INDUCED MINORS AND SUBPOLYNOMIAL TREEWIDTH

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ABSTRACT. Given a family $\mathcal{H}$ of graphs, we say that a graph G is $\mathcal{H}$-induced-minor-free if no induced minor of G is isomorphic to a member of $\mathcal{H}$, We denote by $W_{t\times t}$ the t-by-t hexagonal grid, and by $K_{t,t}$ the complete bipartite graph with both sides of the bipartition of size t. We show that the class of $\{K_{t,t}, W_{t\times t}\}$-induced minor-free graphs with bounded clique number has subpolynomial treewidth. Specifically, we prove that for every integer t there exist $\epsilon \in (0, 1]$ and $c \in \mathbb{N}$ such that every n-vertex $\{K_{t,t}, W_{t\times t}\}$-induced minor-free graph with no clique of size t has treewidth at most $2^{c \log^{1-\epsilon} n}$.

1. INTRODUCTION

All graphs in this paper are finite and simple, and all logarithms are base 2. For standard graph theory terminology that is not defined here we refer to reader to [14]. Let $G = (V(G), E(G))$ be a graph. For a set $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X, and by $G \setminus X$ the subgraph of G induced by $V(G) \setminus X$. In this paper, we use induced subgraphs and their vertex sets interchangeably. For subsets $X, Y \subseteq V(G)$ we say that X is *complete* to Y if X and Y are disjoint and every vertex of X is adjacent to every vertex of Y, and that X is *anticomplete* to Y if X and Y are disjoint and every vertex of X is non-adjacent to every vertex of Y.

For graphs G and H, we say that H is an *induced minor* of G if there exist disjoint connected induced subgraphs $\{X_v\}_{v\in V(H)}$ of G such that X_u is anticomplete to X_v if and only if u is non-adjacent to v in H; in this case we say that G *contains an H-induced-minor*. Given a family $\mathcal{H}$ of graphs, we say that a graph G is *$\mathcal{H}$-induced-minor-free* if no induced minor of G is isomorphic to a member of $\mathcal{H}$.

For a graph G, a *tree decomposition* (T, χ) of G consists of a tree T and a map $\chi\colon V(T) \to 2^{V(G)}$ with the following properties:

(1) For every $v \in V(G)$, there exists $t \in V(T)$ such that $v \in \chi(t)$.
(2) For every $v_1v_2 \in E(G)$, there exists $t \in V(T)$ such that $v_1, v_2 \in \chi(t)$.
(3) For every $v \in V(G)$, the subgraph of T induced by $\{t \in V(T) \mid v \in \chi(t)\}$ is connected.

For each $t \in V(T)$, we refer to $\chi(t)$ as a *bag of* (T, χ). The *width* of a tree decomposition (T, χ), denoted by *width*(T, χ), is $\max_{t\in V(T)} |\chi(t)| - 1$. The *treewidth* of G, denoted by $tw(G)$, is the minimum width of a tree decomposition of G. Graphs of bounded treewidth are well-understood both structurally [21] and algorithmically [5].

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Let $f : \mathbb{N} \to \mathbb{N}$ be a function. A class $\mathcal{C}$ of graphs has *treewidth bounded by* f if every n-vertex graph in $\mathcal{C}$ has treewidth at most $f(n)$. If f can be taken to be a constant function, $\mathcal{C}$ is said to have *bounded treewidth*. The question of which classes defined by forbidden induced subgraphs or minors have bounded treewidth has received significant attention in recent years, including [6], [16], [18], [19], a series of papers involving some of the authors of this manuscript, and others. However, a recent result of [4] suggests that this question is unlikely to have a nice answer. On the other hand, classes whose treewidth is bounded by a slow-growing function seem to be better behaved, and are still of interest from the algorithmic perspective.

A *clique* in a graph is a set of pairwise adjacent vertices, and a *stable (or independent) set* is a set of pairwise non-adjacent vertices. Given a graph G with weights on its vertices, the Maximum Weight Independent Set (MWIS) problem is the problem of finding a stable set in G of maximum total weight. We will discuss MWIS here, but much of what we say applies to a wide variety of algorithmic questions, as is explained in [9]. MWIS is known to be NP-hard [15], but it can be solved in polynomial time on graph classes whose treewidth is bounded by a logarithmic function, in quasi-polynomial time on graph classes whose treewidth is bounded by a poly-logarithmic function, and in sub-exponential time on graph classes whose treewidth is bounded by a subpolynomial function. This suggests that MWIS is unlikely to be NP-hard on these classes of graphs.

We denote by $W_{t\times t}$ the t-by-t hexagonal grid, and by $K_{t,t}$ the complete bipartite graph with both sides of the bipartition of size t. For a positive integer t, we denote by $\mathcal{C}_t$ the class of $\{K_{t,t}, W_{t\times t}\}$-induced-minor-free graphs, and by $\mathcal{C}_t^*$ be the subclass of $\mathcal{C}_t$ consisting of all graphs with no clique of size t. The following conjecture has become known in the area:

Conjecture 1.1. *For every $t \in \mathbb{N}$, there is an integer $d = d(t)$ such that every n-vertex graph $G \in \mathcal{C}_t^*$ satisfies $tw(G) \leq \log^d n$.*

Here we prove a weakening of this, replacing the poly-logarithmic bound on treewidth by a subpolynomial one:

Theorem 1.2. *For every $t \in \mathbb{N}$, there exist $\epsilon = \epsilon(t) \in (0, 1]$, $c = c(t) \in \mathbb{N}$ and $d \in \mathbb{N}$ such that every n-vertex graph G in $\mathcal{C}_t^*$ satisfies $tw(G) \leq 2^{a \log^{1-\epsilon} n}$.*

We remark that in view of Lemma 3.6 of [1] and the main theorem of [12], Theorem 1.2 can be restated in the language of forbidden induced subgraphs instead of induced minors, but we will not do it here.

Theorem 1.2 has the following corollary:

Corollary 1.3. *For every $t \in \mathbb{N}$ and for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that every n-vertex graph $G \in \mathcal{C}_t^*$ with $n > N$ satisfies $tw(G) < n^\epsilon$.*

Since for $G \in \{W_{n\times n}, K_{n,n}, K_n\}$, $tw(G) \geq |V(G)|^{\frac{1}{2}}$, every induced-minor-closed class of graphs satisfying the conclusion of Corollary 1.3 is contained in $\mathcal{C}_s^*$ for some $s \in \mathbb{N}$. Moreover, for every induced-minor-closed graph class $\mathcal{C}$, either $\mathcal{C}$ has subpolynomial treewidth, or for every $N \in \mathbb{N}$ there is a graph $G \in \mathcal{C}$ with $|V(G)| > N$ such that $tw(G) \geq |V(G)|^{\frac{1}{2}}$.

1.1. **Definitions and notation.** We continue with a few more definitions that will be used throughout the paper. Let G be a graph. We denote by $cc(G)$ the set of connected components of G. For a vertex $v \in V(G)$ we denote by $N(v)$ the set of neighbors of v, and $N[v]$ denotes $N(v) \cup \{v\}$. We denote the set of vertices in G at distance exactly 2 from v by $N^2(v)$. For a set $X \subseteq V(G)$ we denote by $N(X)$ the set of all vertices of $G \setminus X$ that have a neighbor in X, and we let $N[X] = N(X) \cup X$. A *path* in G is an induced subgraph that is a path. The *length* of a path is the number of edges in it. We denote by $P = p_1\text{-}\dots\text{-}p_k$ a path in G where $p_ip_j \in E(G)$ if and only if $|j - i| = 1$. We say that p_1 and p_k are the *ends* of P. The *interior* of P, denoted by P^*, is the set $P \setminus \{p_1, p_k\}$. For $i, j \in \{1, \dots k\}$ we denote by $p_i\text{-}P\text{-}p_j$ the subpath of P with ends p_i, p_j.

Let G be a graph and let $A, B \subseteq G$ be disjoint. We say that a set $X \subseteq V(G) \setminus (A \cup B)$ *separates* A from B if for every connected component D of $G \setminus X$, $D \cap A = \emptyset$ or $D \cap B = \emptyset$. Let $a, b \in V(G)$ be non-adjacent. A set $X \subseteq V(G) \setminus \{a, b\}$ *separates* a from b if for every connected component D of $G \setminus X$, $|D \cap \{a, b\}| \leq 1$. We also call X an *a-b-separator*. We denote by $conn_G(a, b)$ the minimum size of an a-b-separator in G.

1.2. **Proof outline and organization.** First we prove Lemma 2.1 that states that the number of edges in a bipartite graph in $\mathcal{C}_t$ is linear in the number of vertices on the smaller side of the bipartition, provided no two vertices on the other side are twins (this last assumption is necessary because of the example of a large star). The proof uses a result of [7] and some probabilistic arguments. This lemma is separate from the rest of the proof, and we believe it to be of independent interest.

Let us now describe the main proof. For this informal presentation we find it easier to go through the proof in reverse order. First we use a result of [6] that states that the edges of every graph in $\mathcal{C}_t^*$ can be partitioned into a small number of star forests. Then, following the ideas introduced in [18] and developed in [6], we reduce the problem of bounding the treewidth of graphs in $\mathcal{C}_t^*$ to a subclass of consisting of what we call "(F, r)-based" graphs. A graph G is *(F, r)-based* if there exists an induced star forest F with no isolated vertices in G, such that G admits a tree decomposition in which every bag consists of at most r objects, each of which is a star of F or a single vertex; moreover, this phenomenon persists in all induced subgraphs of G (with F modified appropriately). (F, r)-based graphs are defined at the start of Section 6. This is the only place in the proof where we explicitly use the bound on the clique number; the rest of the proof only assumes that the graph at hand is (F, r)-based for some F and r. This reduction is done in Section 8.

The next observation is that graphs in $\mathcal{C}_t$ are *(p, q)-slim* for appropriately chosen parameters p and q. This means that in every stable set of p vertices there exists a pair a, b such that there are no q disjoint and pairwise anticomplete a-b-paths in G (we call such a pair *q-slim*). The main ingredient of this proof is a lemma, essentially proved in [11], applied to an appropriate graph. Analyzing the outcomes of that lemma through the lens of the main theorem of [12] gives the result. The details are explained in Section 5.

The next step is to reduce the task of bounding the treewidth of an (F, r)-based graph in $\mathcal{C}_t$ to the question of separating q-slim pairs of vertices. This is immediate from a theorem in [2] if the clique number is bounded, but here we present a different proof, which is more in the spirit of [10], that does not use this assumption. For precise definitions of the terms below, see Section 7. Let G be an (F, r)-based graph in $\mathcal{C}_t$; it is enough to show that every normal weight function w on G admits a small w-balanced separator. Since G is (F, r)-based, we can find (by repeatedly using the existence of the special tree decomposition in a sequence of induced subgraphs of G) pairwise anticomplete sets $Y_1, \dots, Y_p$, each of size at most r, and such that $N[Y_i]$ is a balanced separator in $G \setminus \bigcup_{j<i} Y_j$. Next, assume that for every q-slim pair of vertices $y_i \in Y_i$ and $y_j \in Y_j$, there is a small y_i-y_j-separator $S_{y_iy_j}$ in G. Let C be the union of all such $S_{y_iy_j}$. Then C is still small. We may assume that some component D of $G \setminus (C \cup \bigcup_{i=1}^{p} Y_i)$ has $w(D) > \frac{1}{2}$. From the choice of $Y_1, \dots, Y_p$, each Y_i contains a vertex v_i with a neighbor in D. Since G is (p, q)-slim, some pair (v_i, v_j) is q-slim. But $S_{v_iv_j} \subseteq C$, and yet there is a v_i-v_j-path with interior in D, which is a contradiction. This proof is done in Section 7.

Our next and final goal is to prove Theorem 6.3, asserting that every q-slim pair in an (F, r)-based graph in $\mathcal{C}_t$ admits a small separator. This is the most novel part of the paper, where new ideas are needed. Let G be an (F, r)-based graph in $\mathcal{C}_t$, and let (a, b) be a q-slim pair of vertices in G, and assume that no small a-b-separator exists. Using the special tree decomposition and a lemma from [9], we can find a very large (but with size bounded as a function of r and t) collection of a-b-separators $S_1, \ldots, S_x$, all pairwise disjoint and anticomplete to each other, such that each S_i has the following properties:

- S_i has a partition $D_i \cup Y_i \cup X_i$.
- Y_i is stable.
- $X_i \subseteq N(Y_i)$.
- $|D_i \cup Y_i| \leq r$.

This is done in Section 6. Let $D = \bigcup_{i=1}^{x} D_i$ and $M = \bigcup_{i=1}^{x} Y_i$. Then the sizes of D and M are still under control. Note that M is a stable set. Next, we use Lemma 2.1 and averaging arguments to produce an induced subgraph of H of $G \setminus D$ and a a subset C of M such that

- the size of H is a small proportion of the size of G.
- the size of C is bounded as a function of t and r.
- every a-b-path in $G \setminus (D \cup M)$ contains a subpath with interior in H such that at least p vertices of C have neighbors in the subpath.

Such a triple $(H, C, D \cup M)$ is called a *good a-b-barrier* and its existence is proved in Section 4.

Since G is (p, q)-slim, it follows from the definition of a good barrier that every a-b-path in $G \setminus (D \cup M)$ contains a subpath Q such that

- $Q \subseteq H$, and
- there is a q-slim pair (u, v) with $u, v \in C$ such that Q contains a subpath with ends u, v.

Inductively (on the size of H) there is a small (subpolynomial in $V(H)$) set $S \subseteq V(H)$ such that every q-slim pair (u, v) with $u, v \in C$ is separated in $H \setminus S$. But now $S \cup M \cup D$ is an a-b-separator in G. This argument is carried out in Section 3 and it completes the proof. See Fig. 1 for a diagrammatic depiction of the outline of the proof of Theorem 6.3.

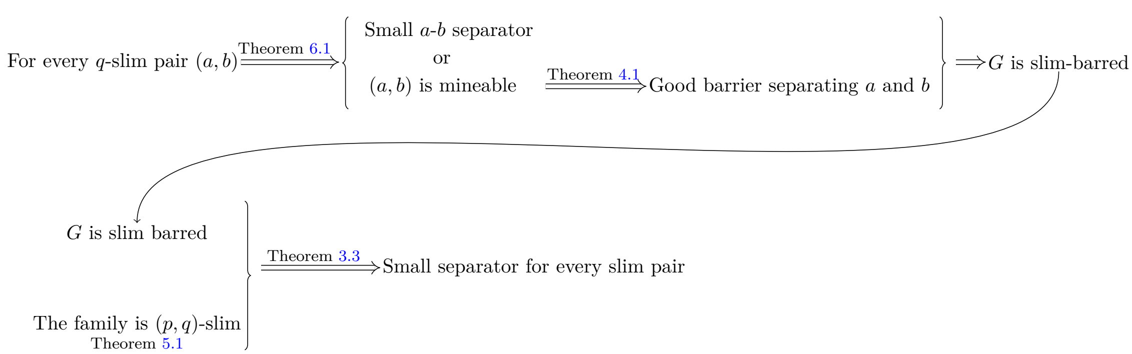

FIGURE 1. Outline of the proof of the existence of small separators for q-slim pairs in (F, r)-based graphs in $\mathcal{C}_t$.

Finally, we remark that the results in Section 3 and Section 4 are stated in greater generality than what we need for the proof of Theorem 1.2, as we expect them to be useful for other families of graphs that admit balanced separators with small domination number.

2. The bipartite lemma

Given a graph G, we say that $u, v \in V(G)$ are *twins in G* if $N(u) \setminus \{v\} = N(v) \setminus \{u\}$. If G is clear from the context, we will simply say that u and v are twins. A *twin class* in G is a maximal set of vertices, every two of which are twins. It is not hard to see that every graph has a unique partition into twin classes.

Lemma 2.1. *There exists a function f such that the following holds. Let G be a bipartite $K_{t,t}$-induced-minor-free graph with bipartition (A, B) such that no two vertices in B are twins. Then $|E(G)| \leq f(t)|A|$.*

Proof. It follows from the main result of [7] that there exists $\Delta = \Delta(t)$ such that G is Δ-degenerate. We may assume that $\Delta > 2$. [7] also implies the existence of $\Delta' = \Delta'(t)$ as a degeneracy bound for graphs with no K_{2t}-subgraph and no induced subdivision of $K_{t,t}$. Let $f(t) = 2^{40\Delta^2\Delta'}\Delta + (\Delta+1)\Delta$. Let us take a degenerate ordering of $V(G)$, that is an ordering $v_1, \dots, v_n$ in which each v_i has at most Δ neighbors in $\{v_{i+1}, \dots, v_n\}$. We may assume that the ordering $v_1, \dots, v_n$ was chosen so that the vertices of A appear as late as possible: if $v_i \in A$ then all the vertices of degree at most Δ in $G \setminus \{v_1, \dots, v_{i-1}\}$ are in A.

(1) *If $1 \leq i < j \leq n$, $v_i \in A$ and $\{v_{i+1}, \dots, v_j\} \cap A = \emptyset$ then $j - i \leq \Delta$.*

Let $i < l \leq j$. It follows from our choice of the ordering that $|N(v_l) \setminus \{v_1, \dots, v_{i-1}\}| > \Delta$. Moreover, since $v_{i+1}, \dots, v_{l-1} \in B$ and B is stable, we have that $\Delta \geq |N(v_l) \setminus \{v_1, \dots, v_{l-1}\}| = |N(v_l) \setminus \{v_1, \dots, v_i\}|$. This implies that $v_l \in N(v_i) \setminus \{v_1, \dots, v_{i-1}\}$. Since $|N(v_i) \setminus \{v_1, \dots, v_{i-1}\}| \leq \Delta$, (1) follows.

Let $s \in \mathbb{N}$ be minimal such that $v_s \in A$. Let $B_1 = B \cap \{v_s, \dots, v_n\}$. Let $G' = G \setminus B_1$ and $B' = B \setminus B_1$. Note that every vertex in B' has a degree at most Δ in G'.

(2) $|E(G')| \geq |E(G)| - |A|(\Delta+1)\Delta$.

Since G is bipartite, $E(G) \setminus E(G') = E(G[\{v_s, \dots, v_n\}])$. By (1), $|\{v_s, \dots, v_n\}| \leq |A|(\Delta+1)$. Since $G[\{v_s, \dots, v_n\}]$ is Δ-degenerate, $|E(G')| \geq |E(G)| - |A|(\Delta+1)\Delta$. This proves (2).

Let $r = 40\Delta^2\Delta'$. Let $k \in \mathbb{N} \cup \{0\}$ be maximal such that there exist $a_1, \dots, a_k \in A$ satisfying that, for all $1 \leq i \leq k$, $|N^2_{G' \setminus N_{G'}[\{a_1, \dots, a_{i-1}\}]}(a_i)| \leq r$. Let $A_1 = \{a_1, \dots, a_k\}$ and let $A'' = A \setminus A_1$ and $B'' = B' \setminus N_{G'}(A_1)$. Let $G'' = G[A'' \cup B'']$.

(3) $|E(G'')| \geq |E(G')| - 2^r\Delta|A_1|$

Since there are no twins in B, we have that for every $i \leq k$, $|N_{G'}(a_i) \setminus N[\{a_1, \dots, a_{i-1}\}]| \leq 2^r$. Thus, since every vertex in B' has a degree at most Δ in G', we deduce that $|E(G'')| \geq |E(G')| - 2^r\Delta|A_1|$, proving (3).

If $|A''| = 0$ then (2) and (3) imply that

$$|E(G)| \leq (2^r\Delta + (\Delta+1)\Delta)\,|A|$$

and so we are done. Therefore, we may assume for a contradiction that $|A''| > 0$.

(4) *Let* $S = \{\{u, v\}$ *s.t* $u, v \in A''$ *and* $u \in N^2_{G''}(v)\}$, *then* $|S| \geq \frac{r|A''|}{2}$

Since for every $v \in A''$, we have $|N^2_{G''}(v)| > r$, there are at least $r|A''|$ ordered pairs of vertices in A'' at distance 2. Dividing by 2 accounts for the double counting. This proves (4).

Sample $X \subseteq A''$ by iterating over every element of A'' and including it in X with probability $p = \frac{1}{\Delta}$. We will say that $u, v \in X$ is a *good pair* if there exists $b \in B''$ for which $N(b) \cap X = \{u, v\}$. We now show that the expected number of good pairs is fairly high. Let $\{u, v\} \in S$. We have that

$$\mathbb{P}\left(u, v \text{ is a good pair}\right) \geq \frac{1}{\Delta^2}\left(1 - \frac{1}{\Delta}\right)^{\Delta-2} \geq \frac{1}{10\Delta^2},$$

as this bounds the probability of the event that $N(b) \cap X = \{u, v\}$ for $b \in B$ such that $\{u, v\} \subseteq N(b)$. Therefore, by (4), $\mathbb{E}\left[\#\text{ good pairs}\right] = \sum_{\{u,v\}\in S} \mathbb{P}\left(u, v \text{ is a good pair}\right) \geq \frac{1}{10\Delta^2}\frac{r|A''|}{2}$. So there exists a choice of X^* with at least $\frac{r|A''|}{20\Delta^2}$ good pairs. Let Γ be the graph with vertex set X^*, and where u is adjacent to v if and only if u, v is a good pair.

(5) $E(\Gamma) \leq \Delta'|X^*|$.

Since Γ is an induced minor of G, it follows that Γ is $K_{t,t}$-induced-minor-free, and in particular no induced subgraph of Γ is a subdivision of $K_{t,t}$. Next suppose that there is a clique K of size $2t$ in Γ; let $K = \{k_1, \ldots, k_{2t}\}$. It follows from the definition of a good pair that for every $1 \leq j < j \leq 2t$ there exists $b_{ij} \in B$ such that $N(b_{ij}) \cap K = \{k_i, k_j\}$. But then $K \cup \{b_{ij}\}_{1\leq i<j\leq 2t}$ is an induced subdivision of K_{2t} in G, contrary to the fact that G is $K_{t,t}$-induced-minor-free. This proves that Γ has no clique of size $2t$. Now the main result of [7] implies that Γ is Δ'-degenerate, and therefore $|E(\Gamma)| \leq \Delta'|X^*|$. This proves (5).

On the other hand, by the choice of X^* we have that $|E(\Gamma)| \geq \frac{r|A''|}{20\Delta^2} \geq \frac{r|X^*|}{20\Delta^2} \geq 2\Delta'|X^*|$, contrary to (5). Hence $|A''| = 0$, concluding the proof. ■

3. Separating slim pairs in barred graphs

Let $G = (V, E)$ be a graph, let $a, b \in V$ be non-adjacent and let $t, s \in \mathbb{N}$. We say that the pair (a, b) is *s-wide* if there exist s internally anticomplete a-b-paths in G; a pair of non-adjacent vertices that is not s-wide is said to be *s-slim*. We say that G is *(t, s)-slim* if for every stable set $S \subseteq V$ of size t, there exist $a, b \in S$ such that (a, b) is s-slim. Similarly, we say that a graph class $\mathcal{F}$ is *(t, s)-slim* if every graph in $\mathcal{F}$ is (t, s)-slim.

In this section, we take the first step to our next goal: showing that an s-slim pair can be separated by a small subset of vertices. To do so we define the notion of a "barrier". Loosely speaking, a barrier separating a from b is a relatively small induced subgraph F of G such that, in order to separate a from b in G, it suffices to delete a few vertices from $G \setminus F$ and then separate a few slim pairs from each other in F. Now fix a slim pair (a, b). Assuming that such a barrier exists for *every* s-slim pair for an appropriately chosen s (which is a property we called "slim-barred"), we design a recursive procedure to obtain a "small" a-b-separator in G. This reduces the problem of separating slim pairs of vertices to the problem of finding good barriers. In this section, we define and analyze this reduction.

Let $G = (V, E)$ be a graph, let C, X, Y, Z be disjoint (and possibly empty) subsets of V and let $t, p \in \mathbb{N}$. Let $G' = G \setminus C$. We say that $B = (X, Y, Z, C)$ is a (t, p)-*barrier* if the following hold:

(1) Every X-Z path P in G' contains an X-Z path P' such that the interior of P' is contained in Y.
(2) For every X-Z path P in G', $|\{\mathcal{C} \in cc(C) \mid N(P) \cap \mathcal{C} \neq \emptyset\}| \geq t$.
(3) Every connected component of C has at most p vertices.

We say that two (t, p)-barriers $B = (X, Y, Z, C)$ and $B' = (X', Y', Z', C')$ are *disjoint* if $X \cup Y \cup Z$ is disjoint from $X' \cup Y' \cup Z'$. Similarly, we say that two (t, p)-barriers $B = (X, Y, Z, C)$ and $B' = (X', Y', Z', C')$ are *anticomplete* if $X \cup Y \cup Z$ is anticomplete to $X' \cup Y' \cup Z'$. Let $u, v \in V \setminus (X \cup Y \cup Z \cup C)$. We say that B separates u from v (in G) if both $X \cup C$ and $Z \cup C$ separate u from v in G. See Fig. 2 for an illustration.

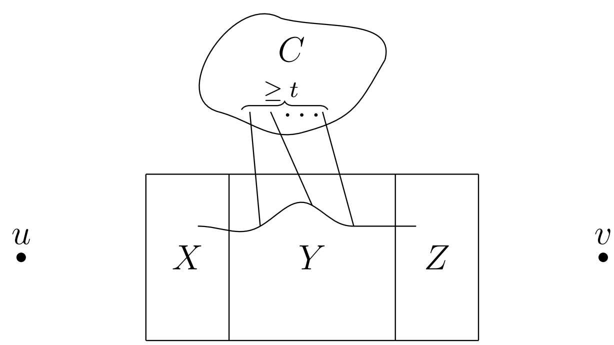

FIGURE 2. Visualization of a barrier separating u from v.

We say that a (t, p)-barrier $B = (X, Y, Z, C)$ is *reduced* if every connected component of $X \cup Y \cup Z$ intersects both X and Z.

Lemma 3.1. *Let $B = (X, Y, Z, C)$ be a (t, p)-barrier separating u from v. Then there exist $X' \subseteq X, Y' \subseteq Y, Z' \subseteq Z$ such that (X', Y', Z', C) is a reduced (t, p)-barrier separating u from v.*

Proof. Let Γ be the union of all connected components in $X \cup Y \cup Z$ meeting both X and Z. Let $X' = X \cap \Gamma, Y' = Y \cap \Gamma, Z' = Z \cap \Gamma$.

(6) *(X', Y', Z', C) is a (t, p)-barrier.*

Condition 1 holds as every X'-Z' path in $G \setminus C$ contains a path with interior in Y, and therefore in Y'. Condition 2 holds as every X'-Z' path is also a X-Z path. Condition 3 holds since C is unchanged. This proves (6).

(7) *$X' \cup C$ separates u from v.*

Suppose not. Then there is a u-v-path P with $P \cap (X' \cup C) = \emptyset$. Since (X, Y, Z, C) separates u from v, it follows that $P \cap X \neq \emptyset$ and $P \cap Z \neq \emptyset$, and consequently P contains an X-Z path Q. Since (X, Y, Z, C) is a barrier, we may assume that $Q^* \subseteq Y$. But then $Q \cap X' \neq \emptyset$, a contradiction. This proves (7).

Similarly, $Z' \cup C$ separates u from v and, thus, (X', Y', Z', C) separates u from v. ■

Let $p, s, t \in \mathbb{N}$ and let $c, f, g : \mathbb{N} \to \mathbb{N}$ be functions such that $c(n)+f(n)+g(n) < n$. We say that an n-vertex graph G is (s,t,p,c,f,g)-*slim-barred* if for every s-slim pair (u,v) there exist disjoint subsets X, Y, Z, C, M of $V \setminus \{u,v\}$ such that

(1) $B = (X,Y,Z,C)$ is a (t,p)-barrier in $G \setminus M$ that separates u from v.
(2) $|cc(C)| \leq c(n)$.
(3) $|M| \leq f(n)$.
(4) $|X \cup Y \cup Z| \leq g(n)$.

Similarly, we say that a graph class $\mathcal{F}$ is (s,t,p,c,f,g)-*slim-barred* if every graph in $\mathcal{F}$ is (s,t,p,c,f,g)-slim-barred.

Lemma 3.2. *Let $s, t \in \mathbb{N}$ and let $c, f, g : \mathbb{N} \to \mathbb{N}$ be functions such that $g(n) < n$. Let $\mathcal{H}$ be a hereditary graph class that is (t,s)-slim and (s,t,p,c,f,g)-slim-barred. Let*

$$h(n) = \max_{G \in \mathcal{H},\, |V(G)| \leq n} \; \max_{(a,b) \text{ is an } s\text{-slim pair in } G} conn_G(a,b).$$

Then, h obeys the following recursive inequality.

$$h(n) \leq \begin{cases} n & \text{if } n \leq 10 \\ f(n) + 3(c(n)p)^2 + (c(n)p)^2 h(g(n)) & \text{otherwise} \end{cases}$$

Proof. We proceed by induction on n. If $n \leq 10$, then the statement holds trivially. Let G be an n-vertex graph in $\mathcal{H}$ and (a,b) be an s-slim pair in G. Let X, Y, Z, C, M be disjoint subsets of $V \setminus \{a,b\}$ as in the definition of (s,t,p,c,f,g)-slim-barred for a and b. Let $I = \Big\{\{u,v\} \Big| u, v \in C$ and u,v is an s-slim pair in $X \cup Y \cup Z \cup \{u,v\}\Big\}$. For every pair $\{u,v\} \in I$, let $M_{u,v}$ be a u-v separator in $X \cup Y \cup Z \cup \{u,v\}$ of size $conn_{X \cup Y \cup Z \cup \{u,v\}}(u,v)$. Let $M' = \bigcup_{\{u,v\} \in I} M_{u,v}$.

(8) *$M \cup M' \cup C$ separates a from b in G.*

Suppose not and let P be a path from a to b in $G \setminus (M \cup M' \cup C)$. Since (X,Y,Z,C) separates a from b in G, both $P \cap X$ and $P \cap Z$ are non-empty. Therefore, P contains an X-Z path. Since (X,Y,Z,C) is a (t,p)-barrier, it follows that $N(P)$ meets at least t connected components of C. Since $\mathcal{H}$ is (t,s)-slim, there exist $u, v \in N(P) \cap C$ such that the pair (u,v) is s-slim. Then $M_{u,v} \subseteq M'$. But since both u and v have neighbors in P, there is a u-v-path with interior in P. This is a contradiction, and (8) follows.

(9) *For all $\{u,v\} \in I$, $|M_{u,v}| \leq h(g(n)) + 2$.*

If $|M_{u,v}| \leq 2$, the statement trivially holds, so we may assume that $|M_{u,v}| \geq 2$. Let $c, d \in M_{u,v}$ and let $M'_{u,v}$ be a u-v separator in $(X \cup Y \cup Z \cup \{u,v\}) \setminus \{c,d\}$ with $|M'_{u,v}|$ minimum. Then, by induction, we have $|M_{u,v}| \leq 2 + |M'_{u,v}| \leq 2 + h(g(n))$. This proves (9).

Since C contains at most $|C|^2 \leq (c(n)p)^2$ slim pairs, and using both (8) and (9), we get an a-b separator of size

$$|M| + |C| + |M'| \leq f(n) + c(n)p + (c(n)p)^2(h(g(n)) + 2) \leq f(n) + 3(c(n)p)^2 + (c(n)p)^2 h(g(n)).$$

■

Theorem 3.3. *Let $s, t \in \mathbb{N}$ and let $c, f, g : \mathbb{N} \to \mathbb{N}$ be increasing functions such that $g(n) < n$ and such that the ratio $\frac{n}{g(n)}$ is non-increasing. Let $\mathcal{H}$ be a hereditary graph class that is (t, s)-slim and (s, t, p, c, f, g)-slim-barred. Let*

$$h(n) = \max_{G \in \mathcal{H},\, |V(G)| \leq n} \; \max_{(a,b) \text{ is an } s\text{-slim pair in } G} conn_G(a, b).$$

Then, $h(n) \leq 20(f(n) + 3c(n)^2p^2)(c(n)p)^{\frac{2\log(n)}{\log\left(\frac{n}{g(n)}\right)}}$

Intuitively, Theorem 3.3 follows by analyzing the recursion tree for the function h obtained by Lemma 3.2. It has a depth of at most $\frac{\log(n)}{\log\left(\frac{n}{g(n)}\right)}$ and each of its nodes has at most $(c(n)p)^2$ children; consequently it has a most $2(c(n)p)^{\frac{2\log(n)}{\log\left(\frac{n}{g(n)}\right)}}$ vertices. The contribution of each vertex of the recursion tree is, at most, $10(f(n) + 3c(n)^2p^2)$. We now proceed with a formal proof.

Proof of Theorem 3.3. Let $N \in \mathbb{N}$. Let

$$H(n) = \begin{cases} n & \text{if } n \leq 10 \\ f(N) + 3(c(N)p)^2 + (c(N)p)^2 H(g(n)) & \text{otherwise} \end{cases}.$$

By Lemma 3.2 and since f and c are increasing, we have that $h(n) \leq H(n)$ for all $n \leq N$, so it is sufficient to prove that $H(N) \leq 20(f(N) + 3c(N)^2p^2)(c(N)p)^{\frac{2\log(N)}{\log\left(\frac{N}{g(N)}\right)}}$.

We prove the following slightly stronger statement: for all $n \leq N$, we have

$$H(n) \leq 20(f(N) + 3c(N)^2p^2)(c(N)p)^{\frac{2\log(n)}{\log\left(\frac{n}{g(n)}\right)}} - \frac{(f(N) + 3c(N)^2p^2)}{c(N)^2p^2 - 1}.$$

Let $K = f(N) + 3(c(N)p)^2$ and $z = c(N)^2p^2$. We proceed by induction on n. If $n \leq 10$, then $20K - \frac{K}{z-1} \geq 19K \geq 19 \geq n$.

Therefore, we may assume that $n > 10$ and that for all $n' \leq n$, $H(n') \leq 20K\, z^{\frac{\log(n')}{\log\left(\frac{n'}{g(n')}\right)}} - \frac{K}{z-1}$.

Now we have that

$$\begin{aligned}
H(n) &\leq K + z\ H(g(n)) \\
&\leq K + z\left(20K\ z^{\frac{\log(g(n))}{\log\left(\frac{g(n)}{g(g(n))}\right)}} - \frac{K}{z-1}\right) \\
&\leq K + z\left(20K\ z^{\frac{\log(g(n))}{\log\left(\frac{n}{g(n)}\right)}} - \frac{K}{z-1}\right) \\
&= K + z\left(20Kz^{\frac{\log\left(n\frac{g(n)}{n}\right)}{\log\left(\frac{n}{g(n)}\right)}} - \frac{K}{z-1}\right) \\
&= K + z\left(20Kz^{\frac{\log(n)}{\log\left(\frac{n}{g(n)}\right)}-1} - \frac{K}{z-1}\right) \\
&= K + 20K\ z^{\frac{\log(n)}{\log\left(\frac{n}{g(n)}\right)}} - \frac{zK}{z-1} \\
&= 20K\ z^{\frac{\log(n)}{\log\left(\frac{n}{g(n)}\right)}} - \left(\frac{z}{z-1} - 1\right)K \\
&= 20K\ z^{\frac{\log(n)}{\log\left(\frac{n}{g(n)}\right)}} - \frac{K}{z-1}
\end{aligned}$$

as required.

■

4. From mines to barriers

In this section, we introduce the notion of "mineable" pairs of vertices. Informally, a pair a, b is mineable if there exist many "almost disjoint" a-b-separators each with a small dominating set (a "core"), and these cores are pairwise disjoint and anticomplete. We show that mineable pairs of vertices admit good barriers separating them (after deleting a small set of vertices from the graph). This is then used to prove that our graph class is slim-barred. We now proceed with formal definitions. In this paper the we will only apply mineability with $z = p = 1$. We include the more general form for potential future applications.

Let $x, y, z, p \in \mathbb{N}$ and let G be a graph. For $a, b \in V(G)$, we say that (a, b) is (x, y, z, p)-mineable in G if there exist disjoint $Y_1, \ldots, Y_x \subseteq V \setminus \{a, b\}$ for which the following hold:

(1) For every i, there exists non-empty $X_i \subseteq N(Y_i) \setminus \{a, b\}$ such that $X'_i = Y_i \cup X_i$ is an a-b separator in $G \setminus \bigcup_{j<i} Y_j$.
(2) a and b belong the the same component of $G \setminus \bigcup_{j=1}^{x} Y_j$; and in particular for every i, a and b belong to the same component of $G \setminus \bigcup_{j<i} Y_j$.
(3) For distinct $i, j \in [x]$, Y_i and Y_j are anticomplete.
(4) For every i, $|cc(Y_i)| \leq y$.
(5) For every i, every component of Y_i has size at most p.
(6) Every vertex of G is contained in at most z of the sets $X'_1, \ldots, X'_x$.

The goal of this section is to show that every mineable pair (with appropriately chosen parameters) in a $K_{t,t}$-induced-minor-free graphs can be separated by a barrier with certain properties:

Theorem 4.1. *There exists a function $\phi : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $t, p, x, y, z \in \mathbb{N}$. Let G be an n-vertex $K_{t,t}$-induced-minor-free graph and let (a, b) be an (x, y, z, p)-mineable pair in G. Then there exist disjoint subsets X, Y, Z, C, M of $V(G) \setminus \{a, b\}$ such that*

(1) $B = (X, Y, Z, C)$ is a (t, p)-barrier in $G \setminus M$ that separates a from b.
(2) $|cc(C)| \leq \frac{100}{99}(4z + 2t)\phi(t)yt$.
(3) $|M| \leq xyp$.
(4) $|X \cup Y \cup Z| \leq \frac{100n(4z+2t)^2}{x}$.

We start with a lemma, which roughly states that the property of (a, b) being (x, y, z, p)-mineable implies that we can find a sufficiently large number of small, pairwise anticomplete (t, p)-barriers in G each of which separate a from b.

Lemma 4.2. *Let G be a graph with $n = |V(G)|$, let $x, y, z, t \in \mathbb{N}$, and let $w = \lceil \frac{99}{100} \frac{x}{4z+2t} \rceil$. Let $a, b \in V(G)$ and assume that (a, b) is (x, y, z, p)-mineable in G. Then there exists a set $C \subseteq V(G) \setminus \{a, b\}$ with $|cc(C)| \leq xy$ and a collection $\mathcal{B} = \{B_i = (I_i, J_i, K_i, C)\}_{i \in [w]}$ of (t, p)-barriers, such that*

- *For every i, (I_i, J_i, K_i, C) separates a from b.*
- *For every i, $|I_i \cup J_i \cup K_i| \leq \frac{100n(4z+2t)^2}{x}$.*
- *For all distinct $i, j \in [w]$, B_i and B_j are anticomplete.*

See Fig. 3 for an illustration of the outcome of Lemma 4.2.

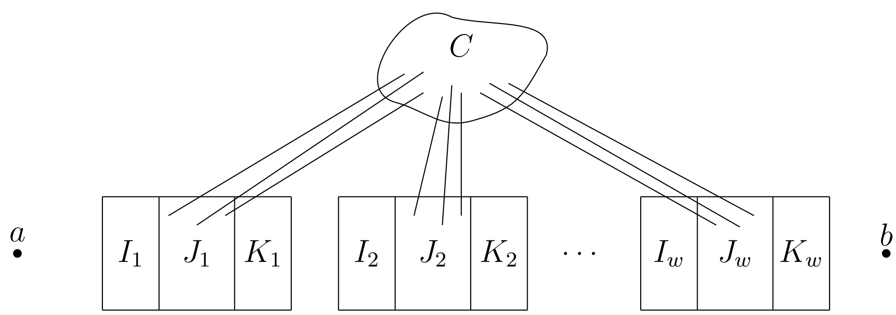

Figure 3. Visualization of the result of Lemma 4.2.

The proof of Lemma 4.2 proceeds by a "distance layering" argument. We obtain a set C and a-b separators $X_1, \ldots, X_x$ in $G \setminus C$ using that (a, b) is mineable. We define the "distance from a" for every vertex v to be the number of separators X_j we need to pass through in order to get from a to v in $G \setminus C$. This acts much like a distance function, with an approximate triangle inequality and the property that walking along any edge changes the distance by at most z. This lets us define "distance layers" as all vertices that have a certain distance from a. Our barriers will be unions of not too many consecutive distance layers. The bound on the size of the layers follows from an averaging argument.

Proof. Let $Y_1, \ldots, Y_x$ and $X_1, \ldots, X_x$ be sets as in the definition of (x, y, z, p)-mineable, with $X'_i = Y_i \cup X_i$ for each i. Let $C = \bigcup_{i=1}^{x} Y_i$; observe that $|cc(C)| \leq xy$ as $|cc(Y_i)| \leq y$ for each i. For each subset $W \subseteq V(G) \setminus C$, let $\psi(W) = |\{i \colon W \cap X_i \neq \emptyset\}|$. For every vertex $v \in G \setminus C$ we define

$$d_a(v) := \min_{a\text{-}v\text{-path } P \text{ in } G \setminus C} \psi(P),$$

where $d_a(v) = \infty$ if there is no a-v path in $G \setminus C$. For $j \in \mathbb{N}$ we denote by S_j the set of vertices $v \in \bigcup_{i=1}^{x} X_i$ such that $d_a(v) = j$.

The following properties regarding ψ and d_a are immediate from the definitions, and are used implicitly in the analysis that follows.

- For $W, W' \subseteq V(G) \setminus C$ we have $\psi(W \cup W') \leq \psi(W) + \psi(W')$. In particular, for distinct $v_1, v_2 \in V(G)$ and a path P with ends v_1 and v_2, we have $d_a(v_2) \leq d_a(v_1) + \psi(P)$.
- If W is a walk from a to v in $G \setminus C$, then $d_a(v) \leq \psi(W)$.
- If $v_1, v_2 \in V(G) \setminus C$ are adjacent and $d_a(v_1) < d_a(v_2)$, then $v_2 \in S_k$ for some $k \in [d_a(v_1), d_a(v_1) + z]$.

We now define $m = \left\lfloor \frac{x}{2z+t} \right\rfloor$, and for $j \in [m]$ we let $i_j^- = (j-1)(2z+t)+1$ and $i_j^+ = i_j^- + z$. Write $W_j = \bigcup_{k=i_j^-}^{i_j^+} S_k$. Then the sets W_j are pairwise disjoint. We let

$$L_j = \left\{ v \in G \setminus (C \cup W_j \cup W_{j+1}) : i_j^- \leq d_a(v) < i_{j+1}^- \right\}.$$

Next we prove the following.

(10) *For $j \in [m-1]$, $B_j = (W_j, L_j, W_{j+1}, C)$ is a (t,p)-barrier.*

Recall that, for each j, no component of Y_j has more than p vertices. As C is the union of the sets Y_j, and the sets Y_j are pairwise anticomplete, it follows that B_j satisfies condition (3) in the definition of a (t,p)-barrier.

Next, we show that B_j satisfies condition (1) in the definition of a (t,p)-barrier. Let P be a W_j-W_{j+1} path in $G \setminus C$; we show that there is a subpath P' of P that is a W_j-W_{j+1} path with $(P')^* \in L_j$. If P consists of a single edge then the statement holds trivially, so we may assume this is not the case. Write $P = v_1\text{-}v_2\text{-}\ldots\text{-}v_r$. If $d_a(v_i) \geq i_j^-$ for every $1 \leq i \leq r$, let $\alpha = 1$, otherwise let $\alpha = \max\left\{ i \colon 1 \leq i \leq r,\ d_a(v_i) < i_j^- \right\} + 1$. Note that $\alpha \leq n$ as $d_a(v_n) \geq i_{j+1}^- > i_j^-$ since P is a W_j-W_{j+1} path.

We show that $v_\alpha \in W_j$. If $\alpha = 1$ then this is true because P is a W_j-W_{j+1} path. Otherwise, by the choice of α, we have $d_a(v_{\alpha-1}) < d_a(v_\alpha)$, so $v_\alpha \in S_k$ for some $k \in [d_a(v_{\alpha-1}), d_a(v_{\alpha-1}) + z]$. Furthermore, by the choice of α we have $v_\alpha \geq i_j^-$, and since $d_a(v_{\alpha-1}) < i_j^-$ we have $d_a(v_{\alpha-1}) + z < i_j^+$. Thus $k \in [i_j^-, i_j^+]$, so $v_\alpha \in W_j$.

We further observe that $\alpha < r$, since $d_a(v_\alpha) \leq i_j^+ < i_{j+1}^- \leq d_a(v_r)$. We may thus define $\beta = \min\left\{ i \colon \alpha < i \leq r, d(v_i) \geq i_{j+1}^- \right\}$. An argument analogous to the previous paragraph shows that $v_\beta \in W_{j+1}$.

It follows immediately from the definitions of α and β that $i_j^- \leq d_a(v_i) < i_{j+1}^-$ for each i such that $\alpha < i < \beta$. In particular, we have that v_α-P-v_β is a W_j-W_{j+1} path contained in $W_j \cup L_j \cup W_{j+1}$. It follows that v_α-P-v_β contains a subpath (possibly itself) that is a W_j-W_{j+1} path with interior in L_j. Thus, B_j satisfies condition (1) in the definition of a (t,p)-barrier.

It remains to check condition (2) in the definition of a (t,p)-barrier. In view of the first condition, it is enough to show that every W_j-W_{j+1} path P with $P^* \subseteq L_j$ satisfies the second condition. Observe that if $\psi(P) \geq t$, then the second condition is satisfied as the sets Y_i are pairwise anticomplete.

We now show that $\psi(P) \geq t$. Let $u \in W_j$ and $v \in W_{j+1}$ be the ends of P, and let P_1 be a a-u-path in $G \setminus C$ achieving $\psi(P_1) = d_a(u) \leq i_j^+$. Then appending P to P_1 yields an a-v walk W

in $G \setminus C$. We thus have have

$$d_a(v) \leq \psi(W) \leq d_a(u) + \psi(P) \leq i_j^+ + \psi(P).$$

On the other hand, since $v \in W_{j+1}$ we know that $d_a(v) \geq i_{j+1}^- \geq i_j^+ + t$. Consequently, $i_j^+ + t \leq d_a(v) \leq i_j^+ + \psi(P)$, thus $\psi(P) \geq t$, and condition (2) in the definition of a (t,p)-barrier is satisfied. This proves (10).

(11) *For $j \in [m-1]$, B_j separates a and b.*

It suffices to show that $W_\ell \cup C$ separates a from b for every $\ell \in [m]$. First, since each X_k' separates a from b in $G \setminus C$, it follows that every a-b path in $G \setminus C$ meets X_k' for all $k \in [x]$, and therefore $d_a(b) \geq x$.

Now let $\ell \in [m]$, and let $P = v_1\text{-}\dots\text{-}v_r$ be an a-b path in $G \setminus C$, where $v_1 = a$ and $v_r = b$. Observe that by the definition of i_ℓ^- it holds that $i_\ell^- < x$. Since $d_a(b) \geq x$, we can define $\alpha \in [r]$ to be maximal such that $d_a(v_\alpha) < i_\ell^-$; note that $\alpha \leq r-1$. We now have that $v_{\alpha+1} \in S_k$ for some $k \in [d_a(v_\alpha), d_a(v_\alpha)+z]$. By the choice of α we have $k \geq i_j^-$, and since $d_a(v_\alpha) < i_\ell^-$ we have $d_a(v_\alpha) + z < i_\ell^- + z = i_\ell^+$, so $v_{\alpha+1} \in W_\ell$. Since every a-b path P in $G \setminus C$ meets W_ℓ, it follows that $W_\ell \cup C$ separates a from b in G, proving (11).

Finally, we show that a large enough subset of the B_j are pairwise anticomplete and have sufficiently small size. First, we put $m' = \lfloor \frac{m}{2} \rfloor = \lfloor \frac{x}{4z+2t} \rfloor$ and $\mathcal{B}' = \{B_{2i} \colon i \in [m']\}$.

(12) *For distinct $\alpha, \beta \in [m']$, $B_{2\alpha}$ and $B_{2\beta}$ are anticomplete.*

Assume without loss of generality that $\alpha < \beta$. We have $d_a(v) \leq i_{2\alpha+1}^+$ for all $v \in (W_{2\alpha} \cup I_{2\alpha} \cup W_{2\alpha+1})$ and $d_a(v) \geq i_{2\beta}^-$ for all $v \in (W_{2\beta} \cup I_{2\beta} \cup W_{2\beta+1})$. Since $\alpha < \beta$ we have $2\beta - (2\alpha+1) > 0$, and thus

$$i_{2\beta}^- - i_{2\alpha+1}^+ = (2\beta - (2\alpha+1))(2z+t) - z \geq 2z + t - z > 0.$$

Since d_a takes values at most $i_{2\alpha+1}$ on vertices in $(W_{2\alpha} \cup L_{2\alpha} \cup W_{2\alpha+1})$ and values at least $i_{2\beta}^-$ on vertices in $(W_{2\beta} \cup L_{2\beta} \cup W_{2\beta+1})$, and $i_{2\alpha+1}^+ < i_{2\beta}^-$, we conclude that $B_{2\alpha}$ and $B_{2\beta}$ are disjoint. Furthermore, suppose that $v_\alpha \in (W_{2\alpha} \cup L_{2\alpha} \cup W_{2\alpha+1})$ and $v_\beta \in (W_{2\beta} \cup L_{2\beta} \cup W_{2\beta+1})$ are adjacent. Then we have $v_\beta \in S_k$ for some $k \in [d_a(v_\alpha), d_a(v_\alpha)+z]$, and consequently $d_a(v_\beta) \in [d_a(v_\alpha), d_a(v_\alpha)+z]$. But $d_a(v_\alpha) + z < i_{2\beta}^- \leq d_a(v_\beta)$, a contradiction. It follows that $B_{2\alpha}$ and $B_{2\beta}$ are anticomplete, proving (12).

We now define

$$\mathcal{B}'' = \left\{ B_{2i} \colon i \in [m'],\ |W_{2i} \cup L_{2i} \cup W_{2i+1}| > \frac{100n(4z+2t)^2}{x} \right\} \subseteq \mathcal{B}'$$

and write $\mathcal{B} = \mathcal{B}' \setminus \mathcal{B}''$, so that for each $B_{2i} \in \mathcal{B}$ it holds that $|W_{2i} \cup L_{2i} \cup W_{2i+1}| \leq \frac{100n(4z+2t)^2}{x}$.

(13) *It holds that $|\mathcal{B}| \geq \frac{99}{100} \frac{100n(4z+2t)^2}{x}$.*

Since the B_{2i} (for $i \in [m']$) are pairwise disjoint, we have $\sum_{i=1}^{m'} |W_{2i} \cup I_{2i} \cup W_{2i+1}| \leq n$. Thus, we also have $\sum_{B_{2i} \in \mathcal{B}''} |W_{2i} \cup L_{2i} \cup W_{2i+1}| \leq n$. Since

$$\sum_{B_{2i} \in \mathcal{B}''} |W_{2i} \cup L_{2i} \cup W_{2i+1}| > |\mathcal{B}''| \left(\frac{100n(4z+2t)^2}{x} \right),$$

we have $|\mathcal{B}''| < \frac{x}{100(4z+2t)^2} \leq \frac{x}{100(4z+2t)}$. Letting $\mathcal{B} = \mathcal{B}' \setminus \mathcal{B}''$, we have

$$|\mathcal{B}| = |\mathcal{B}'| - |\mathcal{B}''| > \left\lfloor \frac{x}{4z+2t} \right\rfloor - \frac{x}{100(4z+2t)},$$

and thus $|\mathcal{B}| \geq \frac{99}{100}\frac{x}{4z+2t}$, proving (13).

We may now arbitrarily remove elements from $\mathcal{B}$ so that it has size $\lceil \frac{99}{100}\frac{x}{4z+2t} \rceil$, completing the proof. ■

Next we show that in the collection of barriers produced by Lemma 4.2 we can choose one with vertex set anticomplete to almost all components of C.

Lemma 4.3. *There exists a function $\phi : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $\beta, p, t \in \mathbb{N}$. Let G be an n-vertex $K_{t,t}$-induced-minor-free graph and let $a, b \in V(G)$. Let $\mathcal{B} = \{B_i = (X_i, Y_i, Z_i, C)\}_{i=1}^{\beta}$ be a family of pairwise anticomplete (t,p)-barriers separating a from b. Then there exists $C' \subseteq C$ with $|cc(C')| \leq \frac{\phi(t)|cc(C)|t}{\beta}$, i^* and $X'_{i^*} \subseteq X_{i^*}, Y_{i^*} \subseteq Y'_{i^*}, Z'_{i^*} \subseteq Z_{i^*}$, such that $(X'_{i^*}, Y'_{i^*}, Z'_{i^*}, C')$ is a (t,p)-barrier in $G \setminus M$ where $M = C \setminus C'$.*

See Fig. 4 for an illustration of the outcome of Lemma 4.3.

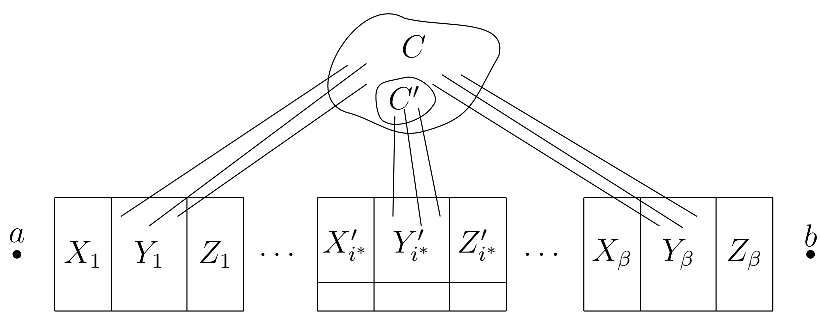

FIGURE 4. Visualization of Lemma 4.3.

Proof. Let ϕ be the function f from Lemma 2.1. Let $\mathcal{B}' = \{B'_i = (X'_i, Y'_i, Z'_i, C)\}_{i=1}^{\beta}$ be the family of reduced pairwise anticomplete (t,p)-barriers obtained by applying Lemma 3.1 on each member of $\mathcal{B}$.

Let $\mathcal{D} = cc\left(\bigcup_{i \leq \beta}(X'_i \cup Y'_i \cup Z'_i)\right)$ and $\mathcal{C} = cc(C)$. Now consider the bipartite graph Γ with bipartition $(\mathcal{D}, \mathcal{C})$ and where there is an edge from $d \in \mathcal{D}$ to $c \in \mathcal{C}$ if there is an edge from d to c in G. Γ is an induced minor of G as it can be obtained by deleting every vertex not in $C \cup \bigcup_{D \in \mathcal{D}} D$ and contracting every component of $\mathcal{D}$ and $\mathcal{C}$. Therefore, Γ is $K_{t,t}$-induced-minor-free. Let Γ' be the induced subgraph of Γ containing exactly one representative for each twin class of vertices in $\mathcal{D}$.

(14) $|E(\Gamma')| \geq \frac{|E(\Gamma)|}{t}$

The second condition of the definition of a (t,p)-barrier together with the fact that all barriers in $\mathcal{B}'$ are reduced imply that $\deg_{\Gamma}(d) \geq t$ for every $d \in \mathcal{D}$. It follows that every twin class in $\mathcal{D}$ contains fewer than t elements, as otherwise, there is a $K_{t,t}$-induced-minor in G. This proves (14).

The graph Γ' satisfies the assumptions of Lemma 2.1, and therefore $|E(\Gamma')| \leq \phi(t,k)|\mathcal{C}|$. By

(14), $|E(\Gamma)| \leq \phi(t,k)|\mathcal{C}|t$. Therefore, there is a (t,p)-barrier B'_{i^*} and a set $\mathcal{C}' \subseteq \mathcal{C}$ such that $|\mathcal{C}'| \leq \frac{\phi(t)|\mathcal{C}|t}{\beta}$ and $N(X'_{i^*} \cup Y'_{i^*} \cup Z'_{i^*}) \subseteq \bigcup_{c \in \mathcal{C}'} c$. Let $C' = \bigcup_{c \in \mathcal{C}'} c$. Noting that each component of C' is a component of C and thus contains at most p vertices, we deduce that $(X'_{i^*}, Y'_{i^*}, Z'_{i^*}, C')$ is a (t,p)-barrier in $G \setminus M$ where $M = C \setminus C'$ as required. ■

Now we summarize what we have shown so far to prove the main result of this section:

Proof of Theorem 4.1. Let ϕ be defined as in Lemma 4.3. The assertion of Theorem 4.1 follows by combining the family of (t,p)-barriers obtained in Lemma 4.2 with Lemma 4.3. ■

5. The class $\mathcal{C}_t$ is slim

The goal of this section is to prove the following:

Theorem 5.1. *For every $t \in \mathbb{N}$ there exist $p, q \in \mathbb{N}$ such that the class $\mathcal{C}_t$ is (p,q)-slim.*

We start with some definitions from [11]. Let G, H be graphs. Let $V(H) = \{v_1, \ldots, v_k\}$. An *induced H-model in G* is a k-tuple $K = (C_1, \ldots, C_k)$ of pairwise disjoint connected induced subgraphs of G such that for all distinct $i, j \in \{1, \ldots, k\}$, the sets C_i and C_j are anticomplete if an only if v_i is non-adjacent to v_j in H. We say that K is *linear* if every C_i is a path in G.

Let G be a graph. For $k, l \in \mathbb{N}$, a *(k,l)-block* in G is a pair $(B, \mathcal{P})$ where $B \subseteq V(G)$ with $|B| \geq k$, and $\mathcal{P}$ is map assigning to each 2-subset $\{x,y\}$ of B a set of at least l pairwise internally disjoint paths in G from x to y. We write $\mathcal{P}_{\{x,y\}} = \mathcal{P}(\{x,y\})$. We denote by $V(\mathcal{P}_{\{x,y\}})$ the union of the interiors of the paths that are elements of $\mathcal{P}_{\{x,y\}}$. We say that $(B, \mathcal{P})$ is *strong* if for all distinct 2-subsets $\{x,y\}, \{x',y'\}$ of B, we have $V(\mathcal{P}) \cap V(\mathcal{P}) = \emptyset$; that is, each path $P \in \mathcal{P}_{\{x,y\}}$ is internally disjoint from each path $P' \in \mathcal{P}_{\{x',y'\}}$.

We need the following result that was essentially proved in [11].

Lemma 5.2. *For all $s, \rho, \sigma \in \mathbb{N}$ there exist positive integers $f = f(s,\rho,\sigma)$ and $g = g(s,\rho,\sigma)$ with the following property. Let G be a graph and let $(B, \mathcal{Q})$ be a strong (f,g)-block in G such that B is a stable set and for every $\{x,y\} \subseteq B$, the paths $(Q^* : Q \in \mathcal{Q}_{\{x,y\}})$ are pairwise anticomplete in G. Then one of the following holds.*

- *(a) There is an induced subgraph of G isomorphic to a proper subdivision of K_s.*
- *(b) There is a linear induced $K_{\rho,\sigma}$-model in G.*

The difference between Lemma 5.2 here and Lemma 3.6 of [11] is that in [11] it is assumed that G is K_{t+1}-free (and t is another parameter in the statement of the theorem), but there is no assumption that the set B is stable. However, the only place in the proof where the bound on the clique number of G is used is an application of Ramsey Theorem to conclude that a large subset of B is stable, so we do not need that assumption here.

We also need the following, which follows immediately from the main result of result of [12]. Following [12], we say that a (s,l)-*constellation* is a graph C in which there is a stable set S_C with $|S_C| = s$, such that $C \setminus S_C$ has exactly l components, every component of $C \setminus S_C$ is a path, and every vertex of S_C has at least one neighbor in each component of $S \setminus S_C$.

Theorem 5.3. *For all $s, l, r \in \mathbb{N}$, there is a positive integer $a = a(s,l,r)$ with the following property. Let G be a graph that that contains a $K_{a,a}$-induced-minor. Then one of the following holds.*

- (a) *There is an induced subgraph of G isomorphic to either $K_{r,r}$, a subdivision of $W_{r \times r}$, or the line graph of a subdivision of $W_{r \times r}$.*

(b) *There is an* (s, l)*-constellation in* G*.*

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. Let $G \in \mathcal{C}_t$. Let $\rho = a(t, t, t)$ from Theorem 5.3. Let $q = g(t^2, \rho, \rho) + f(t^2, \rho, \rho)$ and $p = f(t^2, \rho, \rho)$ as in Lemma 5.2. We will show that G is (p, q)-slim. Suppose that there is a stable set $B \subseteq V(G)$ with $|B| = p$ such that every pair x, y in B is q-wide.

Since $|B| \leq p$, it follows that for every $x, y \in B$ there exists a set $\mathcal{Q}_{\{x,y\}}$ of x-y-paths whose interiors are disjoint from B and pairwise anticomplete with $|\mathcal{Q}_{\{x,y\}}| = g(t^2, \rho, \rho)$. Let G' be the graph obtained from $B \cup \bigcup_{x,y \in B} V(\mathcal{Q}_{\{x,y\}})$ by replacing each vertex v in $\bigcup_{x,y \in B} V(\mathcal{Q}_{\{x,y\}})$ with a clique consisting of vertices $v_{\{x,y\}}$ for every $x, y \in B$ such that v belongs to a path of $\mathcal{Q}_{\{x,y\}}$. Now for every $x, y \in B$, let $\mathcal{Q}'_{\{x,y\}}$ be the set of paths obtained by replacing each vertex v in each path of $\mathcal{Q}_{\{x,y\}}$ by its copy $v_{\{x,y\}}$. Let $\mathcal{Q}' = \bigcup_{x,y \in B} \mathcal{Q}'_{\{x,y\}}$. Then $(B, \mathcal{Q}')$ is an $(f(s, \rho, \rho), g(s, \rho, \rho))$ strong block satisfying the assumptions of Lemma 5.2. It follows from Lemma 5.2 that one of the following holds.

(a) There is an induced subgraph of G' isomorphic to a proper subdivision of K_t.
(b) There is a linear induced $K_{\rho,\rho}$-model in G'.

Suppose first that G' contains an induced subgraph F isomorphic to a proper subdivision of K_t. Then no two vertices of F are adjacent twins, and so F is isomorphic to an induced subgraph of G. It follows that G contains a subdivision of $W_{t,t}$, contrary to the fact that $G \in \mathcal{C}_t$. Thus we deduce that that there is a linear induced $K_{\rho,\rho}$-model $(C_1, \dots, C_{2\rho})$ in G'.

We apply Theorem 5.3 to $G'[\bigcup_{i=1}^{2\rho} C_i]$ to deduce that G' contains an induced subgraph F isomorphic to either $K_{t,t}$, a subdivision of $W_{t \times t}$, or the line graph of a subdivision of $W_{t \times t}$ or a (t, t)-constellation. In all cases, no two vertices of F are adjacent twins, and so F is isomorphic to an induced subgraph of G. But in all cases F contains either an induced $W_{t \times t}$-model, or an induced $K_{t,t}$-model, contrary to the fact that $G \in \mathcal{C}_t$. ■

6. Separating slim pairs of vertices in (F, r)-based graphs in $\mathcal{C}_t$

Let G be a graph and let F be an induced subgraph of G such that every component of F is a star (with at least two vertices). For every star S of F, let *the center* of S, denoted by $c(F)$, be defined as follows. If $|S| = 2$, then let $c(S)$ be an arbitrary vertex of S. If $|S| > 2$, let $c(S)$ be the vertex of degree greater than one in S. Let $l(S) = S \setminus c(S)$. Let $C(F)$ be the set of all centers of stars of F; we call the vertices of $C(F)$ the *centers of* F. Let $L(F) = F \setminus C(F)$; we call the vertices of $L(F)$ the *leaves* of F. Observe that both $C(F)$ and $L(F)$ are stable sets. For a set $X \subset V(G)$, we say that X is F*-based* if $S \subset X$ for every star S of F with $S \cap X \neq \emptyset$. We define the F*-measure* of an F-based subset X, $\mu_F(X)$, as follows. Let $a(X) = |\{S$ such that S is a star of F and $S \cap X \neq \emptyset\}|$. Let $b(X) = |X \setminus F|$. Then $\mu_F(X) = a(X) + b(X)$. Let G' be an induced subgraph of G and let S be a star of F. If $c(S) \in G'$ and $l(S) \cap G' \neq \emptyset$, the *projection of* S *onto* G', $p_{G'}(S)$, is the star $S \cap G'$. We define $F(G')$ to be the induced subgraph of G' whose components are $p_{G'}(S)$ where S is a star of F with $c(S) \in G'$ and $l(S) \cap G' \neq \emptyset$. We define $C(F(G')) = C(F) \cap F(G')$ and $L(F(G')) = L(F) \cap F(G')$. Let r be a positive integer. We say that G is (F, r)*-based* if every induced subgraph G' of G admits a tree decomposition (T', χ') such that for each $t \in V(T')$, $\chi'(t')$ is an $F(G')$-based set with $\mu_{F(G')}(\chi'(t')) \leq r$. We call (T', χ') an (F, r)*-tree decomposition* of G'. The goal of this section is to prove that, for appropriately chosen q, every q-slim pair of vertices in an (F, r)-based graph in $\mathcal{C}_t$ has a small separator.

We start with the following:

Theorem 6.1. *Let G be a graph, let F be an induced subgraph of G such that every component of F is a star (with at least two vertices), and let $r, q, x \in \mathbb{N}$. Assume that G is (F, r)-based. Let $a, b \in V(G)$ be an q-slim pair in G. Then there exists a set $D \subseteq V(G)$ with $|D| \leq r(q-1)x$ such that either*

(1) D is an a-b-separator in G, or
(2) (a, b) is $(x, r(q-1), 1, 1)$-mineable in $G \setminus D$.

We start with a lemma.

Lemma 6.2. *Let G be a graph, let $q \in \mathbb{N}$, and let $a, b \in V(G)$ be a q-slim pair. Let (T, χ) be a tree decomposition of G. Then there exists $I \subseteq V(T)$ with $|I| < q$ such that for every a-b-path P in G we have that $(\bigcup_{t\in T} \chi(t)) \cap P^* \neq \emptyset$.*

For $q = 3$ this is Theorem 2.6 of [9], but the same proof works for all q.
We can now prove Theorem 6.1.

Proof of Theorem 6.1. We may assume that a and b are in the same component of G, for otherwise the empty set is in a-b-separator in G. Let $G_0 = G$. Let (T_0, χ_0) be an (F, r)-tree decomposition of G_0. Let I_0 be as in Lemma 6.2. Let
$C_1 = \bigcup_{t\in I_0}(\chi(t) \setminus F)$,
$Y_1 = \bigcup_{t\in I_0}(\chi(t) \cap C(F))$ and
$X_1 = \bigcup_{t\in I_0}(\chi(t) \cap L(F))$.
Then $Y_1 \subseteq C(F)$, $X_1 \subseteq N(Y_1) \cap L(F)$ and $Y_1 \cup X_1$ is an (a, b)-separator in $G_0 \setminus C_1$. Since $Y_1 \subseteq C(F)$, it follows that Y_1 is a stable set, and so every component of Y_1 has size one. Moreover $|C_1 \cup Y_1| \leq r(q-1)$. Let $G_1 = G_0 \setminus (C_1 \cup Y_1)$. We may assume that a and b are in the same component of G_1, for otherwise the theorem holds setting $D = C_1 \cup Y_1$.

We proceed as follows. Assume that for some $i \geq 1$ we have defined $G_i, C_i, Y_1, \ldots, Y_i, X_1, \ldots, X_i$ with the following properties:

(1) $G_i = G \setminus (C_i \cup \bigcup_{j\leq i} Y_j)$.
(2) a and b are in the same component of G_i.
(3) $|C_i \cup \bigcup_{j\leq i} \cup Y_j)| \leq i(q-1)r$.
(4) $|Y_j| \leq r(q-1)$ for all $j \leq i$.
(5) $Y_1, \ldots, Y_i$ are disjoint subsets of $C(F)$.
(6) $\bigcup_{j=1}^{i} Y_i$ is a stable set.
(7) For every $j < i$, $X_j \subseteq L(F) \cap N(Y_j)$.
(8) $X_1, \ldots, X_i$ are disjoint subsets of $L(F)$.
(9) $Y_i \cup X_i$ is an a-b-separator in $G \setminus (C_i \cup \bigcup_{j<i} Y_j)$.

Note that G_1, C_1, X_1, Y_1 satisfy the conditions above. If $i = x$, we stop. If $i < x$, proceed as follows.

Let (T_i, χ_i) be an (F, r)-tree decomposition of G_i. Let I_i be as in Lemma 6.2. Let
$C_{i+1} = C_i \cup \bigcup_{t\in I_i}(\chi(t) \setminus F(G_i))$,
$Y_{i+1} = \bigcup_{t\in I_i}(\chi(t) \cap C(F(G_i)))$ and
$X_{i+1} = \bigcup_{t\in I_i}(\chi(t) \cap L(F(G_i)))$.
Then $X_{i+1} \subseteq N(Y_{i+1}) \cap L(F)$ and $Y_{i+1} \cup X_{i+1}$ is an (a, b)-separator in $G \setminus (C_{i+1} \cup \bigcup_{j<i+1} Y_j)| = G_i \setminus C_{i+1}$. Moreover, $|Y_{i+1}| \leq r(q-1)$ and

$$|C_{i+1} \cup \bigcup_{j\leq i+1} Y_j| \leq |C_i \cup \bigcup_{j\leq i} Y_j| + |\bigcup_{t\in I_i} \chi(t) \setminus L(F(G_i)| \leq (i-1)r(q-1) + r(q-1) = i(q-1)r.$$

It follows from the definitions of $C(F(G_i))$ and $L(F(G_i))$ that Y_{i+1} is disjoint from $\bigcup_{j\leq i} Y_j$, and X_{i+1} is disjoint from $\bigcup_{j\leq i} X_j$. Let $G_{i+1} = G_i \setminus (C_{i+1} \cup \bigcup_{j\leq i+1} Y_j)$. If a and b are in different components of G_{i+1}, then $C_{i+1} \cup \bigcup_{j\leq i+1} Y_j$ is an a-b-separator of size at most $(i+1)(q-1)r \leq x(q-1)r$ in G, and the theorem holds. Thus we may assume that a and b are in the same component of G_{i+1}. Now $G_{i+1}, C_{i+1}, Y_1, \dots Y_{i+1}, X_1, \dots, X_{i+1}$ satisfy the conditions above, and we can continue the process.

It follows that we may assume that $i = x$. Now, setting $D = C_i$, the sets $Y_1, \dots, Y_x, X_1, \dots, X_x$ show that (a, b) is $(x,(q-1)r, 1, 1)$-mineable in $G \setminus D$, as required. ■

We can now prove the main result of this section. Let ϕ be as in Theorem 4.1. We define $\psi(t, q) = \frac{100^2}{99}(4 + 2t)^2\phi(t)(q - 1)$.

Theorem 6.3. *Let $t \in \mathbb{N}$ and let p, q be as in Theorem 5.1. Let $r \in \mathbb{N}$ be such that $r \geq \psi(t, q)$. Then, for every (F, r)-based graph $G \in \mathcal{C}_t$ on n vertices, and every q-slim pair (a, b) in G, there exists an a-b separator in G of size at most $2^{5(\log(r)+\sqrt{\log n \log r})}$.*

Proof. By Theorem 5.1 G is (p, q)-slim. Let $\phi : \mathbb{N} \to \mathbb{N}$ be as in Theorem 4.1. Let $x = 2^{\sqrt{\log(n)\log(r)}}100(4 + 2t)^2 \leq 2^{\sqrt{\log(n)\log(r)}}r$. We define the following functions.

$$c(n) = \frac{100}{99}(4 + 2t)\phi(t)\, r(q - 1)\, t \ \leq \frac{r^2}{100}.$$

$$f(n) = 2xr(q - 1) \leq \frac{xr^2}{100}.$$

$$g(n) = \frac{100n(4 + 2t)^2}{x} \leq 2^{-\sqrt{\log(n)\log(r)}}.$$

(15) *G is $(q, t, 1, c, f, g)$-slim-barred.*

Let (c, d) be a q-slim pair in G. Since G is (F, r)-based, Theorem 6.1 implies that there is $D \subseteq V(G)$ with $|D| \leq xr(q - 1)$ such that either

(1) D is a c-d-separator in G, or
(2) (c, d) is $(x, r(q - 1), 1, 1)$-mineable in $G \setminus D$.

If D is a c-d-separator in G, then setting $X = Y = Z = C = \emptyset$ and $M = D$ gives a $(t, 1)$ barrier in $G \setminus M$. Thus, we may assume that (c, d) is $(x,(q-1)r, 1, 1)$-mineable in $G \setminus D$. Since (c, d) is q-slim in G and $G \in \mathcal{C}_t$, Theorem 4.1 implies that there exist disjoint subsets X, Y, Z, C, M' of $V(G) \setminus (D \cup \{c, d\})$ such that

(1) $B = (X, Y, Z, C)$ is a t-barrier in $G \setminus M'$ that separates c from d.
(2) $|C| \leq \frac{100}{99}(4 + 2t)\phi(t)\, r(q - 1)\, t$.
(3) $|M'| \leq xr(q - 1)$.
(4) $|X \cup Y \cup Z| \leq \frac{100n(4+2t)^2}{x}$.

Setting $M = M' \cup D$, we get that there exist disjoint subsets X, Y, Z, C, M of $V(G) \setminus \{c, d\}$ such that

(1) $B = (X, Y, Z, C)$ is a t-barrier in $G \setminus M$ that separates c from d.
(2) $|C| \leq \frac{100}{99}(4 + 2t)\phi(t)\, r(q - 1)\, t$.
(3) $|M| \leq 2xr(q - 1)$.
(4) $|X \cup Y \cup Z| \leq \frac{100n(4+2t)^2}{x}$.

This proves (15).

Now since (a, b) is q-slim and G is (q, t, c, f, g)-slim-barred, Theorem 3.3 implies that there is an a-b-separator of size at most $20(f(n) + 3c(n)^2)c(n)^{2^{\frac{\log n}{\log \frac{n}{g(n)}}}}$ in G. We now bound this quantity.

$$\begin{aligned}
20(f(n) + 3c(n)^2)c(n)^{2^{\frac{\log n}{\log \frac{n}{g(n)}}}} &= 20(f(n) + 3c(n)^2)c(n)^{2^{\sqrt{\frac{\log n}{\log r}}}} \\
&\leq 20\left(\frac{xr^2}{100} + \frac{3r^4}{100}\right) c(n)^{2^{\sqrt{\frac{\log n}{\log r}}}} \\
&\leq xr^4 c(n)^{2^{\sqrt{\frac{\log n}{\log r}}}} \\
&\leq xr^4 2^{4\log(r)\sqrt{\frac{\log n}{\log r}}} \\
&\leq xr^4 2^{4\sqrt{\log n \log r}} \\
&\leq r^5 2^{5\sqrt{\log n \log r}} \\
&= 2^{5(\log(r) + \sqrt{\log n \log r})}.
\end{aligned}$$

■

7. Bounding the treewidth of (F, r)-based graphs in $\mathcal{C}_t$

In this section, we complete the treatment of (F, r)-based graphs, proving the following. Recall that $\psi(t, q) = \frac{100^2}{99}(4 + 2t)^2\phi(t)(q - 1)$, where ϕ is as in Theorem 4.1.

Theorem 7.1. *Let $t \in \mathbb{N}$, let p, q be as in Theorem 5.1, and let $r \geq \psi(t, q)$. Then every n-vertex (F, r)-based graph G in $\mathcal{C}_t$ satisfies $tw(G) \leq 2^{9\log(r) + 5\sqrt{\log n \log r}}$.*

We remark that a version of Theorem 7.1 could be proved using Theorem 6.3 and Theorem 6.5 of [2] if the clique number of G is bounded. Here we will include a different proof that does not use this assumption.

For a graph G a function $w : V(G) \to [0, 1]$ is a *normal weight function* on G if $w(V(G)) = 1$, where for $X \subseteq V(G)$ we denote $\sum_{v \in X} w(v)$ by $w(X)$. Let $c \in [0, 1]$ and let w be a normal weight function on G. A set $X \subseteq V(G)$ is a *(w, c)-balanced separator* if $w(D) \leq c$ for every component D of $G \setminus X$. The set X is a *w-balanced separator* if X is a $(w, \frac{1}{2})$-balanced separator.

The following result was originally proved by Robertson and Seymour in [20], and tightened by Harvey and Wood in [17]. It was then restated and proved in the language of (w, c)-balanced separators in [3].

Theorem 7.2. *Let G be a graph, let $c \in [\frac{1}{2}, 1)$, and let d be a positive integer. If for every normal weight function $w : V(G) \to [0, 1]$, G has a (w, c)-balanced separator of size at most d, then $tw(G) \leq \frac{1}{1-c}d$.*

We now prove the main result of this section.

Proof of Theorem 7.1. We start with the following.

(16) *Let $w : V(G) \to [0,1]$ be a normal weight function on G. Then G admits a $(w, \frac{1}{2})$-balanced separator of size at most $rp + (rp)^2 2^{5(\log(r)+\sqrt{\log n \log r})}$.*

Let $G_1 = G$, $X_0 = \emptyset$ and $w_1 = w$. For $i \in \{1, \dots, p\}$, Assume that $X_1, \dots, X_{i-1}, G_1, \dots, G_i$ have already been defined and proceed as follows. Let (T_i, χ_i) be an (F, r)-tree decomposition of G_i. It is well known (see e.g. the proof of Lemma 7.19 in [13] or Theorem 2.7 in [9]) that there exists $t_i \in V(T_i)$ such that $w_i(D) \leq \frac{1}{2}$ for every component D of $G \setminus \chi_i(t_i)$. Let $X_i = \chi_i(t_1) \setminus L(F(G_i))$. Then $|X_i| \leq r$.

If $i < p$, define $G_{i+1} = G_i \setminus X_i$ and $w_{i+1}(v) = \frac{w_i(v)}{1-\Sigma_{u \in X_i} w_i(u)}$ for every $v \in G_{i+1}$. Then w_{i+1} is a normal weight function on G_{i+1}, and we repeat the process to define X_{i+1}. We stop when $i = p$ and $X_1, \dots, X_p$ have been defined.

Next, let $\mathcal{P}$ be the set of all q-slim pairs (v, v') such that $v \in X_i$ and $v' \in X_{i'}$ for some $1 \leq i < i' \leq p$. For every $(v, v') \in \mathcal{P}$, let $X_{vv'}$ be a v-v'-separator in G of size at most $2^{5(\log(r)+\sqrt{\log n \log r})}$ given by Theorem 6.3. Let $X = \bigcup_{i=1}^{p} X_i \cup \bigcup_{(v,v') \in \mathcal{P}} X_{vv'}$. Then $|X| \leq rp + (rp)^2 2^{5(\log(r)+\sqrt{\log n \log r})}$. We show that X is a balanced separator in G. Suppose for contradiction that there is a component D of $G \setminus X$ with $w(D) > \frac{1}{2}$. Then $w_i(D) > \frac{1}{2}$ for every i. Since $\chi(t_i)$ is a w_i-balanced separator in G_i for every i, it follows that for every $i \in \{1, \dots, p\}$ there is $v_i \in X_i \cap C(F(G_i))$ such that v_i has a neighbour in D (in fact, v_i has a neighbour in $D \cap \chi_i(t_i) \cap L(F(G_i))$). By Theorem 5.1 there exist $v, v' \in \{v_1, \dots, v_p\}$ such that $(v, v') \in \mathcal{P}$. But there is a path from v to v' with interior in D, contrary to the fact that $D \subseteq G \setminus X_{vv'}$. This proves (16).

From (16) and Theorem 7.2 we deduce that

$$\begin{aligned} tw(G) &\leq 2\left(rp + (rp)^2 2^{5(\log(r)+\sqrt{\log n \log r})}\right) \\ &\leq 4(rp)^2 2^{5(\log(r)+\sqrt{\log n \log r})} \\ &\leq r^4 2^{5(\log(r)+\sqrt{\log n \log r})} \\ &= 2^{9\log(r)+5\sqrt{\log n \log r}}. \end{aligned}$$

■

8. Bounding the treewidth of $\mathcal{C}_t^*$

In this section, we prove the main result of this paper, which we restate.

Theorem 1.2. *For every $t \in \mathbb{N}$, there exist $\epsilon = \epsilon(t) \in (0, 1]$, $c = c(t) \in \mathbb{N}$ and $d \in \mathbb{N}$ such that every n-vertex graph G in $\mathcal{C}_t^*$ satisfies $tw(G) \leq 2^{a \log^{1-\epsilon} n}$.*

We follow the general road map of [6], but we phrase our arguments in a slightly different language. A *star-coloring* of a graph G is a proper coloring such that the union of every two color classes induces a star forest in G (that is a graph where each component is a star or a singleton). The *star chromatic number* of G is the minimum k such that G admits a star coloring with k color classes. The following is immediate from Theorem 11 of [6] and Ramsey Theorem:

Theorem 8.1. *For every $t \in \mathbb{N}$ there is an integer $d_1 = d_1(t)$ such that every graph in C_t^* has star-chromatic number d_1.*

We will also use the main result of [6]:

Theorem 8.2. *There exists an integer d_2 such that for all $t, \Delta \in \mathbb{N}$, every graph in $\mathcal{C}_t^*$ with maximum degree at most Δ has treewidth at most $(t\Delta)^{d_2}$.*

Finally, we need the following result of [8]:

Theorem 8.3. *There is an integer d_3 such that every graph G contains a subcubic subgraph of treewidth at least $\frac{tw(G)}{\log^{d_3}(tw(G))}$.*

Proof of Theorem 1.2. Let d_1 be as in Theorem 8.1, and let $C_1, \dots, C_{d_1}$ be the color classes of a star coloring of G. For $i, j \in \{1, \dots, d_1\}$ let $F'_{ij} = G[C_i \cup C_j]$. Then F'_{ij} is a star forest. Let F_{ij} be the graph obtained from F'_{ij} be removing all isolated vertices; write $E_{ij} = E(F_{ij})$. Note that $\bigcup_{i,j \in \{1,\dots,d_1\}} E_{ij} = E(G)$; we call $\{E_{ij}\}_{i,j\in\{1,\dots,d_1\}}$ *the corresponding edge partition*. We say that E_{ij} is *active* in G if some vertex of G is incident with more than three edges of E_{ij}. We define the *dimension* of G to be the number of pairs (i, j) for which E_{ij} is active, and denote the dimension of G by $dim(G)$. Let p, q be as in Theorem 5.1, let d_2 be as in Theorem 8.2, and let $r_0 = \max\left\{\psi(t, q), (3td_1^2)^{d_2}\right\}$.

(17) *If $dim(G) = 0$, then $tw(G) \leq r_0$.*

Since $dim(G) = 0$, it follows that the maximum degree of G is at most $3d_1^2$. Now (17) follows from Theorem 8.2.

Let d_3 be as in Theorem 8.3. Next, we recursively define a sequence of integers $r_1, \dots, r_{{d_1}^2}$. Having defined $r_1, \dots, r_i$, let $r_{i+1} = 2^{9\log\left(r_i \log^{d_3} n\right)+5\sqrt{\log\left(r_i\log^{d_3}(n)\right)\log(n))}}$.

We will prove by induction on $dim(G)$ that $tw(G) \leq r_{dim(G)}$. The base case is (17), thus we assume that we have proved the result for graphs of dimension at most i, and that $dim(G) = i+1$. Let $i_0, j_0 \in 1, \dots d_1$ be such that $E_{i_0j_0}$ is active in G.

(18) *Let G_0 be an induced subgraph of G with $tw(G_0) > r_i \log^{d_3} n$. Let G' be the graph obtained from G_0 by contracting the edges of $F_{i_0j_0}(G_0)$. Then $tw(G') \leq r_i \log^{d_3} n$.*

Suppose not. By Theorem 8.3 G' contains a subcubic subgraph G'' with $tw(G'') > r_i$. The third and fourth paragraphs of the proof of Lemma 7 of [6] show how to construct an induced subgraph H of G_0 that contains G'' as a minor, and such that every vertex of H is incident with at most three edges of $E_{i_0j_0}$. It follows that $tw(H) \geq tw(G'') > r_i$.

We now show that $dim(H) \leq i$ and get contradiction. First observe that $C_1 \cap V(H), \dots, C_{d_1^2} \cap V(H)$ is a star coloring of H with corresponding edge partition $\{E_{pq} \cap E(H))\}_{p,q\in\{1,\dots,d_1\}}$. It follows that the set $E_{i_0j_0} \cap V(H)$ is not active in H. Moreover, if the set E_{pq} is not active in G, then the set $E_{pq} \cap V(H)$ is not active in H. Since $E_{i_0j_0}$ is active in G, we deduce that $dim(H) < dim(G) = i+1$, a contradiction. This proves (18).

(19) *G is $(F_{i_0j_0}, r_i \log^{d_3} n)$-based.*

Let G_0 be an induced subgraph of G. We need to show that G_0 admits an $(F_{i_0j_0}, r_i \log^{d_3} n)$-based tree decomposition. Let G' be the graph obtained from G_0 by contracting the edges of $F_{i_0j_0}(G_0)$. Then every vertex of G' is either a vertex of G or corresponds to a component of $F_{i_0j_0}(G_0)$; we call the latter kind of vertex a *star vertex*. Let (T, χ') be a tree decomposition of G' of width $tw(G')$. We construct a tree decompositon (T, χ_0) of G_0, where for every $t \in T$, $\chi_0(t)$ is obtained from

$\chi'(t)$ by replacing every star vertex with the component of $G(F_{i_0 j_0})$ to which it corresponds. Since we know by (18) that $tw(G') \leq r_i \log^{d_3} n$, it follows that (T_0, χ_0) is an $(F_{i_0 j_0}, r_i \log^{d_3} n)$-based tree decomposition of G_0, and (19) follows.

Now, in view of (19), Theorem 7.1 implies that $tw(G) \leq r_{i+1}$, as required. This completes the inductive proof that $tw(G) \leq r_{dim(G)}$.

We now bound r_i. Let $c_0 = r_0 + 18d_3$. Now we show that $r_i \leq 2^{16^i c_0 \log^{1-1/2^i} n}$. It is enough to prove that $\log r_i \leq 16^i c_0 \log^{1-1/2^i} n$. To do so, we proceed by induction. The base case trivially holds as $c_0 \geq r_0$. For $i \geq 1$, let $c_i = 16^i c_0$ and $\epsilon_i = \frac{1}{2^i}$. By the induction hypothesis, we have that

$$
\begin{aligned}
\log r_{i+1} &\leq 9 \log\Big(2^{c_i \log^{(1-\epsilon_i)}(n)} \log^{d_3} n\Big) + 5\sqrt{\Big(c_i \log^{(1-\epsilon_i)}(n) + \log\Big(\log^{d_3}(n)\Big)\Big) \log(n))} \\
&\leq 9c_i \log^{(1-\epsilon_i)}(n) + 9d_3 \log\log n + 5\sqrt{c_i} \log^{(1-\epsilon_i/2)}(n) + 5\sqrt{d_3 \log\log(n) \log(n))} \\
&\leq 14c_i \log^{1-\epsilon_i/2}(n) + 9d_3 \log\log n + 5\sqrt{d_3 \log\log(n) \log(n)} \\
&\leq 14c_i \log^{1-\epsilon_i/2}(n) + c_i \log^{1-\epsilon_i/2}(n) + c_i \log^{1-\epsilon_i/2}(n) \\
&\leq 16c_i \log^{1-\epsilon_i/2}(n)
\end{aligned}
$$

Setting $c = c_{d_1^2}$ and $\epsilon = \epsilon_{d_1^2}$ completes the proof. ■

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