The three-in-a-tree problem

Maria Chudnovsky[1] and Paul Seymour[2]
Princeton University, Princeton NJ 08544

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Abstract

We show that there is a polynomial time algorithm that, given three vertices of a graph, tests whether there is an induced subgraph that is a tree, containing the three vertices. (Indeed, there is an explicit construction of the cases when there is no such tree.) As a consequence, we show that there is a polynomial time algorithm to test whether a graph contains a “theta” as an induced subgraph (this was an open question of interest) and an alternative way to test whether a graph contains a “pyramid” (a fundamental step in checking whether a graph is perfect).

1 Introduction

All graphs in this paper are finite and simple. If G is a graph, its vertex- and edge-sets are denoted $V(G), E(G)$. If $X \subseteq V(G)$, the subgraph with vertex set X and edge set all edges of G with both ends in X is denoted $G|X$, and called the subgraph *induced* on X.

There are many algorithmic questions of interest concerning the existence of an induced subgraph of some specific type containing some specified vertices, but almost all of them seem to be NP-complete, by virtue of the following result of Bienstock [1]:

1.1 *The following problem is NP-complete:*
Input: *A graph G and two edges e, f of G.*
Question: *Is there a subset $X \subseteq V(G)$ such that $G|X$ is a cycle containing e, f?*

Bienstock's result leaves very little room between the trivial problems and the NP-complete problems, but in this paper we report on a problem that falls into the gap. We call the following the "three-in-a-tree" problem:

Input: A graph G, and three vertices v_1, v_2, v_3 of G.
Question: Does there exist $X \subseteq V(G)$ with $v_1, v_2, v_3 \in X$ such that $G|X$ is a tree?

For most graphs one would expect a "yes" answer, but there are interesting graphs for which the answer is "no"; for instance, if e_1, e_2, e_3 are edges of a graph H each incident with a vertex of degree one, and G is the line graph of H, then e_1, e_2, e_3 are vertices of G and there is no induced tree in G containing e_1, e_2, e_3. Nevertheless, we will show that the three-in-a-tree problem can be solved in time $O(|V(G)|^4)$. We shall give an explicit construction of all instances (G, v_1, v_2, v_3) such that the desired tree does not exist, and the proof that all such instances must fall under this construction can be converted to an algorithm to check whether the desired tree exists or not.

2 Thetas, pyramids and prisms

We were led to the three-in-a-tree problem while working on the question of deciding if a graph contains a theta, so let us describe that. First we need some definitions. If G, H are graphs, and H is isomorphic to $G|X$ for some $X \subseteq V(G)$, we say that G contains H as an induced subgraph. A *path* is a graph P whose vertex set and edge set can be labeled as $V(P) = \{v_1, \ldots, v_k\}$ and $E(P) = \{e_1, \ldots, e_{k-1}\}$ for some $k \geq 1$, such that e_i is incident with v_i, v_{i+1} for $1 \leq i \leq k-1$. A *cycle* is a graph C with $V(C) = \{v_1, \ldots, v_k\}$ and $E(C) = \{e_1, \ldots, e_k\}$ for some $k \geq 3$, such that e_i is incident with v_i, v_{i+1} for $1 \leq i \leq k-1$, and e_k is incident with v_1, v_k. The *length* of a path or cycle is the number of edges in it, and a path or cycle if *odd* or *even* if its length is odd or even respectively. A path or cycle *of G* means a subgraph (not necessarily induced) of G that is a path or cycle. A *hole* of G means a cycle in G that is an induced subgraph and has length at least four. A *triangle* is a set of three pairwise adjacent vertices.

Here are three types of graph that will be important to us:

- A *pyramid* is a graph consisting of a vertex a and a triangle $\{b_1, b_2, b_3\}$, and three paths P_1, P_2, P_3, such that: P_i is between a and b_i for $i = 1, 2, 3$; for $1 \leq i < j \leq 3$ P_i, P_j are vertex-disjoint except for a and the subgraph induced on $V(P_i) \cup V(P_j)$ is a cycle; and at most one of P_1, P_2, P_3 has only one edge.

- A *theta* is a graph consisting of two nonadjacent vertices a, b and three paths P_1, P_2, P_3, each joining a, b and otherwise vertex-disjoint, such that for $1 \leq i < j \leq 3$ the subgraph induced on $V(P_i) \cup V(P_j)$ is a cycle.
- A *prism* is a graph consisting of two vertex-disjoint triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, and three paths P_1, P_2, P_3, pairwise vertex-disjoint, such that for $1 \leq i < j \leq 3$ the subgraph induced on $V(P_i) \cup V(P_j)$ is a cycle.

Perhaps the main reason for interest in pyramids, thetas and prisms is that every graph containing a pyramid as an induced subgraph has an odd hole, and every graph containing a theta or prism has an even hole; and there seems to be some parallel between pyramids and "thetas or prisms". Yet although pyramids, thetas and prisms are superficially similar, there is a real difference in the difficulty of detecting their presence. We showed in [3] that

2.1 *There is an algorithm with running time $O(|V(G)|^9)$, that, with input a graph G, tests whether G contains a pyramid as an induced subgraph.*

Motivated by the parallel between pyramids and thetas-or-prisms, Chudnovsky and Kapadia [2, 4] proved the following:

2.2 *There is a polynomial-time algorithm to test whether a graph G contains either a theta or a prism as an induced subgraph.*

In contrast, Maffray and Trotignon[5] showed that

2.3 *It is NP-complete to test whether a graph contains a prism as an induced subgraph.*

There are two useful applications of the three-in-a-tree algorithm here. First, until now the complexity of testing whether G contains a theta has been open; but it can be solved in polynomial time as follows.

2.4 *There is an algorithm to test if a graph G contains a theta as an induced subgraph, with running time $O(|V(G)|^{11})$.*

Proof. Enumerate all four-tuples (a, b_1, b_2, b_3) of distinct vertices such that a is adjacent to b_1, b_2, b_3 and b_1, b_2, b_3 are pairwise nonadjacent. For each such four-tuple (a, b_1, b_2, b_3), enumerate all subsets $X \subseteq V(G)$ such that a has no neighbour in X, and b_1, b_2, b_3 each have exactly one neighbour in X, and each member of X is adjacent to at least one of b_1, b_2, b_3 (it follows that $|X| \leq 3$). For each such choice of X, let G' be obtained from G by deleting a and all vertices adjacent to one of a, b_1, b_2, b_3 except for the members of $\{b_1, b_2, b_3\} \cup X$; and test whether there is an induced tree in G' containing all of b_1, b_2, b_3. Thus we have to run the three-in-a-tree algorithm at most $|V(G)|^7$ times, and each one takes time $O(|V(G)|^4)$. It is easy to see that there is some choice of a, b_1, b_2, b_3 and X such that the tree exists, if and only if G contains a theta. ■

Second, the algorithm of 2.1 depended heavily on some fortuitous properties of the smallest pyramid in a graph, and this was a little disturbing because testing for a pyramid was a crucial step in our algorithm to test whether a graph is perfect. The three-in-a-tree algorithm can be used to give another, less miraculous, way to test for pyramids, as follows.

2.5 *There is an algorithm to test if a graph G contains a pyramid as an induced subgraph, with running time $O(|V(G)|^{10})$.*

Proof. Enumerate all six-tuples $(a_1, a_2, a_3, b_1, b_2, b_3)$ of distinct vertices such that $\{b_1, b_2, b_3\}$ is a triangle, and a_i is adjacent to b_j if and only if $i = j$ (for $1 \leq i, j \leq 3$). For each such six-tuple, let G' be obtained from G by deleting all vertices with a neighbour in $\{b_1, b_2, b_3\}$ except for a_1, a_2, a_3, and test whether there is an induced tree in G' that contains all of a_1, a_2, a_3. It is easy to see that G contains a pyramid if and only if for some six-tuple there is an induced tree as described. ■

3 Strip structures

Let G be a connected graph. We say that $Z \subseteq V(G)$ is *constricted* if $|Z \cap V(T)| \leq 2$ for every induced tree T of G. We wish to study which three-vertex subsets of $V(G)$ are constricted; but we might as well study which sets are constricted in general, because that question is no more difficult, and does seem to be strictly more general. We will prove that if $|Z| \geq 2$, then Z is constricted if and only if G admits a certain decomposition with respect to Z, that we call an "extended strip decomposition". Our next goal is to define this; but before we do so, let us motivate the definition a little.

First, let us say a *leaf edge* of a graph is an edge incident with a vertex of degree one. We observe that for any graph H, with line graph G say, if $Z \subseteq E(H) = V(G)$ and every member of Z is a leaf edge of H, then Z is constricted in G. Our main result is an attempt at a converse of this; that for any graph G, if $Z \subseteq V(G)$ is constricted then G is a sort of modified line graph of a graph H in which the members of Z are the leaf edges.

Second, to see that there is some hope of a converse, let us mention the following "toy" version of our main theorem. (This is essentially a reformulation of 5.1, and we postpone the proof.)

3.1 *Let G be a connected graph and let $Z \subseteq V(G)$ be constricted with $|Z| \geq 2$. Suppose that there is no proper subset $X \subset V(G)$ with $Z \subseteq X$ such that $G|X$ is connected. Then there is a tree H with $E(H) = X$, such that G is the line graph of H, and Z is the set of leaf edges of H.*

Thus, we need to modify 3.1, eliminating the hypothesis about the nonexistence of X, and modifying the conclusion appropriately. It is natural to try changing the conclusion from "there is a tree H..." to "there is a connected graph H...", but this is not sufficiently general; the graph G might admit a 2-join with all the members of Z on one side, and if so then we do not have much information or control about the other side of the 2-join. Such far sides of 2-joins must be incorporated into the theorem as general pieces of graph about which we know nothing, except we know that they attach to the controlled part of the graph via a 2-join structure. (We call these pieces "strips").

Now let us define an "extended strip decomposition". Let G be a graph and $Z \subseteq V(G)$. Let H also be a graph, let W be the set of vertices of H that have degree one in H, and let η be a map with domain the union of $E(H)$ and the set of all pairs (e, v) where $e \in E(H), v \in V(H)$ and e incident with v, satisfying the following conditions:

- for each edge $e \in E(H)$, $\eta(e) \subseteq V(G)$, and for each $v \in V(H)$ incident with e, $\eta(e, v) \subseteq \eta(e)$
- $\eta(e) \cap \eta(f) = \emptyset$ for all distinct $e, f \in E(H)$

- for all distinct $e, f \in E(H)$, let $x \in \eta(e)$ and $y \in \eta(f)$; then x, y are adjacent in G if and only if e, f share an end-vertex v in H, and $x \in \eta(e, v)$ and $y \in \eta(f, v)$
- $|Z| = |W|$, and for each $z \in Z$ there is a vertex $v \in W$ such that $\eta(e, v) = \{z\}$, where e is the (unique) edge of H incident with v.

Let us call such a map η an *H-strip structure* in (G, Z). (Thus, we have now incorporated the "far sides of 2-joins", as discussed above. Unfortunately, this still is not enough, and we need a little more, as follows.) Let η be an H-strip structure, and let us extend the domain of η by adding to it the union of $V(H)$ and the set of all triangles of H, as follows. For each vertex $v \in V(H)$, let $\eta(v) \subseteq V(G)$, and for each triangle D of H let $\eta(D) \subseteq V(G)$, satisfying the following:

- all the sets $\eta(e)$ ($e \in E(H)$), $\eta(v)$ ($v \in V(H)$) and $\eta(D)$ (for all triangles D of H) are pairwise disjoint, and their union is $V(G)$
- for each $v \in V(H)$, if $x \in \eta(v)$ and $y \in V(G) \setminus \eta(v)$ are adjacent in G then $y \in \eta(e, v)$ for some $e \in E(H)$ incident in H with v
- for each triangle D of H, if $x \in \eta(D)$ and $y \in V(G) \setminus \eta(D)$ are adjacent in G then $y \in \eta(e, u) \cap \eta(e, v)$ for some distinct $u, v \in D$, where e is the edge uv of H.

In this case we say that η is an *extended H-strip decomposition* of (G, Z). Our main theorem asserts the following:

3.2 *Let G be a connected graph and let $Z \subseteq V(G)$ with $|Z| \geq 2$. Then Z is constricted if and only if for some graph H, (G, Z) admits an extended H-strip decomposition.*

Proof of the "if" half of 3.2.

Suppose that Z is not constricted, and yet (G, Z) admits an extended H-strip-decomposition η. Choose an induced tree T in G with $|V(T) \cap Z| \geq 3$, with $V(T)$ minimal. It follows that every vertex of T that has degree one in T belongs to Z (for otherwise it could be deleted from T); and since there is such a vertex, and it cannot be deleted from T, it follows that $|V(T) \cap Z| = 3$. Now either T is an induced path with both end-vertices in Z (in this case there is a unique vertex of $Z \cap V(T)$ that is an internal vertex of P, say y) or the three members of $Z \cap V(T)$ all have degree one in T, and T has a unique vertex y of degree three (and possibly some vertices of degree two). If $z \in Z \cap V(T) \setminus \{y\}$, a path P of T is said to be a *z-limb* if $y \notin V(P)$ and $z \in V(P)$ (and consequently z is an end-vertex of P, and no other vertex of P belongs to Z).

For each $v \in V(H)$, let $N(v)$ denote the union of all the sets $\eta(e, v)$, as e ranges over all edges of H incident with v.

(1) *For each $v \in V(H)$, there do not exist distinct $z_1, z_2 \in Z \cap V(T) \setminus \{y\}$ such that some z_1-limb contains a vertex in $N(v)$ and some z_2-limb contains a vertex in $N(v)$.*

For suppose this is false; then for $i = 1, 2$ we may choose a z_i-limb Q_i, with end-vertices z_i, y_i say, and $v \in V(H)$, such that $y_1, y_2 \in N(v)$. Choose Q_1, Q_2, v so that the sum of the lengths of Q_1, Q_2 is minimum. It follows that for $i = 1, 2$, no vertex of Q_i belongs to $N(v)$ except y_i. Since y_1 is not adjacent to y_2 (from the definition of z_1-, z_2-limb) it follows that for some $e \in E(H)$,

$y_1, y_2 \in \eta(e, v)$. Let e be incident with $v, u \in V(H)$ say. Since $|\eta(e, v)| > 1$, it follows that not both $z_1, z_2 \in \eta(e)$, so we may assume that $z_1 \notin \eta(e)$. Let R_1 be a maximal z_1-limb such that R_1 is a subpath of Q_1 and no vertex of R_1 belongs to $\eta(e)$. Let R_1 have ends r_1, z_1 say; and let s_1 be the neighbour of r_1 in Q_1 that does not belong to R_1. Thus $s_1 \in \eta(e)$. If $z_2 \notin \eta(e)$ define R_2, r_2, s_2 similarly. Since $r_1 \notin \eta(e)$, it follows that either $r_1 \in \eta(D)$ for some triangle D of H, or $r_1 \in \eta(w)$ for some $w \in V(H)$, or $r_1 \in \eta(f)$ for some $f \in E(H) \setminus \{e\}$.

Suppose that the first holds, that is, $r_1 \in \eta(D)$ for some triangle D of H. Since $s_1 \in \eta(e)$ and r_1, s_1 are adjacent, it follows that $u, v \in D$, and $s_1 \in \eta(e, u) \cap \eta(e, v)$. Let $D = \{u, v, w\}$ say. Since $z_1 \notin \eta(D)$, there is an edge ab of R_1 such that $a \in \eta(D)$ and $b \notin \eta(D)$. Consequently b belongs to one of

$$\eta(uv, u) \cap \eta(uv, v), \eta(vw, v) \cap \eta(vw, w), \eta(uw, u) \cap \eta(uw, w).$$

But from the choice of R_1, $b \notin \eta(e)$, and from the choice of Q_1, $b \notin N(v)$. Hence $b \in \eta(uw, u) \cap \eta(uw, w)$; but then $s_1 \in \eta(uv, u)$ and $b \in \eta(uw, u)$, and so s_1, b are adjacent, contradicting that T is an induced tree. Thus the first case cannot occur.

Suppose that the second holds, and $r_1 \in \eta(w)$ for some $w \in V(H)$. Since r_1, s_1 are adjacent, it follows that $s_1 \in N(w)$, and so $s_1 \in \eta(f, w)$ for some edge f incident with w; and since $s_1 \in \eta(e)$, it follows that $f = e$, so w is one of u, v. Moreover, since $z_1 \notin \eta(w)$, there is an edge ab of R_1 such that $a \in \eta(w)$ and $b \notin \eta(w)$, and therefore $b \in N(w)$. Since $b \notin \eta(v)$ it follows that $w \neq v$, and so $w = u$, and hence $s_1 \in \eta(e, u)$. Moreover, since T is an induced tree it follows that b and s_1 are not adjacent. Since $s_1, b \in N(u)$, it follows that $b \in \eta(e, u)$, contrary to the choice of R_1. Thus the second case cannot hold.

We deduce that the third holds, and so $r_1 \in \eta(f)$ for some $f \in E(H) \setminus \{e\}$. Since r_1, s_1 are adjacent, it follows that there is a vertex w of H incident with e, f such that $s_1 \in \eta(e, w)$ and $r_1 \in \eta(f, w)$. Since $r_1 \notin N(v)$, it follows that $w = u$, and so $s_1 \in \eta(e, u)$.

In particular, $\eta(e, u) \neq \{z_2\}$, and since $\eta(e, v) \neq \{z_2\}$, it follows that $z_2 \notin \eta(e)$, and so R_2, r_2, s_2 are defined. By the same argument with z_1, z_2 exchanged, it follows that $s_2 \in \eta(e, u)$. Since $r_1 \in \eta(f, u)$ for some $f \in E(H) \setminus \{e\}$, and $s_2 \in \eta(e, u)$, it follows that r_1, s_2 are adjacent, a contradiction. This proves (1).

Let $Z \cap V(T) = \{z_1, z_2, z_3\}$.

(2) *$y \in \eta(e)$ for some $e \in E(H)$.*

For either $y \in \eta(D)$ for some triangle D of H, or $y \in \eta(v)$ for some $v \in V(H)$, or $y \in \eta(e)$ for some $e \in E(H)$. Suppose that the first holds. Now for $i = 1, 2, 3$, $z_i \notin \eta(D)$; let R_i be a maximal z_i-limb containing no vertex of $\eta(D)$, with ends z_i, r_i say, and r_i has a neighbour in $\eta(D)$. Let $D = \{u, v, w\}$; then for $i = 1, 2, 3$, r_i belongs to at least two of $N(u), N(v), N(w)$. Hence one of $N(u), N(v), N(w)$ meets both of R_1, R_2, contrary to (1). Next suppose the second holds, and $y \in \eta(v)$ for some $v \in V(H)$. For $i = 1, 2, 3$, since $z_i \notin \eta(v)$, there is a maximal z_i-limb R_i with no vertex in $\eta(v)$, with ends z_i, r_i say, and r_i has a neighbour in $\eta(v)$. Hence r_1, r_2, r_3 all belong to $N(v)$, contrary to (1). Thus the third holds, that is, $y \in \eta(e)$ for some $e \in E(H)$. This proves (2).

Let e be as in (2). For $i = 1, 2, 3$, if $z_i \notin \eta(e)$, let R_i be a maximal z_i-limb containing no vertex of $\eta(e)$, with ends z_i, r_i say, and let r_i be adjacent in T to $s_i \in \eta(e)$. Let e be incident in H with

$u, v \in V(H)$.

(3) *For $i = 1,2,3$, if $z_i \notin \eta(e)$, then there exists $w \in \{u, v\}$ and an edge $f \in E(H) \setminus \{e\}$ incident with w such that $r_i \in \eta(f, w)$ and $s_i \in \eta(e, w)$.*

For either $r_i \in \eta(D)$ for some triangle D of H, or $r_i \in \eta(w)$ for some $w \in V(H)$, or $r_i \in \eta(f)$ for some $f \in E(H) \setminus \{e\}$. Suppose that the first holds, and $r_i \in \eta(D)$. Since $s_i \notin \eta(D)$, and r_i, s_i are adjacent, and $s_i \in \eta(e)$, it follows that $u, v \in D$, and $s_i \in \eta(e, u) \cap \eta(e, v)$. Let $D = \{u, v, w\}$ say. Since $z_i \notin \eta(D)$, there is an edge ab of R_i with $a \in \eta(D)$ and $b \notin \eta(D)$. Consequently b belongs to one of

$$\eta(uv, u) \cap \eta(uv, v), \eta(vw, v) \cap \eta(vw, w), \eta(uw, u) \cap \eta(uw, w).$$

Since b, s_i are nonadjacent, it follows that $b \notin \eta(uw, u) \cup \eta(vw, v)$, and since $b \notin \eta(e)$, this is impossible. Thus the first case cannot occur.

Suppose that the second holds, and $r_i \in \eta(w)$ for some $w \in V(H)$. Since r_i, s_i are adjacent, and $s_i \in \eta(e)$, it follows that $w \in \{u, v\}$ and $s_i \in \eta(e, w)$; and we may assume that $w = v$. Since $z_i \notin \eta(v)$, there is an edge ab of R_i with $a \in \eta(v)$ and $b \notin \eta(v)$. Hence $b \in N(v)$, and so $b \in \eta(f, v)$ for some edge $f \in E(H)$ incident with v. Since s_i, b are nonadjacent, it follows that $f = e$, contrary to the definition of R_i. Thus the second case cannot hold. We deduce that the third holds, and $r_i \in \eta(f)$ for some $f \in E(H) \setminus \{e\}$. Since r_i, s_i are adjacent, and $s_i \in \eta(e)$, there exists $w \in V(H)$ incident with both e, f, such that $r_i \in \eta(f, w)$ and $s_i \in \eta(e, w)$. Hence $w \in \{u, v\}$. This proves (3).

Now if none of z_1, z_2, z_3 belong to $\eta(e)$, then by (3), we may assume that $r_i \in \eta(f, u)$ and $s_i \in \eta(e, u)$ for $i = 1, 2$, contrary to (1). Thus we may assume that $z_3 \in \eta(e)$, and therefore we may assume that $N(u) = \eta(e, u) = \{z_3\}$. If $z_1, z_2 \notin \eta(e)$, then for $i = 1, 2$, $r_i \notin N(u)$, and so $r_i \in N(v)$ by (3), contrary to (1). Consequently one of $z_1, z_2 \in \eta(e)$, and so we may assume that $N(v) = \eta(e, v) = \{z_2\}$. But then $z_1 \notin \eta(e)$, and yet $r_1 \notin N(u) \cup N(v)$, contrary to (3). This completes the proof. ■

4 A lemma

In this section we prove a lemma needed for the "only if" half of 3.2. If G' is a subgraph of a graph G, and C is a subgraph of $G \setminus V(G')$, and $v \in V(G')$ has a neighbour in $V(C)$, we say that v is an *attachment* of C (in G'). A *separation* of a graph K is a pair (A, B) of subsets of $V(K)$ with union $V(K)$, such that no edge joins a vertex in $A \setminus B$ and a vertex in $B \setminus A$. We call $|A \cap B|$ the *order* of the separation. Let $W \subseteq V(K)$. We say that (K, W) is a *frame* if

- every vertex in W has degree one in K
- $|W| \geq 3$
- K is connected
- for every separation (A, B) of K of order at most two with $W \subseteq B \neq V(K)$, we have that $|A \cap B| = 2$ and $K|A$ is a path between the two members of $A \cap B$.

If (K, W) is a frame, we see that W is the set of all vertices of K that have degree one. A *branch* of (K, W) is a path of K with distinct ends, such that both its ends have degree in K different from two, and all its internal vertices have degree two in K. Since (K, W) is a frame, it follows that every branch is an induced subgraph of K, and every edge of K belongs to a unique branch. If $v \in V(K)$, $\delta_K(v)$ or $\delta(v)$ denotes the set of all edges of K incident with v.

Let (K, W) be a frame and let $F \subseteq E(K)$. An *F-line* is a path P in K such that one end of P belongs to W and some edge of P belongs to F. A *double F-line* is a path P such that both ends of P belong to W and exactly one edge of P belongs to F. We say that $F \subseteq E(K)$ is *focused* if

- there do not exist three F-lines that are pairwise vertex-disjoint, and
- there do not exist an F-line and a double F-line that are vertex-disjoint.

We need to study which subsets F are focused. We shall prove the following:

4.1 *Let (K, W) be a frame and let $F \subseteq E(K)$ be focused. Then either:*

1. *there exists $x \in V(K)$ with $F \subseteq \delta(x)$, or*
2. *$|F| = 3$ and the three edges of F form a triangle, or*
3. *there exist $x, y \in V(K)$, not in the same branch of K, such that $F = \delta(x) \cup \delta(y)$, or*
4. *there is a branch B of (K, W) with $F \subseteq E(B)$, or*
5. *there is a branch B of (K, W) with ends x, y such that $x \notin W$ and $F \setminus E(B) = \delta(x) \setminus E(B)$ and $F \cap E(B) \not\subseteq \delta(x)$, or*
6. *there is a branch B of (K, W) with ends x, y such that $x, y \notin W$ and $F \setminus E(B) = (\delta(x) \cup \delta(y)) \setminus E(B)$.*

Proof. We proceed by induction on $|E(K)|$.

(1) *We may assume that for every $X \subseteq V(K)$ with $|X| \leq 2$, there is an F-line with no vertex in X.*

For suppose that $X \subseteq V(K)$ with $|X| \leq 2$, and every F-line has a vertex in X. Choose such a set X with $|X|$ minimum. Since all vertices in W have degree one and their neighbours are not in W, we may also choose X with $X \cap W = \emptyset$. If $|X| = 2$ and the two members of X belong to the same branch, let B_0 be the path in this branch between the two members of X, and otherwise let B_0 be the subgraph with vertex set X and no edges. Suppose that there exists $f \in F \setminus E(B_0)$ not incident with any member of X. There is no path between f and W in $K \setminus X$, from the property of X, and so there is a separation (A, B) of K with $A \cap B = X$ and $W \subseteq B$ such that both ends of f belong to A. Since f has no end in X, it follows that $B \neq V(K)$, and so $|X| = 2$ and $K|A$ is a path between the two members of X; but then this path is B_0 and contains f, a contradiction. This proves that every member of F either belongs to $E(B_0)$ or is incident with a member of X. If $|X| \leq 1$ then the first outcome holds, so we may assume that $X = \{x_1, x_2\}$ say. For $i = 1, 2$, let V_i be the set of all neighbours of x_i that are not in $V(B_0)$, let Y_i be the set of all $y \in V_i$ such that the edge yx_i belongs to F, and let $Z_i = V_i \setminus Y_i$. If one of Y_1, Z_1 is empty, and one of Y_2, Z_2 is empty, then one of the outcomes of the theorem holds; so we may assume that Y_1, Z_1 are both nonempty. Let $J = K \setminus V(B_0)$. Since $X \cap W = \emptyset$, it follows that $W \subseteq V(J)$. If G is a graph and $X, Y \subseteq V(G)$, we denote by $\kappa(G, X, Y)$ the minimum order of a separation (A, B) of G with $X \subseteq A$ and $Y \subseteq B$. Since (K, W) is a frame, it follows that

- $\kappa(J, Y_1, W) \geq 1$, for otherwise there would be a separation (A, B) of K with $A \cap B = X$ and $V(B_0) \subseteq A$ and $Y_1 \subseteq A$, which is impossible since (K, W) is a frame and no branch of (K, W) includes $V(B_0) \cup Y_1$, since $Z_1 \neq \emptyset$
- $\kappa(J, Z_1, W) \geq 1$, similarly
- $\kappa(J, V_2, W) \geq 1$; indeed, $\kappa(J, V_2, W) = \kappa(K \setminus \{x_1\}, V_2, W)$, and therefore is at least two unless $|V_2| = 1$
- $\kappa(J, Y_1 \cup Z_1, W) \geq 2$, similarly
- $\kappa(J, Y_1 \cup V_2, W), \kappa(J, Z_1 \cup V_2, W) \geq 2$, since no branch of (K, W) includes both $V(B_0)$ and one of Y_1, Z_1
- $\kappa(J, Y_1 \cup Z_1 \cup V_2, W) \geq 3$, since $\kappa(J, Y_1 \cup Z_1 \cup V_2, W) = \kappa(K, V_1 \cup V_2, W)$, and the latter is at least three since no branch of K includes x_1, x_2 and all their neighbours.

From this and Menger's theorem (applied to the graph obtained from J by adding three new vertices with neighbour sets Y_1, Z_1, V_2 respectively, and asking for three vertex-disjoint paths between the three new vertices and W), we deduce that there are three vertex-disjoint paths P_1, P_2, P_3 of J, from some $y_1 \in Y_1$ to $w_1 \in W$, from $z_1 \in Z_1$ to $w_2 \in W$, and from $v_2 \in V_2$ to $w_3 \in W$ respectively. We may assume that v_2 is the only vertex of P_3 in V_2. The path w_1-P_1-y_1-x_1-z_1-P_2-w_2 (with the obvious notation) is a double F-line Q say. Hence the path x_2-v_2-P_3-w_3 is not an F-line, since F is focused, and so $v_2 \in Z_2$. For $y_2 \in Y_2$, the path y_2-x_2-v_2-P_3-w_3 is an F-line, and therefore is not disjoint from Q; and so $Y_2 \subseteq V(Q)$; and by a similar argument, every edge of B_0 in F is incident with x_1. Thus if $Y_2 = \emptyset$ then the first outcome of the theorem holds, so we assume that $Y_2 \neq \emptyset$. Let $y_2 \in Y_2$; then we have seen that $y_2 \in V(P_1) \cup V(P_2)$. If $y_2 \in V(P_2)$, then the path w_3-P_3-v_2-x_2-y_2-P_2-w_2 is a double F-line (here the notation y_2-P_2-w_2 at the end of this sequence of concatenations means that we take the subpath of P_2 between y_2 and w_2; we will use this and similar notation repeatedly without further explanation); and this double F-line is vertex-disjoint from the F-line x_1-y_1-P_1-w_1, a contradiction. Thus $y_2 \in V(P_1)$. If $y_2 \neq y_1$, then w_3-P_3-v_2-x_2-y_2-P_1-w_1 is a double F-line vertex-disjoint from the F-line y_1-x_1-z_1-P_2-w_2, a contradiction; and so $y_2 = y_1$. This proves that $Y_2 = \{y_1\}$. Since $Y_2, Z_2 \neq \emptyset$, this restores the symmetry between x_1, x_2, and so it follows that $Y_1 = \{y_1\}$, and every edge of B_0 in F is incident with x_2 (as well as with x_1). If no edge of B_0 belongs to F, then every edge in F is incident with y_1, contrary to the minimality of X; so $F \cap E(B_0) \neq \emptyset$, and therefore B_0 is a path of length one, and the second outcome holds. This proves (1).

Henceforth, therefore, we make the assumption of (1), and will obtain a contradiction.

(2) *K is not a tree.*

For suppose K is a tree. It follows that for any set $\mathcal{C}$ of trees of K, either there are k members of $\mathcal{C}$ pairwise vertex-disjoint, or there is a set $X \subseteq V(K)$ with $|X| < k$ meeting every member of $\mathcal{C}$. Since there do not exist three pairwise disjoint F-lines, we deduce (by taking $\mathcal{C}$ to be the set of all F-lines, and $k = 3$) that there exists $X \subseteq V(K)$ with $|X| \leq 2$, such that every F-line contains a member of X. But this contradicts (1), and so proves (2).

(3) *There is a branch B of (K, W) such that, if K' denotes the graph obtained from K by deleting the edges and internal vertices of B, then (K', W) is a frame.*

For let T be a minimal connected subgraph of K with $W \subseteq V(T)$; then T is a tree, and it is easy to see that (T, W) is a frame. Since K is not a tree, it follows that $T \neq K$, and so there is a frame (K', W) with K' a proper subgraph of K. Choose such a frame with as many branches as possible. Suppose that there exist $u, v \in V(K')$ that are joined by a path P of K such that no edges or internal vertices of P belong to K', such that u, v are not in the same branch of (K', W). It follows that $(K' \cup P, W)$ is a frame, and from our choice of (K', W), we deduce that $K' \cup P = K$. But then P is a branch of (K, W), and (2) holds taking $B = P$. We may therefore assume that for every two vertices u, v that are joined by a path with no edges or internal vertices in K', some branch of (K', W) contains u, v. In particular every edge of K that does not belong to $E(K')$ but has both ends in $V(K')$ is between two vertices in the same branch. For any component C of $K \setminus V(K')$, we have seen that every two attachments of C belong to the same branch. If for every such C there is a branch of (K', W) that contains every attachment of C, this contradicts that (K, W) is a frame. Thus there is a component C of $K \setminus V(K')$ such that no branch of (K', W) contains all attachments of C. In particular C has at least two attachments, and every two of them belong to a branch; let B_1 be a branch, with ends x_2, x_3, containing at least two attachments v_2, v_3 of C. Let v_1 be an attachment of C that is not in B_1. Since some branch B_2 contains v_1, v_3, it follows that v_3 is one of x_2, x_3, say $v_3 = x_3$, and B_2 has ends x_1, x_3 say. Similarly there is a branch B_3 containing v_1, v_2; so $v_2 = x_2$, and v_1 is a common end of B_2, B_3. Since v_1, v_2, v_3 are attachments of C, we may choose a vertex c of C and three paths P_1, P_2, P_3 from c to v_1, v_2, v_3 respectively, pairwise vertex-disjoint except for c, such that $V(P_i) \subseteq V(C) \cup \{v_i\}$. But then if K'' denotes the graph obtained from K' by deleting the edges and internal vertices of B_1, and adding $P_1 \cup P_2 \cup P_3$, then (K'', W) is a frame, and $K'' \neq K$, since the edges of B_1 do not belong to K'', contrary to the choice of K'. This proves (3).

Henceforth, let B, K' be as in (3). Let the ends of B be x_1, x_2. Let $F' = F \cap E(K')$.

(4) *There does not exist $v \in V(K')$ such that $F' \subseteq \delta_{K'}(v)$.*

For suppose that v has this property. Let Y be the set of neighbours y of v in K such that the edge vy belongs to F. By (1) there is an F-line disjoint from $\{x_1, x_2\}$, and in particular, $v \notin V(B)$, and $Y \not\subseteq V(B)$. Since (K, W) is a frame, there are three paths P_1, P_2, P_3 from x_1, x_2, v respectively to W, pairwise vertex-disjoint. Consequently P_3, B are vertex-disjoint. For $i = 1, 2, 3$ let w_i be the end of P_i in W. By (1), no vertex of B meets every edge in $E(B) \cap F$ (since otherwise this vertex together with v would meet every edge in F), and so there are two disjoint edges in $E(B) \cap F$, and therefore there are two vertex-disjoint F-lines Q_1, Q_2, both subpaths of $P_1 \cup B \cup P_2$. Hence P_3 is not an F-line, and for each $y \in Y \setminus V(B)$, y-v-P_3-w_3 is not disjoint from both Q_1, Q_2, since y-v-P_3-w_3

includes an F-line. In particular, $Y \subseteq V(P_1 \cup B \cup P_2)$. We have already seen that there exists $y \in Y \setminus V(B)$, and we may therefore assume that $y \in V(P_1) \setminus \{x_1\}$. The path w_3-P_3-v-y-P_1-w_1 is a double F-line, and it is disjoint from the F-line $B \cup P_2$, a contradiction. This proves (4).

(5) *It is not the case that $|F'| = 3$, and the three edges in F' form a triangle.*

For suppose there is a triangle $\{v_1, v_2, v_3\}$ of K' such that F' consists of the three edges v_1v_2, v_2v_3, v_3v_1. (Possibly v_1, v_2, v_3, x_1, x_2 are not all distinct.) By (1) (applied to $\{v_1, v_2\}$) there is an edge of B in F. For any $M \subseteq \{v_1, v_2, v_3, x_1, x_2\}$ with $|M| = 3$, there is no branch of (K, W) including all members of M, and so, since (K, W) is a frame, there exist three vertex-disjoint paths in K between M and W. Consequently there are three vertex-disjoint paths P_1, P_2, P_3 from $\{v_1, v_2, v_3, x_1, x_2\}$ to W, such that at least one of them has first vertex in $\{x_1, x_2\}$, and at least two of them have first vertex in $\{v_1, v_2, v_3\}$. Choose three such paths P_1, P_2, P_3 with minimal union, and let P_i be between u_i and $w_i \in W$ say. We may assume that $u_1 = x_1, u_2 = v_2$ and $u_3 = v_3$. Moreover, from the minimality of $P_1 \cup P_2 \cup P_3$, v_1 is not a vertex of $P_2 \cup P_3$, and x_2 is not a vertex of P_1. Hence B is not a path of any of P_1, P_2, P_3. The path w_2-P_2-v_2-v_3-P_3-w_3 is a double F-line, disjoint from $P_1 \cup B \setminus \{x_2\}$; so the latter is not an F-line. But some edge of B is in F, and so the edge of B incident with x_2 is the unique edge of B in F. Since the F-line $B \cup P_1$ is not disjoint from the double F-line w_2-P_2-v_2-v_3-P_3-w_3, it follows that x_2 belongs to one of P_2, P_3, say P_2. The double F-line w_1-P_1-x_1-B-x_2-P_2-w_2 is not vertex-disjoint from the F-line v_2-v_3-P_3-w_3, so $v_2 = x_2$. But then every edge in F is incident with one of v_2, v_3, contrary to (1). This proves (5).

(6) *There do not exist two vertices $x_3, x_4 \in V(K')$ such that $F' = \delta_{K'}(x_3) \cup \delta_{K'}(x_4)$.*

For suppose such x_3, x_4 exist. By (1) (applied to $\{x_3, x_4\}$) there is an edge f of B in F and incident with neither of x_3, x_4. Also by (1) $\{x_1, x_2\} \neq \{x_3, x_4\}$, so there are three pairwise vertex-disjoint paths P_2, P_3, P_4 from $\{x_1, x_2, x_3, x_4\}$ to W, where P_i is from $u_i \in \{x_1, x_2, x_3, x_4\}$ to $w_i \in W$, and $u_3 = x_3, u_4 = x_4$, and $u_2 \in \{x_1, x_2\}$. Choose such paths with P_2 minimal; then we may assume that $u_2 = x_2$, and $x_1 \notin V(P_2)$. Hence B is not a path of any of P_2, P_3, P_4. Moreover, P_3, P_4 are disjoint F-lines, and $P_2 \cup B \setminus \{x_1\}$ is disjoint from both of them; so the latter is not an F-line. Hence f is incident with x_1, and therefore $x_1 \neq x_3, x_4$, and so $x_1, \ldots, x_4$ are all distinct. Since $P_2 \cup B$ is an F-line, it meets one of P_3, P_4, and so $x_1 \in V(P_3) \cup V(P_4)$, and we may assume that $x_1 \in V(P_3)$ say. But then the double F-line w_2-P_2-x_2-B-x_1-P_3-w_3 is disjoint from the F-line P_4, a contradiction. This proves (6).

(7) *There is no branch B' of (K', W) such that $F' \subseteq E(B')$.*

For suppose that B' is such a branch, with ends x_3, x_4. First suppose that one of x_1, x_2, say x_1, belongs to $V(B')$. Since (K, W) is a frame, it follows that $x_2 \notin V(B')$, and there are three paths P_i of K from x_i to $w_i \in W$ for $i = 2, 3, 4$, pairwise vertex-disjoint. Consequently $P2, P3, P4$ contain no edges of B or of B'. For $i = 3, 4$, let Q_i be the subpath of B' between x_1 and x_i, and let $Q_2 = B$. Now $F \subseteq E(Q_2) \cup E(Q_3) \cup E(Q_4)$. By (1), not every edge in F is incident with x_1, so we may assume that $P_i \cup Q_i \setminus \{x_1\}$ is an F-line for some $i \in \{2, 3, 4\}$. Let $\{i, j, k\} = \{2, 3, 4\}$. Consequently there do not exist two vertex-disjoint F-lines in $P_j \cup Q_j \cup Q_k \cup P_k$, and so there are at most two edges in

$F \cap E(Q_j \cup Q_k)$, and if there are two then they have a common end. By (1) (applied to $\{x_1, x_i\}$) at least one edge of $Q_j \cup Q_k$ is in F, and if there is only one then $P_j \cup Q_j \cup Q_k \cup P_k$ is a double F-line, disjoint from the F-line $P_i \cup Q_i \setminus \{x_1\}$, a contradiction. Hence exactly two edges of $Q_j \cup Q_k$ belong to F, and they have a common end $y_j \in V(Q_j)$ say. By (1) applied to $\{x_1, x_i\}$ it follows that $y_j \neq x_1$, so y_j is an internal vertex of Q_j, and in particular $F \cap E(Q_k) = \emptyset$, and $F \cap E(Q_j) = \delta(y_j)$. Then $P_j \cup Q_j \setminus \{x_1\}$ is an F-line, and so by the same argument there is an internal vertex y_i of Q_i such that $F \cap E(Q_i) = \delta(y_i)$. But this contradicts (1) (applied to $\{y_i, y_j\}$). This completes the proof of (7) in the case when one of x_1, x_2 belongs to B'.

Thus we may assume that B, B' are vertex-disjoint. There is symmetry between B and B' (for we will not use any more that (K', W) is a frame). By two applications of (1), both of B, B' contain an edge in F. There are three vertex-disjoint paths between $\{x_1, \ldots, x_4\}$ and W, and we may assume that none of them has an internal vertex in $\{x_1, \ldots, x_4\}$; and from the symmetry we may assume that these paths are P_1, P_2, P_3, where P_i is between x_i and $w_i \in W$. Now $P_3 \cup B'$ is an F-line, and so $P_1 \cup B \cup P_2$ is not a double F-line and does not include two disjoint F-lines; so there is an internal vertex y of B such that $F \cap E(B) = \delta(y)$. There are three disjoint paths Q_2, Q_3, Q_4 from $\{x_1, \ldots, x_4\}$ to W, such that for $i = 3, 4$, Q_i has first vertex x_i; choose them with Q_2 minimal, then we may assume that Q_2 has first vertex x_2 and $x_1 \notin V(Q_2)$ (possibly $x_1 \in V(Q_3 \cup Q_4)$). The path y-B-x_2-Q_2-w'_2 (where Q_2 is from x_2 to $w'_2 \in W$) is an F-line, disjoint from the path $Q_3 \cup B' \cup Q_4$; so the latter is not a double F-line, and does not include two disjoint F-lines. Hence there is an internal vertex y' of B' such that $F \cap E(B') = \delta(y')$; but this contradicts (1) (applied to $\{y, y'\}$). This proves (7).

(8) *There is no branch B' of (K', W) with ends x_3, x_4, such that $F' \setminus E(B') = \delta_{K'}(x_3)$.*

For suppose B' is such a branch. Again there are two cases depending whether B, B' are vertex-disjoint or not. First suppose that $x_1 \in V(B')$ say, and as in (7) we may choose three paths P_i of K from x_i to $w_i \in W$ for $i = 2, 3, 4$, pairwise vertex-disjoint. For $i = 3, 4$, let Q_i be the subpath of B' between x_1 and x_i, and let $Q_2 = B$. Suppose that $x_1 = x_3$. By (1) (applied to $\{x_2, x_3\}$), there is an edge of $B' \setminus \{x_3\}$ in F, and similarly an edge of $B \setminus \{x_3\}$ in F. But then $P_2 \cup Q_2 \setminus \{x_3\}, P_3, P_4 \cup Q_4 \setminus \{x_3\}$ are three disjoint F-lines, a contradiction. Thus $x_1 \neq x_3$. By (1) (applied to $\{x_1, x_3\}$), at least one edge of $Q_2 \cup Q_4$ is in F and not incident with x_1. Since P_3 is an F-line, the path $P_2 \cup Q_2 \cup Q_4 \cup P_4$ is not a double F-line, and does not include two disjoint F-lines; so exactly two edges of $Q_2 \cup Q_4$ belong to F and they have a common end $y \neq x_1$. From the symmetry we may assume that y belongs to the interior of Q_2 say, and so $F \cap E(Q_4) = \emptyset$. By (1) (applied to $\{x_3, y\}$) there is an edge of $Q_3 \setminus \{x_3\}$ in F; but then $P_2 \cup Q_2 \setminus \{x_1\}, P_3, P_4 \cup Q_4 \cup Q_3 \setminus \{x_3\}$ are three vertex-disjoint F-lines, a contradiction. This proves (8) in the case that B, B' are not disjoint.

We may therefore assume that B, B' are disjoint. By (1) (applied to $\{x_3, x_4\}$) at least one edge of B is in F. By (4) at least one edge of $B' \setminus \{x_3\}$ belongs to F. There are three vertex-disjoint paths P_2, P_3, P_4 from $\{x_1, \ldots, x_4\}$ to W, such that for $i = 3, 4$, P_i is from x_i to $w_i \in W$ say. We may assume that P_2 is from x_2 to w_2, and $x_1 \notin V(P_2)$ (possibly $x_1 \in V(P_3 \cup P_4)$). Thus $P_3, P_4 \cup B' \setminus \{x_3\}$ are disjoint F-lines, and so $P_2 \cup B \setminus \{x_1\}$ is not an F-line; and therefore the edge of B incident with x_1 is the unique edge of B in F. Since $B \cup P_2$ is an F-line, it follows that x_1 belongs to one of P_3, P_4. If $x_1 \in V(P_3)$, then w_2-P_2-x_2-B-x_1-P_3-w_3 is a double F-line, and $B' \cup P_4$ is an F-line, and they are disjoint, a contradiction. If $x_1 \in V(P_4)$, then w_2-P_2-x_2-B-x_1-P_4-w_4 is a double F-line, and P_3 is an X-line, and they are disjoint, a contradiction. This proves (8).

(9) *There is no branch B' of (K',W) with ends x_3,x_4, such that $F' \setminus E(B') = \delta_{K'}(x_3) \cup \delta_{K'}(x_4)$.*

For suppose that B' is such a branch. Again there are two cases depending whether B,B' are vertex-disjoint or not. First suppose that $x_1 \in V(B')$ say, and as in (7) we may choose three paths P_i of K from x_i to $w_i \in W$ for $i = 2,3,4$, pairwise vertex-disjoint. For $i = 3,4$, let Q_i be the subpath of B' between x_1 and x_i, and let $Q_2 = B$. Thus P_3,P_4 are F-lines. Suppose that $x_1 = x_3$. Then from (1) (applied to $\{x_3,x_4\}$) there is an edge of $Q_2 \setminus \{x_3\}$ in F, and so $P_2 \cup Q_2 \setminus \{x_3\}$ is an F-line disjoint from P_3,P_4, a contradiction. Thus $x_1 \neq x_3$, and similarly $x_1 \neq x_4$. By (1) (applied to $\{x_3,x_4\}$) there is an edge of F in $Q_2 \cup Q_3 \cup Q_4$ not incident with either of x_3,x_4; but hence there is an F-line in $P_2 \cup Q_2 \cup Q_3 \cup Q_4 \setminus \{x_3,x_4\}$, and it is disjoint from P_3,P_4, a contradiction. This proves (9) in the case that B,B' are not disjoint.

Thus we may assume that B,B' are disjoint. By (1) (applied to $\{x_3,x_4\}$) at least one edge of B is in F. There are three vertex-disjoint paths P_2,P_3,P_4 from $\{x_1,\ldots,x_4\}$ to W, such that for $i = 3,4$, P_i is from x_i to $w_i \in W$ say. We may assume that P_2 is from x_2 to w_2, and $x_1 \notin V(P_2)$ (possibly $x_1 \in V(P_3 \cup P_4)$). Thus P_3,P_4 are disjoint F-lines, and so $P_2 \cup B \setminus \{x_1\}$ is not an F-line; and therefore the edge of B incident with x_1 is the unique edge of B in F. Since $B \cup P_2$ is an F-line, it follows that x_1 belongs to one of P_3,P_4, and we may assume it belongs to P_3 from the symmetry. Then w_2-P_2-x_2-B-x_1-P_3-w_3 is a double F-line, and P_4 is an F-line, and they are disjoint, a contradiction. This proves (9).

But (4)–(9) are contrary to the inductive hypothesis applied to the frame (K',W). This proves that our assumption of (1) was false, and so proves 4.1. ■

5 The main proof

In this section we prove the "only if" half of 3.2. We need to show that if G is a connected graph and $Z \subseteq V(G)$ with $|Z| \geq 2$ is constricted, then (G,Z) admits an extended H-strip decomposition for some graph H. The result is trivial if $|Z| = 2$, so we may assume that $|Z| \geq 3$. Therefore, throughout this section we assume that G is a connected graph, $Z \subseteq V(G)$ with $|Z| \geq 3$, and Z is constricted in G. We shall prove a series of lemmas about the pair (G,Z).

Let (K,W) be a frame. We say it is a *frame for* (G,Z) if $E(K) \subseteq V(G)$, and

- for all distinct $e,f \in E(K)$, e,f have a common end in K if and only if $e,f \in V(G)$ are adjacent in G
- Z is the set of edges of K incident with a vertex in W.

We begin with:

5.1 *There is a frame for (G,Z).*

Proof. Since G is connected, we may choose $X \subseteq V(G)$ with $Z \subseteq X$, minimal such that $G|X$ is connected.

(1) *For each $v \in X \setminus Z$, $G|(X \setminus \{v\})$ has exactly two components, and they both contain at least one vertex of Z.*

For $G|(X \setminus \{v\})$ is not connected, from the minimality of X. Let its components be $C_1, \ldots, C_k$ say where $k \geq 2$. If $C_i \cap Z = \emptyset$, let $X' = X \setminus V(C_i)$; then $G|X'$ is connected and $Z \subseteq X'$, contrary to the minimality of X. Thus each C_i contains at least one vertex of Z. Suppose that $k \geq 3$, and choose $z_i \in V(C_i) \cap Z$ for $i = 1, 2, 3$. Since $G|X$ is connected, there are paths P_1, P_2, P_3 of $G|X$ between v and z_1, z_2, z_3 respectively, with $V(P_i) \subseteq C_i \cup \{v\}$, and if we choose P_1, P_2, P_3 with minimal union then their union is an induced tree of G, containing three members of Z, contradicting that Z is constricted in G. This proves (1).

For each $v \in X \setminus Z$, let A_v, B_v be the vertex sets of the two components of $G|(X \setminus \{v\})$.

(2) *For each $v \in X \setminus Z$, the set of neighbours of v in A_v is a clique, and so is the the set of neighbours of v in B_v.*

For suppose that $u_1, u_2 \in A_v$ are nonadjacent, and are both adjacent to v. Choose $z_3 \in B_v \cap Z$, and let P_3 be an induced path between v and z_3 with vertex set in $B_v \cup \{v\}$. From the minimality of X, for $i = 1, 2$ there exists $z_i \in Z$ such that every path of $G|X$ between v and z_i contains u_i; let P_i be some such path, induced. Consequently $u_1 \notin V(P_2)$, since P_2 is induced and $u_2 \in V(P_2)$, and similarly $u_2 \notin V(P_1)$. Hence $z_1 \neq z_2$, and $V(P_1), V(P_2) \subseteq A_v \cup \{v\}$. Since every path of $G|X$ between v and z_1 contains u_1, it follows that $V(P_1) \setminus \{v, u_1\}$ is disjoint from $V(P_2) \setminus \{v\}$, and there is no edge between these two sets. Similarly there is no edge between $V(P_1) \setminus \{v\}$ and $V(P_2) \setminus \{v, u_2\}$; and therefore there is no edge between $V(P_1) \setminus \{v\}$ and $V(P_2) \setminus \{v\}$, since u_1, u_2 are nonadjacent. Hence $P_1 \cup P_2 \cup P_3$ is an induced tree in G, contradicting that Z is constricted. This proves (2).

(3) *For each $z \in Z$, the set of neighbours of z in $X \setminus \{z\}$ is a clique.*

The proof is similar to that of (2). Suppose that $u_1, u_2 \in X \setminus \{z\}$ are nonadjacent, and both adjacent to z. From the minimality of X, there exist $z_i \in Z$ such that every path of $G|X$ between z and z_i contains u_i; let P_i be such a path, induced, for $i = 1, 2$. Then $z_1, z_2 \neq z$, and as in (2), there are no edges between $V(P_1) \setminus \{z\}$ and $V(P_2) \setminus \{z\}$. But then $P_1 \cup P_2$ is an induced tree containing z, z_1, z_2, a contradiction. This proves (3).

From (2) and (3) it follows that $G|X$ is the line graph of a tree K; thus $E(K) = X$, and for $x, y \in X$, x, y are adjacent in G if and only if some vertex of K is incident with them both. Let W be the set of vertices of K that have degree one in K. By (3), every $z \in Z$ is incident in K with a member of W. Moreover, if $x \in E(K)$ is incident with a member of W, and $x \notin Z$, then one of A_x, B_x is empty, which is impossible; so Z is equal to the set of edges of K incident with members of W. But then (K, W) is a frame for (G, Z). This proves 5.1. ■

Let η be an H-strip structure in (G, Z). If $e \in E(H)$ with ends u, v, an *e-rung* of η means an induced path $G|\eta(e)$ with vertices $p_1, \ldots, p_k$ in order, where for $1 \leq i \leq k$, $p_i \in \eta(e, u)$ if and only if $i = 1$, and $p_i \in \eta(e, v)$ if and only if $i = k$. (Possibly $k = 1$.) An H-strip structure η in (G, Z) is said to be *connected* if for every $e \in E(H)$, $\eta(e)$ is nonempty, and $\eta(e)$ is the union of the vertex sets of the e-rungs of η.

5.2 *There is a graph H with the following properties, where W denotes the set of vertices of H of degree one:*

- *(H, W) is a frame*
- *no vertex of H has degree two*
- *there is a connected H-strip structure in (G, Z)*
- *subject to these three conditions, $|E(H)|$ is maximum.*

Proof. By 5.1, there is a frame (K, W) for (G, Z). Let W_2 be the set of vertices of K that have degree two, and let W_3 be the set that have degree at least three; thus W, W_2, W_3 are pairwise disjoint and have union $V(K)$. Let H be the graph with vertex set $W \cup W_3$, in which vertices u, v are adjacent if there is a branch of K with ends u, v. Hence for each edge $e \in E(H)$ there is a branch B_e of K with the same ends as e. Thus (H, W) is a frame (though no longer a frame for (G, Z), in general), and no vertex of H has degree two. Define η as follows:

- for each $e \in E(H)$, $\eta(e) = E(B_e) \subseteq V(G)$
- for each $e \in E(H)$ incident with $v \in V(H)$, $\eta(e, v) = \{f\}$ where $f \in E(B_e) \subseteq V(G)$ is the edge of B_e incident with v.

It follows that η is a connected H-strip structure in (G, Z). Thus the first three conditions of the theorem are satisfied. Since the sets $\eta(e)$ ($e \in E(H)$) are nonempty (since η is connected) and pairwise disjoint, it follows that $|E(H)| \leq |V(G)|$ for every choice of H satisfying the first three conditions above, and therefore the fourth can also be satisfied. This proves 5.2. ■

Henceforth in the section, H, W will be as in 5.2. Moreover, η will be a connected H-strip structure in (G, Z), chosen with $\cup\eta$ maximal, where $\cup\eta$ denotes the union of all the sets $\eta(e)$ ($e \in E(H)$). For each $e = uv \in E(H)$, define $M(e) = \eta(e, u) \cap \eta(e, v)$. For each $v \in V(H)$, define

$$N(v) = \bigcup_{e \in \delta_H(v)} \eta(e, v),$$

and for every triangle $D = \{v_1, v_2, v_3\}$ of H, define

$$N(D) = M(v_1v_2) \cup M(v_2v_3) \cup M(v_3v_1).$$

5.3 *Let $p \in V(G) \setminus \cup\eta$, and let Y denote the set of all neighbours of p in $\cup\eta$. Then either*

- *there is an edge e of H such that $Y \subseteq \eta(e)$, or*
- *there is a vertex v of H such that $Y \subseteq N(v)$, or*
- *there is a triangle D of H such that $Y \subseteq N(D)$.*

Proof.

(1) *For each $e \in E(H)$, let R_e be an e-rung. Let R be the union of all the sets $V(R_e)$ ($e \in E(H)$). Then one of the following holds:*

- *there exists $e \in E(H)$ with $Y \cap R \subseteq V(R_e)$, or*
- *there exists $v \in V(H)$ such that $Y \cap R \subseteq N(v)$, or*
- *there is a triangle $\{u,v,w\}$ of H such that R_{uv}, R_{vw}, R_{wu} all have length zero, and $Y \cap R = V(R_{uv}) \cup V(R_{vw}) \cup V(R_{wu})$, or*
- *there exists $e = uv \in E(H)$ such that $u \notin W$ and $Y \cap (R \setminus V(R_e)) = N(u) \cap (R \setminus V(R_e))$ and $Y \cap V(R_e) \not\subseteq N(u)$, or*
- *there exists $e = uv \in E(H)$ such that $u, v \notin W$ and $Y \cap (R \setminus V(R_e)) = (N(u) \cup N(v)) \cap (R \setminus V(R_e))$.*

For let K be obtained from H by replacing each edge $e \in E(H)$ by a path with edges the vertices of R_e in order, in the natural way, so that $G|R$ is the line graph of K. Thus (K, W) is a frame for (G, Z). Let $F = R \cap Y$; then $F \subseteq E(K)$. Moreover, F is focused, since Z is constricted in G. Hence one of the six outcomes of 4.1 holds. If 4.1.3 holds and x, y are as in 4.1.3, then there is a frame (K', W) for G, where K' is obtained from K by adding p to K as a new edge incident with x, y, contrary to the maximality of $|E(H)|$. If 4.1.1 holds, then the second outcome of (1) holds. Similarly if one of 4.1.2, 4.1.4, 4.1.5, 4.1.6 holds then respectively the third, first, fourth and fifth outcome of (1) holds. This proves (1).

(2) *If there is an edge $e = v_1v_2$ of H such that $Y \subseteq \eta(e) \cup N(v_1) \cup N(v_2)$ then the theorem holds.*

Suppose there is such an edge $e = v_1v_2$. For each $f \in E(H)$ choose an f-rung R_f. For $i = 1, 2$, let E_i be the set of all edges of H that are incident with v_i and different from e; thus $|E_i| \neq 1$. Let A_i be the set of all $f \in E_i$ such that Y contains the end of R_f in $N(v_i)$, and let $B_i = E_i \setminus A_i$.

Suppose first that $Y \cap N(v_2) \subseteq \eta(e)$. If also $Y \cap N(v_1) \subseteq \eta(e)$ then $Y \subseteq \eta(e)$ and the theorem holds, so we may assume that $Y \cap N(v_1) \not\subseteq \eta(e)$. Moreover, we may assume that $Y \cap \eta(e) \not\subseteq N(v_1)$, for otherwise $Y \subseteq N(v_1)$ and the theorem holds. Hence we may choose the rungs R_f ($f \in E(H)$) such that there exists $a_1 \in A_1$ and $Y \cap V(R_e) \not\subseteq N(v_1)$. By (1), it follows routinely that $B_1 = \emptyset$. Since this holds for all choices of R_f ($f \neq a_1, e$), we deduce that $Y \cap \eta(f) = \eta(f, v_1)$ for all $f \in E(H) \setminus \{a_1, e\}$ incident with v_1. Since $|E_1| \neq 1$, it follows by exchanging the roles of a_1 and some other member of E_1 that $Y \cap \eta(a_1) = \eta(a_1, v_1)$, and so $N(v_1) \setminus \eta(e) = Y \setminus \eta(e)$. But then a can be added to $\eta(e)$ and to $\eta(e, v_1)$, contrary to the maximality of $\cup\eta$. Thus we may assume that $Y \cap N(v_2) \not\subseteq \eta(e)$, and similarly $Y \cap N(v_1) \not\subseteq \eta(e)$. Hence we may choose the R_f ($f \in E(H)$) such that A_1, A_2 are both nonempty.

Suppose that there is a choice of the R_f ($f \in E(H)$) such that for some $a_1 \in A_1$ and $a_2 \in A_2$, either a_1, a_2 are disjoint edges of H, or not both R_{a_1}, R_{a_2} have length zero. By (1), B_1, B_2 are both empty. Since this holds for all choices of R_f ($f \neq a_1, a_2$), we deduce that $Y \cap \eta(f) = \eta(f, v_i)$ for $i = 1, 2$ and for all $f \in E(G) \setminus \{e, a_1, a_2\}$ incident with v_i. Now there exist $a_1' \in A_1$ and $a_2' \in A_2$ such that $a_1' \neq a_1$ and a_1', a_2' are disjoint edges of H, since $|E_1|, |E_2| \geq 2$ (possibly $a_2' = a_2$). We have seen that we can choose $R_{a_1'}, R_{a_2'}$ such that they both meet Y, and so by the same argument with a_1, a_2 replaced by a_1', a_2', it follows that $Y \cap \eta(a_1) = \eta(a_1, v_1)$, and so $N(v_i) \setminus \eta(e) \subseteq Y$ for $i = 1$, and also for $i = 2$ by the symmetry. But then a can be added to $\eta(e), \eta(e, v_1)$ and $\eta(e, v_2)$, contrary to the maximality of $\cup\eta$.

Hence for every choice of the R_f ($f \in E(H)$) with A_1, A_2 both nonempty, we have that $|A_1| = |A_2| = 1$, say $A_i = \{a_i\}$ for $i = 1, 2$, and R_{a_1}, R_{a_2} both have length zero, and a_1, a_2 have a common end w in H. Take some such choice of the R_f ($f \in E(H)$). For $f \in B_1$, we deduce that we cannot replace R_f with some other f-rung that meets Y, and therefore $Y \cap (N(v_1) \setminus \eta(e)) \subseteq \eta(a_1, v_1)$. Moreover, there is no a_1-rung that meets Y with length greater than zero, and so $Y \cap \eta(a_1) \subseteq \eta(a_1, v_1) \cap \eta(a_1, w)$. Similarly $Y \cap (N(v_2) \setminus \eta(e)) \subseteq \eta(a_2, v_2)$, and $Y \cap \eta(a_2) \subseteq \eta(a_2, v_2) \cap \eta(a_2, w)$. We may assume that $Y \not\subseteq N(w)$, and so $Y \cap \eta(e) \neq \emptyset$. Choose R_e with $Y \cap V(R_e)$ nonempty. By (1), every such choice of R_e has length zero. But then $Y \subseteq N(D)$ where D is the triangle $\{v_1, v_2, w\}$, and the theorem holds. This proves (2).

(3) *For each $e \in E(H)$, let R_e be an e-rung. If either the fourth or fifth outcome of (1) holds, then the theorem holds.*

For let R be the union of the sets $V(R_e)$ ($e \in E(H)$). Suppose first that there exists $e = uv \in E(H)$ such that $u \notin W$ and $Y \cap (R \setminus V(R_e)) = N(u) \cap (R \setminus V(R_e))$ and $Y \cap V(R_e) \not\subseteq N(u)$. Then (1) implies that $\eta(f) \cap Y = \emptyset$ for all $f \in E(H)$ not incident with u (because otherwise we could make another choice of R_f so that (1) was violated); and $\eta(f) \cap Y \subseteq N(u)$ for each $f \neq e$ incident with u, for the same reason. But then $Y \subseteq \eta(e) \cup N(u) \cup N(v)$, and so the theorem holds by (2).

Next suppose that there exists $e = uv \in E(H)$ such that $u, v \notin W$ and $Y \cap (R \setminus V(R_e)) = (N(u) \cup N(v)) \cap (R \setminus V(R_e))$. Again by (1), $Y \cap \eta(f) = \emptyset$ for each edge f of H not incident with u, v, and $Y \cap \eta(f) \subseteq \eta(f, u)$ for every $f \neq e$ incident with u, and a similar result holds with u, v exchanged. But then $Y \subseteq \eta(e) \cup N(u) \cup N(v)$ and the theorem holds by (2). This proves (3).

Suppose that there exists $e = v_1v_2 \in E(H)$ such that $Y \cap \eta(e) \not\subseteq \eta(e, v_1) \cup \eta(e, v_2)$. Choose an e-rung R_e such that some internal vertex of R_e belongs to Y. By (1) and (3), $Y \cap \eta(f) = \emptyset$ for all $f \in E(H) \setminus \{e\}$, and so the theorem holds. Hence every vertex in Y belongs to at least one of the sets $N(v)$ ($v \in V(H)$). Suppose that some $y \in Y$ belongs to exactly one such set; say $y \in \eta(e, v) \setminus M(e)$, where $e = uv$. By (1), $Y \setminus \eta(e) \subseteq N(v)$, and so the theorem holds by (2). Thus we may assume that every vertex in Y belongs to $M(e)$ for some $e \in E(H)$. Let F be the set of all $f \in E(H)$ with $Y \cap M(f) \neq \emptyset$. If there exist $e, f \in F$ with no common end in H, then the theorem holds by (1) and (3); if there is some vertex v of H incident with every edge in F, then $Y \subseteq N(v)$ and the theorem holds; and if neither of these hold, then $|F| = 3$, and the three members of F are the edges of a triangle D of H, and $Y \subseteq N(D)$ and the theorem holds. This proves 5.3. ■

5.4 *Let $X \subseteq V(G) \setminus \cup\eta$ such that $G|X$ is connected, and let Y be the set of all attachments of $G|X$ in $\cup\eta$. Then either*

- *there is an edge e of H such that $Y \subseteq \eta(e)$, or*
- *there is a vertex v of H such that $Y \subseteq N(v)$, or*
- *there is a triangle D of H such that $Y \subseteq N(D)$.*

Proof. Suppose that this is false for some X, and choose X minimal such that 5.4 is false for X.

(1) *There exist y_1, y_2 in Y such that $\{y_1, y_2\} \not\subseteq \eta(e)$ for each $e \in E(H)$, and $\{y_1, y_2\} \not\subseteq N(v)$*

for each $v \in V(H)$, and $\{y_1, y_2\} \not\subseteq N(D)$ for each triangle D of H.

For suppose first that for some $e = v_1v_2 \in E(H)$, there exists $y_1 \in Y \cap \eta(e)$ with $y_1 \notin \eta(e, v_1) \cup \eta(e, v_2)$. Now $Y \not\subseteq \eta(e)$; choose $y_2 \in Y \setminus \eta(e)$, and then y_1, y_2 satisfy (1). We may therefore assume that $Y \subseteq \cup_{v \in V(H)} N(v)$.

Next suppose that some $y_1 \in Y$ belongs to exactly one of the sets $N(v)$ $(v \in V(H))$; say $y_1 \in \eta(e, v_1) \setminus M(e)$, where $e = v_1v_2 \in E(H)$. Since $Y \not\subseteq N(v_1)$, there exists $y_2 \in Y \setminus N(v_2)$. If also $y_2 \notin \eta(e)$, then the pair y_1, y_2 satisfies (1), so we may assume that $y_2 \in \eta(e) \setminus N(v_1)$. We already assumed that every member of Y belongs to one of the sets $N(v)$ $(v \in V(H))$, and so $y_2 \in \eta(e, v_2) \setminus M(e)$. This restores the symmetry between v_1 and v_2. Since $Y \not\subseteq \eta(e)$, there exists $y_3 \in Y$ with $y_3 \notin \eta(e)$. Since we may assume that the pair y_1, y_3 does not satisfy (1), it follows that $y_3 \in N(v_1)$, and similarly $y_3 \in N(v_2)$. Let $f \in E(H)$ with $y_3 \in \eta(f)$; then f is incident with v_1 since $y_3 \in N(v_1)$, and similarly f is incident with v_2, and so $f = e$, a contradiction since $y_3 \notin \eta(e)$. We may therefore assume that every $y \in Y$ belongs to $M(e)$ for some $e \in E(H)$.

Let F be the set of all edges $f \in E(H)$ such that $Y \cap M(f) \neq \emptyset$. Suppose that there exist $e, f \in F$ with no common end in H. Choose $y_1 \in Y \cap M(e)$ and $y_2 \in Y \cap M(f)$; then y_1, y_2 satisfy (1). Thus we may assume that every two edges in F share an end. Consequently either there is a vertex $v \in V(H)$ incident with every member of F, or $|F| = 3$ and the three edges in F form a triangle D of H. In the first case $Y \subseteq N(v)$, and in the second $Y \subseteq N(D)$, in either case a contradiction. This proves (1).

For each $p \in X$, let $Y(p)$ denote the set of all $v \in \cup\eta$ adjacent to p; and for $P \subseteq X$, let $Y(P) = \cup_{p \in P} Y(p)$. Thus $Y = Y(X)$. Let y_1, y_2 be as in (1). Then y_1, y_2 are nonadjacent. Since $G|X$ is connected, there is an induced path of G with vertices y_1-p_1-p_2-$\cdots$-p_k-y_2 in order. By 5.3 it follows that $k > 1$. From the minimality of X, $X = \{p_1, \ldots, p_k\}$. Let $P_1 = X \setminus \{p_k\}$, and $P_2 = X \setminus \{p_1\}$. Then for $i = 1, 2$, the minimality of X implies that either

- there is an edge e of H such that $Y(P_i) \subseteq \eta(e)$, or
- there is a vertex v of H such that $Y(P_i) \subseteq N(v)$, or
- there is a triangle D of H such that $Y(P_i) \subseteq N(D)$.

Thus there are three possibilities for $Y(P_1)$ and three for $Y(P_2)$, and we need to check these nine possibilities individually. For each $e \in E(H)$ choose an e-rung R_e (in some cases we shall need to choose the e-rungs subject to some further conditions), and let K be the graph obtained from H by replacing every edge e of H by a path whose edges are the vertices of R_e in the corresponding order. Then (K, W) is a frame for (G, Z). (Thus K depends on the choice of the rungs R_e. In what follows it is sometimes useful to change one or more of the paths R_e, and K is assumed to change correspondingly, although we may not say so explicitly.) We remark that the subgraph of G induced on the union of the sets $V(R_e)$ $(e \in E(H))$ is the line graph of K.

(2) *There do not exist $v_1, v_2 \in V(H)$ such that $Y(P_1) \subseteq N(v_1)$ and $Y(P_2) \subseteq N(v_2)$.*

For suppose that such v_1, v_2 exist. Since $Y = Y(P_1) \cup Y(P_2)$, it follows that $v_1 \neq v_2$. Now v_1, v_2

may or may not be adjacent in H. If they are adjacent, let $f = v_1v_2 \in E(H)$, and otherwise f is undefined. For $1 < i < k$,

$$Y(p_i) \subseteq Y(P_1) \cap Y(P_2) \subseteq N(v_1) \cap N(v_2) = M(f),$$

(where $M(f) = \emptyset$ if f is not defined) and so $Y \subseteq Y(p_1) \cup Y(p_k) \cup M(f)$. Suppose first that for $i = 1, 2$, either $N(v_i) \setminus \eta(f, v_i) \subseteq Y(p_i)$ or $(N(v_i) \setminus \eta(f, v_i)) \cap Y(p_i) = \emptyset$ (where $\eta(f, v_i) = \emptyset$ if f is undefined). If $(N(v_i) \setminus \eta(f, v_i)) \cap Y(p_i) = \emptyset$ for $i = 1, 2$, then $Y \subseteq \eta(f)$ (where $\eta(f) = \emptyset$ if f is undefined), a contradiction, so we may assume that $N(v_1) \setminus \eta(f, v_1) \subseteq Y(p_1)$ say. If $N(v_2) \setminus \eta(f, v_2) \cap Y(p_k) = \emptyset$, then f is defined since $Y(p_k) \neq \emptyset$, and we can add $p_1, \dots, p_k$ to $\eta(f)$, and add p_1 to $\eta(f, v_1)$, contrary to the maximality of $\cup\eta$; if $N(v_2) \setminus \eta(f, v_2) \subseteq Y(p_k)$ and f is defined, we can add $p_1, \dots, p_k$ to $\eta(f)$, add p_1 to $\eta(f, v_1)$, and add p_k to $\eta(f, v_2)$, again contrary to the maximality of $\cup\eta$; so we may assume that $N(v_2) \setminus \eta(f, v_2) \subseteq Y(p_k)$ and f is undefined. Thus $Y(p_1) = N(v_1)$ and $Y(p_k) = N(v_2)$. Let K' be obtained from K by adding a path between v_1, v_2 with edges $p_1, \dots, p_k$ in order; then (K', W) is a frame for (G, Z), contrary to the maximality of $|E(H)|$.

Thus we may assume that for some $i \in \{1, 2\}$, $N(v_i) \setminus \eta(f, v_i) \not\subseteq Y(p_i)$ and $(N(v_i) \setminus \eta(f, v_i)) \cap Y(p_i) \neq \emptyset$. For $i = 1, 2$, let E_i be the set of edges of K incident with v_i if f is undefined, and let E_i is the set of edges of K incident with v_i not in the branch of K between v_1, v_2 if f is defined. Therefore we may choose the paths R_e ($e \in E(H)$) such that at least three of the sets A_1, B_1, A_2, B_2 are nonempty, where for $i = 1, 2$, $A_i = E_i \cap Y(P_i)$, and $B_i = E_i \setminus A_i$. We may also assume that for $i = 1, 2$, either $y_i \in E(K)$, or $|B_i| = 1$ and one of A_j, B_j is empty, where $\{i, j\} = \{1, 2\}$. (To see this, observe that if say $y_1 \notin E(K)$, let $y_1 \in \eta(e)$ say, and choose an e-rung R'_e containing y_1; then if either $|B_1| \neq 1$, or A_2, B_2 are both nonempty, we may replace R_e by R'_e, and still satisfy all the other requirements.) From the symmetry we may assume that $A_1, B_1 \neq \emptyset$, and hence $y_2 \in E(K)$. Since (K, W) is a frame, there are three paths Q_1, Q_2, Q_3 of K such that

- Q_1 is between v_1 and $w_1 \in W$,
- Q_2 is between v_1 and $w_2 \in W$,
- Q_3 is between v_2 and $w_3 \in W$,
- $V(Q_1 \cap Q_2) = \{v_1\}$,
- Q_3 is vertex-disjoint from both Q_1, Q_2,
- the edge $a_1 = u_1v_1$ of Q_1 incident with v_1 belongs to A_1,
- the edge of Q_2 incident with v_1 belongs to B_1, and
- Q_3 is an induced subgraph of K.

If the edge of Q_3 incident with v_2 belongs to A_2, then

$$G|(X \cup E(Q_1) \cup E(Q_2) \cup E(Q_3))$$

is an induced tree containing three vertices of Z, a contradiction. Thus the first edge of Q_3 is in B_2. If $y_2 \notin A_2$, then f is defined and y_2 belongs to the branch of K between v_1, v_2, and this branch has length at least two (since $y_2 \notin N(v_1)$); but then

$$G|(X \cup E(Q_1) \cup E(Q_2) \cup E(Q_3) \cup \{y_2\})$$

is an induced tree containing three vertices of Z, a contradiction. Thus $y_2 \in A_2$. Let $a_2 \in A_2$, with ends v_2, u_2 say. Then $u_2 \notin V(Q_3)$, since Q_3 is an induced subgraph of K. If $u_2 \notin V(Q_1) \cup V(Q_2)$, then

$$G|(X \cup E(Q_1) \cup E(Q_2) \cup E(Q_3) \cup \{a_2\})$$

is an induced tree containing three vertices of Z, a contradiction. Thus $u_2 \in V(Q_1) \cup V(Q_2)$. If $u_2 \in V(Q_1) \setminus \{u_1, v_1\}$, let Q_1' be the path of Q_1 between u_2 and w_1; then

$$G|(X \cup E(Q_1') \cup E(Q_2) \cup E(Q_3) \cup \{a_1, a_2\})$$

is an induced tree containing three vertices of Z, a contradiction. If $u_2 \in V(Q_2) \setminus \{v_1\}$, let Q_2' be the path of Q_2 between u_2 and w_2; then

$$G|(X \cup E(Q_1) \cup E(Q_2') \cup E(Q_3) \cup \{a_1, a_2\})$$

is an induced tree containing three vertices of Z, a contradiction. Thus $u_2 \in \{u_1, v_1\}$. But $u_2 \neq v_1$ from the definition of A_2, and so $u_2 = u_1$. Hence $|A_2| = 1$, and so $A_2 = \{y_2\}$, and $B_2 \neq \emptyset$. This restores the symmetry between v_1, v_2. From the same argument with v_1, v_2 exchanged, applied to the paths v_2-u_2-Q_1-w_1, Q_3 and Q_2, it follows that $A_1 = \{y_1\}$, contradicting that y_1, y_2 are nonadjacent in G. This proves (2).

(3) *There do not exist $v_1 \in V(H)$ and $e \in E(H)$ such that $Y(P_1) \subseteq N(v)$ and $Y(P_2) \subseteq \eta(e)$.*

For suppose that such v_1, e exist. Then e may or may not be incident with v_1. If e is not incident with v_1, then $Y = Y(p_1) \cup Y(p_2)$, while if e is incident with v_1 then $Y \subseteq Y_1 \cup Y_2 \cup \eta(e, v_1)$. Suppose first that $N(v_1) \subseteq Y(P_1) \cup \eta(e)$. Since $Y(P_2) \not\subseteq N(v_1)$, we may choose the e-rung R_e such that at least one vertex of $V(R_e) \cap Y(P_2)$ does not belong to $N(v_1)$. If e is incident with v_1, then we can add $p_1, \dots, p_k$ to $\eta(e)$ and add p_1 to $\eta(e, v_1)$, contrary to the maximality of $\cup\eta$. Thus e is not incident with v_1, and so $Y(p_1) = N(v_1)$. Let e be incident with v_2, v_3 in H. There are three vertex-disjoint paths Q_1, Q_2, Q_3 of K such that for $i = 1, 2, 3$, Q_i is between v_i and some $w_i \in W$. If exactly one vertex of R_e is in $Y(p_2)$, then

$$G|(E(Q_1) \cup E(Q_2) \cup E(Q_3) \cup V(R_e) \cup \{p_1, \dots, p_k\})$$

is an induced tree of G containing three members of Z, a contradiction. If there are two nonadjacent vertices of R_e that are both in $Y(p_2)$, let S_2, S_3 be minimal subpaths of R_e that meet both $Y(p_2)$ and $N(v_2), N(v_3)$ respectively; then there is no edge between $V(S_2)$ and $V(S_3)$, and

$$G|(E(Q_1) \cup E(Q_2) \cup E(Q_3) \cup V(S_2) \cup V(S_3) \cup \{p_1, \dots, p_k\})$$

is an induced tree of G containing three members of Z, a contradiction. Thus there are exactly two vertices in R_e that belong to $Y(p_2)$, say x, y, and they are adjacent in G. There is a branch of K with edge set the vertex set of R_e; let t be the vertex of this branch that is incident with x, y in K. Let K' be obtained from K by adding a path between v_1, t with edges $p_1, \dots, p_k$ in order; then (K', W) is a frame for (G, Z), contrary to the maximality of $|E(H)|$. This proves that $N(v_1) \not\subseteq Y(P_1) \cup \eta(e)$.

Hence we may choose the R_f ($f \in E(H)$) such that both A_1, B_1 are nonempty, where E_1 denotes the set of edges of K incident with v_1 and not in $V(R_e)$, and $A_1 = E_1 \cap Y(p_1)$, and $B_1 = E_1 \setminus Y(p_1)$.

Let e be incident with v_2, v_3 in H. Since (K, W) is a frame, there are three paths Q_1, Q_2, Q_3 of K, such that Q_1 is from v_1 to some $w_1 \in W$, Q_2 is from v_1 to some $w_2 \in W$, Q_3 is from one of v_2, v_3 to some $w_3 \in W$, $V(Q_1) \cap V(Q_2) = \{v_1\}$, Q_3 is vertex-disjoint from both Q_1 and Q_2, the edge of Q_1 incident with v_1 belongs to A_1, and the edge of Q_2 incident with v_1 belongs to B_1. Moreover, we may assume that only one of v_2, v_3 belongs to $V(Q_3)$, say v_3. It follows that $v_1 \neq v_3$. Since none of Q_1, Q_2, Q_3 contain both v_2, v_3, we may alter our choice of R_e without affecting Q_1, Q_2, Q_3; and since $Y(P_2) \not\subseteq N(v_2)$ by (2), we may choose R_e such that some vertex of R_e belongs to $Y(P_2) \setminus N(v_2)$. Let S be a minimal subpath of R_e that meets both $Y(P_2)$ and $N(v_3)$. Then

$$G|(E(Q_1) \cup E(Q_2) \cup E(Q_3) \cup V(S) \cup \{p_1, \ldots, p_k\})$$

is an induced tree of G containing three vertices of Z, a contradiction. This proves (3).

(4) *There do not exist edges e_1, e_2 of H such that $Y(P_i) \subseteq \eta(e_i)$ for $i = 1, 2$.*

For suppose that such edges exist. Then $e_1 \neq e_2$; let e_i have ends u_i, v_i for $i = 1, 2$. For $i = 1, 2$, let T_i be the branch of K with edge set $V(R_{e_i})$. Since $\eta(e_1) \cap \eta(e_2) = \emptyset$, it follows that $Y = Y(p_1) \cup Y(p_2)$. We may assume that $v_1 \neq u_2, v_2$ and $v_2 \neq u_1, v_1$; that is, u_1, u_2, v_1, v_2 are all distinct except that possibly $u_1 = u_2$. For $i = 1, 2$, choose R_{e_i} such that some vertex of R_{e_i} belongs to $Y(P_i)$, and in addition, choose R_{e_i} such that some vertex of R_{e_i} belongs to $Y(P_i)$ and not to $N(u_i)$ (this is possible since $Y(P_i) \not\subseteq N(u_i)$ by (3)). Suppose first that for $i = 1, 2$, exactly two vertices x_i, y_i of R_{e_i} belong to $Y(P_i)$ and they are adjacent. Thus x_i, y_i are edges of T_i with a common end t_i say. Let K' be obtained from K by adding a new path between t_1, t_2 with edges $p_1, \ldots, p_k$ in order; then (K', W) is a frame for (G, Z), contrary to the maximality of $|E(H)|$. We may therefore assume that either exactly one vertex of R_{e_1} belongs to $Y(p_1)$, or two nonadjacent vertices of R_{e_1} belong to $Y(p_1)$. Let Q_1, Q_2, Q_3 be vertex-disjoint paths of K between $\{u_1, v_1, v_2\}$ and W, where Q_1 is between u_1 and some $w_1 \in W$, and Q_2 is between v_1 and some w_2 in W, and Q_3 is between v_2 and some $w_3 \in W$. (Possibly u_2 belongs to one of these paths.) Then some edge of T_2 belongs to $Y(p_2)$ and is not incident with u_2, from the choice of R_{e_2}. Hence there is a path S_3 of $T_2 \cup Q_3$, with first vertex in $V(T_2)$ and last vertex w_3, such that the first edge and no other edge of S_3 belongs to $Y(p_2)$, and Q_1, Q_2, S_3 are pairwise vertex-disjoint. If only one vertex of R_{e_1} is in $Y(p_1)$, then

$$G|(E(Q_1) \cup E(Q_2) \cup E(S_3) \cup \{p_1, \ldots, p_k\})$$

is an induced tree of G containing three members of Z, a contradiction. Thus there are two nonadjacent vertices in $V(R_{e_1}) \cap Y(p_1)$, and so there are vertex-disjoint subpaths S_1, S_2 of $Q_1 \cup T_1 \cup Q_2$, such that for $i = 1, 2$, S_i has first vertex in $V(T_1)$, first edge and no other edge in $Y(p_1)$, and last vertex w_i. But then

$$G|(E(S_1) \cup E(S_2) \cup E(S_3) \cup \{p_1, \ldots, p_k\})$$

is an induced tree of G containing three members of Z, a contradiction. This proves (4).

From (2),(3),(4), we may assume that $Y(P_2) \subseteq N(D)$ for some triangle $D = \{u_1, u_2, u_3\}$ of H, and that $M(u_1u_2), M(u_2u_3), M(u_1u_3)$ all contain at least one member of $Y(P_2)$. Let e_1, e_2, e_3 be the edges u_2u_3, u_3u_1, u_1u_2 of H respectively. Choose R_{e_1} of length zero such that its vertex (r_1 say) is in $Y(P_2)$, and choose R_{e_2}, R_{e_3} similarly. Thus r_1 is the edge of K joining u_2, u_3. For $i = 1, 2, 3$, let Q_i be a path of K between u_i and some $w_i \in W$, such that Q_1, Q_2, Q_3 are pairwise vertex-disjoint.

(5) $Y(P_1) \cap E(K) \subseteq \{r_1, r_2, r_3\}$.

For let $e \in Y(P_1) \cap E(K)$, and suppose first that at most one of Q_1, Q_2, Q_3 contains an end of e. Since K is connected, we may choose a path S of K with first edge in $Y(p_1)$ such that S meets one of Q_1, Q_2, Q_3; and by choosing S minimal, we may assume that S meets Q_3 and not Q_1, Q_2, and only its first edge is in $Y(p_1)$. In particular, $u_1, u_2 \notin V(S)$, and so no edge of S is in $N(D)$; and therefore no edge of S except the first is adjacent in G to any of $p_1, \ldots, p_k$. Let S' be a path of $S \cup Q_3$ between the first vertex of S and w_3. Then

$$G|(E(Q_1) \cup E(Q_2) \cup E(S') \cup \{p_1, \ldots, p_k\})$$

is an induced tree in G containing three members of Z, a contradiction. This proves that two of Q_1, Q_2, Q_3 contain ends of e. Let $e = v_1v_2$ where $v_1 \in V(Q_1)$ and $v_2 \in V(Q_2)$ say. Suppose that $v_2 \neq u_2$. For $i = 1, 2$, let S_i be a subpath of Q_i between v_i and w_i. Then

$$G|(E(S_1) \cup E(S_2) \cup E(Q_3) \cup \{e, r_1\} \cup \{p_1, \ldots, p_k\})$$

is an induced tree in G containing three members of Z, a contradiction. Thus $v_2 = u_2$, and similarly $v_1 = u_1$ and so $e = r_3$. This proves (5).

From (5), we deduce that $Y(P_1) \subseteq \eta(e_1) \cup \eta(e_2) \cup \eta(e_3)$ (for otherwise we could make a choice of the R_f for $f \neq u_1u_2, u_1u_3, u_2u_3$ that would violate (5)). Since $Y(P_1) \not\subseteq N(D)$, we may assume that there is some e_3-rung R'_{e_3} such that some vertex of R'_{e_3} belongs to $Y(P_1)$ and not to $\eta(e_3, u_2)$ (and so R'_{e_3} has length at least one). Let S be a minimal subpath of R'_{e_3} that meets both $Y(P_1)$ and $\eta(e_3, u_1)$. Then

$$G|(E(Q_1) \cup E(Q_2) \cup E(Q_3) \cup V(S) \cup \{r_1\} \cup \{p_1, \ldots, p_k\})$$

is an induced tree in G containing three members of Z, a contradiction. This proves 5.4. ■

Proof of 3.2. We have already seen the proof of the "if" half. To prove the "only if" half, we may assume that $|Z| \geq 3$. We choose H, W as in 5.2. Choose η as before; that is, η is a connected H-strip structure in (G, Z), chosen with $\cup\eta$ maximal. Let $\mathcal{C}$ be the set of all vertex sets of components of $G \setminus \cup\eta$. For $C \in \mathcal{C}$ we define its *home* as follows. Let Y be the set of attachments of $G|C$ in $G|\cup\eta$. We say that $e \in E(H)$ is the home of C if $Y \subseteq \eta(e)$; $v \in V(H)$ is the home of C if $Y \subseteq N(v)$ and no edge of H is the home of C; and a triangle D of H is the home of C if $Y \subseteq N(D)$ and there is no vertex or edge of H that is the home of C. By 5.4, each $C \in \mathcal{C}$ has a (unique) home. For each $e \in E(H)$, let $\eta'(e)$ be the union of $\eta(e)$ and all $C \in \mathcal{C}$ with home e. Define $\eta'(e, v) = \eta(e, v)$ if $v \in V(H)$ is incident with e. For each $v \in V(H)$, define $\eta'(v)$ to be the union of all $C \in \mathcal{C}$ with home v, and for each triangle D of H, let $\eta'(D)$ be the union of all $C \in \mathcal{C}$ with home D. It follows that η' is an extended H-strip decomposition of (G, Z). This proves 3.2. ■

6 The algorithm

So far, we have proved our main result, the description of the structure of the constricted pairs; now we present a polynomial-time algorithm for the following question:

CONSTRICTED. With input a graph G and $Z \subseteq V(G)$, decide whether Z is constricted.

Our main theorem 3.2 asserts that for a given pair (G, Z), either there is an induced subtree containing three members of Z, or (G, Z) admits an extended H-strip decomposition for some H, and not both. Therefore we could ask for a polynomial algorithm for the following more demanding question:

TREE-OR-DECOMPOSITION. With input a connected graph G and $Z \subseteq V(G)$ with $|Z| \geq 2$, output either an induced subtree of G containing at least three members of Z, or a graph H and an extended H-strip decomposition η of (G, Z).

We will first give an algorithm for CONSTRICTED, and then discuss how to modify it to solve TREE-OR-DECOMPOSITION. It would be nice to use 3.2 to show that some simple algorithm works, but so far we have not been able to do this. The best method we can see is just to convert the proof of the theorem to an algorithm.

Thus, we have input a graph G and a subset $Z \subseteq V(G)$, and we wish to decide whether Z is constricted in G. It is easy to reduce this question to the special case when G is connected and $|Z| \geq 3$, so from now on we assume that. Let $|V(G)| = n$. Let η be a connected H-strip structure in (G, Z), let $X \subseteq V(G) \setminus \cup\eta$, and let Y be the set of all members of $\cup\eta$ that have at least one neighbour in X. We say that X is *local* (with respect to (H, η)) if either $Y \subseteq \eta(e)$ for some $e \in E(H)$, or $Y \subseteq N(v)$ for some $v \in V(H)$, or $Y \subseteq N(D)$ for some triangle D of H (using the notation of 5.3 and 5.4).

It is easy to modify the first part of the proof of 5.4, and the proof of the "only if" half of 3.2, to yield the following subroutine.

6.1 Algorithm.

- **Input:** *A pair* (G, Z) *as above, a graph* H*, and a connected* H*-strip structure* η *in* (G, Z)*.*
- **Output:** *One of the following:*
  - *An extended* H*-strip decomposition of* (G, Z)*, or*
  - *a subset* $X \subseteq V(G) \setminus \cup\eta$*, such that* $G|X$ *is connected,* X *is not local, and* X *is minimal with these properties.*
- **Running time:** $O(n^2)$.

Let η be some connected H-strip structure in (G, Z). We define its *worth* to be $|\cup\eta| + (n+1)|E(H)|$. Thus every H-strip structure has worth at most $(n+1)^2$, since $|\cup\eta| \leq n$ and $|E(H)| \leq n$. We need another subroutine, as follows.

6.2 Algorithm.

- **Input:** *A pair* (G, Z) *as above, a graph* H*, a connected* H*-strip structure* η *in* (G, Z)*, and a subset* $X \subseteq V(G) \setminus \cup\eta$*, such that* $G|X$ *is connected,* X *is not local, and* X *is minimal with these properties.*
- **Output:** *One of the following:*

- *A graph H' and a connected H'-strip structure η' in (G, Z) with worth greater than that of η, or*
- *a (true) statement that Z is not constricted in $G|((\cup\eta) \cup X)$, and therefore not in G.*

- **Running time:** $O(n^2)$.

This can be obtained by modifying the proof of 5.3 and the latter part of the proof of 5.4 appropriately. More exactly, if $|X| = 1$ we modify the proof of 5.3, and otherwise we modify the proof of 5.4. Suppose first that $|X| = 1$. We test whether one of the five statements of step (1) of the proof of 5.3 holds. The first three are impossible since X is not local. If either the fourth or fifth holds, we can add the vertex in X to one of the strips $\eta(e)$ and produce a connected H-strip structure in (G, Z) with worth greater than that of η as required. If none of the outcomes of step (1) of the proof of 5.3 holds, it follows that either Z is not constricted in $G|((\cup\eta) \cup X)$ is not constricted and we are done, or H, η does not satisfy the hypotheses of 5.3; and in this case it is easy to modify the proof of 5.3 to yield a connected H'-strip structure η' in (G, Z) with worth greater than that of η. When $|X| > 1$ we modify the proof of 5.4 in an analogous way.

Combining these two subroutines yields:

6.3 Algorithm.

- **Input:** *A pair (G, Z) as above, a graph H, and a connected H-strip structure η in (G, Z).*
- **Output:** *One of the following:*
  - *An extended H-strip decomposition of (G, Z), or*
  - *a graph H' and a connected H'-strip structure η' in (G, Z) with worth greater than that of η, or*
  - *a statement that Z is not constricted in G.*
- **Running time:** $O(n^2)$.

To make use of 6.3 to solve CONSTRICTED, we first choose some frame (K, W) for (G, Z), by using the method of 5.1. (Or we find some induced tree containing at least three members of Z, and then we output this and stop.) This takes time $O(n^2)$, where $n = |V(G)|$. We convert this to a connected H_1-strip structure η_1 say. Inductively, having produced a graph H_i and a connected H_i-strip structure η_i of worth at least i, we apply 6.3 to H_i, η_i. If we obtain either the first or third output of 6.3 we are done, and if we obtain the second output this defines H_{i+1}, η_{i+1}, of worth at least $i + 1$. Since no η_i has worth more than $(n + 1)^2$, this process iterates at most $(n + 1)^2$ times before terminating, and since each iteration takes time $O(n^2)$, the total running time is $O(n^4)$.

To solve TREE-OR-DECOMPOSITION instead of just CONSTRICTED, note first that the algorithm just described outputs an extended H-decomposition for some H when Z is constricted, so we have half of what we need for free. We just need to modify the last output of 6.2. At that stage, instead of just the statement that Z is not constricted in G, we need to find an induced subtree of G containing three members of Z; and to do so we need to look more carefully at the proofs of 4.1, 5.3 and 5.4 at the appropriate points. Since we will exit the main recursion at this stage, we can afford to spend time $O(n^4)$ instead of just $O(n^2)$ to exhibit the desired tree, and this is easily done (we omit the details). Thus the running time of the algorithm to solve TREE-OR-DECOMPOSITION is also $O(n^4)$.

There is also a simpler method to solve TREE-OR-DECOMPOSITION (but with running time $O(n^5)$). We repeatedly use the algorithm of CONSTRICTED as a subroutine. We may assume that initially Z is not constricted in G (for otherwise the algorithm above for CONSTRICTED outputs an extended H-strip decomposition and we are done). For each vertex v in turn, test whether $Z \setminus \{v\}$ is constricted in $G \setminus \{v\}$, and if not then remove v from G, Z. What remains at the end is the desired tree.

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