TREE INDEPENDENCE NUMBER
III. THETAS, PRISMS AND STARS

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Abstract. We prove that for every $t \in \mathbb{N}$ there exists $\tau = \tau(t) \in \mathbb{N}$ such that every (theta, prism, $K_{1,t}$)-free graph has tree independence number at most τ (where we allow "prisms" to have one path of length zero).

1. Introduction

Graphs in this paper have finite and non-empty vertex sets, no loops and no parallel edges. The set of all positive integers is denoted by $\mathbb{N}$, and for every $n \in \mathbb{N}$, we write $[n]$ for the set of all positive integers no greater than n.

Let $G = (V(G), E(G))$ be a graph. A *clique* in G is a set of pairwise adjacent vertices. A *stable* or *independent* set in G is a set of vertices no two of which are adjacent. The maximum cardinality of a stable set is denoted by $\alpha(G)$, and the maximum cardinality of a clique in G is denoted by $\omega(G)$. For a graph H we say that G *contains* H if H is isomorphic to an induced subgraph of G. We say that G is *H-free* if G does not contain H. For a set $\mathcal{H}$ of graphs, G is *$\mathcal{H}$-free* if G is H-free for every $H \in \mathcal{H}$. For a subset X of $V(G)$, we denote by $G[X]$ the induced subgraph of G with vertex set X, we often use "X" to denote both the set X of vertices and the graph $G[X]$.

Let $X \subseteq V(G)$. We write $N_G(X)$ for the set of all vertices in $G \setminus X$ with at least one neighbor in X, and we define $N_G[X] = N_G(X) \cup X$. When there is no danger of confusion, we omit the subscript "G". For $Y \subseteq G$, we write $N_Y(X) = N_G(X) \cap Y$ and $N_Y[X] = N_Y(X) \cup X$. When $X = \{x\}$ is a singleton, we write $N_Y(x)$ for $N_Y(\{x\})$ and $N_Y[x]$ for $N_Y[\{x\}]$.

Let $x \in V(G)$ and let $Y \subseteq V(G) \setminus \{x\}$. We say that x *is complete to* Y *in* G if $N_Y[x] = Y$, and we say that x *is anticomplete to* Y *in* G if $N_G[x] \cap Y = \emptyset$. In particular, if $x \in Y$, then x is neither complete nor anticomplete to Y in G. For subsets X, Y of $V(G)$, we say that X *and* Y *are complete in* G if every vertex in X is complete to Y in G, and we say that X *and* Y *are anticomplete in* G if every vertex in X is anticomplete to Y in G. In particular, if X and Y are either complete or anticomplete in G, then $X \cap Y = \emptyset$.

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For a graph $G = (V(G), E(G))$, a *tree decomposition* (T, β) of G consists of a tree T and a map $\beta : V(T) \to 2^{V(G)}$ with the following properties:

- For every $v \in V(G)$, there exists $t \in V(T)$ with $v \in \beta(t)$.
- For every $v_1v_2 \in E(G)$, there exists $t \in V(T)$ with $v_1, v_2 \in \beta(t)$.
- $T[\{t \in V(T) \mid v \in \beta(t)\}]$ is connected for all $v \in V(G)$.

The *treewidth* of G, denoted $\text{tw}(G)$ is the smallest integer $w \in \mathbb{N}$ such that G admits a tree decomposition (T, β) with $|\beta(t)| \leq w + 1$ for all $t \in V(T)$. The *tree independence number* of G, denoted tree-$\alpha(G)$, is the smallest integer $s \in \mathbb{N}$ such that G admits a tree decomposition (T, β) with $\alpha(G[\beta(t)]) \leq s$ for all $t \in V(T)$.

Both the treewidth and the tree independence number are of great interest in structural and algorithmic graph theory (see [2, 3, 4, 6, 8] for detailed discussions). They are also related quantitatively because, by Ramsey's theorem [11], graphs of bounded clique number and bounded tree independence number have bounded treewidth. Dallard, Milanič, and Štorgel [8] conjectured that the converse is also true in *hereditary* classes of graphs (meaning classes which are closed under taking induced subgraphs). Let us say that a graph class $\mathcal{G}$ is tw-*bounded* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that every graph $G \in \mathcal{G}$ satisfies $\text{tw}(G) \leq f(\omega(G))$.

Conjecture 1.1 (Dallard, Milanič, and Štorgel [8])**.** *For every hereditary class $\mathcal{G}$ which is* tw*-bounded, there exists $\tau = \tau(\mathcal{G}) \in \mathbb{N}$ such that tree-$\alpha(G) \leq \tau$ for all $G \in \mathcal{G}$.*

Conjecture 1.1 was recently refuted [5] by two of the authors of this paper. It is still natural to ask: which tw-bounded hereditary classes have bounded tree independence number? So far, the list of hereditary classes known to be of bounded tree independence number is not very long (see [2, 7, 8] for a few). More hereditary classes are known to be tw-bounded. The reasons for the existence of the bound are often highly non-trivial, and it is not known whether the corresponding class has bounded tree independence number. A notable instance is the class of all (theta, prism)-free graphs excluding a fixed forest [1], which we will focus on in this paper.

Let us first give a few definitions. Let P be a graph which is a path. Then we write, for $k \in \mathbb{N}$, $P = p_1\text{-}\cdots\text{-}p_k$ to mean $V(P) = \{p_1, \ldots, p_k\}$, and for all $i, j \in [t]$, the vertices p_i and p_j are adjacent in P if and only if $|i - j| = 1$. We call the vertices p_1 and p_k the *ends of* P, and we say that P is a *path from* p_1 *to* p_k or a *path between* p_1 *and* p_k. We refer to $V(P) \setminus \{p_1, p_k\}$ as the *interior of* P and denote it by P^*. The *length* of a path is its number of edges. Given a graph G, by a *path in* G we mean an induced subgraph of G which is a path. Similarly, for $t \in \mathbb{N} \setminus \{1, 2\}$, given a t-vertex graph C which is a cycle, we write $C = c_1\text{-}\cdots\text{-}c_t\text{-}c_1$ to mean $V(C) = \{c_1, \ldots, c_t\}$, and for all $i, j \in [t]$, the vertices c_i and c_j are adjacent in G if and only if $|i - j| \in \{1, t - 1\}$. The *length* of a cycle is its number of edges (which is the same as its number of vertices). For a graph G, a *hole* in G is an induced subgraph of G which is a cycle of length at least four.

A *theta* is a graph Θ consisting of two non-adjacent vertices a, b, called the *ends of* Θ, and three pairwise internally disjoint paths P_1, P_2, P_3 of length at least two in Θ from a to b, called the *paths of* Θ, such that P_1^*, P_2^*, P_3^* are pairwise anticomplete in Θ (see Figure 1). A *prism* is a graph Π consisting of two triangles $\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}$ called the *triangles of* Π, and three pairwise disjoint paths P_1, P_2, P_3 in Π, called the *paths of* Π, such that for each $i \in \{1, 2, 3\}$, P_i has ends a_i, b_i, for all distinct $i, j \in \{1, 2, 3\}$, a_ia_j and b_ib_j are the only edges of Π with an end in P_i and an end in P_j, and for every

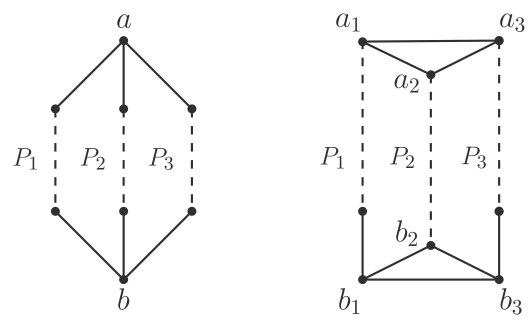

FIGURE 1. A theta (left) and a prism (right). Dashed lines represent paths of arbitrary (possibly zero) length.

$i \neq j \in \{1,2,3\}$ $P_i \cup P_j$ is a hole (see Figure 1). If follows that if P_2 has length zero, then each of P_1, P_3 has length at least two. We remark that the last condition is non-standard; the paths of a prism are usually of non-zero length, and a prism with a length-zero path is sometimes called a "line-wheel." For a graph G, a *theta in* G is an induced subgraph of G which is a theta and a *prism in* G is an induced subgraph of G which is a prism.

The following was proved in [1] to show that the local structure of the so-called "layered wheels" [12] is realized in all theta-free graphs of large treewidth. It also characterizes all forests, and remains true when only the usual "prisms" (with no length-zero path) are excluded:

Theorem 1.2 (Abrishami, Alecu, Chudnovsky, Hajebi, Spirkl [1])**.** *Let F be a graph. Then the class of all (theta, prism, F)-free graphs is* tw*-bounded if and only if F is a forest.*

We propose the following strengthening:

Conjecture 1.3. *For every forest F, there is a constant $\tau = \tau(F) \in \mathbb{N}$ such that for every (theta, prism, F)-free graph G, we have tree-$\alpha(G) \leq \tau$.*

As far as we know, Conjecture 1.3 remains open even for paths. But our main result settles the case of stars. For every $t \in \mathbb{N}$, let $\mathcal{C}_t$ be the class of all (theta, prism, $K_{1,t}$)-free graphs. We prove that:

Theorem 1.4. *For every $t \in \mathbb{N}$, there is a constant $f_{1.4} = f_{1.4}(t) \in \mathbb{N}$ such that every graph $G \in \mathcal{C}_t$ satisfies tree-$\alpha(G) \leq f_{1.4}$.*

2. Outline of the main proof

Like several earlier results [2, 4, 3] coauthored by the first two authors of this work, the proof of Theorem 1.4 deals with "balanced separators." Let G be a graph and let $w : G \to \mathbb{R}^{\geq 0}$. For every $X \subseteq G$, we write $w(X) = \sum_{v \in X} w(v)$. We say that that w is a *weight function on* G if $W(G) = 1$. Given a graph G and a weight function w on G, a subset X of $V(G)$ is called a *w-balanced separator* if for every component D of $G \setminus X$, we have $w(D) \leq 1/2$. The main step in the proof of Theorem 1.4 is the following:

Theorem 2.1. *For every $t \in \mathbb{N}$, there is a constant $f_{2.1} = f_{2.1}(t) \in \mathbb{N}$ with the following property. Let $G \in \mathcal{C}_t$ and let w be a normal weight function on G. Then there exists $Y \subseteq V(G)$ such that $|Y| \leq f_{2.1}$ and $N[Y]$ is a w-balanced separator in G.*

As shown below, Theorem 1.4 follows by combining Theorem 2.1 and the following (this is not a difficult result; see [4] for a proof):

Lemma 2.2 (Chudnovsky, Gartland, Hajebi, Lokshtanov and Spirkl; see Lemma 7.1 in [4])**.** *Let $s \in \mathbb{N}$ and let G be a graph. If for every normal weight function w on G, there is a w-balanced separator X_w in G with $\alpha(X_w) \leq s$, then we have tree-$\alpha(G) \leq 5s$.*

Proof of Theorem 1.4 assuming Theorem 2.1. Let $c = f_{2.1}(t)$. We prove that $f_{1.4}(t) = 5ct$ satisfies the theorem. Let w be a normal weight function on G. By Theorem 2.1, there exists $Y \subseteq V(G)$ such that $|Y| \leq c$ and $X_w = N[Y]$ is a w-balanced separator in G. Assume that there is a stable set S in X_w with $|S| > ct$. Since $S \subseteq N[Y]$, it follows that there is a vertex $y \in Y$ with $|N[y] \cap S| \geq t$. But now G contains $K_{1,t}$, a contradiction. We deduce that $\alpha(X_w) \leq ct$. Hence, by Lemma 2.2, we have tree-$\alpha(G) \leq 5ct = f_{1.4}(t)$. This completes the proof of Theorem 1.4. ■

It remains to prove Theorem 2.1. The idea of the proof is the following. In [3] a technique was developed to prove that separators satisfying the conclusion of Theorem 2.1 exist. It consists of showing that the graph class in question satisfies two properties: being "amiable" and being "amicable." Here we use the same technique. To prove that a graph class is amiable, one needs to analyze the structure of connected subgraphs containing neighbors of a given set of vertices. To prove that a graph is amicable, it is necessary to show that certain carefully chosen pairs of vertices can be separated by well-structured separators. Most of the remainder of the paper is devoted to these two tasks. Section 3 and Section **??** contain structural results asserting the existence of separators that will be used to establish amicability. Section 5 contains definitions and previously known results related to amiability. Section 6 contains the proof of the fact that the class $\mathcal{C}_t$ is amiable. Section 6 uses the results of Sections 3 and 4 to deduce that $\mathcal{C}_t$ is amicable, and to complete the proof of Theorem 2.1.

3. Breaking a wheel

A *wheel* in a graph G is a pair $W = (H, c)$ when H is a hole in G and $c \in G \setminus H$ has at least three neighbors H. We also use W to denote the vertex set $H \cup \{c\} \subseteq G$. A *sector* of the wheel (H, c) is a path of non-zero length in H whose ends are adjacent to c and whose internal vertices are not. A wheel is *special* if it has exactly three sectors, one sector has length one and the other two (called the *long* sectors) have length at least two (see Figure 2 – A special wheel is sometimes referred to as a "short pyramid.")

For a wheel $W = (H, c)$ in a graph G, we define the set $Z(W) \subseteq W$ as follows (see Figure 2). If W is non-special, then $Z(W) = N_H[c]$. Now assume that W is special. Let ab be the sector of length one of W and let d be the neighbor of c in $H \setminus \{a, b\}$. Then we define $Z(W) = \{a, b, c\} \cup N_H[d]$.

Let G be a graph. By a *separation* in G we mean a triple (L, M, R) of pairwise disjoint subsets $V(G)$ with $L \cup M \cup R = V(G)$, such that neither L nor R is empty and L and R are anticomplete in G. Let $x, y \in G$ be distinct. We say that a set $M \subseteq G \setminus \{x, y\}$ *separates x and y in G* if there exists a separation (L, M, R) in G with $x \in L$ and $y \in R$. Also, for disjoint sets $X, Y \subseteq V(G)$, we say that a set $M \subseteq V(G) \setminus (X \cup Y)$ *separates X and Y* if there exists a separation (L, M, R) in G with $X \subseteq L$ and $Y \subseteq R$. If $X = \{x\}$, we say that M *separates x and Y* to mean M separates X and Y.

We have two results in this section; one for the non-special wheels and one for special wheels:

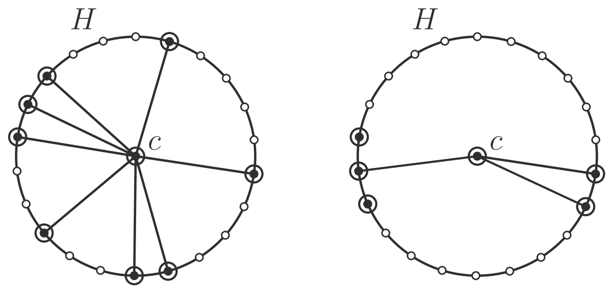

FIGURE 2. A non-special wheel W (left) and a special wheel W (right). Circled nodes represent the vertices in $Z(W)$.

Theorem 3.1. *Let G be a (theta, prism)-free graph, let $W = (H, c)$ be a non-special wheel in G such that H has length at least seven. Let $a, b \in G \setminus N[Z(W)]$ belong to (the interiors of) distinct sectors of W. Then $N[Z(W)]$ separates a and b in G.*

Proof. Let $S = N_H(c)$ and let $T = N[c] \cup (N[S] \setminus H)$. Then $T \subseteq N[Z(W)]$, and so it suffices to show that T separates a and b (note that $a, b \notin T$). We begin with the following:

(1) *Assume that some vertex $v \in G \setminus (W \cup T)$ has either a unique neighbor or two non-adjacent neighbors in some sector $P = p\text{-}\cdots\text{-}p'$ of W. Let b' be the neighbor of p in $W \setminus P$ and b'' be the neighbor of p' in $W \setminus P$. Then $N_H(v) \subseteq P \cup \{b', b''\}$.*

Otherwise, v has a neighbor $d \in H \setminus (P \cup \{b', b''\})$. Also, c has a neighbour $d' \in H \setminus (P \cup \{b', b''\})$, as otherwise W would be a prism or a special wheel. We choose d and d' such that the path Q in $H \setminus P$ from d to d' is minimal. If v has a unique neighbor a in P, then $P \cup Q \cup \{c, v\}$ is a theta in G with ends a and c, a contradiction. Also, if v has two non-adjacent neighbors in P, then $P \cup Q \cup \{v, c\}$ contains a theta with ends c and v. This proves (1).

(2) *For every $v \in G \setminus (W \cup T)$, there exists a sector P of W such that $N_H(v) \subseteq P$.*

Suppose there exists a sector $P = p\text{-}\cdots\text{-}p'$ such that v has two non-adjacent neighbors in P. Then, by (1), we may assume up to symmetry that v is adjacent to b that is the neighbor of p in $H \setminus P$. By (1), b is the unique neighbor of v in some sector Q of W. So the fact that v has at least two neighbors in P contradicts (1) applied to v and Q.

Suppose there exists a sector $P = p\text{-}\cdots\text{-}p'$ such that v has a unique neighbor a in P. By (1), we may assume that $N_H(v) = \{a, b', b''\}$ where b' is the neighbor of p in $W \setminus P$ and b'' is the neighbor of p' in $W \setminus P$ (because $N_H(v) = \{a, b'\}$ or $N_H(v) = \{a, b''\}$ would imply that v and H form a theta). Let $Q = p\text{-}\cdots\text{-}q$ be the sector of W that contains b. By (1) applied to v and Q, we have $ap \in E(G)$ and $b''q \in E(G)$. So, b'' is the unique neighbor of v in the sector $R = p'\text{-}\cdots\text{-}q$ of W. By (1) applied to v and R, we have $ap' \in E(G)$ and $b'q \in E(G)$. So H has length six, a contradiction.

We proved that for every sector P of W, either v has no neighbor in P, or v has two neighbors in P, and those neighbors are adjacent. We may therefore assume that v has neighbors in at least three distinct sectors of W, because if v has neighbor in exactly two of them, then $H \cup \{v\}$ would be a prism. So, suppose that $P = p\text{-}\cdots\text{-}p'$, $Q = q\text{-}\cdots\text{-}q'$ and $R = r\text{-}\cdots\text{-}r'$ are three distinct sectors of W, and v is adjacent to $x, x' \in P$, to

$y, y' \in Q$ and to $z, z' \in R$. Suppose up to symmetry that p, x, x', p', q, y, y', q', r, z, z' and r' appear in this order along H. Then there is a theta in G with ends c, v and paths v-x-P-p-c, v-y-Q-q-c and v-z-R-r-c, a contradiction. This proves (2).

To conclude the proof, suppose for a contradiction that the interiors of two distinct sectors of W are contained in the same connected component of $G \setminus T$. Then there exists a path $Y = v$-$\cdots$-w in $G \setminus T$ and two sectors $P = p$-$\cdots$-p' and $Q = q$-$\cdots$-q' of W such that v has neighbors in P^* and w has neighbors Q^*. By (2), v is anticomplete to $W \setminus P$ and w is anticomplete to $W \setminus Q$ (in particular, Y has length at least one). By choosing such a path Y minimal, we deduce that Y^* is anticomplete to W.

Suppose that v has a unique neighbor, or two distinct and non-adjacent neighbors in P. Next assume that w has a neighbor d in H that is distinct from b' and b'' where b' is the neighbor of p in $W \setminus P$ and b'' is the neighbor of p' in $W \setminus P$, then let d' be a neighbor of c $H \setminus (P \cup \{b', b''\})$ (d' exists for otherwise, W would be a prism or a special wheel). We choose d and d' such that the path R in $H \setminus P$ from d to d' is minimal. We now see that if v has a unique neighbor a in P, then $P \cup Y \cup R \cup \{c\}$ contains a theta with ends a and c, a contradiction. Also, if v has two distinct non-adjacent neighbors in P, then $P \cup Y \cup R \cup \{c\}$ contains a theta with ends c and v. So, w has only two possible neighbors in H, namely, b' and b''. Due to symmetry, we may assume that $b'w \in E(G)$ (so $b''w \notin E(G)$). It follows that b' is non-adjacent to c. If v has a unique neighbor in P, then $H \cup Y$ is a theta in G, so v has a neighbor in P that is non-adjacent to p. In particular, there exists a path R' from v to p' in $P \cup \{v\}$ that contains no neighbor of p. It follows that $R' \cup Q \cup Y \cup \{c\}$ is a theta in G with ends b' and c.

We deduce that v has exactly two neighbor in P, and those neighbors are adjacent. By the same argument, we can prove that w has exactly two neighbors in P that are which are adjacent. But now $H \cup Y$ is a prism in G, a contradiction. This completes the proof of Theorem 3.1. ■

Theorem 3.2. *Let G be a (theta, prism)-free graph and let $W = (H, c)$ be a special wheel in G whose long sectors have lengths at least three. Let $a'', b'' \in G \setminus N[Z(W)]$ belong to (the interiors of) distinct sectors of W. Then $N[Z(W)]$ separates a and b in G*

Proof. Let ab be the sector of length one of W and let d be the neighbor of c in $H \setminus \{a, b\}$. Let a' be the neighbor of d in the long sector of W containing a and let b' be the neighbors of d in the long sector of W containing b. Then $Z(W) = \{a, a', b, b', c, d\}$. Let P be the path in $H \setminus d$ from a to a' and let Q be the path of $H \setminus d$ from b to b'. Assume, without loss of generality, that $a'' \in P^* \setminus N[Z(W)]$ and let $b'' \in Q^* \setminus N[Z(W)]$.

Let $T = N[c] \cup (N[\{a, b, a', b', d\}] \setminus H)$. Then $T \subseteq N[Z(W)]$, and so it suffices to show that T separates a'' and b'' (note that $a'', b'' \notin T$). Suppose not. Then there exists a path $Y = v$-$\cdots$-w in $G \setminus T$ such that v has neighbors in P^*, w has neighbors in Q^*, $Y \setminus v$ is anticomplete to $W \setminus P$ and $Y \setminus w$ is anticomplete to $W \setminus Q$ (note that possibly $v = w$).

Let x be the neighbor of v in P closest to a along P and let x' be the neighbor of v in P closest to a' along P. Let y be the neighbor of w in Q closest to b along P and let y' be the neighbor of w in Q closest to b' along P.

If $x = x'$, then there is a theta in G with ends x and d and paths x-P-a'-d, x-P-a-c-d and x-v-Y-w-y'-Q-b'-d. So, $x \neq x'$, and symmetrically we have $y \neq y'$. If $xx' \notin E(G)$, then there is a theta in G with ends v and d and paths v-x'-P-a'-d, v-x-P-a-c-d and v-Y-w-y'-Q-b'-d. So, $xx' \in E(G)$, and symmetrically we can prove that $yy' \in E(G)$. But now $H \cup Y$ is a prism in G, a contradiction. This completes the proof of Theorem 3.2. ■

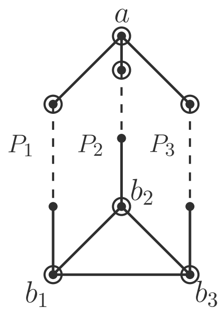

FIGURE 3. A pyramid Σ. Dashed lines represent paths of arbitrary (possibly zero) length, and circled nodes represent the vertices in $Z(\Sigma)$.

4. Breaking a pyramid

A *pyramid* is a graph Σ consisting of a vertex a, a triangle $\{b_1, b_2, b_3\}$ disjoint from a and three paths P_1, P_2, P_3 in Σ of length at least two, such that for each $i \in [3]$, the ends of P_i are a and b_i, and for all distinct $i, j \in [3]$, the sets $V(P_i) \setminus \{a\}$ and $V(P_j) \setminus \{a\}$ are disjoint, b_ib_j is the only edge of G with an end in $V(P_i) \setminus \{a\}$ and an end in $V(P_j) \setminus \{a\}$, and for every $i \neq j \in \{1, 2, 3\}$ $P_i \cup P_j$ is a hole (the assumption that P_1, P_2, P_3 have length at least two is non-standard; usually, one of the paths is allowed to have length 1, and our definition above would refer to a "long" pyramid.)

We say that a is the *apex* of Σ, the triangle $\{b_1, b_2, b_3\}$ is the *base* of Σ, and P_1, P_2, P_3 are the *paths* of Σ. We also define $Z(\Sigma) = N_\Sigma[a] \cup \{b_1, b_2, b_3\}$ (so we have $|Z(\Sigma)| = 7$). For a graph G, by a *pyramid in G* we mean an induced subgraph of G which is a pyramid (see Figure 3).

The main result of this section, Theorem 4.1 below, follows from much more general results of [1]. However, there is also a short and self-contained proof, which we include here:

Theorem 4.1. *Let G be a (theta, prism)-free graph and let Σ be a pyramid in G with apex a, base $\{b_1, b_2, b_3\}$ and paths P_1, P_2 and P_3 as in the definition. Let $u, v \in G \setminus N[Z(\Sigma)]$ belong to distinct paths of Σ. Then $N[Z(\Sigma)]$ separates u and v in G.*

Proof. Suppose not. Then there exist $u, v \in G \setminus N[Z(\Sigma)]$, belonging to distinct paths of Σ, such that $N[Z(\Sigma)]$ does not separate u and v in G. It follows that for distinct $i, j \in [3]$, there exists a path $Q = x\text{-}\cdots\text{-}y$ in $G \setminus (\Sigma \cup N[Z(\Sigma)])$ such that x has a neighbor in P_i and y has a neighbor in P_j. We choose $i, j \in [3]$ and Q subject to the minimality of Q. Due to symmetry, we may assume that $i = 1$ and $j = 2$.

From the minimality of Q and the fact that $Q \subseteq G \setminus (\Sigma \cup N[Z(\Sigma)])$, it follows that:

- $N_{P_1}(x) \subseteq P_1 \setminus Z(\Sigma)$, and $Q \setminus x$ and P_1 are anticomplete in G.
- $N_{P_2}(y) \subseteq P_2 \setminus Z(\Sigma)$, and $Q \setminus y$ and P_2 are anticomplete in G.

Now, if some vertex of Q has a neighbor in P_3, then by the minimality of Q, we must have $x = y$. In particular, x has neighbors in P_1, P_2 and P_3. Since a and x are not adjacent in G, it follows that the three paths in G from a to x with interiors in P_1, P_2 and P_3 form a theta in G with ends a and x, a contradiction. We deduce that Q and P_3 are anticomplete in G.

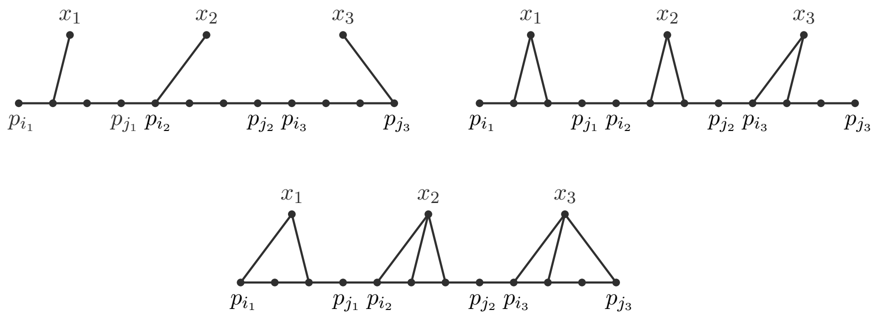

FIGURE 4. A consistent alignment which is spiky (top left), triangular (top right) and wide (bottom).

Let x' be the neighbor of x in P_1 closest to a along P_1 and let x'' be the neighbor of x in P_1 closest to b_1 along P_1. Similarly, let y' be the neighbor of y in P_2 closest to a along P_2 and let y'' be the neighbor of y in P_2 closest to b_2 along P_2. Recall that $x', x'' \in P_1 \setminus Z(\Sigma)$ and $y', y'' \in P_2 \setminus Z(\Sigma)$. If $x' = x''$, then there is a theta in G with ends a, x' and paths a-P_1-x', a-P_2-y'-y-Q-x-x' and a-P_3-b_3-b_1-P_1-x'. Also, if x' and x'' are distinct and adjacent in G, then there is a prism in G with triangles $x''xx'$ and $b_1b_2b_3$ and paths x''-P_1-b_1, x-Q-y-y''-P_2-b_2 and x'-P_1-a-P_3-b_3. Hence, we have $x' \neq x''$ and $x'x'' \notin E(G)$. But now there is a theta in G with ends a, x and paths a-P_1-x'-x, a-P_2-y'-y-Q-x and a-P_3-b_3-b_1-P_1-x''-x, a contradiction. This completes the proof of Theorem 4.1. ■

5. Alignments and Connectifiers

This section covers a number of definitions and a result from [3], which we will use in the proof of Theorem 2.1.

Let G be a graph, let P be a path in G and let $X \subseteq G \setminus P$. We say that (P, X) is an *alignment* if every vertex of X has at least one neighbor in P, and one may write $P = p_1\text{-}\cdots\text{-}p_n$ and $X = \{x_1, \ldots, x_k\}$ for $k, n \in \mathbb{N}$ such that there exist $1 \leq i_1 \leq j_1 < i_2 \leq j_2 < \cdots < i_k \leq j_k \leq n$ where $N_P(x_l) \subseteq p_{i_l}$-$P$-$p_{j_l}$ for every $l \in [k]$. This is a little different from the definition if [3], but the difference is not substantial, and using this definition is more convenient for us here. In this case, we say that $x_1, \ldots, x_k$ is *the order on X given by the alignment* (P, X). An alignment (P, X) is *wide* if each of $x_1, \ldots, x_k$ has two non-adjacent neighbors in P, *spiky* if each of $x_1, \ldots, x_k$ has a unique neighbor in P and *triangular* if each of $x_1, \ldots, x_k$ has exactly two neighbors in P and those neighbors are adjacent. An alignment is *consistent* if it is wide, spiky or triangular. See Figure 4.

By a *caterpillar* we mean a tree C with maximum degree three such that not two branch vertices in C are adjacent, and there exists a path P of C where all branch vertices of C belong to P. We call a maximal such path P the *spine* of C. (We note that our definition of a "caterpillar" is non-standard in multiple ways.) By a *subdivided star* we mean a graph isomorphic to a subdivision of the complete bipartite graph $K_{1,\delta}$ for some $\delta \geq 3$. In other words, a subdivided star is a tree with exactly one branch vertex, which we call its *root*. For a graph H, a vertex v of H is said to be *simplicial* if $N_H(v)$ is a clique. We denote by $\mathcal{Z}(H)$ the set of all simplicial vertices of H. Note that for every tree T, $\mathcal{Z}(T)$ is the set of all leaves of T. An edge e of a tree T is said to be a *leaf-edge* of T if e is incident with a leaf of T. It follows that if H is the line graph of a tree T, then $\mathcal{Z}(H)$ is the set of all vertices in H corresponding to the leaf-edges of T.

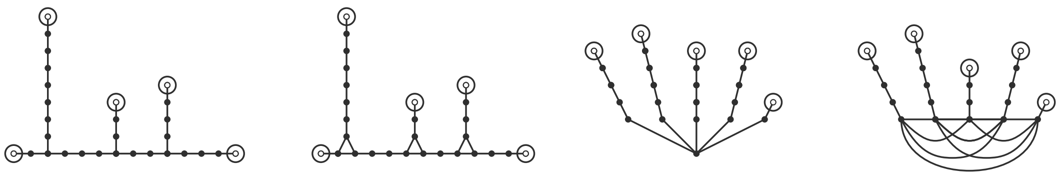

FIGURE 5. Examples of a connectifier. Circled nodes represent the vertices in X.

Let H be a graph that is either a caterpillar, or the line graph of a caterpillar, or a subdivided star with root r, or the line graph of a subdivided star with root r. We define an induced subgraph of H, denoted by $P(H)$, which we will use throughout the paper. If H is a path (possibly of length zero), then let $P(H) = H$. If H is a caterpillar, then let $P(H)$ be the spine of H. If H is the line graph of a caterpillar C, then let $P(H)$ be the path in H consisting of the vertices of H that correspond to the edges of the spine of C. If H is a subdivided star with root r, then let $P(H) = \{r\}$. It H is the line graph of a subdivided star S with root r, let $P(H)$ be the clique of H consisting of the vertices of H that correspond to the edges of S incident with r. The *legs* of H are the components of $H \setminus P(H)$. Let G be a graph and let H be an induced subgraph of G that is either a caterpillar, or the line graph of a caterpillar, or a subdivided star or the line graph of a subdivided star. Let $X \subseteq G \setminus H$ such that every vertex of X has a unique neighbor in H and $N_H(X) = \mathcal{Z}(H)$ (see Figure 5). We call (H, X) a *connectifier.* Also, if H is a single vertex and $X \subseteq N(H)$, we call (H, X) a *connectifer* as well. We say that the connectifier (H, X) is *concentrated* if H is a subdivided star or the line graph of a subdivided star or a singleton.

Let (H, X) be a connectifier in G which is not concentrated. So H is a caterpillar or the line graph of a caterpillar. Let S be the set of vertices of $H \setminus P(H)$ that have neighbors in $P(H)$. Then $(S, P(H))$ is an alignment Let $s_1, \ldots, s_k$ be the corresponding order on X given by $(X, P(H))$. Now, order the vertices of X as $x_1, \ldots, x_k$ where for every $i \in [k]$, the vertex x_i has a neighbor in the leg of H containing s_i. We say that $x_1, \ldots, x_k$ is *the order on X given by* (H, X).

The following was proved in [3]:

Theorem 5.1 (Chudnovsky, Gartland, Hajebi, Lokshtanov and Spirkl; Theorem 5.2 in [3])**.** *For every integer $h \in \mathbb{N}$, there is a constant $f_{5.1} = f_{5.1}(h) \in \mathbb{N}$ with the following property. Let G be a connected graph. Let $S \subseteq G$ such that $|S| \geq f_{5.1}$, the graph $G \setminus S$ is connected and every vertex of S has a neighbor in $G \setminus S$. Then there exists $S' \subseteq S$ with $|S'| = h$ as well as an induced subgraph H of $G \setminus S$ for which one of the following holds.*

- *(H, S') is a connectifier, or*
- *H is a path and every vertex in S' has a neighbor in H.*

6. Amiability

The two notions of "amiability" and "amicability," first introduced in [3], are at the heart of the proof of Theorem 2.1. We deal with the former in this section and leave the latter for the next one.

Let $s \in \mathbb{N}$ and let G be a graph. An *s-trisection in G* is a separation (D_1, Y, D_2) in G such that the following hold.

- Y is stable set with $|Y| = s$.
- $N(D_1) = N(D_2) = Y$.
- D_1 is a path and for every $y \in Y$ there exists $d_y \in D_1$ such that $N_Y(d_y) = \{y\}$.

(The reader may notice that we will never use the third condition in this paper. It was however necessary in [3], so we keep it for easier cross-referencing.)

We say that a graph class $\mathcal{G}$ is *amiable* if there is a function $\sigma : \mathbb{N} \to \mathbb{N}$ with the following property. Let $x \in \mathbb{N}$, let $G \in \mathcal{G}$ and let (D_1, Y, D_2) be a $\sigma(x)$-trisection in G. Then there exist $H \subseteq D_2$ and $X \subseteq Y$ with $|X| = x$ such that the following hold.

- (D_1, X) is a consistent alignment.
- (H, X) is either a connectifier or a consistent alignment.
- If (H, X) is not a concentrated connectifier, then the orders given on X by (D_1, X) and by (H, X) are the same.

In this case, we say that *H and X are given by amiability.* The main result of this section is the following:

Theorem 6.1. *For every $t \in \mathbb{N}$, the class $\mathcal{C}_t$ is amiable. Moreover, with notation as in the definition of amiability, if (H, X) is a connectifier, then we have $|H| > 1$.*

In order to prove Theorem 6.1, first we prove the following lemma:

Lemma 6.2. *Let $d, s \in \mathbb{N}$, let G be a theta-free graph and let Y be a stable set in G of cardinality $3s(d+1)$. Let P be a path in $G \setminus Y$ such that every vertex in Y has a neighbor in P, and each vertex of P has fewer than d neighbors in Y. Assume that for every two vertices $y, y' \in Y$, there is a path R in G from y to y' such that P and R^* are disjoint and anticomplete in G. Then there is an s-subset S of Y such that (P, S) is a consistent alignment.*

Proof. For every vertex $y \in Y$, let P_y be the (unique) path in P with the property that y is complete to the ends of P_y and anticomplete to $P \setminus P_y$. Let I be the graph with $V(I) = Y$ such that two distinct vertices $y, y' \in Y$ are adjacent in I if and only if $P_y \cap P_{y'} \neq \emptyset$. Then I is an interval graph and so I is perfect [10]. Since $|V(I)| = 3s(d + 1)$, it follows that I contains either a clique of cardinality $d + 1$ or a stable set of cardinality $3s$.

Assume that I contains a clique of cardinality $d + 1$. Then there exists $C \subseteq Y$ with $|C| = d + 1$ and $p \in P$ such that $p \in P_y$ for every $y \in C$. Since $p \in P$ has fewer than d neighbors in $C \subseteq Y$, it follows that there are at least two vertices $y, y' \in C \setminus N(p)$. Since $p \in P_y \cap P_{y'}$, it follows that $P \setminus \{p\}$ has two components, and each of y and y' has a neighbor in each component of $P \setminus \{p\}$. It follows that there are two paths P_1 and P_2 from y to y' with disjoint and anticomplete interiors contained in P. On the other hand, there is a path R in G from y to y' such that P and R^* are disjoint and anticomplete in G. It follows that P_1, P_2 and R are pairwise internally disjoint and anticomplete. But now there is a theta in G with ends y, y' and paths P_1, P_2, R, a contradiction.

We deduce that I contains a stable set S' of cardinality $3s$. From the definition of I, it follows that (P, S') is an alignment. Hence, there exists $S \subseteq S' \subseteq Y$ with $|S| = s$ such that (P, S) is a consistent alignment. This completes the proof of Lemma 6.2. ■

Proof of Theorem 6.1. For every $x \in \mathbb{N}$, let

$$s = f_{5.1}(3x^2(t + 1))$$

and let

$$\sigma(x) = 3s(t + 1).$$

We will show that $\mathcal{C}_t$ is amiable with respect to $\sigma : \mathbb{N} \to \mathbb{N}$ as defined above. Let $x \in \mathbb{N}$, let $G \in \mathcal{C}_t$ and let (D_1, Y, D_2) be a $\sigma(x)$-trisection in G. Then Y is a stable set of cardinality $3s(t+1)$, D_1 is a path in $G \setminus Y$ and every vertex in Y has a neighbor in D_1. Moreover, since G is $K_{1,t}$-free, no vertex in D_1 has t or more neighbors in Y, and since $N(D_2) = Y$, it follows that for every two vertices $y, y' \in Y$, there is a path R in G from y to y' with $R^* \subseteq D_2$, and so D_1 and R^* are disjoint and anticomplete in G. By Lemma 6.2, there exists $S \subseteq Y$ with $|S| = s$ such that (D_1, S) is a consistent alignment.

Now, we show that there exists $H \subseteq D_2$ as well as an x-subset X of $S \subseteq Y$ such that H and X satisfy the definition of amiability. Since D_2 is connected and every vertex in $S \subseteq Y$ has a neighbor in D_2, it follows that $D_2 \cup S$ is connected too. Since $|S| = s = f_{5.1}(3x^2(t+1))$, it follows from Theorem 5.1 that there exists $S' \subseteq S$ with $|S'| = 3x^2(t+1)$ and an induced subgraph H_2 of D_2 for which one of the following holds:

- (H_2, S') is a connectifier.
- H_2 is a path and every vertex of S' has a neighbor in H_2.

First, assume that (H_2, S') is a concentrated connectifier. Then, since $|S'| \geq t$ and G is $K_{1,t}$-free, it follows that $|H_2| > 1$. Now, since $|S'| \geq x$, we may choose a concentrated connectifier (H, X) where X is an x-subset of $S' \subseteq S \subseteq Y$ and H is an induced subgraph $H_2 \subseteq D_2$ with $|H| > 1$. In particular, H and X satisfy the definition of amiability.

Next, assume that (H_2, S') is a connectifier which is not concentrated. Consider the orders on S' given by (D_1, S') and by (H_2, S'). Since $|S'| \geq x^2$, it follows from the Erdős-Szekers theorem [9] that there is an x-subset X of $S' \subseteq S \subseteq Y$ as well as an induced subgraph H of $H_2 \subseteq D_2$ such that:

- (D_1, X) is a consistent alignment (because (D_1, S) is);
- (H, X) is a connectifier which is not concentrated; and
- The orders given on X by (D_1, X) and by (H, X) are the same.

It follows that H and X satisfy the definition of amiability.

Finally, assume that H_2 is a path and every vertex in S' has a neighbor in H_2. Let $H = H_2$. Recall that (D_1, S') is an alignment. In particular, S' is a stable set of cardinality $3x^2(t+1)$, and since G is $K_{1,t}$-free, no vertex in H_2 has t or more neighbors in S'. Also, for every two vertices $y, y' \in S$, there is a path R in G from y to y' such that $R^* \subseteq D_1$, and so H and R^* are disjoint and anticomplete in G. By Lemma 6.2, there exists $S'' \subseteq S' \subseteq S$ with $|S''| = x^2$ such that (H, S'') is a consistent alignment. Consider the order on S'' given by (D_1, S'') and by (H, S''). Since $|S''| = x^2$, it follows from the Erdős-Szekers theorem [9] that there is an x-subset X of $S'' \subseteq S' \subseteq S \subseteq Y$ such that such that:

- (D_1, X) is a consistent alignment (because (D_1, S) is);
- (H, X) is a consistent alignment (because (H, S'') is); and
- The orders given on X by (D_1, X) and by (H, X) are the same.

So H and X satisfy the definition of amiability. This completes the proof of Theorem 6.1 ■

7. Amicability

Here we complete the proof of Theorem 2.1, beginning with the following definition.

Let $m \in \mathbb{N}$ and let $\mathcal{G}$ be a graph class $\mathcal{G}$. We say that $\mathcal{G}$ is *m-amicable* if $\mathcal{G}$ is amiable and the following holds. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be as in the definition of amiability for $\mathcal{G}$. Let $G \in \mathcal{G}$ and let (D_1, Y, D_2) be a $\sigma(7)$-trisection in G. Let $X = \{x_1, \dots, x_7\} \subseteq Y$

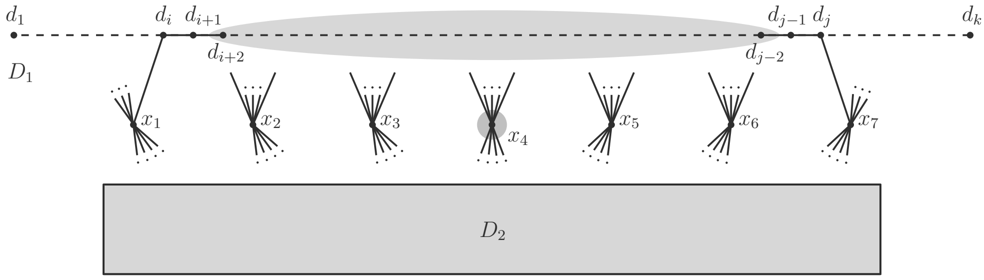

FIGURE 6. Amicability – Note that Z is contained in the highlighted set.

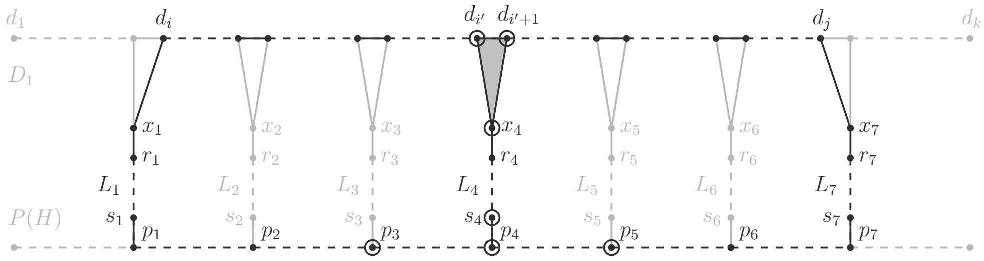

FIGURE 7. H is a caterpillar. Circled nodes depict the vertices in $Z(\Sigma)$.

be given by amiability such that $x_1, \dots, x_7$ is the order on X given by (D_1, X). Let $D_1 = d_1\text{-}\cdots\text{-}d_k$ such that traversing D_1 from d_1 to d_k, the first vertex in D_1 with a neighbor in X is a neighbor of x_1. Let $i \in [k]$ be maximum such that x_1 is adjacent to d_i and let $j \in [k]$ be minimum such that x_7 is adjacent to d_j. Then there exists a subset Z of $D_2 \cup \{d_k : i+2 \leq k \leq j-2\} \cup \{x_4\}$ with $|Z| \leq m$ such that $N[Z]$ separates d_i and d_j. It follows that $N[Z]$ separates $d_1\text{-}D_1\text{-}d_i$ and $d_j\text{-}D_1\text{-}d_k$ (see Figure 6).

We prove that:

Theorem 7.1. *For every $t \in \mathbb{N}$, the class $\mathcal{C}_t$ is $\max\{2t, 7\}$-amicable.*

Proof. By Theorem 6.1, $\mathcal{C}_t$ is amiable, and with notation as in the definition of amiability, if (H, X) is a connectifier, then we have $|H| > 1$. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be as in the definition of amiability for $\mathcal{C}_t$. Let $G \in \mathcal{C}_t$ and let (D_1, Y, D_2) be a $\sigma(7)$-trisection in G. Let $X = \{x_1, \dots, x_7\} \subseteq Y$ be given by amiability such that $x_1, \dots, x_7$ is the order on X given by the consistent alignment (D_1, X). Let $D_1 = d_1\text{-}\cdots\text{-}d_k$ and $i, j \in [k]$ be as in the definition of amicability. Our goal is to show that there exists a subset Z of $D_2 \cup \{d_k : i+2 \leq k \leq j-2\} \cup \{x_4\}$ with $|Z| \leq \max\{2t, 7\}$ such that $N[Z]$ separates d_i and d_j.

Let $i' \in [k]$ be minimum such that x_4 is adjacent to $d_{i'}$, let $j' \in [k]$ be maximum such that x_4 is adjacent to $d_{j'}$, and let H be the induced subgraph of D_2 given by amiability. It follows that $i+2 < i' \leq j' < j-2$, (H, X) is either a connectifier with $|H| > 1$ or a consistent alignment, and if (H, X) is not a concentrated connectifier, then $x_1, \dots, x_7$ is the order on X given by (H, X). When (H, X) is a connectifier with $|H| > 1$, then for each $l \in [7]$, let r_l be the unique neighbor of x_l in H (so $r_l \in \mathcal{Z}(H)$) and let L_l be the (unique) shortest path in H from r_l to a vertex $s_l \in N_H[P(H)]$. It follows that $s_l \in H \setminus P(H)$ unless H is a complete graph, where we have $r_l = s_l \in P(H) = \mathcal{Z}(H) = H$.

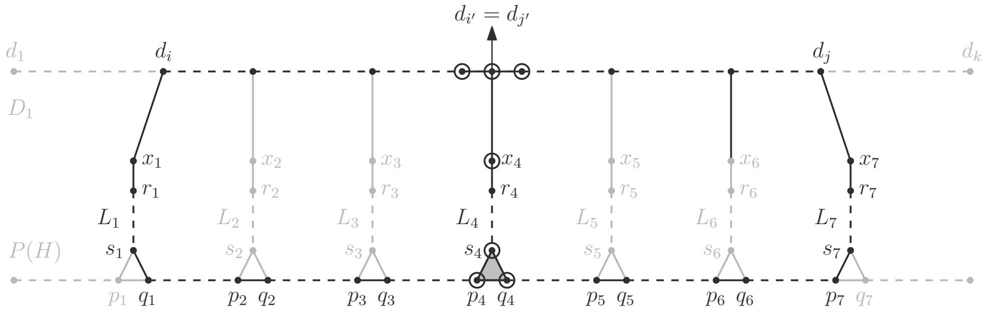

FIGURE 8. H is the line graph of a caterpillar and (D_1, X) is spiky. Circled nodes represent the vertices in $Z(\Sigma)$.

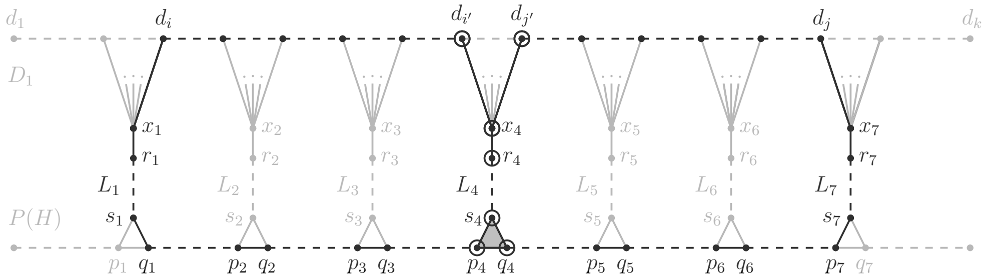

FIGURE 9. H is the line graph of a caterpillar and (D_1, X) is wide. Circled nodes represent the vertices in $Z(\Sigma)$.

First, consider the case when H is a caterpillar. It follows that for each $l \in [7]$, we have $s_l \in H \setminus P(H)$ and s_l has a unique neighbor $p_l \in P(H)$. Since G is theta-free, it follows that (D_1, X) is triangular, and so $j' = i' + 1$ (see Figure 7). Let Σ be the pyramid with apex p_4, base $\{d_{i'}, x_4, d_{j'}\}$ and paths

$$P_1 = p_4\text{-}P(H)\text{-}p_1\text{-}s_1\text{-}L_1\text{-}r_1\text{-}x_1\text{-}d_i\text{-}D_1\text{-}d_{i'};$$

$$P_2 = p_4\text{-}s_4\text{-}L_4\text{-}r_4\text{-}x_4;$$

$$P_3 = p_4\text{-}P(H)\text{-}p_7\text{-}s_7\text{-}L_7\text{-}r_7\text{-}x_7\text{-}d_j\text{-}D_1\text{-}d_{j'}.$$

Then $Z(\Sigma)$ is a 7-subset of $D_2 \cup \{d_k : i+2 \leq k \leq j-2\} \cup \{x_4\}$. Moreover, we have $d_i \in P_1^* \setminus N[Z(\Sigma)]$ and $d_j \in P_3^* \setminus N[Z(\Sigma)]$. Therefore, by Theorem 4.1, $N[Z(\Sigma)]$ separates d_i and d_j, as desired.

Second, consider the case when H is the line graph of a caterpillar. It follows that for each $l \in [7]$, we have $s_l \in H \setminus P(H)$ and s_l has exactly two neighbors $p_l, q_l \in P(H)$, where p_l and q_l are adjacent, and the vertices $p_1, q_1, p_2 q_2, \dots, p_7, q_7$ appear on $P(H)$ in this order. Since G is prism-free, it follows that (D_1, X) is either spiky or wide. Suppose that (D_1, X) is spiky (see Figure 8). Then $i' = j'$. Let Σ be the pyramid with apex $d_{i'} = d_{j'}$, base $\{p_4, s_4, q_4\}$ and paths

$$P_1 = d_{i'}\text{-}D_1\text{-}d_i\text{-}x_1\text{-}r_1\text{-}L_1\text{-}s_1\text{-}q_1\text{-}P(H)\text{-}p_4;$$

$$P_2 = d_{i'}\text{-}x_4\text{-}r_4\text{-}L_4\text{-}s_4;$$

$$P_3 = d_{i'}\text{-}D_1\text{-}d_j\text{-}x_7\text{-}r_7\text{-}L_7\text{-}s_7\text{-}p_7\text{-}P(H)\text{-}q_4.$$

Then $Z(\Sigma)$ is a 7-subset of $D_2 \cup \{d_k : i+2 \leq k \leq j-2\} \cup \{x_4\}$. Moreover, we have $d_i \in P_1^* \setminus N[Z(\Sigma)]$ and $d_j \in P_3^* \setminus N[Z(\Sigma)]$. So by Theorem 4.1, $N[Z(\Sigma)]$ separates d_i

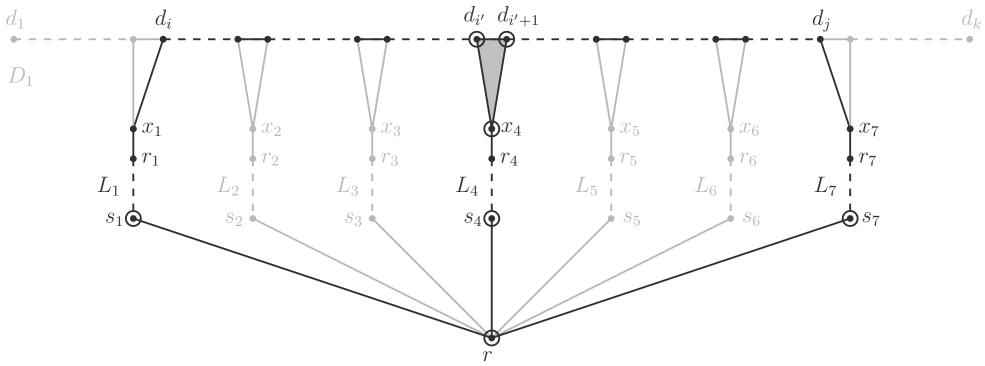

FIGURE 10. H is a subdivided star. Circled nodes represent the vertices in $Z(\Sigma)$.

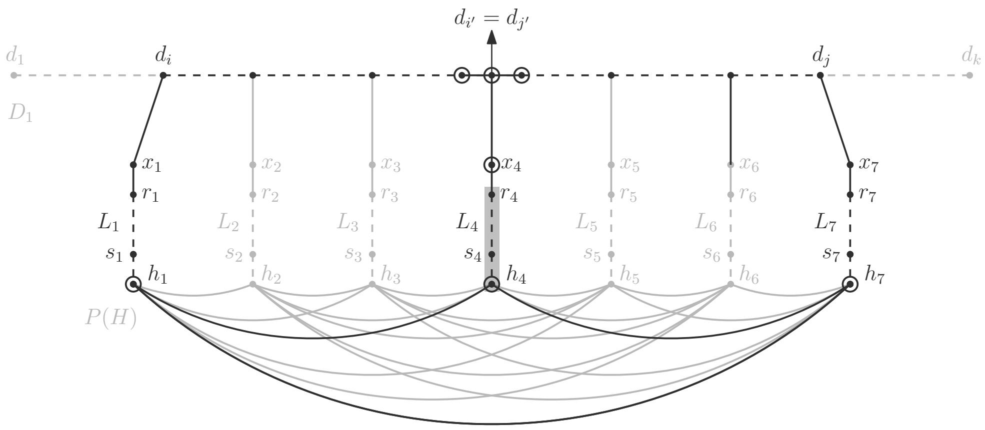

FIGURE 11. H is the line graph of a subdivided star and (D_1, X) is spiky. Circled nodes represent the vertices in $Z(\Sigma)$, and the highlighted path may be of length zero.

and d_j. Now assume that (D_1, X) is wide (see Figure 9). Then $j' - i' > 1$. Let Σ be the pyramid with apex x_4, base $\{p_4, s_4, q_4\}$ and paths

$$P_1 = x_4\text{-}d_{i'}\text{-}D_1\text{-}d_i\text{-}x_1\text{-}r_1\text{-}L_1\text{-}s_1\text{-}q_1\text{-}P(H)\text{-}p_4;$$

$$P_2 = x_4\text{-}r_4\text{-}L_4\text{-}s_4;$$

$$P_3 = x_4\text{-}d_{j'}\text{-}D_1\text{-}d_j\text{-}x_7\text{-}r_7\text{-}L_7\text{-}s_7\text{-}p_7\text{-}P(H)\text{-}q_4.$$

Let $Z = (N(x_4) \cap \Sigma) \cup \{p_4, s_4, q_4\}$. Then $Z(\Sigma)$ is a 7-subset of $D_2 \cup \{d_k : i + 2 \leq k \leq j - 2\} \cup \{x_4\}$. Also, we have $d_i \in P_1^* \setminus N[Z(\Sigma)]$ and $d_j \in P_3^* \setminus N[Z(\Sigma)]$. So by Theorem 4.1, $N[Z(\Sigma)]$ separates d_i and d_j, as required.

Third, consider the case when H is a subdivided star with root r. It follows that $P(H) = \{r\}$ and $H \neq \{r\}$ (because $|H| > 1$). For each $l \in [7]$, we have $r_l, s_l \in H \setminus P(H)$ and r_l is a leaf of H. Since G is theta-free, it follows that (D_1, X) is triangular and so $j' - i' = 1$ (see Figure 10). Let Σ be the pyramid with apex r, base $\{d_{i'}, x_4, d_{j'}\}$ and paths

$$P_1 = r\text{-}s_1\text{-}L_1\text{-}r_1\text{-}x_1\text{-}d_i\text{-}D_1\text{-}d_{i'};$$

$$P_2 = r\text{-}s_4\text{-}L_4\text{-}r_4\text{-}x_4;$$

$$P_3 = r\text{-}s_7\text{-}L_7\text{-}r_7\text{-}x_7\text{-}d_j\text{-}D_1\text{-}d_{j'}.$$

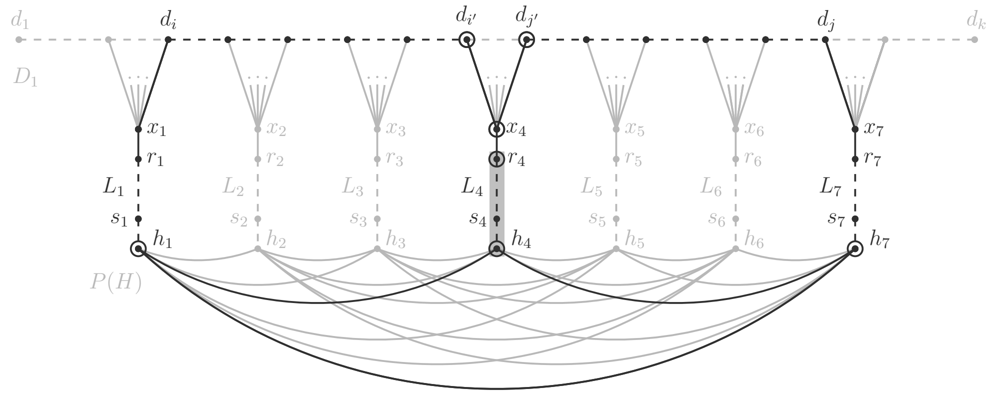

FIGURE 12. H is the line graph of a subdivided star, (D_1, X) is wide and the vertices r_4, s_4, h_4 are not all the same. Circled nodes represent the vertices in $Z(\Sigma)$, and the highlighted path has length at least one.

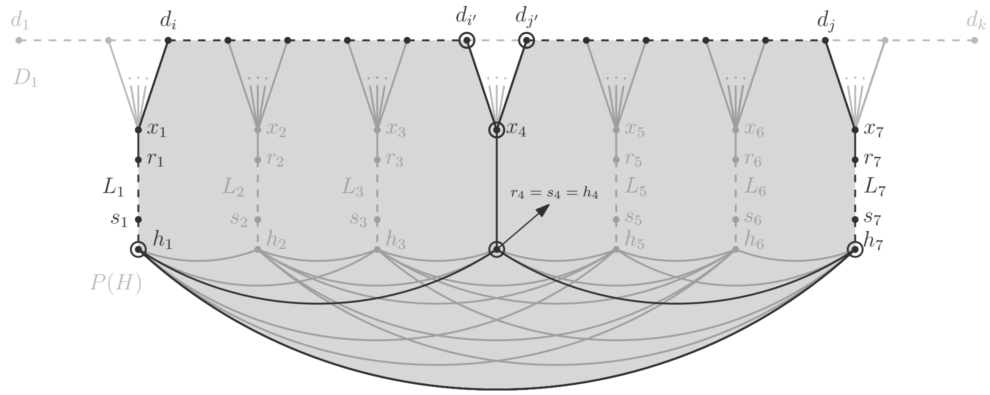

FIGURE 13. H is the line graph of a subdivided star, (D_1, X) is wide and $r_4 = s_4 = h_4$. The hole C is highlighted, and circled nodes represent the vertices in $Z(W)$.

Then $Z(\Sigma)$ is a 7-subset of $D_2 \cup \{d_k : i+2 \leq k \leq j-2\} \cup \{x_4\}$. Also, we have $d_i \in P_1^* \setminus N[Z(\Sigma)]$ and $d_j \in P_3^* \setminus N[Z(\Sigma)]$. So it follows from Theorem 4.1 that $N[Z(\Sigma)]$ separates d_i and d_j, as desired.

Fourth, consider the case when H is the line graph of a subdivided star. It follows that for each $l \in [7]$, either we have $s_l \in P(H)$, in which case we set $h_l = s_l$, or we have $s_l \in H \setminus P(H)$, in which case we choose h_l to be the unique neighbour of s_l in $P(H)$. Since G is prism-free, it follows that (D_1, X) is either spiky or wide. There are now three cases to analyze:

Case 1. Suppose that (D_1, X) is spiky (see Figure 11). Then we have $i' = j'$. Consider the pyramid Σ in G with apex $d_{i'} = d_{j'}$, base $\{h_1, h_4, h_7\}$ and paths

$$P_1 = d_{i'}\text{-}D_1\text{-}d_i\text{-}x_1\text{-}r_1\text{-}L_1\text{-}s_1\text{-}h_1;$$

$$P_2 = d_{i'}\text{-}x_4\text{-}r_4\text{-}L_4\text{-}s_4\text{-}h_4;$$

$$P_3 = d_{i'}\text{-}D_1\text{-}d_j\text{-}x_7\text{-}r_7\text{-}L_7\text{-}s_7\text{-}h_7.$$

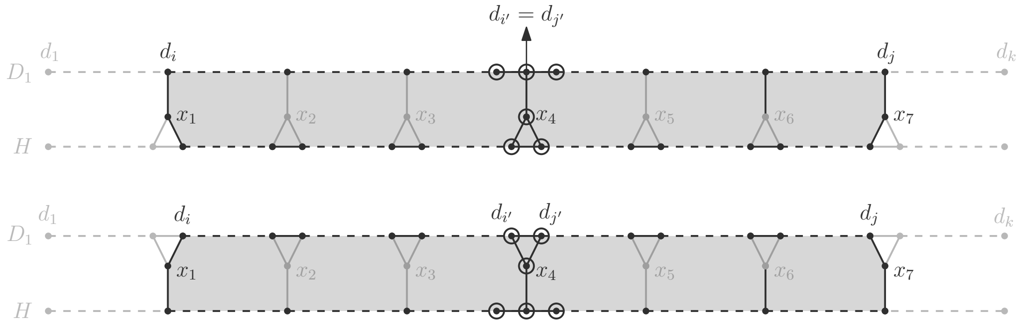

FIGURE 14. One of (D_1, X) and (H, X) is spiky and the other is triangular. The hole C is highlighted, and circled nodes represent the vertices in $Z(W)$.

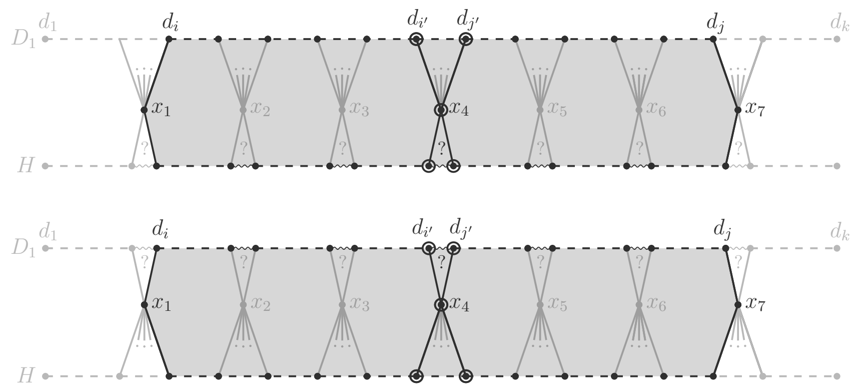

FIGURE 15. One of (D_1, X) and (H, X) is wide. The hole C is highlighted, and circled nodes represent the vertices in $Z(W)$.

Then $Z(\Sigma)$ is a 7-subset of $D_2 \cup \{d_k : i+2 \leq k \leq j-2\} \cup \{x_4\}$. Moreover, we have $d_i \in P_1^* \setminus N[Z(\Sigma)]$ and $d_j \in P_3^* \setminus N[Z(\Sigma)]$. Thus, by Theorem 4.1, $N[Z(\Sigma)]$ separates d_i and d_j.

Case 2. Suppose that (D_1, X) is wide and the vertices r_4, s_4, h_4 are not all the same (see Figure 12). Then $j' - i' > 1$. Let Σ be the pyramid with apex x_4, base $\{h_1, h_4, h_7\}$ and paths

$$P_1 = x_4\text{-}d_{i'}\text{-}D_1\text{-}d_i\text{-}x_1\text{-}r_1\text{-}L_1\text{-}s_1\text{-}h_1;$$

$$P_2 = x_4\text{-}r_4\text{-}L_4\text{-}s_4\text{-}h_4;$$

$$P_3 = x_4\text{-}d_{j'}\text{-}D_1\text{-}d_j\text{-}x_7\text{-}r_7\text{-}L_7\text{-}s_7\text{-}h_7.$$

Then $Z(\Sigma)$ is a 7-subset of $D_2 \cup \{d_k : i+2 \leq k \leq j-2\} \cup \{x_4\}$, and we have $d_i \in P_1^* \setminus N[Z(\Sigma)]$ and $d_j \in P_3^* \setminus N[Z(\Sigma)]$. It follows from Theorem 4.1 that $N[Z(\Sigma)]$ separates d_i and d_j.

Case 3. Suppose that (D_1, X) is wide and $r_4 = s_4 = h_4$ (see Figure 13). Then $j' - i' > 1$. Let $C = x_4\text{-}d_{i'}\text{-}D_1\text{-}d_i\text{-}x_1\text{-}r_1\text{-}L_1\text{-}s_1\text{-}h_1\text{-}h_7\text{-}s_7\text{-}L_7\text{-}r_7\text{-}x_7\text{-}d_j\text{-}D_1\text{-}d_{j'}\text{-}x$. Then C is a hole on more than seven vertices and $W = (C, h_4)$ is a special wheel in G where $Z(W) = \{d_{i'}, d_{j'}, h_1, h_4, h_7, x_4\}$; in particular, $Z(W)$ is a 6-subset of $D_2 \cup \{d_k : i+2 \leq k \leq j-2\} \cup \{x_4\}$. By Theorem 4.1, $N[Z(W)]$ separates d_i and d_j.

Finally, assume that (H, X) is a consistent alignment. Recall that (D_1, X) is also a consistent alignment, and that (D_1, X) and (H, X) give the same order $x_1, \ldots, x_7$ on X. Let R be the unique path in G from x_1 to x_7 with $R^* \subseteq H$. Then $C = d_i\text{-}x_1\text{-}R\text{-}x_7\text{-}d_j\text{-}D_1\text{-}d_i$ is a hole on more than seven vertices in G. Also, since G is (theta, prism)-free, it follows that either one of (D_1, X) and (H, X) is spiky and the other is triangular, or at least one of (D_1, X) and (H, X) is wide. In the former case, $W = (C, x_4)$ is a special wheel (see Figure 14). It follows from Theorem 3.2 that $Z(W)$ is a 6-subset of $D_2 \cup \{d_k : i+2 \leq k \leq j-2\} \cup \{x_4\}$ such that $N[Z(W)]$ separates d_i and d_j. In the latter case, $W = (C, x_4)$ is a non-special wheel (see Figure 15). Since G is $K_{1,t}$-free, it follows that $Z(W) = N_C[x_4] \subseteq D_2 \cup \{d_k : i+2 \leq k \leq j-2\} \cup \{x_4\}$ has cardinality at most $2t$. Moreover, by Theorem 3.1, $N[Z(W)]$ separates d_i and d_j. This completes the proof of Theorem 7.1. ■

We also need the following result from [3]:

Theorem 7.2 (Chudnovsky, Gartland, Hajebi, Lokshtanov, Spirkl [3])**.** *For every $m \in \mathbb{N}$ and every m-amicable graph class $\mathcal{G}$, there is a constant $f_{7.2} = f_{7.2}(\mathcal{G}, m) \in \mathbb{N}$ with the following property. Let $\mathcal{G}$ be a graph class which is m-amicable. Let $G \in \mathcal{C}$ and let w be a normal weight function on G. Then there exists $Y \subseteq V(G)$ such that*

- *$|Y| \leq f_{7.2}$, and*
- *$N[Y]$ is a w-balanced separator in G.*

Now, defining $f_{2.1}(t) = f_{7.2}(\mathcal{C}_t, \max\{2t, 7\})$ for every $t \in \mathbb{N}$, Theorem 2.1 is immediate from Theorems 7.1 and 7.2.

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