INDUCED SUBGRAPHS AND TREE DECOMPOSITIONS VIII. EXCLUDING A FOREST IN (THETA, PRISM)-FREE GRAPHS

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Abstract. Given a graph H, we prove that every (theta, prism)-free graph of sufficiently large treewidth contains either a large clique or an induced subgraph isomorphic to H, if and only if H is a forest.

1. Introduction

All graphs in this paper are finite and simple unless specified otherwise. Let $G = (V(G), E(G))$ be a graph. For a set $X \subseteq V(G)$ we denote by $G[X]$ the subgraph of G induced by X, and we use X and $G[X]$ interchangeably. Given a graph H, we say that G *contains* H if G has an induced subgraph isomorphic to H, and we say G is H*-free* if G does not contain H. For a family $\mathcal{H}$ of graphs we say G is $\mathcal{H}$*-free* if G is H-free for every $H \in \mathcal{H}$. A class of graphs is *hereditary* if it is closed under isomorphism and taking induced subgraphs, or equivalently, if it is the class of all $\mathcal{H}$-free graphs for some family $\mathcal{H}$ of graphs.

For a graph $G = (V(G), E(G))$, a *tree decomposition* (T, χ) of G consists of a tree T and a map $\chi : V(T) \to 2^{V(G)}$ with the following properties:

(i) For every $v \in V(G)$, there exists $t \in V(T)$ such that $v \in \chi(t)$.
(ii) For every $v_1v_2 \in E(G)$, there exists $t \in V(T)$ such that $v_1, v_2 \in \chi(t)$.
(iii) For every $v \in V(G)$, the subgraph of T induced by $\{t \in V(T) \mid v \in \chi(t)\}$ is connected.

For each $t \in V(T)$, we refer to $\chi(t)$ as a *bag of* (T, χ). The *width* of a tree decomposition (T, χ), denoted by width(T, χ), is $\max_{t\in V(T)} |\chi(t)| - 1$. The *treewidth* of G, denoted by tw(G), is the minimum width of a tree decomposition of G.

Treewidth was first popularized by Robertson and Seymour in their graph minors project, and has attracted a great deal of interest over the past three decades. Particularly, graphs of bounded treewidth have been shown to be well-behaved from structural [19] and algorithmic [5] viewpoints.

This motivates investigating the structure of graphs with large treewidth, and especially, the substructures emerging in them. The canonical result in this realm is the Grid Theorem of Robertson and Seymour [19], which describes the unavoidable minors – and the unavoidable subgraphs – of graphs with large treewidth. For a positive integer t, the $(t \times t)$*-wall*, denoted by $W_{t\times t}$, is the t*-by-*t hexagonal grid (see Figure 1; a formal definition can be found in [3]).

Theorem 1.1 (Robertson and Seymour [19])**.** *For every integer $t \geq 1$ there exists $w = w(t) \geq 1$ such that every graph of treewidth more than w contains a subdivision of $W_{t\times t}$ as a subgraph.*

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Ш Supported by NSF-EPSRC Grant DMS-2120644 and by AFOSR grant FA9550-22-1-0083.
¶ Supported by DMS-EPSRC Grant EP/V002813/1.
‖ We acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC), [funding reference number RGPIN-2020-03912]. Cette recherche a été financée par le Conseil de recherches en sciences naturelles et en génie du Canada (CRSNG), [numéro de référence RGPIN-2020-03912]. This project was funded in part by the Government of Ontario.
Date: October 9, 2023.

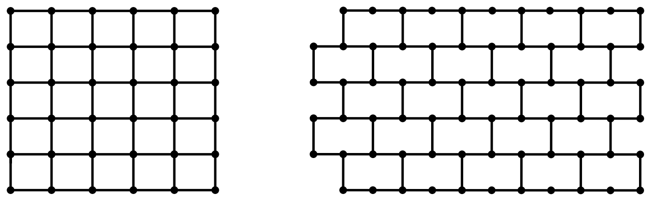

FIGURE 1. The 6-by-6 square grid (left) and the 6-by-6 wall $W_{6\times 6}$ (right).

In contrast, unavoidable induced subgraphs of graphs with large treewidth are far from completely understood. There are some natural candidates though, which we refer to as the "basic obstructions": complete graphs and complete bipartite graphs, subdivided walls mentioned above, and line graphs of subdivided walls, where the *line graph* $L(F)$ of a graph F is the graph with vertex set $E(F)$, such that two vertices of $L(F)$ are adjacent if and only if the corresponding edges of F share an end. In fact, the complete graph K_{t+1}, the complete bipartite graph $K_{t,t}$, and the line graph of every subdivision of $W_{t\times t}$ all have treewidth t [19]. For a positive integer t, let us say a graph H is a *t-basic obstruction* if H is one of the following graphs: K_t, $K_{t,t}$, a subdivision of $W_{t\times t}$, or the line graph of a subdivision of $W_{t\times t}$ (see Figure 2). We say a graph G is *t-clean* if G does not contain a t-basic obstruction.

The basic obstructions do not form a comprehensive list of induced subgraph obstructions to bounded treewidth: there are t-clean graphs of arbitrarily large treewidth for small values of t. A well-known hereditary graph class demonstrating this fact is the class of even-hole-free graphs, where a *hole* is an induced cycle on at least four vertices, the *length* of a hole is its number of edges and an *even hole* is a hole with even length. In fact, complete graphs are the only even-hole-free basic obstruction (see Figure 2). In other words, for every positive integer $t \geq 1$, one may observe that an even-hole-free graph is t-clean if and only if it is K_t-free. It is therefore tempting to ask whether even-hole-free graphs excluding a fixed complete graph have bounded treewidth. Sintiari and Trotignon [20] answered this with a vehement "no", providing a construction of (even-hole, K_4)-free graphs with arbitrarily large treewidth, hence proving that there are t-clean (even-hole-free) graphs of arbitrarily large treewidth for every fixed $t \geq 4$. In addition, graphs from this construction are sparse, in the sense that they exclude short holes.

Theorem 1.2 (Sintiari and Trotignon [20])**.** *For all integers $w, l \geq 1$, there exists an (even-hole, K_4)-free graph $G_{w,l}$ of treewidth more than w and with no hole of length at most l.*

At the same time, 3-clean even-hole-free graphs have treewidth at most five [6]. So one may wonder whether all 3-clean graphs have bounded treewidth. However, another construction by Sintiari and Trotignon [20] shows that in a superclass of even-hole-free graphs, namely the class of theta-free graphs, there are 3-clean graphs with arbitrarily large treewidth (see the next section for the definition of a theta; one may check that every t-basic obstruction for $t \geq 3$ contains either a theta or a triangle – see Figure 2). Indeed, the treewidth of theta-free graphs remains unbounded even when forbidding short cycles.

Theorem 1.3 (Sintiari and Trotignon [20])**.** *For all integers $w, g \geq 1$, there exists a theta-free graph $G_{w,g}$ of treewidth more than w and girth more than g.*

A natural question to ask, then, is what further conditions must be imposed to guarantee bounded treewidth in even-hole-free graphs. For instance, graphs from both Theorems 1.2 and 1.3 have vertices of arbitrarily large degree, and so it was conjectured in [1] that (theta, triangle)-free graphs of bounded maximum degree have bounded treewidth and that even-hole-free graphs of bounded maximum degree have bounded treewidth. These were proved in [3] and [4], respectively. In the same paper [1], a stronger conjecture was made, asserting that basic obstructions are in fact the only obstructions to bounded treewidth in graphs of bounded maximum degree. This was later proved in [15], which closed the line of inquiry into graph classes of bounded maximum degree.

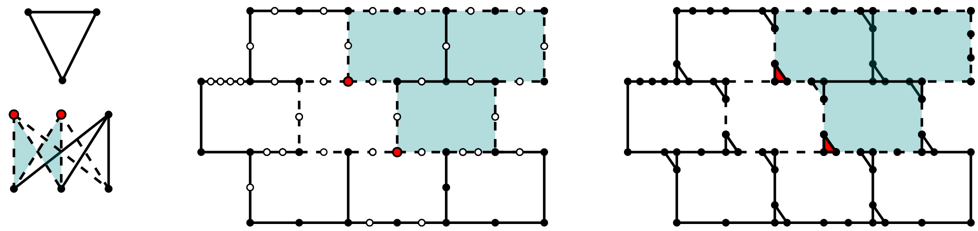

Figure 2. The 3-basic obstructions. A theta in $K_{3,3}$ (left), a theta in a subdivision of $W_{3\times 3}$ (middle) and a prism in the line-graph of the same subdivision of $W_{3\times 3}$ (right) are all depicted with dashed lines. An even hole in each theta and prism is also highlighted.

Theorem 1.4 (Korhonen [15])**.** *For all integers $t, \delta \geq 1$, there exists $w = w(t, \delta)$ such that every t-clean graph of maximum degree at most δ has treewidth at most w.*

Despite its generality, the proof of Theorem 1.4 is surprisingly short. However, the case of proper hereditary classes containing graphs of unbounded maximum degree seems to be much harder. For graph classes $\mathcal{G}$ and $\mathcal{H}$, let us say $\mathcal{H}$ *modulates* $\mathcal{G}$ if for every positive integer t, there exists a positive integer $w(t)$ (depending on $\mathcal{G}$ and $\mathcal{H}$) such that every t-clean $\mathcal{H}$-free graph in $\mathcal{G}$ has treewidth at most $w(t)$. An induced subgraph analogue to Theorem 1.1 is therefore equivalent to a full characterization of graph classes $\mathcal{H}$ which modulate the class of all graphs. This remains out of reach, but the special case where $|\mathcal{H}| = 1$ turns out to be more approachable. For a graph H and a graph class $\mathcal{G}$, let us say H *modulates* $\mathcal{G}$ if $\{H\}$ modulates $\mathcal{G}$. Building on a method from [16], recently we characterized all graphs H which modulate the class of all graphs:

Theorem 1.5 (Abrishami, Alecu, Chudnovsky, Hajebi and Spirkl [2])**.** *Let H be a graph. Then H modulates the class of all graphs if and only if H is a subdivided star forest, that is, a forest in which every component has at most one vertex of degree more than two.*

In general, for a hereditary class $\mathcal{G}$ containing t-clean graphs of arbitrarily large treewidth for small t, one can ask for a characterization of graphs H modulating $\mathcal{G}$. Given Theorem 1.2, a natural class $\mathcal{G}$ to consider is the class of even-hole-free graphs. Note that Theorem 1.2 shows that a graph H modulates even-hole-free graphs only if H is a chordal graph (that is, a graph with no hole) of clique number at most three. As far as we know, the converse may also be true, that every chordal graph of clique number at most three modulates even-hole-free graphs. In fact, in this paper we narrow the gap, showing that every chordal graph of clique number at most two, that is, every forest, modulates the class of even-hole-free graphs.

Theorem 1.6. *For every forest H and every integer $t \geq 1$, every even-hole-free graph of sufficiently large treewidth contains either H or a clique of cardinality t.*

This aligns with the observation [21] that every forest is contained in some graph $G_{w,l}$ from Theorem 1.2. As mentioned above, one way to improve on Theorem 1.6 is to push H towards being an arbitrary chordal graph of clique number three. Another way to strengthen Theorem 1.6 is to find a superclass $\mathcal{G}$ of even-hole-free graphs for which forests are the only graphs modulating $\mathcal{G}$. While the former remains open, we provide an appealing answer to the latter: our main result shows that forests are exactly the graphs which modulate the class of (theta, prism)-free graphs (see the next section for the definition of a prism; again one may check that in (theta, prism)-free graphs, being t-clean is equivalent to being K_t-free for every positive integer t).

Theorem 1.7. *Let H be a graph. Then H modulates (theta, prism)-free graphs if and only if H is a forest. In other words, given a graph H, for every integer $t \geq 1$, every (theta, prism)-free graph of sufficiently large treewidth contains either H or a clique of cardinality t, if and only if H is a forest.*

We remark that, since graphs from both Theorems 1.2 and 1.3 are rather complicated in structure [20], it is therefore natural to look for less involved constructions of sparse theta-free (or even-hole-free) graphs of large treewidth. On the other hand, by Theorem 1.7, theta-free graphs of large girth and large (enough) treewidth induce *all* forests. This can be viewed as an indication of structural richness, showing that in fact *every* construction satisfying Theorems 1.2 and 1.3 is necessarily complex in one way or another.

Let $\mathcal{C}$ be the class of all (theta, prism)-free graphs. It is easily seen that $\mathcal{C}$ contains all even-hole-free graphs, and so Theorem 1.7 implies Theorem 1.6. Note that the "only if" direction of Theorem 1.7 follows immediately from Theorem 1.3 as prisms contain triangles. Since every forest is an induced subgraph of a tree, in order to prove Theorem 1.7, it suffices to prove Theorem 1.8 below, which we do in Section 7. For a positive integer t and a tree F, we denote by $\mathcal{C}_t$ the class of all graphs in $\mathcal{C}$ with no clique of cardinality t (that is, t-clean graphs in $\mathcal{C}$), and by $\mathcal{C}_t(F)$ the class of all F-free graphs in $\mathcal{C}_t$.

Theorem 1.8. *For every tree F and every integer $t \geq 1$, there exists an integer $\tau(F,t) \geq 1$ such that every graph in $\mathcal{C}_t(F)$ has treewidth at most $\tau(F,t)$.*

We conclude this introduction by sketching our proofs (the terms we use here are defined in later sections). The proof of Theorem 1.8 begins with a two-step preparation. As the first step, inspired by a result from [8], we show that for every graph $G \in \mathcal{C}$ which contains a pyramid with certain conditions on the apex and its neighbors, G admits a construction which we call a "(T,a)-strip-structure," where a is the apex of the pyramid and T is an optimally chosen tree. Roughly speaking, we show that $G \setminus \{a\}$ can be partitioned into two induced subgraphs H and J where H is more or less similar to the line graph of the tree T and every vertex in J with a neighbor in H attaches at a pyramid lurking in H in a restricted way; we call the latter vertices "jewels." The proof of this theorem occupies Sections 3 and 4. The second step is to employ the previous result to show that if $G \in \mathcal{C}_t$ admits a (C,a)-strip-structure where C is a caterpillar, then every vertex in $G \setminus N_G[a]$ can be separated from a by removing a few vertices (our proof works more generally when C is any tree of bounded maximum degree, but the caterpillar case suffices for our application). We prove this in Section 6. The central difficulty in the proof is to deal with the jewels separately. This is surmounted in Section 5 where we prove several results concerning the properties of jewels. Most notably, we show that jewels only attach at "local areas of the line-graph-like part" of G, and that only a few jewels attach at each local area. This concludes the preparation for proving Theorem 1.8.

Next, we embark on the proof of Theorem 1.8. We assume that $G \in \mathcal{C}_t$ has large treewidth, which together with results from Section 2 implies that G contains two vertices x, y joined by many pairwise internally disjoint induced paths $P_1, \ldots, P_m$. Now we analyze the structure of the graph $G[P_1 \cup \cdots \cup P_m]$. It turns out that, if m is large enough, then either

- there are many paths among P_i's whose union H admits a (C,x)-strip-structure for some caterpillar C, or
- for some large value of d, $G[P_1 \cup \cdots \cup P_m]$ contains a tree S isomorphic to the complete bipartite graph $K_{1,d}$, such that x is the vertex of degree d in S, and for every leaf l of S, there are many pairwise internally disjoint induced paths between l and y, such that in addition, paths corresponding to distinct leaves of S are also pairwise internally disjoint.

The former case implies that y can be separated from x by removing few vertices, which using a result from Section 6, yields a contradiction with Menger's theorem. The latter case is the first step towards building the large tree in G as a subgraph. We now iterate the argument we just described, applying it to each leaf l of S and y, obtaining larger and larger trees. The process is stopped once we reach a sufficiently large tree as a subgraph of G. This, combined with the fact that $G \in \mathcal{C}_t$ and a result of Kierstead and Penrice [14], yields the desired tree F as an induced subgraph of G.

This paper is organized as follows. Section 2 covers preliminary definitions as well as some results from the literature used in our proofs. Section 3 investigates the behavior of pyramids in

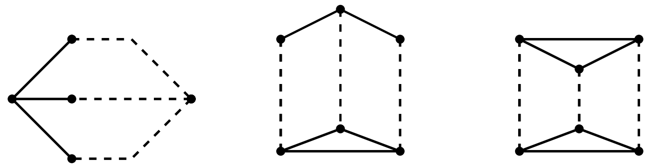

Figure 3. Theta, pyramid and prism. The dashed lines represent paths of length at least one.

graphs from $\mathcal{C}$. Section 4 is devoted to defining strip-structures and jewels, and showing how they arise from pyramids in graphs in $\mathcal{C}$. Section 5 takes a closer look at jewels for the strip-structures obtained in Section 4. In Section 6 we show that admitting certain strip-structures weakens the connectivity of most vertices to the apex. Finally, in Section 7, we prove Theorem 1.8.

2. Preliminaries and results from the literature

Let $G = (V(G), E(G))$ be a graph. For $X \subseteq V(G) \cup E(G)$, $G \setminus X$ denotes the subgraph of G obtained by removing X. Note that if $X \subseteq V(G)$, then $G \setminus X$ denotes the subgraph of G induced by $V(G) \setminus X$. In this paper, we use induced subgraphs and their vertex sets interchangeably.

Let $x \in G$ and let d be a positive integer. We denote by $N_G^d(x)$ the set of all vertices in G at distance d from some x, and by $N_G^d[x]$ the set of all vertices in G at distance at most d from x. We write $N_G(x)$ for $N_G^1(x)$ and $N_G[x]$ for $N_G^1[x]$. For an induced subgraph H of G, we define $N_H(x) = N_G(x) \cap H$, $N_H[x] = N_G[x] \cap H$. Also, for $X \subseteq G$, we denote by $N_G(X)$ the set of all vertices in $G \setminus X$ with at least one neighbor in X, and define $N_G[X] = N_G(X) \cup X$.

Let $X, Y \subseteq G$ be disjoint. We say X is *complete* to Y if all edges with an end in X and an end in Y are present in G, and X is *anticomplete* to Y if no edges between X and Y are present in G.

A *path in* G is an induced subgraph of G that is a path. If P is a path in G, we write $P = p_1\text{-}\cdots\text{-}p_k$ to mean that $V(P) = \{p_1, \dots, p_k\}$ and p_i is adjacent to p_j if and only if $|i-j| = 1$. We call the vertices p_1 and p_k the *ends of* P, and say that P is *from* p_1 *to* p_k. Accordingly, for two vertices $x, y \in V(P)$, we write $x\text{-}P\text{-}y$ for the unique path in P from x to y. The *interior of* P, denoted by P^*, is the set $P \setminus \{p_1, p_k\}$. The *length* of a path is its number of edges (so a path of length at most one has empty interior). Similarly, if C is a cycle, we write $C = c_1\text{-}\cdots\text{-}c_k\text{-}c_1$ to mean that $V(C) = \{c_1, \dots, c_k\}$ and c_i is adjacent to c_j if $|i-j| \in \{1, k-1\}$. The *length* of a cycle is its number edges (or equivalently, vertices.)

A *theta* is a graph Θ consisting of two non-adjacent vertices a, b, called the *ends of* Θ, and three pairwise internally disjoint paths P_1, P_2, P_3 from a to b of length at least two, called the *paths of* Θ, such that P_1^*, P_2^*, P_3^* are pairwise anticomplete to each other. For a graph G, by a *theta in* G we mean an induced subgraph of G which is a theta.

A *prism* is a graph Π consisting of two disjoint triangles $\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}$ called the *triangles of* Π, and three pairwise disjoint paths P_1, P_2, P_3 called the *paths of* Π, where P_i has ends a_i, b_i for each $i \in \{1, 2, 3\}$, and for distinct $i, j \in \{1, 2, 3\}$, a_ia_j and b_ib_j are the only edges between P_i and P_j. For a graph G, by a *prism in* G we mean an induced subgraph of G which is a prism.

A *pyramid* is a graph Σ consisting of a vertex a, a triangle $\{b_1, b_2, b_3\}$ and three paths P_1, P_2, P_3 of length at least one with P_i from a to b_i for each $i \in \{1, 2, 3\}$ and otherwise pairwise disjoint, such that for distinct $i, j \in \{1, 2, 3\}$, b_ib_j is the only edge between $P_i \setminus \{a\}$ and $P_j \setminus \{a\}$, and at most one of P_1, P_2, P_3 has length exactly one. We say that a is the *apex* of Σ, $b_1b_2b_3$ is the *base* of Σ, and P_1, P_2, P_3 are the *paths* of Σ. The pyramid Σ is said to be *long* if all its paths have lengths more than one. For a graph G, by a *(long) pyramid in* G we mean an induced subgraph of G which is a (long) pyramid.

Let us now mention a few results from the literature which we will use in this paper. Let G be a graph. By a *separation* in G we mean a triple (L, M, R) of pairwise disjoint subsets of vertices in G with $L \cup M \cup R = G$, such that neither L nor R is empty and L is anticomplete to R in G. Let $x, y \in G$ be distinct. We say a set $M \subseteq G \setminus \{x, y\}$ *separates x and y* if there exists a separation (L, M, R) in G with $x \in L$ and $y \in R$. Also, for disjoint sets $X, Y \subseteq V(G)$, we say a set $M \subseteq G \setminus (X \cup Y)$ *separates X and Y* if there exists a separation (L, M, R) in G with $X \subseteq L$ and $Y \subseteq R$. If $X = \{x\}$, we say that M *separates x and Y* to mean M separates X and Y. Recall the following well-known theorem of Menger [17]:

Theorem 2.1 (Menger [17])**.** *Let $k \geq 1$ be an integer, let G be a graph and let $x, y \in G$ be distinct and non-adjacent. Then either there exists a set $M \subseteq G \setminus \{x, y\}$ with $|M| < k$ such that M separates x and y, or there are k pairwise internally disjoint paths in G from x to y.*

Let k be a positive integer and let G be a graph. A *strong k-block* in G is a set B of at least k vertices in G such that for every 2-subset $\{x, y\}$ of B, there exists a collection $\mathcal{P}_{\{x,y\}}$ of at least k distinct and pairwise internally disjoint paths in G from x to y, where for every two distinct 2-subsets $\{x, y\}, \{x', y'\} \subseteq B$ of G, and every choice of paths $P \in \mathcal{P}_{\{x,y\}}$ and $P' \in \mathcal{P}_{\{x',y'\}}$, we have $P \cap P' = \{x, y\} \cap \{x', y'\}$.

For a tree T and $xy \in E(T)$, we denote by $T_{x,y}$ the component of $T - xy$ containing x. Let G be a graph and (T, χ) be a tree decomposition for G. For every $S \subseteq T$, let $\chi(S) = \bigcup_{x \in S} \chi(x)$. By an *adhesion* of (T, χ) we mean the set $\chi(x) \cap \chi(y) = \chi(T_{x,y}) \cap \chi(T_{y,x})$ for some $xy \in E(T)$. For every $x \in V(T)$, by the *torso at x*, denoted by $\hat{\chi}(x)$, we mean the graph obtained from the bag $\chi(x)$ by, for each $y \in N_T(x)$, adding an edge between every two non-adjacent vertices $u, v \in \chi(x) \cap \chi(y)$. In [2], we used Theorem 1.4 and the following result from [12]:

Theorem 2.2 (Erde and Weißauer [12], see also [13])**.** *Let r be a positive integer, and let G be a graph containing no subdivision of K_r as a subgraph. Then G admits a tree decomposition (T, χ) for which the following hold.*

- *Every adhesion of (T, χ) has cardinality less than r^2.*
- *For every $x \in V(T)$, either $\hat{\chi}(x)$ has fewer than r^2 vertices of degree at least $2r^4$, or $\hat{\chi}(x)$ has no minor isomorphic to K_{2r^2}.*

to prove the following.

Theorem 2.3 (Abrishami, Alecu, Chudnovsky, Hajebi and Spirkl [2])**.** *Let $k, t \geq 1$ be integers. Then there exists an integer $w = w(k, t) \geq 1$ such that every t-clean graph with no strong k-block has treewidth at most w.*

Again, recall that for every $t \geq 3$, every subdivision of $W_{t\times t}$ contains a theta and the line graph of every subdivision of $W_{t\times t}$ contains a prism (see Figure 2). It follows that for every $t \geq 1$, every graph in $\mathcal{C}_t$ is t-clean, and so the following is immediate from Theorem 2.3:

Corollary 2.4. *For all integers $k, t \geq 1$, there exists an integer $\beta = \beta(k, t)$ such that every graph in $\mathcal{C}_t$ with no strong k-block has treewidth at most $\beta(k, t)$.*

A vertex v in a graph G is said to be a *branch vertex* if v has degree more than two. By a *caterpillar* we mean a tree C with maximum degree three such that there is a path P in C containing all branch vertices of C (our definition of a caterpillar is non-standard for two reasons: a caterpillar is often allowed to be of arbitrary maximum degree, and the path P from the definition often contains all vertices of degree more than one). By a *subdivided star* we mean a graph isomorphic to a subdivision of the complete bipartite graph $K_{1,\delta}$ for some $\delta \geq 3$. In other words, a subdivided star is a tree with exactly one branch vertex, which we call its *root*. For every graph H, a vertex v of H is said to be *simplicial* if $N_H(v)$ is a clique. We denote by $\mathcal{Z}(H)$ the set of all simplicial vertices of H. Note that for every tree T, $\mathcal{Z}(T)$ is the set of all leaves of T. An edge e of a tree T is said to be a *leaf-edge* of T if e is incident with a leaf of T. It follows that if H is the line graph of a tree T, then $\mathcal{Z}(H)$ is the set of all vertices in H corresponding to the leaf-edges of T. The following is proved in [2] based on (and refining) a result from [10].

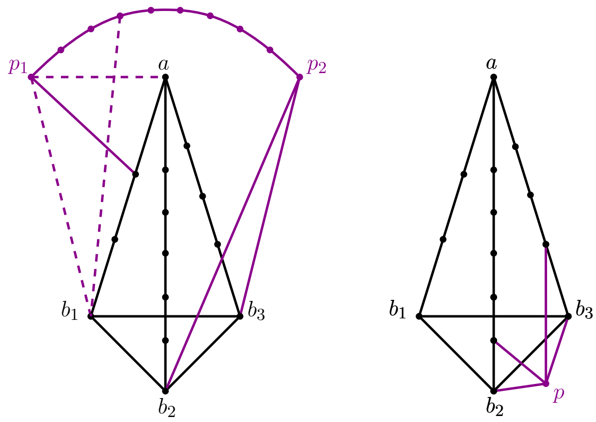

FIGURE 4. A corner at b_1 (left – dashed lines represent possible edges) and a jewel at b_1 (right).

Theorem 2.5 (Abrishami, Alecu, Chudnovsky, Hajebi and Spirkl [2]). *For every integer $h \geq 1$, there exists an integer $\mu = \mu(h) \geq 1$ with the following property. Let G be a connected graph with no clique of cardinality h and let $S \subseteq G$ such that $|S| \geq \mu$. Then either some path in G contains h vertices from S, or there is an induced subgraph H of G with $|H \cap S| = h$ for which one of the following holds.*

- *H is either a caterpillar or the line graph of a caterpillar with $H \cap S = \mathcal{Z}(H)$.*
- *H is a subdivided star with root r such that $\mathcal{Z}(H) \subseteq H \cap S \subseteq \mathcal{Z}(H) \cup \{r\}$.*

3. Jumps and jewels on pyramids with trapped apices

For a graph G, an induced subgraph H of G and a vertex $a \in H$, we say a is *trapped* in H if

- we have $N^2_G[a] \subseteq H$; and
- every vertex in $N_H(a) = N_G(a)$ has degree two in H (and so in G).

The goal of this section is, for a graph $G \in \mathcal{C}$, $H \subseteq G$ and a pyramid Σ in H, to investigate the adjacency between Σ and a path in $G \setminus H$, assuming that the apex of Σ is trapped in H. This will be of essential use in the next section.

We begin with a few definitions. Let G be a graph and let Σ be a pyramid in G with apex a, base $b_1b_2b_3$ and paths P_1, P_2, P_3. A set $X \subseteq \Sigma$ is said to be *local (in Σ)* if either $X \subseteq P_i$ for some $i \in \{1, 2, 3\}$ or $X \subseteq \{b_1, b_2, b_3\}$. Let P be a path in $G \setminus \Sigma$ with (not necessarily distinct) ends p_1, p_2. For $i \in \{1, 2, 3\}$, we say P is a *corner path for Σ at b_i* if

- p_1 has at least one neighbor in $P_i \setminus \{b_i\}$;
- p_2 is complete to $\{b_1, b_2, b_3\} \setminus \{b_i\}$; and
- except for the edges between $\{p_1, p_2\}$ and Σ described in the above two bullets, there is no edge with an end in P and an end in $\Sigma \setminus \{b_i\}$.

See Figure 4. By a *corner path for Σ* we mean a corner path for Σ at one of b_1, b_2 or b_3.

Let $p \in G \setminus \Sigma$. Then p is said to be *narrow for Σ* if $N_\Sigma(p)$ is local in Σ. Otherwise, we say p is *wide for Σ*. For $i \in \{1, 2, 3\}$, we say p is a *jewel for Σ at b_i* if p is anticomplete to P_i (in particular, p is anticomplete to a), and for every $j \in \{1, 2, 3\} \setminus \{i\}$, we have $N_{P_j}(p) = N_{P_j}[b_j]$ (see Figure 4). By a *jewel for Σ* we mean a jewel for Σ at one of b_1, b_2 or b_3. Note that if p is either a corner path or a jewel for Σ, then p is wide for Σ. The following lemma establishes a converse to this fact for graphs in $\mathcal{C}$ and pyramids with a trapped apex.

Lemma 3.1. *Let $G \in \mathcal{C}$ be a graph, let $H \subseteq G$, and let $a \in V(H)$ be trapped in H. Let Σ be a pyramid in H with apex a, base $b_1b_2b_3$, and paths P_1, P_2, P_3. Let $p \in G \setminus H$. Then p is wide for Σ if and only if p is either a corner path for Σ or a jewel for Σ.*

Proof. We only need to prove the "only if" direction. Assume that $p \in G \setminus H$ is wide for Σ and p is not a corner path for Σ. Since a is trapped in H, every vertex in $N_\Sigma(a)$ has degree two in G. It follows that Σ is long, and since $p \in G \setminus H$, it follows that p is anticomplete to $N_\Sigma[a]$. First, we show that:

(1) *There exists $i \in \{1,2,3\}$ for which p is anticomplete to P_i.*

Suppose for a contradiction that p has a neighbor in each of P_1, P_2, P_3. Since p is wide for Σ and p is not a corner path for Σ, we may assume without loss of generality that p has a neighbor in P_1^* and a neighbor in P_2^*. For each $i \in \{1,2,3\}$, traversing P_i from a to b_i, let x_i be the first neighbor of p in P_i. Since a is trapped, it follows that $x_1 \in P_1^*$, $x_2 \in P_2^*$ and $x_3 \in P_3 \setminus N_\Sigma[a]$ (in particular, $x_3 = b_3$ is possible). But then G contains a theta with ends a, p and paths a-P_i-x_i-p for $i \in \{1,2,3\}$, a contradiction. This proves (1).

By (1) and without loss of generality, we may assume that p is anticomplete to P_3. Note that since p is wide for Σ, it follows that for every $j \in \{1,2\}$, p has a neighbor in P_j, and there exists $j \in \{1,2\}$ for which p has a neighbor in P_j^*. For each $j \in \{1,2\}$, traversing P_j from a to b_j, let x_j and y_j be the first and the last neighbor of p in P_j, respectively. Then we have $x_j \in P_j^* \setminus N_{P_j}(a)$ for some $j \in \{1,2\}$. In fact, the following holds.

(2) *For every $j \in \{1,2\}$, we have $x_j \in P_j^* \setminus N_{P_j}(a)$.*

Suppose not. Since p is wide for Σ and p is anticomplete to $N_\Sigma(a)$, we may assume without loss of generality that p has a neighbor in P_1^* and $x_2 = y_2 = b_2$. But now G contains a theta with ends a, b_2 and paths a-P_1-x_1-p-b_2, a-P_2-b_2 and a-P_3-b_3-b_2, a contradiction. This proves (2).

(3) *For every $j \in \{1,2\}$, $N_{P_j}(p)$ is a clique of cardinality two.*

Suppose not. Then we may assume without loss of generality that either $x_1 = y_1$ or x_1 and y_1 are distinct and non-adjacent. By (2), for every $j \in \{1,2\}$, we have $x_j \in P_j^* \setminus N_{P_j}(a)$. Therefore, if $x_1 = y_1$, then G contains a theta with ends a, x_1 and paths a-P_1-x_1, a-P_2-x_2-p-x_1 and a-P_3-b_3-b_1-P_1-x_1, which is impossible. Thus, x_1 and y_1 are distinct and non-adjacent. But now G contains a theta with ends a, p and paths a-P_1-x_1-p, a-P_2-x_2-p and a-P_3-b_3-b_1-P_1-y_1-p, a contradiction. This proves (3).

The proof is almost concluded. By (3), for every $j \in \{1,2\}$, we have $N_{P_j}(p) = \{x_j, y_j\}$ and x_j is adjacent to y_j. If $y_j \in P_j^*$ for some $j \in \{1,2\}$, then G contains a prism with triangles $x_j y_j p$ and $b_1 b_2 b_3$ and paths x_j-P_j-a-P_3-b_3, y_j-P_j-b_j and p-y_{3-j}-P_{3-j}-b_{3-j}, a contradiction. Hence, we have $y_j = b_j$ for every $j \in \{1,2\}$, and so p is a jewel corner for Σ at b_i. This completes the proof of Lemma 3.1. ■

We can now prove the main result of this section.

Theorem 3.2. *Let $G \in \mathcal{C}$ be a graph, let $H \subseteq G$, and let $a \in V(H)$ be trapped in H. Let Σ be a pyramid in H with apex a, base $b_1b_2b_3$, and paths P_1, P_2, P_3. Let P be a path in $G \setminus H$. Then one of the following holds.*

- *$N_\Sigma(P)$ is local in Σ.*
- *P contains a corner path for Σ.*
- *P contains a jewel for Σ.*

Proof. Suppose for a contradiction that there exists a path P in $G \setminus H$ for which none of the outcomes of Theorem 3.2 hold. We choose such a path P with $|P|$ minimum. It follows that $N_\Sigma(P)$ is not local in Σ, $N_\Sigma(X)$ is local in Σ for every connected set $X \subsetneq P$, P contains no corner path for Σ and P contains no jewel for Σ. Therefore, by Lemma 3.1, we have $|P| > 1$. Since a is trapped in H, every vertex in $N_\Sigma(a)$ has degree two in G. It follows that Σ is long, and since $P \subseteq G \setminus H$, it follows that p is anticomplete to $N_\Sigma[a]$. For every $i \in \{1,2,3\}$, let $P_i' = P_i \setminus N_{P_i}[a]$. Let p_1 and p_2 be the ends of P (which are in fact distinct). Since $N_\Sigma(P)$ is not local and P is minimal subject to this property, we may assume without loss of generality that

- $N_\Sigma(p_1) \subseteq P'_1$ and p_1 has a neighbor in $P'_1 \setminus \{b_1\}$; and
- p_2 has a neighbor in P'_2, and either $N_\Sigma(p_2) \subseteq P'_2$ or $N_\Sigma(p_2) \subseteq \{b_1, b_2, b_3\}$.

It follows from the choice of P that P^* is anticomplete to $\Sigma \setminus \{b_1\}$. For each $i \in \{1, 2\}$, traversing P_i from a to b_i, let x_i and y_i be the first and the last neighbor of p_i in P_i, respectively. So we have $x_1 \in P'_1 \setminus \{b_1\}$, $y_1 \in P'_1$ and $x_2, y_2 \in P'_2$. In fact, the following holds.

(4) *We have $x_2 \in P'_2 \setminus \{b_2\}$.*

Suppose not. Then we have $x_2 = y_2 = b_2$, and so $b_2 \in N_\Sigma(p_2) \subseteq \{b_1, b_2, b_3\}$. Since P is not a corner path for Σ at b_1, it follows that p_2 is not adjacent to b_3. But now G contains a theta with ends a, b_2 and paths a-P_1-x_1-p_1-P-p_2-b_2, a-P_2-b_2 and a-P_3-b_3-b_2, a contradiction. This proves (4).

In view of (4) and the choice of P, we conclude that P^* is anticomplete to Σ, and for every $i \in \{1, 2\}$, we have $N_\Sigma(p_i) = N_{P'_i}(p_i)$, $x_i \in P'_i \setminus \{b_i\}$ and $y_i \in P'_i$.

(5) *For every $i \in \{1, 2\}$, x_i and y_i are distinct and adjacent.*

Suppose not. Then we may assume without loss of generality that either $x_1 = y_1$ or x_1 and y_1 are distinct and non-adjacent. In the former case, G contains a theta with ends a, x_1 and paths a-P_1-x_1, a-P_2-x_2-p_2-P-p_1-x_1 and a-P_3-b_3-b_1-P_1-x_1, a contradiction. It follows that x_1 and y_1 are distinct and non-adjacent. But then G contains a theta with ends a, p_1 and paths a-P_1-x_1-p_1, a-P_2-x_2-p_2-P-p_1 and a-P_3-b_3-b_1-P_1-y_1-p_1, again a contradiction. This proves (5).

By (5), for every $i \in \{1, 2\}$, we have $N_{P_i}(p) = \{x_i, y_i\}$ and x_i is adjacent to y_i. But now G contains a prism with triangles $p_1x_1y_1$ and $p_2x_2y_2$ and paths P, x_1-P_1-a-P_2-x_2 and y_1-P_1-b_1-b_2-P_2-y_2, a contradiction. This completes the proof of Theorem 3.2. ■

4. Strip structures with an ornament of jewels

The main result of this section, Theorem 4.2, provides a description of the structure of graphs in $\mathcal{C}$ which contain a pyramid with a trapped apex.

We first set up a framework that allows us to think of a pyramid with apex a as a special case of a construction similar to the line graph of a tree T, which we call a "(T, a)-strip-structure." We start with an induced subgraph W of G that admits an "optimal" (T, a)-strip-structure in G in a certain sense, and show that the rest of the graph fits into the same construction, except for vertices which are jewels for certain canonically positioned pyramids in W.

First, we need to properly define a strip-structure (this is similar to [7], [8], and [9]). A tree T is said to be *smooth* if T has at least three vertices and every vertex of T is either a branch vertex or a leaf. Let G be a graph, let $a \in G$, let T be a smooth tree, and let $\eta : V(T) \cup E(T) \cup (E(T) \times V(T)) \to 2^{G \setminus \{a\}}$ be a function. For every $S \subseteq V(T)$, we define $\eta(S) = \bigcup_{v \in S, e \in E(T[S])}(\eta(v) \cup \eta(e))$ and $\eta^+(S) = \eta(S) \cup \{a\}$. For every vertex $v \in V(T)$, we define $B_\eta(v)$ to be the union of all sets $\eta(e, v)$ taken over all edges $e \in E(T)$ incident with v (we often omit the subscript η unless there is ambiguity).

The function η is said to be a *(T, a)-strip-structure in G* if the following conditions are satisfied.

(S1) For all distinct $o, o' \in V(T) \cup E(T)$, we have $\eta(o) \cap \eta(o') = \emptyset$.
(S2) If $l \in V(T)$ is a leaf of T, then $\eta(l)$ is empty.
(S3) For all $e \in E(T)$ and $v \in V(T)$, we have $\eta(e, v) \subseteq \eta(e)$, and $\eta(e, v) \neq \emptyset$ if and only if e is incident with v.
(S4) For all distinct edges $e, f \in E(T)$ and every vertex $v \in V(T)$, $\eta(e, v)$ is complete to $\eta(f, v)$, and there are no other edges between $\eta(e)$ and $\eta(f)$. In particular, if e and f share no end, then $\eta(e)$ is anticomplete to $\eta(f)$.

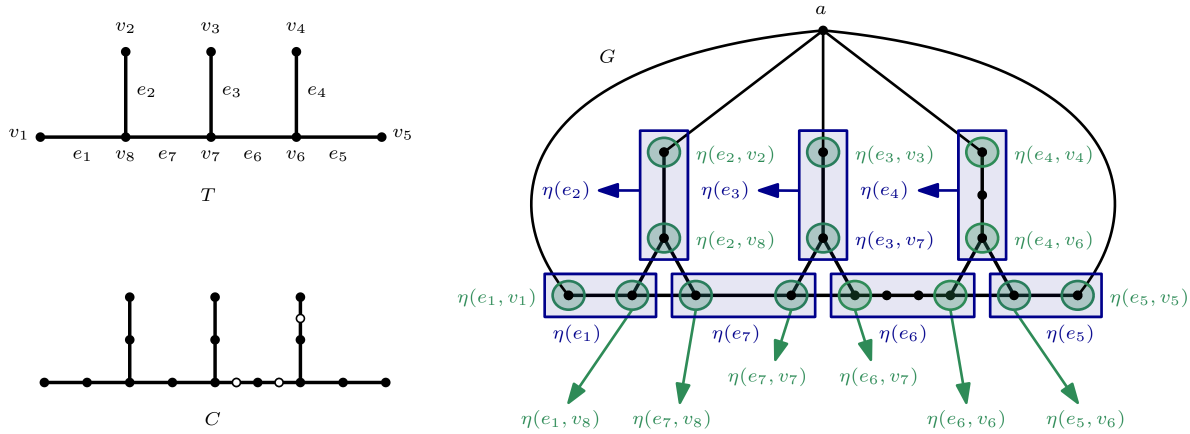

Figure 5. A smooth tree T (top left – note that T is a caterpillar), a proper subdivision C of T (bottom left – note that C is also a caterpillar), and a graph G admitting a (T, a)-strip-structure (right – note that $G[\eta(T)]$ is isomorphic to the line graph of C).

(S5) For every $e \in E(T)$ with ends u, v, define $\eta^\circ(e) = \eta(e) \setminus (\eta(e, u) \cup \eta(e, v))$. Then for every vertex $x \in \eta(e)$, either
- we have $x \in \eta(e, u) \cap \eta(e, v)$; or
- there is a path in $\eta(e)$ from x to a vertex in $\eta(e, u) \setminus \eta(e, v)$ with interior contained in $\eta^\circ(e)$, and there is a path in $\eta(e)$ from x to a vertex in $\eta(e, v) \setminus \eta(e, u)$ with interior contained in $\eta^\circ(e)$.

(S6) For all $v \in V(T)$ and $e \in E(T)$, $\eta(v)$ is anticomplete to $\eta(e) \setminus \eta(e, v)$. In other words, we have $N_{\eta(T)}(\eta(v)) \subseteq B_\eta(v)$.

(S7) For every $v \in V(T)$ and every connected component D of $\eta(v)$, we have $N_{B_\eta(v)}(D) \neq \emptyset$.

(S8) For every leaf $l \in V(T)$ of T, assuming $e \in E(T)$ to be the leaf-edge of T incident with l, a is complete to $\eta(e, l)$. Also, a has no other neighbors in $\eta(T)$.

See Figure 5. Let $S \subseteq \eta(T)$. We say that S is *local in* η if $S \subseteq \eta(e)$ for some $e \in E(T)$ or $S \subseteq B_\eta(v) \cup \eta(v)$ for some $v \in V(T)$. The following lemma shows that every non-local set contains a non-local 2-subset.

Lemma 4.1. *Let G be a graph and $a \in V(G)$. Let T be a smooth tree and η be a (T, a)-strip-structure in G. Assume that $C \subseteq \eta(T)$ is not local in η. Then there is a 2-subset of C which is not local in η.*

Proof. Suppose not. Then we claim that:

(6) $C \subseteq \bigcup_{v \in V(T)} (B(v) \cup \eta(v))$.

For suppose there exists a vertex $x \in C \cap \eta^\circ(e)$ for some $e \in E(T)$. Then by (S1), we have $x \notin \eta(f)$ for all $f \in E(T) \setminus \{e\}$, and also we have $x \notin B(v) \cup \eta(v)$ for every $v \in V(T)$. Thus, since C is not local, it follows that there exists a vertex $y \in C \setminus \eta(e)$. But then $\{x, y\}$ is a 2-subset of C which is not local in η, a contradiction. This proves (6).

By (6), and since the empty set is local in η, we may pick $x \in C$, $v \in V(T)$ and $e = uv \in E(T)$ such that $x \in \eta(e, v) \cup \eta(v)$. Now we deduce:

(7) $C \subseteq B(u) \cup \eta(u) \cup B(v) \cup \eta(v)$.

Suppose for a contradiction that there exists a vertex $y \in C \setminus (B(u) \cup \eta(u) \cup B(v) \cup \eta(v))$. By (6), we have $C \setminus (B(u) \cup \eta(u) \cup B(v) \cup \eta(v)) = C \setminus (\eta(e) \cup B(u) \cup \eta(u) \cup B(v) \cup \eta(v))$, and so $y \in C \setminus (\eta(e) \cup B(u) \cup \eta(u) \cup B(v) \cup \eta(v))$. But then $\{x, y\}$ is a 2-subset of C which is not local in η, a contradiction. This proves (7).

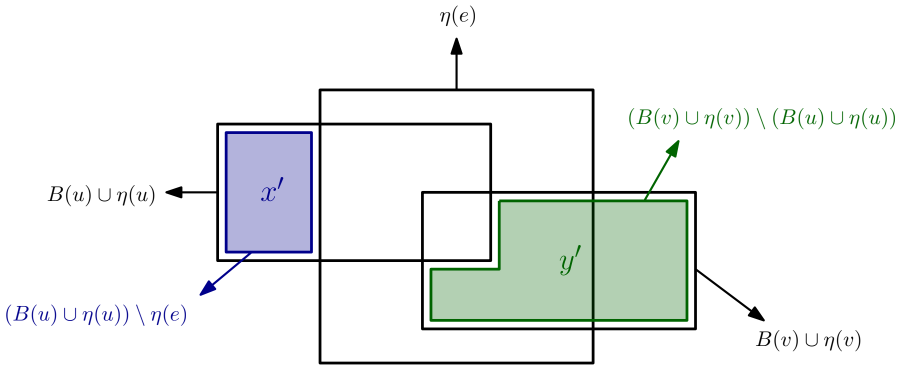

FIGURE 6. Proof of Lemma 4.1.

Now, since C is not local, it follows that $C \not\subseteq \eta(e)$, and by (7) and without loss of generality, we may assume that there exists a vertex $x' \in (B(u) \cup \eta(u)) \setminus \eta(e) \subseteq (B(u) \cup \eta(u)) \setminus (B(v) \cup \eta(v))$ such that $\{x', y'\} \subseteq C$. Similarly, since C is not local, it follows that $C \not\subseteq B(u) \cup \eta(u)$, and so by (7), there exists a vertex $y' \in (B(v) \cup \eta(v)) \setminus (B(u) \cup \eta(v))$. But then $\{x', y'\}$ is a 2-subset of C which is not local in η, a contradiction (see Figure 6). This completes the proof of Lemma 4.1. ■

In order to state and prove the main result of this section, we need to define several notions related to strip-structures. From here until the statement of Theorem 4.2, let us fix a graph G, a vertex $a \in G$, a smooth tree T, and a (T, a)-strip-structure η in G.

For every edge $e \in E(T)$ with ends u, v, by an $\eta(e)$-*rung*, we mean a path P in $\eta(e) \subseteq \eta(T)$ for which either $|P| = 1$ and $P \subseteq \eta(e, u) \cap \eta(e, v)$, or P has an end in $\eta(e, u) \setminus \eta(e, v)$, an end in $\eta(e, v) \setminus \eta(e, u)$, and $P^* \subseteq \eta^\circ(e)$. Equivalently, a path P in $\eta(e)$ is an $\eta(e)$-rung if P has an end in $\eta(e, u)$, an end in $\eta(e, v)$, and $|P \cap \eta(e, u)| = |P \cap \eta(e, v)| = 1$. It follows from (S5) that every vertex in $\eta(e) \setminus \eta^\circ(e)$ is contained in an $\eta(e)$-rung. In particular, if $\eta(e, u) \subseteq \eta(e, v)$, then we have $\eta(e, u) = \eta(e, v)$ (for otherwise each vertex $\eta(e, v) \setminus \eta(e, u)$ fails to satisfy both bullet conditions of (S5)). Similarly, if $\eta(e, v) \subseteq \eta(e, u)$, then we have $\eta(e, u) = \eta(e, v)$. An $\eta(e)$-rung is said to be *long* if it is of non-zero length.

For every edge $e \in E(T)$, let $\tilde{\eta}(e)$ be the set of vertices in $\eta(e)$ that are not in any $\eta(e)$-rung (so $\tilde{\eta}(e) \subseteq \eta^\circ(e)$.) We say that η is *tame* if

- $\eta(v) = \emptyset$ for every $v \in V(T)$; and
- $\tilde{\eta}(e) = \emptyset$ for every $e \in E(T)$.

In other words, η is tame if and only if every vertex in $\eta(T)$ is in an $\eta(e)$-rung for some $e \in E(T)$.

For a (T, a)-strip-structure η' in G, we write $\eta \leq \eta'$ to mean that for every $o \in V(T) \cup E(T) \cup (E(T) \times V(T))$, we have $\eta(o) \subseteq \eta'(o)$. We say that a (T, a)-strip-structure η is *substantial* if for every $e \in E(T)$, there exists a long $\eta(e)$-rung in G. Equivalently, η is substantial if for every edge $e \in E(T)$ with ends u, v, we have $\eta(e, u) \neq \eta(e, v)$, and so $\eta(e, u) \setminus \eta(e, v), \eta(e, v) \setminus \eta(e, u) \neq \emptyset$. One may observe (using (S4) and the smoothness of T, in particular) that if η is substantial and $\eta \leq \eta'$, then η' is substantial too.

We say η is *rich* if

- a is trapped in $\eta^+(T)$; and
- for every leaf $l \in V(T)$ of T, assuming $e \in E(T)$ to be the leaf-edge of T incident with l, we have $|\eta(e, l)| = 1$.

In particular, T has exactly $|N_G(a)|$ leaves. Also, for every edge $e = lv \in E(T)$ where l is a leaf, since T is smooth, it follows that v has degree more than two in T, which in turn implies that $\eta(e, v) \cap \eta(e, l) = \emptyset$ (for otherwise we have $\eta(e, l) \subseteq \eta(e, v)$, and the single neighbor of a in $\eta(e, l)$ violates the assumption that a is trapped in $\eta^+(T)$).

By a *seagull in T* we mean a triple (v, e_1, e_2) where $v \in V(T)$ and e_1, e_2 are two distinct edges of T incident with v. By a *claw in T* we mean a 4-tuple (v, e_1, e_2, e_3) where $v \in V(T)$ and e_1, e_2, e_3 are three distinct edges of T incident with v.

Let (v, e_1, e_2, e_3) be a claw in T. By an *η-pyramid at (v, e_1, e_2, e_3)*, we mean a pyramid Σ with apex a, base $b_1b_2b_3$ and paths P_1, P_2, P_3, satisfying the following for each $i \in \{1, 2, 3\}$.

- $b_i \in \eta(e_i, v)$.
- There exists a leaf l_i of T with the following properties:
  - l_i belongs to the component of $T \setminus \{e_i\}$ not containing v.
  - Let Λ_i be the unique path in T from v to l_i (so $e_i \in E(\Lambda_i)$). Then $P_i = \Gamma_i \cup \{a\}$, where Γ_i is a path in $\bigcup_{e \in E(\Lambda_i)} \eta(e)$ such that $R_i = \Gamma_i \cap \eta(e_i)$ is a long $\eta(e_i)$-rung and $\Gamma_i \cap \eta(e)$ is a $\eta(e)$-rung for each $e \in E(\Lambda_i) \setminus \{e_i\}$.

In particular, assuming u_i to be the ends of e_i distinct from v and c_i to be the unique vertex in $N_{R_i}(b_i) = N_{P_i}(b_i)$ for each $i \in \{1, 2, 3\}$, we have $b_i \in \eta(e_i, v) \setminus \eta(e_i, u_i)$ and $c_i \in \eta(e_i) \setminus \eta(e_i, v)$.

For a branch vertex $v \in V(T)$, by an *η-pyramid at v* we mean an η-pyramid at (v, e_1, e_2, e_3) for some claw (v, e_1, e_2, e_3) in T. Also, by an *η-pyramid* we mean an η-pyramid at v for some branch vertex $v \in V(T)$. It follows that every η-pyramid is a long pyramid (recall that a pyramid is long if all its paths have lengths more than one). Also, if η is substantial, then for every claw (v, e_1, e_2, e_3) in T there is a η-pyramid at (v, e_1, e_2, e_3).

Let (v, e_1, e_2) be a seagull in T. A vertex $p \in G \setminus \eta^+(T)$ is said to be a *jewel for η at (v, e_1, e_2)* if for some edge $e_3 \in E(T) \setminus \{e_1, e_2\}$ incident with v, there exists an η-pyramid Σ at (v, e_1, e_2, e_3) with base $b_1b_2b_3$ where $b_i \in \eta(e_i, v)$ for each $i \in \{1, 2, 3\}$, such that p is a jewel for Σ at b_3. In particular, for each $i \in \{1, 2\}$, p is adjacent to b_i and the unique vertex c_i in $N_{P_i}(b_i)$. Therefore, since Σ is an η-pyramid at (v, e_1, e_2, e_3), assuming u_i to be the end of e_i distinct from v, it follows that p has a neighbor $b_i \in \eta(e_i, v) \setminus \eta(e_i, u_i)$ and a neighbor $c_i \in \eta(e_i) \setminus \eta(e_i, v)$.

For a vertex $v \in V(T)$, by a *jewel for η at v* we mean a jewel for η at (v, e_1, e_2) for some seagull (v, e_1, e_2) in T. Also, by a *jewel for η* we mean a jewel for η at v for some branch vertex $v \in V(T)$. We denote by $\mathcal{J}_\eta$ the set of all jewels for η. It follows that $\mathcal{J}_\eta \subseteq G \setminus \eta^+(T)$.

We are now in a position to prove the main result of this section:

Theorem 4.2. *Let $G \in \mathcal{C}$, let $a \in V(G)$ and let T be a smooth tree. Suppose that there exists a tame, substantial, and rich (T, a)-strip-structure in G. Then there is a substantial and rich (T, a)-strip-structure ζ in G such that $\eta \leq \zeta$ and $G \setminus (\zeta^+(T) \cup \mathcal{J}_\zeta)$ is anticomplete to $\zeta^+(T)$.*

Proof. Let η be a tame, substantial, and rich (T, a)-strip-structure in G such that $\eta(T)$ is maximal with respect to inclusion. Let $M = G \setminus (\eta^+(T) \cup \mathcal{J}_\eta)$.

(8) *Let P be a path in M with ends p_1 and p_2 such that there exist $x_1 \in N_{\eta(T)}(p_1)$ and $x_2 \in N_{\eta(T)}(p_2)$ for which $\{x_1, x_2\}$ is not local in η, and such that $|P| \geq 1$ is minimum subject to this property. Then there exists $\{j_1, j_2\} = \{1, 2\}$ and $f = v_1v_2 \in E(T)$ such that $x_{j_1} \in B(v_{j_1}) \setminus \eta(f)$ and $x_{j_2} \in (B(v_{j_2}) \cup \eta(f)) \setminus B(v_{j_1})$.*

Suppose not. For each $i \in \{1, 2\}$, let $e_i \in E(T)$ be such that $x_i \in \eta(e_i)$ (hence $e_1 \neq e_2$) and let s_i be an end of e_i such that there exists a path Λ_0 (possibly of length zero) from s_1 to s_2 in $T \setminus \{e_1, e_2\}$. We claim that there is a vertex $v \in \Lambda_0$ such that $B(v) \cap \{x_1, x_2\} = \emptyset$. Suppose first that $s_1 \neq s_2$. Let v_1 be the unique neighbor of s_1 in Λ_0. Then we have $x_1 \notin B(v_1)$ and $x_2 \notin B(s_1)$. Also, since $f = s_1v_1$ does not satisfy (8), we have either $x_1 \notin B(s_1)$ or $x_2 \notin B(v_1)$. But then either $v = s_1$ or $v = v_1$ satisfies the claim. Thus, we may assume that $v = s_1 = s_2$. Note that since neither e_1 nor e_2 satisfies (8), we have $x_1 \notin B(s_1)$ and $x_2 \notin B(s_2)$. In other words, we have $B(v) \cap \{x_1, x_2\} = \emptyset$, and the claim follows. Henceforth, let v be as promised by the above claim. For each $i \in \{1, 2\}$, let u_i be the end of e_i distinct from s_i (hence $u_1 \neq u_2$). Let $\Lambda = u_1\text{-}s_1\text{-}\Lambda_0\text{-}s_2\text{-}u_2$ and let u'_1, u'_2 be the neighbors of v in Λ such that Λ traverses u_1, u'_1, v, u'_2, u_2 in this order (so possibly $u_1 = u'_1$ or $u_2 = u'_2$). Let $e'_i = u'_iv$ for each $i \in \{1, 2\}$. Since T is smooth, there exists a vertex $u'_3 \in N_T(v) \setminus \Lambda$. Let $e'_3 = u'_3v$. For each $i \in \{1, 2, 3\}$, let T_i be the component of $T \setminus (N_T(v) \setminus \{u'_i\})$ containing v (so $e'_i \in E(T_i)$). Then since $B(v) \cap \{x_1, x_2\} = \emptyset$

and since η is tame and substantial, there exists an η-pyramid Σ at (v, e_1', e_2', e_3') with apex a, base $b_1b_2b_3$ and paths P_1, P_2, P_3 such that we have

- $b_i \in \eta(e_i', v)$ and $P_i \setminus \{a, b_i\} \subseteq \eta(T_i) \setminus B(v)$ for each $i \in \{1, 2, 3\}$; and
- $x_i \in P_i^*$ for each $i \in \{1, 2\}$.

In particular, the second bullet above implies that $N_\Sigma(P)$ is not local in Σ and P is not a corner path for Σ. Since $P \subseteq M$, we have $P \cap \mathcal{J}_\eta = \emptyset$. Thus, since Σ is an η-pyramid, it follows that P contains no jewel for Σ. Also, since η is rich, a is trapped in $\eta^+(T)$. Therefore, applying Theorem 3.2 to G, $H = \eta^+(T)$, a, Σ and P, we deduce that P contains a corner path for Σ. On the other hand, note that by the second bullet above, for every vertex $x \in \Sigma \setminus \{a\}$, either $\{x, x_1\}$ or $\{x, x_2\}$ is not local in η. From this, the minimality of $|P|$ and the fact that η is rich, it follows that P^* is anticomplete to Σ. But then P is a corner path for Σ, a contradiction. This proves (8).

(9) *Let P be a path in M with ends p_1 and p_2 such that there exist $x_1 \in N_{\eta(T)}(p_1)$ and $x_2 \in N_{\eta(T)}(p_2)$ for which $\{x_1, x_2\}$ is not local in η, and such that $|P| \geq 1$ is minimum subject to this property. Let $f = v_1v_2 \in E(T)$ and $\{j_1, j_2\} = \{1, 2\}$ be as guaranteed by (8) applied to P, x_1 and x_2. Then we have $N_{\eta(T)}(P^*) \subseteq \eta(f, v_{j_1})$ and $N_{\eta(T)}(\{p_1, p_2\}) \subseteq \eta(f) \cup B(v_1) \cup B(v_2)$.*

Suppose not. Without loss of generality, we may assume that $j_1 = 1$ and $j_2 = 2$. Note that by the minimality of $|P|$, we have $N_{\eta(T)}(P^*) \subseteq \eta(f, v_1)$. Therefore, one of p_1 and p_2 has a neighbor in $\eta(T) \setminus (\eta(f) \cup B(v_1) \cup B(v_2))$; say p_1 is adjacent to $x_1' \in \eta(T) \setminus (\eta(f) \cup B(v_1) \cup B(v_2))$. For each $i \in \{1, 2\}$, let T_i be the component of $T \setminus \{f\}$ containing v_i. It follows that there exists $j \in \{1, 2\}$ such that $x_1' \in \eta(T_j) \setminus B(v_j)$. Assume that $|P| > 1$. By the minimality of $|P|$, we have $j = 1$. But then P, x_1' and x_2 violate (8). We deduce that $|P| = 1$. But now P, x_1' and x_{3-j} violate (8). This proves (9).

(10) *Let P be a path in M with ends p_1 and p_2 such that there exist $x_1 \in N_{\eta(T)}(p_1)$ and $x_2 \in N_{\eta(T)}(p_2)$ for which $\{x_1, x_2\}$ is not local in η, and such that $|P| \geq 1$ is minimum subject to this property. Suppose that there exist $\{k_1, k_2\} = \{1, 2\}$, $f = v_1v_2 \in E(T)$ and $e_1 \in E(T) \setminus \{f\}$ incident with v_{k_1} such that p_{k_1} has a neighbor in $\eta(e_1, v_{k_1})$ and p_{k_2} has a neighbor in $(B(v_{k_2}) \cup \eta(f)) \setminus B(v_{k_1})$. Then p_{k_1} is complete to $B(v_{k_1}) \setminus (\eta(e_1, v_{k_1}) \cup \eta(f))$.*

Due to symmetry, we may assume that $k_1 = 1$ and $k_2 = 2$. Let $e_3 \in E(T) \setminus \{e_1, f\}$ be incident with v_1 and let $b_3 \in \eta(e_3, v_1)$ be arbitrary. We need to show that p_1 is adjacent to b_3. Suppose for a contradiction that p_1 and b_3 are non-adjacent. Let $b_1 \in \eta(e_1, v_1)$ be adjacent to p_1 and let $x \in (B(v_2) \cup \eta(f)) \setminus B(v_1)$ be adjacent to p_2. Let T_2 be the component of $T \setminus (N_T(v_1) \setminus \{v_2\})$ containing v_1 (so $f \in E(T_2)$). Also, for each $i \in \{1, 3\}$, let u_i be the end of e_i distinct from v_1 and let T_i be the component of $T \setminus (N_T(v_1) \setminus \{u_i\})$ containing v_1 (so $e_i \in E(T_i)$). By (8) and (9), there exists an edge $f' = v_1'v_2' \in E(T)$ such that $N_{\eta(T)}(\{p_1, p_2\}) \subseteq \eta(f') \cup B(v_1') \cup B(v_2')$. This, along with the minimality of $|P|$, implies that p_1 is anticomplete to $(\eta(T_1) \cup \eta(T_3)) \setminus B(v_1)$, $P \setminus \{p_1\}$ is anticomplete to $\eta(T_1) \cup \eta(T_3)$, and $P \setminus \{p_2\}$ is anticomplete to $\eta(T_2) \setminus B(v_1)$. Since p_2 has a neighbor $x \in (B(v_2) \cup \eta(f)) \setminus B(v_1)$ and since η is tame, there exists a path P_2 in G from a to p_2 with $P_2^* \subseteq \eta(T_2) \setminus B(v_1)$. Also, for each $i \in \{1, 3\}$, there exists a path P_i in G from a to b_i with $P_i^* \subseteq \eta(T_i) \setminus B(v_1)$. Note that since η is rich, it follows that P is anticomplete to $N_G[a]$; in particular, P_1 has length at least two. But now G contains a theta with ends a and b_1 and paths P_1, a-P_2-p_2-P-p_1-b_1 and a-P_3-b_3-b_1, a contradiction. This proves (10).

The following is immediate from (10) and the fact that T is smooth.

(11) *Let P be a path in M with ends p_1 and p_2 such that there exists $x_1 \in N_{\eta(T)}(p_1)$ and $x_2 \in N_{\eta(T)}(p_2)$ for which $\{x_1, x_2\}$ is not local in η, and such that $|P| \geq 1$ is minimum subject to this property. Suppose that there exist $\{k_1, k_2\} = \{1, 2\}$ and $f = v_1v_2 \in E(T)$ such that $x_{k_1} \in B(v_{k_1}) \setminus (\eta(f))$ and $x_{k_2} \in (B(v_{k_2}) \cup \eta(f)) \setminus B(v_{k_1})$. Then p_{k_1} is complete to $B(v_{k_1}) \setminus \eta(f)$.*

We now deduce:

(12) *Let D be a component of M. Then $N_{\eta(T)}(D)$ is local in η.*

Suppose not. By Lemma 4.1, there exist $x_1, x_2 \in N_{\eta(T)}(D)$ such that $\{x_1, x_2\}$ is not local in η. For each $i \in \{1, 2\}$, let p_i be a neighbor of x_i in D. Since D is connected, there exists a path P in $D \subseteq M$ from p_1 to p_2. In other words, there exists a path P in M with ends p_1, p_2 along with $x_1 \in N_{\eta(T)}(p_1)$ and $x_2 \in N_{\eta(T)}(p_2)$ such that $\{x_1, x_2\}$ is not local in η. Now, let P be a path in M with ends p_1 and p_2 such that there exists $x_1 \in N_{\eta(T)}(p_1)$ and $x_2 \in N_{\eta(T)}(p_2)$ for which $\{x_1, x_2\}$ is not local in η, and such that $|P| \geq 1$ is minimum subject to this property. So we can apply (8) to P, x_1 and x_2. Let $\{j_1, j_2\} = \{1, 2\}$ and $f = v_1v_2 \in E(T)$ be as in (8). We may assume without loss of generality that $j_1 = 1$ and $j_2 = 2$; thus, v_1 is a branch vertex of T. It follows from (9) that $N_{\eta(T)}(P^*) \subseteq \eta(f, v_1)$ and $N_{\eta(T)}(\{p_1, p_2\}) \subseteq \eta(f) \cup B(v_1) \cup B(v_2)$. By (11) applied to $k_1 = 1$ and $k_2 = 2$, p_1 is complete to $B(v_1) \setminus \eta(f)$. Also, from (11) applied to $k_1 = 2$ and $k_2 = 1$, it follows that either p_2 is complete to $B(v_2) \setminus \eta(f)$ and $B(v_2) \setminus \eta(f) \neq \emptyset$, or p_2 is anticomplete to $B(v_2) \setminus \eta(f)$. Note that if $|P| > 1$, then by the minimality of $|P|$, we have $N_{\eta(T)}(p_1) \subseteq B(v_1)$ and $N_{\eta(T)}(p_2) \subseteq (B(v_2) \cup \eta(f)) \setminus B(v_1)$. Let us define $\eta' : V(T) \cup E(T) \cup (E(T) \times V(T)) \subseteq 2^{G \setminus \{a\}}$ as follows. Let $\eta'(f) = \eta(f) \cup P$ and let $\eta'(f, v_1) = \eta(f, v_1) \cup \{p_1\}$. Let

- $\eta'(f, v_2) = \eta(f, v_2) \cup \{p_2\}$ if p_2 is complete to $B(v_2) \setminus \eta(f)$ and $B(v_2) \setminus \eta(f) \neq \emptyset$; and
- $\eta'(f, v_2) = \eta(f, v_2)$ if p_2 is anticomplete to $B(v_2) \setminus \eta(f)$.

Let $\eta' = \eta$ elsewhere on $V(T) \cup E(T) \cup (E(T) \times V(T))$. Then since η is tame, substantial and rich, and p_2 is adjacent to $x_2 \in B(v_2) \cup \eta(f)) \setminus B(v_1)$, it is straightforward to check that η' is also a tame, substantial and rich (T, a)-strip-structure. But we have $\eta'(T) = \eta(T) \cup P$, a contradiction with the maximality of $\eta(T)$. This proves (12).

The proof is almost concluded. Let X be the union of all the components D of M such that D is anticomplete to $\eta^+(T)$. Since η is rich, it follows that a is anticomplete to $M \setminus X$, as well. Thus, for every component D of $M \setminus X$, $N_{\eta^+(T)}(D) = N_{\eta(T)}(D)$ is non-empty. By (12), for every component D of $M \setminus X$, $N_{\eta(T)}(D)$ is local in η. Let $\mathcal{D}$ be the set of all components D of $M \setminus X$ for which we have $N_{\eta^+(T)}(D) \subseteq B_\eta(v)$ for some $v \in V(T)$. Breaking the ties arbitrarily and by the definition of X, we may write $\mathcal{D} = \bigcup_{v \in V(T)} \mathcal{D}_v$, where

- for all distinct $u, v \in V(T)$, we have $\mathcal{D}_u \cap \mathcal{D}_v = \emptyset$; and
- for all $v \in V(T)$ and every $D \in \mathcal{D}_v$, we have $N_{\eta^+(T)}(D) \subseteq B_\eta(v)$ and $N_{\eta^+(T)}(D) \neq \emptyset$.

Also, for every $e = uv \in E(T)$, let $\mathcal{D}_e$ be the set of all components D of $M \setminus X$ for which we have $N_{\eta^+(T)}(D) \subseteq \eta(e)$ and

- either $N_{\eta(T)}(D) \cap \eta^\circ(e) \neq \emptyset$, or;
- $N_{\eta(T)}(D) \cap (\eta(e, u) \setminus \eta(e, v)) \neq \emptyset$ and $N_{\eta(T)}(D) \cap (\eta(e, v) \setminus \eta(e, v)) \neq \emptyset$.

From the definition of X, it follows that every component of $M \setminus X$ belongs to exactly one of the sets $\{\mathcal{D}_v, \mathcal{D}_e : v \in V(T), e \in E(T)\}$ (note that since η is rich, a is anticomplete to each such component).

Let $\zeta : V(T) \cup E(T) \cup (E(T) \times V(T)) \subseteq 2^{G \setminus \{a\}}$ be defined as follows. For all $v \in V(T)$ and $e \in E(T)$, let

- $\zeta(v) = \bigcup_{D \in \mathcal{D}_v} D$;
- $\zeta(e) = \eta(e) \cup (\bigcup_{D \in \mathcal{D}_e} D)$; and
- $\zeta(e, v) = \eta(e, v)$.

It is easily seen that ζ satisfies the conditions (S1-S8) from the definition of a (T, a)-strip-structure. In particular, since η is rich, ζ satisfies (S2), and from the definitions of X, $\mathcal{D}_v$ and $\mathcal{D}_e$, it follows that ζ satisfies (S5) and (S7). Also, it is readily observed that $\eta \leq \zeta$.

Now, since η is substantial and rich, since $\eta \leq \zeta$, and from the definitions of X and ζ, it follows that ζ is a substantial and rich (T, a)-strip-structure with $\mathcal{J}_\zeta = \mathcal{J}_\eta$. Moreover, note that we have $\zeta^+(T) = \eta(T)^+ \cup (M \setminus X)$, and so $G \setminus (\zeta^+(T) \cup \mathcal{J}_\zeta) = G \setminus (\zeta^+(T) \cup \mathcal{J}_\eta) = X$ is anticomplete to $\zeta^+(T)$. This completes the proof of Theorem 4.2. ■

5. Jewels under the loupe

Here we revisit jewels for strip-structures, establishing several results about their properties in various settings. This will help attune Theorem 4.2 for its application in the proof of Theorem 6.1.

First we need to introduce some notations. Let G be a graph and let $a \in G$. Let T be a smooth tree and let ζ be a (T, a)-strip-structure in G. Let $v \in V(T)$ and let $e \in E(T)$ be incident with v. We denote by $\zeta_e(v)$ the set of all components D of $\zeta(v)$ for which we have $N_{B(v)}(D) \subseteq \zeta(e, v)$, or equivalently, $N_{\zeta(T)\setminus\zeta(e,v)}(D) = \emptyset$.

Let (v, e_1, e_2) be a seagull in T and let u_i be the end of e_i distinct from v for each $i \in \{1, 2\}$. We define

$$\zeta(v, e_1, e_2) = \zeta(e_1) \cup \zeta(e_2) \cup \zeta_{e_1}(u_1) \cup \zeta_{e_2}(u_2) \cup \zeta(v).$$

We denote by $\mathcal{J}_{\zeta,(v,e_1,e_2)}$ the set of all jewels for ζ at (v, e_1, e_2), and for every vertex $v \in V(T)$, $\mathcal{J}_{\zeta,v}$ stands for the set of all jewels for ζ at v. It follows that $\mathcal{J}_{\zeta,v} = \emptyset$ if v is a leaf of T.

The first result in this section describes, for a (T, a)-strip-structure in a theta-free graph, the attachments of jewels at a vertex of T.

Theorem 5.1. *Let G be a theta-free graph and let $a \in V(G)$. Let T be a smooth tree and let ζ be a (T, a)-strip-structure in G. Let (v, e_1, e_2) be a seagull in T and let $x \in \mathcal{J}_{\zeta,(v,e_1,e_2)}$. Then the following hold.*

- *We have $N_{\zeta^+(T)}(x) \subseteq \zeta(v, e_1, e_2)$, and so $N_{\zeta^+(T)}(\mathcal{J}_{\zeta,(v,e_1,e_2)}) \subseteq \zeta(v, e_1, e_2)$. Consequently, for every vertex $v \in V(T)$, we have $N_{\zeta^+(T)}(\mathcal{J}_{\zeta,v}) \subseteq \zeta(N_T[v])$, and for every two distinct vertices $v, v' \in V(T)$, we have $\mathcal{J}_{\zeta,v} \cap \mathcal{J}_{\zeta,v'} = \emptyset$.*
- *Assume that ζ is rich. Let $i \in \{1, 2\}$ and let R be a long $\zeta(e_i)$-rung, let r be the end of R in $\zeta(e_i, v)$ and let r' be the unique neighbor of r in R. Then either x is anticomplete to R or $N_R(x) = \{r, r'\}$.*

Proof. Note that v is a branch vertex of T. For each $i \in \{1, 2\}$, let u_i be the end of e_i distinct from v and let T_i be the component of $T \setminus (N_T(v) \setminus \{u_i\})$ containing v. Let T' be the component of $T \setminus \{u_1, u_2\}$ containing v. Let $x \in \mathcal{J}_{\zeta,(v,e_1,e_2)}$. Since $x \in \mathcal{J}_{\zeta,(v,e_1,e_2)}$ is a jewel for ζ, there exist an edge $e_3 \in E(T) \setminus \{e_1, e_2\}$ incident with v and a ζ-pyramid Σ at (v, e_1, e_2, e_3) with apex a, base $b_1b_2b_3$, and paths P_1, P_2, P_3 such that x is a jewel for Σ at b_3. In particular, for each $j \in \{1, 2, 3\}$, $P_j \cap \zeta(e_j)$ is a long $\zeta(e_j)$-rung R_j with b_j as its end in $\zeta(e_j, v)$. Also, x is anticomplete to P_3 (and so x is not adjacent to a), and for each $j \in \{1, 2\}$, assuming c_j to be the unique vertex in $N_{R_j}(b_j) = N_{P_j}(b_j)$, x is adjacent to $b_j \in \zeta(e_j, v) \setminus \zeta(e_j, u_j)$ and $c_j \in \zeta(e_j) \setminus \zeta(e_j, v)$. Therefore, there exist paths Q_i, S_i of length more than one in G from a to x for which we have $b_i \in Q_i^* \subseteq (\zeta(T') \setminus \zeta(v)) \cup (\zeta(e_i, v) \setminus \zeta(e_i, u_i))$ and $c_i \in S_i^* \subseteq \zeta(T_i) \setminus (B(v) \cup \zeta(u_i) \cup \zeta(v))$.

To prove the first assertion of Theorem 5.1, assume for a contradiction that x has a neighbor $y \in \zeta^+(T) \setminus \zeta(v, e_1, e_2)$. Since x is not adjacent to a, we have $y \in \zeta(T) \setminus \zeta(v, e_1, e_2)$. First, assume that $y \in \zeta(T') \setminus \zeta(v)$. Then by (S5) and (S7) from the definition of a strip-structure, there exists a path Q' of length more than one in G from a to x with $Q'^* \subseteq \zeta(T') \setminus \zeta(v)$. But now G contains a theta with ends a, x and paths a-S_1-x, a-S_2-x and a-Q'-x, a contradiction. It follows that $y \in \zeta(T_1 \cup T_2) \setminus \zeta(v, e_1, e_2)$. In other words, for some $i \in \{1, 2\}$, we have $y \in \zeta(T_i) \setminus (\zeta(e_i) \cup \zeta_{e_i}(u_i) \cup \zeta(v))$. As a result, by (S5) and (S7) from the definition of a strip-structure, and by the definition of $\zeta_{e_i}(u_i)$, there exists a path S'_i of length more than one in G from a to x with $S'^*_i \subseteq \zeta(T_i) \setminus (\zeta(e_i) \cup \zeta_{e_i}(u_i) \cup \zeta(v))$. But now assuming $i' \in \{1, 2\}$ to be distinct from i, G contains a theta with ends a, x and paths a-Q_i-x, a-S'_i-x and a-$S_{i'}$-x, a contradiction. This proves the first assertion.

Next we prove the second assertion of Theorem 5.1. By symmetry, we may assume that $i = 1$. Assume that x has a neighbor $y \in R$. Let $P'_1 = (P_1 \setminus R_1) \cup R$. Let Σ' be the pyramid with apex a, base rb_2b_3, and paths P'_1, P_2 and P_3. Recall that since ζ is rich, a is trapped in $\zeta^+(T)$. Also, Σ' is a pyramid in $\zeta^+(T)$, x is adjacent to $y \in P'_1$, x is adjacent to $b_2, c_2 \in P_2$, and x is anticomplete to P_3. It follows that x is a wide vertex for Σ' which is not a corner path for Σ'.

Now applying Lemma 3.1 to G, a, $H = \zeta^+(T)$, Σ' and $p = x$, we deduce that x is a jewel for Σ' at b_3, and so $N_R(x) = N_{P_1'}(x) = \{r, r'\}$. This completes the proof of Theorem 5.1. ■

Our next goal is to show that for every rich (T, a)-strip-structure in a graph $G \in \mathcal{C}_t$, there are only a few jewels at each vertex of T. Let us begin with a lemma, asserting that for a rich (T, a)-strip-structure ζ in a theta-free graph, each set $B_\zeta(v)$ is almost a clique.

Lemma 5.2. *Let G be a theta-free graph and $a \in V(G)$. Let T be a smooth tree and ζ be a rich (T, a)-strip-structure in G. Then for every $v \in V(T)$, there exists at most one edge $f \in E(T)$ such that $\zeta(f, v)$ is not a clique.*

Proof. Suppose for a contradiction that there are two distinct edges $f_1, f_2 \in E(T)$ incident with v, and for each $i \in \{1, 2\}$, there exist $x_i, y_i \in \zeta(f_i, v)$ such that x_i is not adjacent to y_i. Then v is not a leaf of T and $H = x_1\text{-}x_2\text{-}y_1\text{-}y_2\text{-}x_1$ is a hole of length four in G. Since ζ is rich, a is anticomplete to H. Let $f_1 = u_1v$. Let l_1 be a leaf of T which belongs to the component of $T \setminus \{v\}$ containing u_1, and let Λ_1 be the unique path in T from v to l_1 (so $f_1 \in E(\Lambda_1)$). Let R_{x_1} be a $\zeta(f_1)$-rung containing x_1 and let R_{y_1} be a $\zeta(f_1)$-rung containing y_1. Since ζ is rich, $H_1 = R_{x_1} \cup R_{y_1} \cup B(u_1)$ is a connected induced subgraph of G, and so there is a path Q in H_1 from x_1 to y_1. It follows that Q has length more than one and $Q^* \subseteq (B(u_1) \cup \zeta(f_1)) \setminus B(v)$. But now G contains a theta with ends x_1, y_1 and paths Q, $x_1\text{-}x_2\text{-}y_1$ and $x_1\text{-}y_2\text{-}y_1$, a contradiction. This completes the proof of Lemma 5.2. ■

Recall the following classical result of Ramsey.

Theorem 5.3 (Ramsey [18])**.** *For all integers $a, b \geq 1$, there exists an integer $R = R(a, b) \geq 1$ such that every graph G on at least $R(a, b)$ vertices contains either a clique of cardinality a or a stable set of cardinality b.*

We can now prove the second main result of this section.

Theorem 5.4. *For every integer $t \geq 1$, there exists an integer $j = j(t)$ with the following property. Let $G \in \mathcal{C}_t$ be a graph, let $a \in G$ and let T be a smooth tree. Let ζ be a rich (T, a)-strip-structure in G. Then for every vertex $v \in V(T)$, we have $|\mathcal{J}_{\zeta,v}| < j$.*

Proof. Let $j = j(t) = \binom{t}{2}R(t, 3)$ with $R(\cdot, \cdot)$ as in Theorem 5.3. Note that since G is K_t-free, T has maximum degree less than t. It follows that $t \geq 4$, and in order to prove $|\mathcal{J}_{\zeta,v}| < j$, it is enough to show that $|\mathcal{J}_{\zeta,(v,e_1,e_2)}| < R(t, 3)$ for every seagull (v, e_1, e_2) in T. Suppose for a contradiction that $|\mathcal{J}_{\zeta,(v,e_1,e_2)}| \geq R(t, 3)$ for some seagull (v, e_1, e_2) in T. Then v is a branch vertex of T. For each $i \in \{1, 2\}$, let u_i be the end of e_i different from v. Since $G \in \mathcal{C}_t$, it follows from Theorem 5.3 that $\mathcal{J}_{\zeta,(v,e_1,e_2)}$ contains a stable set X of cardinality three. For every $x \in X$, since x is a jewel for ζ at (v, e_1, e_2), it follows that for every $i \in \{1, 2\}$, there exists a long $\zeta(e_i)$-rung R_i^x such that $Q_i^x = R_i^x \setminus \zeta(e_i, v)$ is a path in $\zeta(e_i) \setminus \zeta(e_i, v)$ from a neighbor of x to a vertex in $\zeta(e_i, u_i) \setminus \zeta(e_i, v)$; in particular, R_i^x contains a neighbor of x. Therefore, for each $i \in \{1, 2\}$, we may pick a non-empty set $\mathcal{R}_i$ of long $\zeta(e_i)$-rungs such that every vertex in X has a neighbor in at least one rung in $\mathcal{R}_i$, and with $\mathcal{R}_i$ minimal with respect to inclusion. We deduce:

(13) *There exists $i \in \{1, 2\}$ with $|\mathcal{R}_i| > 1$.*

Suppose not. Then for every $i \in \{1, 2\}$, there exists a long $\zeta(e_i)$-rung S_i such that every vertex in X has a neighbor in S_i. Let s_i be the end of S_i in $\zeta(e_i, v)$ and s_i' be unique neighbor of s_i in S_i. By the second assertion of Theorem 5.1, X is complete to $\{s_1', s_2'\}$. But now $X \cup \{s_1', s_2'\}$ is a theta in G with ends s_1', s_2', a contradiction. This proves (13).

By (13) and due to symmetry, we may assume that $|\mathcal{R}_1| > 1$. This, together with the minimality of $\mathcal{R}_1$, implies that there exist distinct vertices $x, y \in X$ as well as distinct long $\zeta(e_1)$-rungs $R_x, R_y \in \mathcal{R}_1$ such that x has a neighbor in R_x, y has a neighbor in R_y, x is anticomplete to R_y, and y anticomplete to R_x. Let r_x and r_y be the ends of R_x and R_y in $\zeta(e_1, v)$, respectively. Let r_x' be the unique neighbor of r_x in R_x and r_y' be the unique neighbor of r_y in R_y. So we have $r'_x, r'_y \in \zeta(e_1) \setminus \zeta(e_1, v)$. By the second assertion of Theorem 5.1, we have $N_{R_x \cup R_y}(x) = \{r_x, r'_x\}$ and $N_{R_x \cup R_y}(y) = \{r_y, r'_y\}$. It follows that $r_x, r'_x \in R_x \setminus R_y$ and $r_y, r'_y \in R_y \setminus R_x$. Also, r_x is anticomplete to $R_y \setminus \{r_y\}$, as otherwise $(R_y \setminus \{r_y\}) \cup \{r_x\}$ contains a long $\zeta(e_1)$-rung R with $N_R(x) = \{r_x\}$, which violates the second assertion of Theorem 5.1. Similarly, r_y is anticomplete to $R_x \setminus \{r_x\}$.

Now, let $G_1 = G[(B(u_1)\setminus\zeta(e_1, u_1))\cup((R_x\cup R_y)\setminus\{r_x, r_y\})]$ and let $G_2 = G[(B(u_2)\setminus\zeta(e_2, u_2))\cup Q_2^x \cup Q_2^y]$. Since ζ is rich, the second bullet in the definition of a rich strip-structure implies that G_1 and G_2 are connected. Consequently, there exists a path Q_1 in G_1 from r'_x to r'_y, and there exists a path Q_2 from x to y with $Q_2^* \subseteq G_2$. Also, since v is a branch vertex of T, we may choose an edge $e_3 \in E(T) \setminus \{e_1, e_2\}$ incident with v. By the first assertion of Theorem 5.1, $\{x, y\}$ is anticomplete to $\zeta(e_3, v)$. Let Q_3 be a path from r_x to r_y with $Q_3^* \subseteq \zeta(e_3, v)$ (thus $|Q_3| \in \{2, 3\}$). But now G contains a prism with triangles $xr_xr'_x$ and $yr_yr'_y$ and paths Q_1, Q_2, Q_3, a contradiction. This completes the proof of Theorem 5.4. ■

Our last theorem in this section examines the connectivity within $G \setminus \zeta^+(T)$ for a (T, a)-strip-structure ζ arising from Theorem 4.2. We need the following lemma, the proof of which is similar to that of Theorem 5.1.

Lemma 5.5. *Let G be a theta-free graph and let $a \in V(G)$. Let T be a smooth tree and let ζ be a (T, a)-strip-structure in G. Let $v, v' \in V(T)$ be distinct and let P be a path in $G \setminus \zeta^+(T)$ with ends x, x' such that $x \in \mathcal{J}_{\zeta,v}$, $x' \in \mathcal{J}_{\zeta,v'}$, and P^* is anticomplete to $\zeta^+(T)$. Then v and v' are adjacent in T.*

Proof. Suppose not. Note that by Theorem 5.1, x and x' are distinct. Let Λ be the path in T from v to v'. Then Λ has length more than one, and so there are two distinct edges $f, f' \in E(\Lambda)$ such that f is incident with v and f' is incident with v'. Let u be the end of f distinct from v and u' be the end of f' distinct from v'. Let (v, e_1, e_2) and (v', e'_1, e'_2) be two seagulls in G such that $x \in \mathcal{J}_{\zeta,(v,e_1,e_2)}$ and $x' \in \mathcal{J}_{\zeta,(v',e'_1,e'_2)}$. For each $i \in \{1, 2\}$, let u_i be the end of e_i distinct from v and let u'_i be the end of e'_i distinct from v'. Without loss of generality, we may assume that $u_2, u'_2 \notin \Lambda$. Let T_2 be the component of $T \setminus (N_T(v) \setminus \{u_2\})$ containing v and let T'_2 be the component of $T \setminus (N_T(v') \setminus \{u'_2\})$ containing v'. Let T' be the component of $T \setminus \{u', u'_2\}$ containing v'. Since x is a jewel for ζ at (v, e_1, e_2), it follows that x is not adjacent to a, and x has a neighbor $c \in \zeta(e_2) \setminus \zeta(e_2, v) \subseteq \zeta(T_2) \setminus (B(v) \cup \zeta(u_2) \cup \zeta(v))$. Therefore, there exists a path Q of length more than one in G from a to x for which we have $c \in Q^* \subseteq \zeta(T_2) \setminus (B(v) \cup \zeta(u_2) \cup \zeta(v))$. Also, since x' is a jewel for ζ at (v', e'_1, e'_2), it follows that x' is not adjacent to a, and x' has a neighbor $b' \in B(v') \setminus (\zeta(f', u') \cup \zeta(e'_2, v'))$ and a neighbor $c' \in \zeta(e'_2) \setminus \zeta(e'_2, v') \subseteq \zeta(T'_2) \setminus (B(v') \cup \zeta(u'_2) \cup \zeta(v'))$. Therefore, there exist paths P', Q' of length more than one in G from a to x' for which we have $b' \in P'^* \subseteq (\zeta(T') \setminus \zeta(v')) \cup (\zeta(f', v') \setminus \zeta(f', u'))$ and $c' \in Q'^* \subseteq \zeta(T'_2) \setminus (B(v') \cup \zeta(u_2) \cup \zeta(v'))$. But now G contains a theta with ends a, x' and paths a-P'-x', a-Q'-x', and a-Q-x-P-x', a contradiction. This proves Lemma 5.5. ■

Theorem 5.6. *Let $t \geq 1$ be an integers and let $j(t)$ be as in Theorem 5.4. Let $G \in \mathcal{C}_t$ be a graph and let $a \in V(G)$. Let T be a smooth tree and let $v \in V(T)$. Let ζ be a rich (T, a)-strip-structure in G such that $G \setminus (\zeta^+(T) \cup \mathcal{J}_\zeta)$ is anticomplete to $\zeta^+(T)$. Let $x \in G \setminus (\zeta^+(T) \cup \mathcal{J}_\zeta)$. Then there exists $S_x \subseteq G \setminus (\zeta^+(T) \cup \{x\})$ such that $|S_x| < 2j(t)$ and S_x separates x and $\mathcal{J}_\zeta \setminus S_x$ in $G \setminus \zeta^+(T)$. Consequently, S_x separates x and $\zeta^+(T)$ in G.*

Proof. By Theorem 5.1, $\{\mathcal{J}_{\zeta,v} : v \in V(T)\}$ is a partition of $\mathcal{J}_\zeta$. Let G' be the graph obtained from $G \setminus \zeta^+(T)$ by contracting the set $\mathcal{J}_{\zeta,v}$ into a vertex z_v for each $v \in V(T)$ with $\mathcal{J}_{\zeta,v} \neq \emptyset$, and then adding a new vertex z such that $N_{G'}(z) = \{z_v : v \in V(T), \mathcal{J}_{\zeta,v} \neq \emptyset\}$. We claim that there is a set $Y \subseteq G' \setminus \{x, z\}$ of cardinality at most two which separates x and z in G'. Suppose not. By Theorem 2.1, there are three pairwise internally disjoint paths in G' from x to z. Thus, there exist $S \subseteq T$ with $|S| = 3$ as well as three paths $\{P_v : v \in S\}$ in $G \setminus \zeta^+(T)$ all having x as an end and otherwise disjoint, such that for each $v \in S$, P_v has an end $y_v \in \mathcal{J}_{\zeta,v}$ distinct from x, and we have $P_v^* \subseteq G \setminus (\zeta^+(T) \cup \mathcal{J}_\zeta)$. As a result, for all distinct $v, v' \in S$, $P_{v,v'} = y_v\text{-}P_v\text{-}x\text{-}P_{v'}\text{-}y_{v'}$ is a path in $G \setminus \zeta^+(T)$ from $y_v \in \mathcal{J}_{\zeta,v}$ to $y_{v'} \in \mathcal{J}_{\zeta,v'}$ such that $P^*_{v,v'} \subseteq G \setminus (\zeta^+(T) \cup \mathcal{J}_\zeta)$. In particular, $P^*_{v,v'}$ is anticomplete to $\zeta^+(T)$. But then by Lemma 5.5, S is a clique in T, which is impossible. The claim follows.

Let Y be as in the above claim. For each $y \in Y$, if $y = z_v$ for some $v \in V(T)$, then let $A_y = \mathcal{J}_{\zeta,v}$. Otherwise, let $A_y = \{y\}$. Let $S_x = \bigcup_{y \in Y} A_y$. Then $S_x \subseteq G \setminus (\zeta^+(T) \cup \{x\})$ separates x and $\mathcal{J}_\zeta \setminus S_x$ in $G \setminus \zeta^+(T)$. Also, by Theorem 5.4, we have $|S_x| < 2j(t)$. This completes the proof of Theorem 5.6. ■

6. Strip structures and connectivity

In this section, we investigate the connectivity implications of the presence of certain (T, a)-strip-structures in graphs from $\mathcal{C}_t$. The main result is the following.

Theorem 6.1. *For every integer $t \geq 1$, there exists an integer $\sigma = \sigma(t) \geq 1$ with the following property. Let $G \in \mathcal{C}_t$ be a graph and let $a \in V(G)$. Let T be a smooth tree and let ζ be a rich (T, a)-strip-structure in G such that $G \setminus (\zeta^+(T) \cup \mathcal{J}_\zeta)$ is anticomplete to $\zeta^+(T)$. Then for every vertex $x \in G \setminus N_G[a]$, there exists a set $S_x \subseteq G \setminus \{a, x\}$ with $|S_x| < \sigma$ such that S separates a and x in G.*

Proof. Let $j(t)$ be as in Theorem 5.4. We claim that

$$\sigma = \sigma(t) = 2t(j(t) + t)$$

satisfies Theorem 6.1. Note that since G is K_t-free, T has maximum degree less than t, and so $t \geq 4$. For every vertex $v \in V(T)$, we define $C_v = B(v)$ if v is a leaf of T and $C_v = \emptyset$ otherwise. Also, for every vertex $v \in V(T)$, let K_v be a maximal clique of G contained in $B(v)$. Thus, we have $|K_v| < t$. Moreover, if v is a leaf of T, then by the richness of ζ, we have $K_v = B(v) = C_v$ (and so $|K_v| = 1$), and if v is a branch vertex of T, then by Lemma 5.2, K_v contains all but possibly one of the sets $\zeta(f, v)$ for $f \in E(T)$. For every $S \subseteq T$, we define

$$\mathcal{M}_S = \bigcup_{w \in N_T(S)} \mathcal{J}_{\zeta,w},$$

$$\mathcal{N}_S = \bigcup_{w \in N_T(S)} K_w.$$

Also, we write $\mathcal{M}_v$ for $\mathcal{M}_{\{v\}}$ and $\mathcal{N}_v$ for $\mathcal{N}_{\{v\}}$. For every $v \in V(T)$, let $\mathcal{O}_v = \mathcal{M}_v \cup \mathcal{N}_v$. The following is immediate from Theorems 5.1 and 5.4 and Lemma 5.5.

(14) *For every $v \in V(T)$, we have*
- *$\mathcal{O}_v \subseteq G \setminus (\mathcal{J}_{\zeta,v} \cup \{a\})$;*
- *$|\mathcal{O}_v| < t(j(t) + t) \leq \sigma$; and*
- *$\mathcal{O}_v$ separates a and $\mathcal{J}_{\zeta,v}$ in G.*

Now, we define S_x for every $x \in G \setminus N_G[a]$. First, assume that $x \in \zeta(T) \setminus N_G[a]$. Then either $x \in \zeta(e)$ for some edge $e = uv \in E(T)$, or $x \in \zeta(v)$ for some branch vertex $v \in V(T)$. In the former case, let

$$\mathcal{E}_x = \mathcal{M}_u \cup \mathcal{M}_v,$$

$$\mathcal{I}_x = \mathcal{N}_{\{u,v\}} \cup C_u \cup C_v.$$

In the latter case, let

$$\mathcal{E}_x = \mathcal{M}_v \cup \mathcal{J}_{\zeta,v}$$

$$\mathcal{I}_x = \mathcal{N}_v.$$

Let $S_x = \mathcal{E}_x \cup \mathcal{I}_x$. Observe that since $x \in G \setminus N_G[a]$, we have $S_x \subseteq G \setminus \{a, x\}$. Also, by Theorem 5.4, we have $|\mathcal{E}_x| \leq 2tj(t)$ and so $|S_x| < 2t(j(t) + t) = \sigma$. Moreover, from Theorem 5.1 and the fact that ζ is rich, it is easy to check that for every path P in G from a to x, if

$P \subseteq \zeta^+(T)$, then P contains a vertex from $\mathcal{I}_x$, and otherwise P contains a vertex from either $\mathcal{I}_x$ or $\mathcal{E}_x$. Therefore, S_x separates a and x in G.

Next, assume that $x \in \mathcal{J}_\zeta$. Then by Theorem 5.1, there exists a unique vertex $v \in V(T)$ such that $x \in \mathcal{J}_{\zeta,v}$. Let $S_x = \mathcal{O}_v$. Then by (14), we have $S_x \subseteq G \setminus \{a, x\}$, $|S_x| < \sigma$ and S_x separates a and x in G.

Finally, assume that $x \in G \setminus (\zeta^+(T) \cup \mathcal{J}_\zeta)$. Then letting S_x to be as in Theorem 5.6, it follows from Theorem 5.6 that $S_x \subseteq G \setminus \{a, x\}$, $|X| < 2j(t) \leq \sigma$ and S_x separates a and x in G. This completes the proof of Theorem 6.1. ■

Our application of Theorem 6.1 is confined to the case where T is a caterpillar. More precisely, for a graph G and a vertex $a \in G$, an induced subgraph $H \subseteq G \setminus \{a\}$ is said to be an *a-seed* in G if the following hold.

- There exists a smooth caterpillar T such that H is the line graph of a proper subdivision of T (where a *proper subdivision of* T is a graph obtained from T by replacing each edge with a path of length at least two), and $N_G(a) = \mathcal{Z}(H)$.
- The vertex a is trapped in $H \cup \{a\}$.

It follows that $\mathcal{Z}(H)$ is the set of all degree-one vertices of H. We now combine Theorems 4.2 and 6.1 to deduce the following.

Theorem 6.2. *For every integer $t \geq 1$, there exists an integer $s = s(t) \geq 1$ with the following property. Let $G \in \mathcal{C}_t$ be a graph and let $a \in V(G)$. Assume that there is an a-seed in G. Then for every vertex $x \in G \setminus N_G[a]$, there exists $S_x \subseteq G \setminus \{a, x\}$ with $|S_x| < s$ such that S_x separates a and x in G.*

Proof. Let $\sigma(\cdot,\cdot)$ be as in Theorem 6.1. We show that $s = s(t) = \sigma(t, 3)$ satisfies Theorem 6.2. Pick an a-seed H in G. Let T be the unique smooth caterpillar with $|N_G(a)|$ leaves. Then T has maximum degree three. Also, one may immediately observe that there is a tame, substantial, and rich (T, a)-strip-structure η in G with $\eta(T) = H$ (see Figure 5). Now we can apply Theorem 4.2 to G, a, and T, deducing that there exists a substantial and rich (T, a)-strip-structure ζ in G such that $G \setminus (\zeta^+(T) \cup \mathcal{J}_\zeta)$ is anticomplete to $\zeta^+(T)$. Hence, by Theorem 6.1 applied to G, a, T, and ζ, for every vertex $x \in G \setminus N_G[a]$, there exists $S_x \subseteq G \setminus \{a, x\}$ with $|S_x| < s$ such that S_x separates a and x in G. This completes the proof of Theorem 6.2. ■

7. From blocks to trees

In this section, we prove Theorem 1.8. We begin with a result which captures the use of Theorem 6.2 in the proof of Theorem 1.8. For a positive integer n, we write $[n] = \{1, \dots, n\}$.

Theorem 7.1. *For all integers $t, \nu \geq 1$, there exists an integer $\psi = \psi(t, \nu) \geq 1$ with the following property. Let $G \in \mathcal{C}_t$, let $a, b \in V(G)$ be distinct and non-adjacent, and let $\mathcal{P}$ be a collection of pairwise internally disjoint paths in G from a to b with $|\mathcal{P}| \geq \psi$. For each $P \in \mathcal{P}$, let a_P be the neighbor of a in P (so $a_P \neq b$). Then there exists $P_1, \dots, P_\nu \in \mathcal{P}$ such that:*

- *$\{a_{P_1}, \dots, a_{P_\nu}, b\}$ is a stable set in G; and*
- *for all $i, j \in [\nu]$ with $i < j$, a_{P_i} has a neighbor in $P_j^* \setminus \{a_{P_j}\}$.*

Proof. Let $s = s(t)$ be as in Theorem 6.2 and let $\mu = \mu(\max\{2s + 1, t\})$, where $\mu(\cdot)$ is as in Theorem 2.5. Let $R(\cdot,\cdot)$ be as in Theorem 5.3. For every integer $p \geq 1$, let $R_{tourn}(p)$ be the smallest positive integer n such that every tournament on at least n vertices contains a transitive tournament on p vertices; the existence of $R_{tourn}(p)$ follows easily from Theorem 5.3 (in fact, one may observe that $R_{tourn}(p) \leq R(p, p)$). Let $\gamma = R(R_{tourn}(\nu + 1), \mu)$. We prove that

$$\psi = \psi(t, \nu) = R(t, \gamma)$$

satisfies Theorem 7.1. Let us choose ψ distinct paths $P_1, \dots, P_\psi \in \mathcal{P}$, and for each $i \in [\nu]$, let us write $a_i = a_{P_i}$. Since G is K_t-free, it follows from Theorem 5.3 and the definition of ψ that there exists a stable set $N \subseteq \{a_i : i \in [\psi]\}$ in G with $|N| = \gamma$; we may assume without loss of generality that $N = \{a_i : i \in [\gamma]\}$.

Let D be a directed graph with $V(D) = N$ such that for distinct $i, j \in [\gamma]$, there is an arc from a_i to a_j in D if and only if a_i has a neighbor in $P_j^* \setminus \{a_j\}$. Note that D may contain both arcs (a_i, a_j) and (a_j, a_i), and so the undirected underlying graph of D might not be simple. Let D^- be the simple graph obtained from the undirected underlying graph of D by removing one of every two parallel edges.

(15) *D^- contains no stable set of cardinality μ.*

Suppose for a contradiction that D^- contains a stable set S of cardinality μ. We may assume without loss of generality that $S = \{a_1, \dots, a_\mu\}$. Let $G_1 = G[(\bigcup_{j=1}^{\mu} P_j) \setminus \{a\}]$. Note that by the definition of D, for every $i \in [\mu]$, we have $N_{G_1}(a_i) = N_{P_i}(a_i) \setminus \{a\}$, and in particular $|N_{G_1}(a_i)| = 1$. Since G_1 is connected and K_t-free, and since $|S| = \mu = \mu(\max\{2s+1, t\})$, we can apply Theorem 2.5 to G_1 and S. Note that every vertex in S has a unique neighbor in G_1, and so no path in G_1 contains $\max\{2s+1, t\} \geq 3$ vertices from S. Consequently, there is an induced subgraph H_1 of G_1 with $|H_1 \cap S| = 2s+1$ for which one of the following holds.

- H_1 is either a caterpillar or the line graph of a caterpillar with $H_1 \cap S = \mathcal{Z}(H_1)$.
- H_1 is a subdivided star with root r_1 such that $\mathcal{Z}(H_1) \subseteq H_1 \cap S \subseteq \mathcal{Z}(H_1) \cup \{r_1\}$.

If H_1 is a caterpillar, then $G[H_1 \cup \{a\}]$ contains a theta with ends a and a' for every vertex $a' \in H_1$ of degree more than two, a contradiction. Also, if the second bullet above holds, then since every vertex in S is of degree one in G_1, we have $H_1 \cap S = \mathcal{Z}(H_1)$, and so r_1 is not adjacent to a. But then $G[H_1 \cup \{a\}]$ contains a theta with ends a and r_1, a contradiction. It follows that H_1 is the line graph of a caterpillar with $|H_1 \cap S| = 2s+1$ and $H_1 \cap S = \mathcal{Z}(H_1)$. This, together with the fact that every vertex in $H_1 \cap S \subseteq S$ has a unique neighbor in $H_1 \subseteq G$, implies that H_1 contains the line graph H_2 of a proper subdivision of a caterpillar with $|H_2 \cap S| = s$ and $H_2 \cap S = \mathcal{Z}(H_2)$. Let $S_2 = H_2 \cap S = \mathcal{Z}(H_2)$; then S_2 is the set of all vertices of degree one in H_2, and we may assume without loss of generality that $S_2 = \{a_1, \dots, a_s\}$. Let $G_2 = G[H_2 \cup (\bigcup_{j=1}^{s} P_j)]$. It follows that $G_2 \in \mathcal{C}_t$, $N_{G_2}(a) = S_2 = \mathcal{Z}(H_2)$ and a is trapped in $H_2 \cup \{a\}$. Therefore, H_2 is an a-seed in G_2. Since $b \in G_2 \setminus N_{G_2}[a]$, applying Theorem 6.2 to G_2 and a, we deduce that there exists $S_b \subseteq G_2 \setminus \{a, b\}$ such that $|S_b| < s$ and S_b separates a and b in G_2. But $P_1, \dots, P_s$ are s pairwise internally disjoint paths in G_2 from a to b, a contradiction with Theorem 2.1. This proves (15).

By (15), Theorem 5.3, and the definition of γ, D^- contains a clique of cardinality $R_{tourn}(\nu+1)$. This, along with the definition of $R_{tourn}(\cdot)$, implies that D contains (as a subdigraph) a transitive tournament K on $\nu + 1$ vertices. We may assume without loss of generality that $V(K) = \{a_1, \dots, a_{\nu+1}\}$ such that for distinct $i, j \in [\nu+1]$, (a_i, a_j) is an arc in K if $i < j$. From the definition of D, it follows that $\{a_2, \dots, a_{\nu+1}, b\}$ is a stable set in G, and for all $i, j \in \{2, \dots, \nu+1\}$ with $i < j$, a_i has a neighbor in $P_j^* \setminus \{a_j\}$. Hence, $I = \{2, \dots, \nu+1\}$ satisfies Theorem 7.1. This completes the proof. ■

For positive integers d and r, let T_d^r denote the rooted tree in which every leaf is at distance r from the root, the root has degree d, and every vertex that is neither a leaf nor the root has degree $d+1$. We need a result from [14]:

Theorem 7.2 (Kierstead and Penrice [14])**.** *For all integers $d, r, s, t \geq 1$, there exists an integer $f = f(d, r, s, t) \geq 1$ such that if G contains T_f^f as a subgraph, then G contains one of $K_{s,s}$, K_t and T_d^r as an induced subgraph.*

The following lemma is the penultimate step in the proof of Theorem 1.8.

Lemma 7.3. *For all integers $d, r, t \geq 1$, there exists an integer $m = m(d, r, t)$ with the following property. Let $G \in \mathcal{C}_t$ be a graph, let $a, b \in V(G)$ be non-adjacent, and let $\{P_i : i \in [m]\}$ be a collection of m pairwise internally disjoint paths in G from a to b. Then $G[\bigcup_{j=1}^{m} P_j]$ contains a* *subgraph J isomorphic to T_d^r such that $a \in J$, a has degree d in J (that is, a is the root of J), and $b \notin J$.*

Proof. Let $d, t \geq 1$ be fixed. Let $m_1 = d$. For every integer $r > 1$, let $m_r = \psi(t, (m_{r-1}+1)d)$ where $\psi(\cdot,\cdot)$ is as in Theorem 7.1. We prove by induction on $r \geq 1$ that $m(d,r,t) = m_r$ satisfies Lemma 7.3. Let $P_1, \ldots, P_{m_r}$ be m_r pairwise internally disjoint paths in G from a to b. Since a and b are not adjacent, it follows that for each $i \in [m_r]$, we have $P_i^* \neq \emptyset$. Let a_i be the neighbor of a in P_i. In particular, we have $b \notin \{a_i : i \in [m_r]\}$. Suppose first that $r = 1$. Then we have $|\{a_i : i \in [m_1]\}| = m_1 = d$, and so $G[\{a_i : i \in [m_r]\} \cup \{a\}]$ contains a (spanning) subgraph J isomorphic to T_d^1 such that $a \in J$ and a has degree d in J, and we have $b \notin J$, as desired. Therefore, we may assume that $r \geq 2$. Since $m_r = \psi(t, (m_{r-1}+1)d)$, we can apply Theorem 7.1 to a, b, and $\mathcal{P} = \{P_i : i \in [m_r]\}$. Without loss of generality, we may deduce that $\{a_1, \cdots, a_{(m_{r-1}+1)d}, b\}$ is a stable set in G, and for all $i, j \in [(m_{r-1}+1)d]$ with $i < j$, a_i has a neighbor in $P_j^* \setminus \{a_j\}$. For every $i \in [d]$, let $a_i' = a_{(i-1)m_{r-1}+i}$ and let

$$A_i = \{(i-1)m_{r-1}+i+1, \ldots, (i-1)m_{r-1}+i+m_{r-1}\}.$$

In particular, we have $|A_i| = m_{r-1}$. Then for each $i \in [d]$ and each $j \in A_i$, a_i' has a neighbor in $P_j^* \setminus \{a_j\}$, and so there exists a path Q_j in G from a_i' to b with $Q_j^* \subseteq P_j^*$. Now, for every $i \in [d]$, a_i' and b are non-adjacent, and $\{Q_j : j \in A_i\}$ is a collection of m_{r-1} pairwise internally disjoint paths in G from a_i' to b. It follows from the induction hypothesis that $G[\bigcup_{j \in A_i} Q_j]$ contains a subgraph J_i isomorphic to T_d^{r-1} such that $a_i' \in J_i$, a_i' has degree d in J_i, and $b \notin J_i$. But now $G[(\bigcup_{i=1}^d V(J_i)) \cup \{a\}] \subseteq G[\bigcup_{j=1}^{m_r} P_j]$ contains a (spanning) subgraph J isomorphic to T_d^r such that $a \in J$, a has degree d in J, and $b \notin J$. This completes the proof of Lemma 7.3. ■

Finally, we prove Theorem 1.8, which we restate:

Theorem 1.8. *For every tree F and every integer $t \geq 1$, there exists an integer $\tau(F,t) \geq 1$ such that every graph in $\mathcal{C}_t(F)$ has treewidth at most $\tau(F,t)$.*

Proof. Let d and r be the maximum degree and the radius of F, respectively. It follows that T_d^r contains F as an induced subgraph. Let $f = f(d,r,3,t)$ be as in Theorem 7.2 and let $m = m(f,f,t)$ be as in Lemma 7.3. Let $\beta(\cdot,\cdot)$ be as in Corollary 2.4. We claim that $\tau(F,t) = \beta(\max\{m, t+1\}, t)$ satisfies Theorem 1.8. Suppose for a contradiction that $\mathrm{tw}(G) > \tau$ for some $G \in \mathcal{C}_t(F)$. By Corollary 2.4, G contains a strong $\max\{m, t+1\}$-block B. Consequently, since G is K_t-free, there are two distinct and non-adjacent vertices $a, b \in B$, and m pairwise internally disjoint paths $P_1, \ldots, P_m$ in G from a to b. It follows from Lemma 7.3 that G contains T_f^f as a subgraph. Also, since $G \in \mathcal{C}_t(F) \subseteq \mathcal{C}_t$, G is $(K_{3,3}, K_t)$-free. But now by Theorem 7.2, G contains T_d^r, and so F, as an induced subgraph, a contradiction. This completes the proof. ■

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