}$ and let $G' = G \setminus (S_1 \cup S_2)$. Since $G|(S_1 \cup S_2)$ is 2-colorable, $\chi(G') \geq \chi(G) - 2$. Then G' has a clique minor of size $\lceil \frac{2}{3}\chi(G') \rceil \geq \lceil \frac{2}{3}(\chi(G) - 2) \rceil \geq \lceil \frac{2}{3}\chi(G) \rceil - 2$. This means that there exists a set $\mathcal{S}$ of $\lceil \frac{2}{3}\chi(G) \rceil - 2$ connected, disjoint subgraphs of G' that are pairwise adjacent.

By 7.3, $\{u_1, u_2, v_1\}$ dominates $V(H) \setminus \{u_3\}$ in H. Now, since if $\{u, v\} \in F$ then u is nonadjacent to v, it follows that S_1 dominates $V(G) \setminus Y_{u_3}$. Similarly, S_2 dominates $V(G) \setminus Y_{v_1}$.

Suppose that no member of $\mathcal{S}$ is a subgraph of $G|Y_{v_1}$ or $G|Y_{u_3}$. Since $G|S_1$ is adjacent to $G|S_2$, it follows that $\mathcal{S} \cup \{G|S_1, G|S_2\}$ is a set of $\lceil \frac{2}{3}\chi(G) \rceil$ connected, disjoint subgraphs of G that are pairwise adjacent, and so G has the desired clique minor.

Hence, we may assume that some member T of $\mathcal{S}$ is a subgraph of $G|Y_{v_1}$ or $G|Y_{u_3}$. From symmetry, we may assume that $V(T) \subseteq Y_{v_1}$. It follows from the definition of a fuzzy antiprismatic graph that Y_{v_1} is anticomplete to Y_{u_3}, and therefore no member of $\mathcal{S}$ is a subgraph of $G|Y_{u_3}$. Let $S'_1 = \{v_1\}$ and $S'_2 = \{v'_2, v'_3, u'_2, u'_3\}$. Then S'_2 dominates $V(G)$ since X_{u_2} is complete to Y_{v_1}. Since $N_{G'}(T) \subseteq N_G(S'_1)$ and $G|S'_1$ is adjacent to $G|S'_2$, it follows that $\mathcal{S} \cup \{G|S'_1, G|S'_2\}$ is a set of $\lceil \frac{2}{3}\chi(G) \rceil$ connected, disjoint subgraphs of G that are pairwise adjacent. This proves (2).

In view of (2), we may assume that there are no two disjoint triads in H. Then by 7.2, there is a vertex v meeting all triads of H. We claim that v can be chosen so that no member of F contains v. If there is more than one triad in H, then v is in more than one triad, and the claim follows from the fact that (H, F) is antiprismatic. So we may assume that there is a unique triad $\{x, y, z\}$ in H. But then, again since (H, F) is antiprismatic, it follows that if for some pair $\{u, v\} \in F$ and $\{u, v\} \cap \{x, y, z\} \neq \emptyset$, then $\{u, v\} \subseteq \{x, y, z\}$. Now, since every vertex of H is in at most one pair of F, we deduce that at least one of x, y, z does not belong to any member of F. This proves the claim. Thus we may assume that no member of F contains v.

Let $N = N_H(v)$, and let $M = V(H) \setminus (N \cup \{v\})$. Since $\alpha(G) > 2$, and since if $\{u, w\} \in F$ then u is nonadjacent to w in H, it follows that there exist nonadjacent $m_1, m_2 \in M$. It follows from the claim that $\bigcup_{n \in N} X_n$ is a cutset in G, and so 3.1 implies that $\bigcup_{n \in N} X_n$ is not a clique of G. Since if $\{u, w\} \in F$ then u is nonadjacent to w in H, it follows that N is not a clique of H.

Since H is antiprismatic, every vertex of N is adjacent to exactly one of m_1, m_2, and no pair $\{n, m_1\}$ or $\{n, m_2\}$, where $n \in N$, belongs to F. For $i = 1, 2$, let N_i be the set of neighbors of m_i in N. Then N_1 is anticomplete to m_2, and N_2 is anticomplete to m_1. Since $H|(N_1 \cup \{m_2\})$ contains no triad, it follows that N_1, and similarly N_2, is a clique of H. Therefore, for $i = 1, 2$, $Y_i = \bigcup_{n \in N_i} X_n$ is a clique of G, and Y_i is complete to X_{m_i}. Since by 4.3, we may assume that G is a reduced thickening of (H, F), it follows that we may assume from the symmetry that there exist $m'_1 \in X_{m_1}$ that is anticomplete to X_{m_2}. Since $Y_1 \cup (X_{m_1} \setminus \{m'_1\})$ is a clique of G, 3.1 implies that m'_1 has a neighbor in $V(G) \setminus (Y_1 \cup (X_{m_1} \setminus \{m'_1\}))$, and consequently $M \neq \{m_1, m_2\}$. Let $m_3 \in M$. Since (H, F) is antiprismatic, it follows that m_3 is adjacent to both m_1, m_2. Since N is not a clique of H, and since both N_1, N_2 are cliques, it follows that there exist $n_1 \in N_1$ and $n_2 \in N_2$ nonadjacent.

Let $v' \in X_v$, for $i \in \{1, 2\}$, let $v_i \in X_{n_i}$ and $u_i \in X_{m_i}$, and let $w \in X_{m_3}$, such that v_1 is nonadjacent to v_2, and u_1 is nonadjacent to u_2. Then v_1 is nonadjacent to u_2, and v_2 to u_1. Let $S_1 = \{v', v_1, v_2\}$, $S_2 = \{u_1, u_2, w\}$, and $G' = G \setminus (S_1 \cup S_2)$. It follows that $G|(S_1 \cup S_2)$ is 3-colorable, and so $\chi(G') \geq \chi(G) - 3$. By assumption, G' has a clique minor of size $\lceil \frac{2}{3}\chi(G') \rceil \geq \lceil \frac{2}{3}(\chi(G) - 3) \rceil \geq \lceil \frac{2}{3}\chi(G) \rceil - 2$. This means that there exists a set $\mathcal{S}$ of $\lceil \frac{2}{3}\chi(G) \rceil - 2$ connected, disjoint subgraphs of G' that are pairwise adjacent. By 7.3, since no triad of H contains n_1 and n_2, it follows that $\{n_1, n_2, v\}$ dominates $V(H)$, and $\{m_1, m_2, m_3\}$ dominates $V(H) \setminus \{v\}$. Since if $\{u, v\} \in F$ then u is nonadjacent to v in H, we deduce that S_1 dominates $V(G)$, and S_2 dominates $V(G) \setminus X_v$. Thus, if no member of $\mathcal{S}$ is a subgraph of $G|X_v$, then, since $G|S_1$ is adjacent to $G|S_2$, it follows that $\mathcal{S} \cup \{G|S_1, G|S_2\}$ is a set of $\lceil \frac{2}{3}\chi(G) \rceil$ connected, disjoint subgraphs of G that are pairwise adjacent. So we may assume that some member T of $\mathcal{S}$ is a subgraph of $G|X_v$. Let $S'_1 = \{v'\}$ and $S'_2 = \{v_1, v_2, u_1, u_2, w\}$. Then S'_2 dominates $V(G)$. Since $N_G(T) \subseteq N_G(S'_1)$ and $G|S'_1$ is adjacent to $G|S'_2$, it follows that $\mathcal{S} \cup \{G|S'_1, G|S'_2\}$ is a set of $\lceil \frac{2}{3}\chi(G) \rceil$ connected, disjoint subgraphs of G that are pairwise adjacent. This proves 7.4. ■

8 Nontrivial strip-structures

In this section we prove 1.1 for graphs G that admit non-trivial strip structures and appear in 2.1.

8.1 *Suppose that G admits a nontrivial strip-structure such that $|Z| = 1$ for some strip (J, Z) of (H, η). Then either G is a clique or G admits a clique cutset.*

Proof. Let $F \in E(H)$ such that the strip (J, Z) of (H, η) at F has $|Z| = 1$. Then $|\overline{F}| = 1$, so let $\overline{F} = \{h\}$.

(1) If $\eta(F) \neq \eta(F, h)$, then 8.1 holds.

Let $v \in \eta(F) \setminus \eta(F, h)$. Suppose that v has a neighbor u that is not in $\eta(F)$. Then $u \in \eta(F')$ for some $F' \neq F$. By **(SD3)** there exists $h' \in \overline{F} \cap \overline{F'}$ such that $v \in \eta(F, h')$ and $u \in \eta(F', h')$. But h is the only member of F and so it follows that $v \in \eta(F, h)$, a contradiction. Hence, $N_G(v) \subseteq \eta(F)$ and so by **(SD2)** $\eta(F, h)$ is a clique cutset. This proves (1).

So we may assume that $\eta(F) = \eta(F, h)$. Let $v \in \eta(F, h)$. Then by **(SD3)**, v is adjacent only to members of $\eta(F', h)$ for F' with $h \in F'$. Hence by **(SD2)**, $N_G(v)$ is a clique and so either G is a clique or G admits a clique cutset. This proves 8.1.

■

Let (J, Z) be a strip. We say that (J, Z) is a *line graph strip* if $|V(J)| = 3$, $|Z| = 2$ and Z is complete to $V(J) \setminus Z$.

8.2 *Let G be a graph that admits a nontrivial strip-structure (H, η) such that for every $F \in E(H)$, the strip of (H, η) at F is a line graph strip. Then G is a line graph.*

Proof. Since $|Z| = 2$ for every strip (J, Z) of (H, η), it follows that H is a graph. Since all strips of (H, η) are line graph strips, it follows that $|\eta(F)| = 1$ for every $F \in E(H)$. Moreover, $\eta(F) = \eta(F, h)$ for every $F \in E(H)$ and $h \in \overline{F}$. We claim that G is the line graph of H. We need to show that there exists a bijection $\phi : E(H) \to V(G)$ such that for $F_1, F_2 \in E(H)$, $\phi(F_1)$ is adjacent to $\phi(F_2)$ in G if and only if F_1, F_2 share an end in H. Let $F \in E(H)$ and let v be the unique vertex of $\eta(F)$. We define $\phi(F) = v$. Clearly ϕ is a bijection.

Let $F_1, F_2 \in E(H)$. Then, $\eta(F_i, h) = \eta(F_i)$ for $i = 1, 2$ and $h \in \overline{F_i}$. Now **(SD2)** and **(SD3)** imply that there exists $h \in \overline{F_1} \cap \overline{F_2}$ if and only if $\eta(F_1) \cup \eta(F_2)$ is a clique in G, which means that $\phi(F_1)$ is adjacent to $\phi(F_2)$. This proves 8.2.

■

For two disjoint subsets U, W of $V(G)$ and a coloring c of G, let $m_c(U, W)$ denote the number of *repeated colors* on U and W (the number of colors i such that $i \in c(U) \cap c(W)$). We can now prove the main result of this section.

8.3 *Let G be a connected, claw-free graph with chromatic number χ. Assume that for every claw-free graph G' with either $|V(G')| < |V(G)|$, or $|V(G')| = |V(G)|$ and $|E(G')| < |E(G)|$, G' has a clique minor of size $\lceil \frac{2}{3}\chi(G') \rceil$. Suppose that G admits a nontrivial strip-structure (H, η) such that for each strip (J, Z) of (H, η), $1 \leq |Z| \leq 2$, and if $|Z| = 2$ then either $|V(J)| = 3$ and Z is complete to $V(J) \setminus Z$, or (J, Z) is a member of $\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4 \cup \mathcal{Z}_5$. Then G has a clique minor of size $\lceil \frac{2}{3}\chi \rceil$.*

Proof. Suppose that G admits a nontrivial strip-structure (H, η) such that for each strip (J, Z) of (H, η), $1 \leq |Z| \leq 2$. Further suppose that $|Z| = 1$ for some strip (J, Z). Then by 8.1 either G is a clique or G admits a clique cutset; in the former case 8.3 follows from 7.1, and in the latter case 8.3 follows from 3.1. Hence we may assume that $|Z| = 2$ for all strips (J, Z).

Let k be the number of strips of (H, η) that are not line graph strips. If $k = 0$, the result follows from [12] and 8.2. So we may assume $k > 0$ and some strip (J_1, Z_1) is not a line graph strip. Let $Z_1 = \{a_1, b_1\}$. Let $A_1 = N_{J_1}(a_1)$, $B_1 = N_{J_1}(b_1)$, $A_2 = N_G(A_1) \setminus V(J_1)$, and $B_2 = N_G(B_1) \setminus V(J_1)$. Let $C_1 = V(J_1) \setminus (A_1 \cup B_1)$ and $C_2 = V(G) \setminus (V(J_1) \cup A_2 \cup B_2)$. Then $V(G) = A_1 \cup B_1 \cup C_1 \cup A_2 \cup B_2 \cup C_2$.

(1) If $C_2 = \emptyset$ and $A_2 = B_2$, then 8.3 holds.

Note that $V(G) = A_1 \cup B_1 \cup C_1 \cup A_2$. Since $|Z_1| = 2$ and (J_1, Z_1) is not a line graph strip, it follows that (J_1, Z_1) is a member of $\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4 \cup \mathcal{Z}_5$. We consider the cases separately:

1. (J_1, Z_1) is a member of $\mathcal{Z}_1$. In this case J_1 is a fuzzy linear interval graph and so G is a fuzzy long circular interval graph and 8.3 follows from [5].

2. (J_1, Z_1) is a member of $\mathcal{Z}_2$, $\mathcal{Z}_3$, or $\mathcal{Z}_4$. In all of these cases, A_1, B_1, and C_1 are all cliques and so $V(G)$ is the union of three cliques, namely $A_1 \cup A_2$, B_1, and C_1. Hence, 8.3 follows from 6.1

3. (J_1, Z_1) is a member of $\mathcal{Z}_5$. Let $v_1, \ldots, v_{12}, X, H, H', F$ be as in the definition of $\mathcal{Z}_5$ and for $1 \leq i \leq 12$ let X_{v_i} be as in the definition of a thickening. Then A_2 is complete to $X_{v_1} \cup X_{v_2} \cup X_{v_4} \cup X_{v_5}$. Let H'' be the graph obtained from H' by adding a new vertex a_2, adjacent to v_1, v_2, v_4 and v_5. Then H'' is an antiprismatic graph. Moreover, no triad of H'' contains v_9 or v_{10}. Thus the pair (H', F) is antiprismatic, and G is a thickening of (H', F), so 8.3 follows from 7.4.

This proves (1).

Therefore, we may assume that either $C_2 \neq \emptyset$, or $A_2 \neq B_2$. For $i = 1, 2$, let $G_i = G|(A_i \cup B_i \cup C_i)$.

Let n be the maximum size of a clique minor in G. Then $n < \lceil \frac{2}{3}\chi \rceil$. Without loss of generality, we may assume that $|A_1 \cup A_2| \leq |B_1 \cup B_2|$. Then, by 3.4, there exist $|A_1 \cup A_2|$

vertex disjoint paths between $A_1 \cup A_2$ and $B_1 \cup B_2$ in G. From the definitions of A_1, A_2, B_1, and B_2 it follows that for $i = 1, 2$, $|A_i| \leq |B_i|$ and that there exist $|A_i|$ vertex disjoint paths from A_i to B_i in G_i.

Let G'_1 be the graph obtained from $G|(A_1 \cup B_1 \cup C_1 \cup A_2)$ by making A_2 complete to B_1. Then since there exist $|A_2|$ vertex disjoint paths between A_2 and B_2 in G_2, it follows that G'_1 is a minor of G. We claim that G'_1 is a claw-free graph. For $v \in C_1$, $G|(N_G(v)) = G'_1|(N_{G'_1}(v))$, and so there is no triad in $N_{G'_1}(v)$. For $v \in A_1$, $N_G(v) = N_{G'_1}(v)$, and if two vertices of $N_{G'_1}(v)$ are adjacent in G, then they are also adjacent in G'_1. Therefore there is no triad in $N_{G'_1}(v)$. Next let $v \in B_1$, and suppose that in G'_1 there is a triad $\{x, y, z\}$ among the neighbors of v. Since G is claw-free, we may assume that $x \in A_2$. Consequently, $y, z \in C_1$. Let $b \in B_2$. Now, $\{b, y, z\}$ is a triad among the neighbors of v in G, contrary to the fact that G is claw-free. Finally, for $v \in A_2$, the set of neighbors of v in G'_1 is the union of two cliques, namely $A_1 \cup A_2 \setminus \{v\}$ and B_1. This proves the claim that G'_1 is a claw-free graph.

Similarly, let G'_2 be the graph obtained from $G|(A_2 \cup B_2 \cup C_2 \cup A_1)$ by making A_1 complete to B_2. Then G'_2 is also a claw-free graph and a minor of G. Since (S_1, a_1, b_1) is not a line-graph strip, it follows that $|V(G'_2)| < |V(G)|$; and since either $C_2 \neq \emptyset$, or $A_2 \neq B_2$, it follows that $|V(G'_1)| < |V(G)|$. Since G'_1 and G'_2 are minors of G, it follows that they contain no clique minors of size greater than n, and since for $i = 1, 2$, $|V(G'_i)| < |V(G)|$, it follows that $\lceil \frac{2}{3}\chi(G'_1) \rceil, \lceil \frac{2}{3}\chi(G'_2) \rceil \leq n < \lceil \frac{2}{3}\chi(G) \rceil$.

Let $n' = \max(\chi(G'_1), \chi(G'_2), |B_1| + |B_2|)$. Since $B_1 \cup B_2$ is a clique of G, we may assume that that $n' < \chi(G)$. Next we show that G can be properly colored with n' colors, thus obtaining a contradiction.

For $i = 1, 2$, let c'_i be an n'-coloring of G'_i. Further, let $m_i = m_{c'_i}(A_i, B_i)$, $a_i = |A_i| - m_i$, and $b_i = |B_i| - m_i$. Then $m_1 + a_1 + b_1 + |A_2| = m_1 + a_1 + b_1 + m_2 + a_2 \leq n'$ and $m_2 + a_2 + b_2 + |A_1| = m_2 + a_2 + b_2 + m_1 + a_1 \leq n'$.

Suppose that $b_1 \leq a_2$. Then since $a_1 \leq b_1$ and $a_2 \leq b_2$, it follows that $a_1 \leq b_2$. Notice that c'_2 induces an n'-coloring c_2 of G_2 with $m_{c_2}(A_2, B_2) = m_2$. Let $T = \{1, \ldots, n'\}$. Without loss of generality, $c_2(V(G_2)) \subseteq T$. Construct the following coloring c of G. For $v \in A_2 \cup B_2 \cup C_2$ let $c(v) = c_2(v)$. Next, use m_1 colors of $T \setminus c(A_2 \cup B_2)$ on both A_1 and B_1 (this is possible since $|c(A_2 \cup B_2)| = a_2 + b_2 + m_2$ and $a_2 + b_2 + m_1 + m_2 \leq n'$). Next, use a_1 colors of $c(B_2) \setminus c(A_2)$ on the remaining vertices of A_1 and b_1 colors of $c(A_2) \setminus c(B_2)$ on the remaining vertices of B_1 (this is possible because $b_1 \leq a_2$ and $a_1 \leq b_2$). Now since $m_{c'_1}(A_1, B_1) = m_1$ it follows that the coloring constructed so far can be extended to an n'-coloring of G_1 using the colors of T. We see that $c(A_2)$ is disjoint from $c(A_1)$ and $c(B_2)$ is disjoint from $c(B_1)$. Thus c is an n'-coloring of G, a contradiction.

Hence, $b_1 > a_2$. Similarly, the argument of the previous paragraph with the roles of G'_1 and G'_2 reversed shows that $b_2 > a_1$. Let T be as before. We construct the following coloring c of G. We use $|B_1| + |B_2|$ distinct colors of T on $B_1 \cup B_2$. Next, we use m_1 colors of $c(B_1)$ and a_1 colors of $c(B_2)$ to color A_1. Then we use m_2 colors of $c(B_2) \setminus c(A_1)$ and a_2 colors of $c(B_1) \setminus c(A_1)$ to color A_2 (this is possible because $b_2 > a_1$ and $b_1 > a_2$). Now since $m_{c'_1}(A_1, B_1) = m_1$ we can extend c to an n'-coloring of G_1 using the colors of T and since $m_{c'_2}(A_2, B_2) = m_2$ we can extend c to an n'-coloring of G_2 using the colors of T. Once again we see that $c(A_2)$ is disjoint from $c(A_1)$ and $c(B_2)$ is disjoint from $c(B_1)$. But now c is an n'-coloring of G, a contradiction. This proves 8.3. ■

We are now ready to prove the main result of this paper.

Proof of 1.1. Let G be a claw-free graph. We may assume that if G' is a claw-free graph, and either $|V(G')| < |V(G)|$, or $|V(G')| = |V(G)|$ and $|E(G') < |E(G)|$, then G' has a clique minor of size $\lceil \frac{2}{3}\chi(G') \rceil$. Consequently, we may assume that G is connected. If $V(G)$ is the union of three cliques, the result follows from 6.1. If G admits a nontrivial strip-structure as in 2.1, then the result follows from 8.3. So by 2.1, we may assume that G is a member of $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$. If G is a member of $\mathcal{T}_1 \cup \mathcal{T}_2$, then the result follows from 5.1 and 7.4. Hence, we may assume that G is a fuzzy long circular interval graph. But fuzzy long circular interval graphs are quasi-line graphs and so the result follows from [5]. This completes the proof of 1.1.

9 Conclusion

The natural question to ask at this point is how far are we from proving the full Hadwiger's conjecture for claw-free graphs. One thing to note is that all graphs with $\alpha = 2$ are claw-free and there has been much research and multiple papers written on the subject of Hadwiger's conjecture for this class of graphs with only minimal progress [4, 8]. So the next best thing would be to prove Hadwiger's conjecture for all claw-free graphs with $\alpha > 2$. However, even that seems very difficult since most of our proofs are by induction and graphs with $\alpha = 2$ often appear in the base case.

We were however able to make some progress towards proving Hadwiger's conjecture for claw-free graphs. Many of the results in this paper proving that certain classes of graphs are not minimal counterexamples to 1.1 can be minimally modified to show that these same classes are not minimal counterexamples to Hadwiger's conjecture. In particular, the proofs showing that graphs from the icosahedron and graphs that admit non-trivial strip structures both belong to this category. Also, Hadwiger's conjecture for fuzzy long circular interval graphs has been proved in [5]. This leaves antiprismatic graphs (which include graphs with $\alpha = 2$) and graphs whose vertex sets are the union of three cliques. But even in these cases we can say a lot about what a possible minimal counterexample could look like. In particular, for antiprismatic graphs we can prove that a minimal counterexample does not have two disjoint triads and for graphs whose vertex sets are the union of three cliques we can likewise narrow the possibilities for the minimal counterexample based on where the triads are.

References

[1] K. Appel and W. Haken, Every planar map is four colorable. Part I: Discharging, *Illinois J. Math.* 21 (1977), 429-490.

[2] K. Appel, W. Haken and J. Koch, Every planar map is four colorable. Part II. Reducibility, *Illinois J. Math.* 21 (1977), 491–567.

[3] C. Berge, Sur le couplage maximum d'un graphe, *C.R. Acad. Sci. Paris*, 247 (1958), 258-259.

[4] J.Blasiak, A special case of Hadwiger's conjecture, *J. Combin. Theory, Ser. B*, 97 (2007), 1056–1073.

[5] M. Chudnovsky and A.O. Fradkin, Hadwiger's conjecture for quasi-line graphs, *J. Graph Theory* 59 (2008), 17-33.

[6] M. Chudnovsky and P. Seymour, Claw-free graphs I. Orientable prismatic graphs, *J. Combin. Theory B* 97 (2007), 866-901.

[7] M. Chudnovsky and P. Seymour, Claw-free graphs V. Global structure, *J. Combin. Theory B* 98 (2008), 1373-1410.

[8] M. Chudnovsky and P. Seymour, Packing seagulls, *submitted for publication*

[9] H. Hadwiger, Über eine Klassifikation der Streckenkomplexe, *Vierteljschr. Naturforsch. Ges. Zürich*, 88 (1943), 133-142.

[10] D. König, "Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre", *Math. Ann.* 77 (1916), 453-465.

[11] K. Menger, Zur allgemeinen Kurventheorie, *Fund. Math.* 10 (1927), 95-115.

[12] B. Reed and P. Seymour, Hadwiger's conjecture for line graphs, *European J. Combin.* 25 (2004), no. 6, 873-876.

[13] N. Robertson, P. Seymour, and R. Thomas, Hadwiger's conjecture for K_6-free graphs, *Combinatorica* 13 (1993), 279-361.

[14] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, *Math. Ann.* 114 (1937).