Large Cliques and Stable Sets in Undirected Graphs

Maria Chudnovsky [*]
Columbia University, New York NY 10027

May 4, 2014

Abstract

The *cochromatic number* of a graph G is the minimum number of stable sets and cliques of G covering the vertex-set of G. In this paper we survey some resent results and techniques developed in an attempt to answer the question: excluding which induced subgraphs causes a graph to have bounded cochromatic number?

1 Introduction

All graphs in this paper are finite and simple. Let G be a graph. We denote by $V(G)$ the vertex-set of G. A *tournament* is a directed graph, where for every two vertices u, v exactly one of the (ordered) pairs uv and vu is an edge. For tournaments S and T, we say that T is *S-free* if no subtournament of T is isomorphic to S. If $\mathcal{S}$ is a family of tournaments, then T is *$\mathcal{S}$-free* if T is S-free for every $S \in \mathcal{S}$. A tournament is *transitive* if it has no directed cycles (or, equivalently, no cyclic triangles). For a tournament T, we denote by $\alpha(T)$ the maximum number of vertices of a transitive subtournament of T. Finally, the *chromatic number* of T is the smallest number of transitive subtournaments of T whose vertex-sets have union $V(T)$.

We say that a tournament S is a *hero* if there exists $d > 0$ such that every S-free tournament has chromatic number at most d, and S is a celebrity if there exists $0 < c \leq 1$ such that every S-free tournament T has $\alpha(T) \geq c|V(T)|$. Heroes and celebrities are studied in [1]. Somewhat surprisingly, it turns out that a tournament is a hero if and only if it is a celebrity (the "only if" implication is clear, but the "if" is non-trivial). The main result of [1] says that all heroes (and equivalently celebrities) can be constructed starting from single vertices by repeatedly applying two growing operations; and every tournament constructed in that way is a hero.

Similar questions make sense for undirected graphs as well as tournaments, and the goal of this paper is to survey recent progress on this topic.

2 Heroes without direction

Let G be an undirected graph. For a subset X of $V(G)$ we denote by $G|X$ the subgraph of G induced by X. The *complement* G^c of G is the graph with vertex set $V(G)$, such that two vertices

[*]Partially supported by NSF grants DMS-1001091 and IIS-1117631.

are adjacent in G if and only if they are non-adjacent in G^c. A *clique* in G is a set of vertices all pairwise adjacent. A *stable set* in G is a set of vertices all pairwise non-adjacent (thus a stable set in G is a clique in G^c). The largest size of a clique in G is denoted by $\omega(G)$, and the largest size of a stable set by $\alpha(G)$. The *chromatic number* of G is the smallest number of stable sets of G with union $V(G)$. Given a graph H, we say that G is *H-free* if G has no induced subgraph isomorphic to H. If G is not H-free, we say that G contains H. For a family $\mathcal{F}$ of graphs, we say that G is $\mathcal{F}$-free if G is F-free for every $F \in \mathcal{F}$.

As with tournaments, one might ask what graphs H have the property that every H-free graph G has chromatic number bounded by a constant d (where d depends on H, but not on G). However, this question does not have an interesting answer. The complete graph on n vertices is H-free for every H that is not a complete graph, and has chromatic number n. On the other hand, for every $k > 0$ there exist graphs with no clique of size three and with chromatic number at least k (this is a theorem of Erdős [4]), and so only graphs with at most two vertices have the property.

Let us modify the question a little. The *cochromatic number* of a graph G is the minimum number of stable sets and cliques of G with union $V(G)$. We denote the cochromatic number of G by $co\chi(G)$. Let us say that a family $\mathcal{H}$ is *heroic* if there exists a constant $d(\mathcal{H}) > 0$ such that $co\chi(G) \leq d(\mathcal{H})$ for every every $\mathcal{H}$-free graph G, and it is *celebrated* if there exists a constant $0 < c(\mathcal{H}) \leq 1$ such that every $\mathcal{H}$-free graph G contains either a clique or a stable set of size at least $c(\mathcal{H})|V(G)|$. Clearly, if $\mathcal{H}$ is heroic, then it is celebrated. This turns out to be a better undirected analogue of the concepts discussed in the Introduction. Heroic and celebrated families of undirected graphs were studied in [3].

Let G be a complete multipartite graph with m parts, each of size m. Then G has m^2 vertices, and no clique or stable set of size larger than m; and the same is true for G^c. Thus every celebrated family contains a complete multipartite graph and the complement of one. The *girth* of a graph is the smallest length of a cycle in it. Recall that for every positive integer g there exist graphs with girth at least g and no linear-size stable set (this is a theorem of Erdős [4]). Consequently, every celebrated family must also contain a graph of girth at least g, and, by taking complements, a graph whose complement has girth at least g. Thus, for a finite family of graphs to be celebrated, it must contain a forest and the complement of one. In particular, if a celebrated family only contains one graph H, then $|V(H)| \leq 2$. The following conjecture, proposed in [3], states that these necessary conditions for a finite family of graphs to be celebrated are in fact sufficient for being heroic.

Conjecture 2.1 *A finite family of graphs is heroic if and only if it contains a complete multipartite graph, the complement of a complete multipartite graph, a forest, and the complement of a forest.*

We remark that this is closely related to a well-known conjecture made independently by Gyárfás [5] and Sumner [10], that can be restated as follows in the language of heroic families:

Conjecture 2.2 *For every complete graph K and every forest T, the family $\{K, T\}$ is heroic.*

For partial results on Conjecture 2.2 see [6, 7, 8, 9].

Since a complete graph is a multipartite graph, the complement of one, and the complement of a forest, we deduce that Conjecture 2.1 implies Conjecture 2.2. The main result of [3] is that Conjecture 2.1 and Conjecture 2.2 are in fact equivalent. A graph G is c-split if $V(G) = X \cup Y$, where

- $\omega(X) \leq c$, and
- $\alpha(Y) \leq c$.

The fact that Conjecture 2.2 implies Conjecture 2.1 is a consequence of the following theorem of [3]:

Theorem 2.1 *Let K and J be graphs, such that both K and J^c are complete multipartite. Then there exists a constant $c(K, J)$ such that every $\{K, J\}$-free graph is $c(K, J)$-split.*

Now let $\mathcal{F}$ be a family of graphs that contains a complete multipartite graph, the complement of a complete multipartite graph, a forest, and the complement of a forest, and let G be an $\mathcal{F}$-free graph. By Theorem 2.1, there exists a constant c such that G is c-split. Now applying Conjecture 2.2 to $G|X$ and $G^c|Y$, the assertion of Conjecture 2.1 follows.

3 Cographs

The goal of this section is to discuss a generalization of Theorem 2.1. A *cograph* is a graph obtained from one-vertex graphs by repeatedly taking disjoint unions and disjoint unions in the complement. In particular, complete graphs, their complements, complete multipartite graphs, and their complements are all cographs. Thus Theorem 2.1 says that excluding a pair of cographs (a complete multipartite graph and the complement of one) from a graph G guarantees that G has a partition into two parts, each of which excludes a cograph that is in some sense simpler (a complete graph or its complement). It turns out that this idea can be generalized to all cographs.

Let us make this more precise. We say that a graph G is *anticonnected* of G^c is connected. A *component* of G is a maximal non-empty connected subgraph of G, and an *anticomponent* of G is a maximal non-empty anticonnected induced subgraph of G.

First we observe that for every cograph G with at least two vertices, exactly one of G, G^c is connected. Next we recursively define a parameter, called the *height* of a cograph, that measures its complexity. The height of a one-vertex cograph is zero. If G is a cograph that is not connected, let m be the maximum height of a component of G; then the height of G is $m + 1$. If G is a cograph that is not anticonnected, let m be the maximum height of an anticomponent of G; then the height of G is $m + 1$.

Let G be a graph. Given a pair of graphs H_1, H_2, we say that G is $\{H_1, H_2\}$*-split* if $V(G) = X_1 \cup X_2$, where the subgraph of G induced by X_i is H_i-free for every $i \in \{1, 2\}$. One of the results of [2] is the following:

Theorem 3.1 *Let $k > 0$ be an integer, and let H and J be cographs, each of height $k + 1$, such that H is anticonnected, and J is connected. Then there exist cographs $\tilde{H}$ and $\tilde{J}$, each of height k, such that $\tilde{H}$ is connected, and $\tilde{J}$ is anticonnected, and every $\{H, J\}$-free graph is $(\tilde{H}, \tilde{J})$-split.*

Clearly Theorem 2.1 follows from Theorem 3.1 by taking $k = 1$, and observing the cographs of height one are complete graphs and their complements.

4 Excluding pairs of graphs

Given an integer $P > 0$, a graph G, and a set of graphs $\mathcal{F}$, we say that G *admits an* $(\mathcal{F}, P)$*-partition* if the vertex set of G can be partitioned into P subsets $X_1, \dots, X_P$, so that for every $i \in \{1, \dots, P\}$, either $|X_i| = 1$, or the subgraph of G induced by X_i is F-free for some $F \in \mathcal{F}$ (we remark that the condition that $|X_i| = 1$ is only necessary when all the members of $\mathcal{F}$ are one-vertex graphs).

The proof of Theorem 2.1 in [3] relies on the following fact:

Theorem 4.1 *Let $p > 0$ be an integer. There exists an integer $r > 0$ such that for every graph G, if every induced subgraph of G with at most r vertices is p-split, then G is p-split.*

Here is a weaker statement that would still imply Theorem 2.1 (here K_p is the complete graph on p vertices, and S_p is the complement of K_p):

Theorem 4.2 *Let $p > 0$ be an integer. There exist integers $r, k > 0$ such that for every graph G, if every induced subgraph of G with at most r vertices is p-split, then G admits a $(\{K_p, S_p\}, k)$-partition.*

However, the proof of Theorem 3.1 did not follow the same route, and no result similar to Theorem 4.2 exists in the setting of general cographs, because of the following theorem of [2]:

Theorem 4.3 *Let H, J be graphs, each with at least one edge. Then for every choice of integers r, k there is a graph G such that*

- *for every $S \subseteq V(G)$ with $|S| \leq r$, the graph $G|S$ is $\{H, J\}$-split, and*
- *G has no $(\{H, J\}, k)$-partition.*

Unfortunately, we do not have an easy construction for Theorem 4.3; our proof involves probabilistic arguments.

By taking complements, the conclusion of Theorem 4.3 also holds if each of H and J has a non-edge. Thus Theorem 4.2 is in a sense the strongest theorem of this form possible. In view of this fact, the proof of Theorem 3.1 in [2] takes a different route, and uses a much more general result, which roughly says that excluding a pair of graphs, one of which is not connected and the other not anticonnected, causes a graph to “break apart” into a bounded number of simpler pieces.

Here is a more precise statement. We denote by $c(H)$ the set of components of H, and by $ac(H)$ the set of anticomponents of H. We remark that for every non-null graph G, at least one of $c(G)$ or $ac(G)$ equals $\{G\}$.

Theorem 4.4 *For every pair of graphs (H, J) there exists an integer P such that every $\{H, J\}$-free graph admits a $(c(H) \cup ac(J), P)$-partition.*

Please note that Theorem 4.4 is trivial unless H is not connected and J is not anticonnected. Even though Theorem 4.4 was motivated by trying to prove Theorem 3.1, its generality makes it an interesting result on its own (possibly more so than Theorem 3.1).

To deduce Theorem 3.1 from Theorem 4.4 one just needs to observe the following:

Theorem 4.5 *Let P, k be positive integers. Let $\mathcal{F}$ be a set of connected cographs, all of height at most k. Then there exists a connected cograph C of height k such that for every partition $X_1, \dots, X_P$ of $V(C)$ there exists $i \in \{1, \dots, P\}$ such that $C|X_i$ contains every member of $\mathcal{F}$.*

The proof of Theorem 4.5 is not difficult, and can be found in [2].

5 Back to tournaments

Given tournaments H_1 and H_2 with disjoint vertex sets, we write $H_1 \Rightarrow H_2$ to mean the tournament H with $V(H) = V(H_1) \cup V(H_2)$, and such that $H|V(H_i) = H_i$ for $i = 1, 2$, and every vertex of $V(H_1)$ is adjacent to (rather than from) every vertex of $V(H_2)$. One of the results of [1] is a complete characterization of all heroes. An important and the most difficult step toward that is the following:

Theorem 5.1 *If H_1 and H_2 are heroes, then so is $H_1 \Rightarrow H_2$.*

It turns out that translating the proof of Theorem 4.4 into the language of tournaments gives the following result [2]:

Theorem 5.2 *Let H_1, H_2 be non-null tournaments, and let H be $H_1 \Rightarrow H_2$. Let $m = \max(|V(H_1)|, |V(H_2)|)$. Then every H-free tournament admits an $(\{H_1, H_2\}, 2(m+1)^m)$-partition.*

Theorem 5.2 immediately implies Theorem 5.1. More precisely, we have:

Theorem 5.3 *Let H_1, H_2 be non-null tournaments, and let H be $H_1 \Rightarrow H_2$. Assume that for $i = 1, 2$ every every H_i-free tournament has chromatic number at most d_i. Let $m = \max(|V(H_1)|, |V(H_2)|)$ and let $d = \max(d_1, d_2)$. Then every H-free tournament has chromatic number at most $2(m+1)^m d$.*

We remark that this proof of Theorem 5.1 is much simpler than the one in [1].

6 Acknowledgment

This paper is based on results obtained in joint work with Alex Scott and Paul Seymour. The author thanks Irena Penev for her careful reading of the manuscript, and many helpful suggestions.

References

[1] E. Berger, K. Choromanski, M. Chudnovsky, J. Fox, M. Loebl, A. Scott, P. Seymour and S. Thomassé, "Tournaments and colouring", *Journal of Combinatorial Theory, Ser. B* **103** (2013), 1–20.

[2] M. Chudnovsky, A. Scott and P. Seymour, "Excluding Pairs of Graphs", *to appear in Journal of Combinatorial Theory, Ser B.*

[3] M. Chudnovsky and P. Seymour, "Extending the Gyárfás-Sumner conjecture", *to appear in Journal of Combinatorial Theory, Ser B.*

[4] P. Erdős, "Graph theory and probability", *Canad. J. Math* **11** (1959), 34–38.

[5] A. Gyárfás, "On Ramsey covering-numbers", *Coll. Math. Soc. János Bolyai*, in *Infinite and Finite Sets*, North Holland/American Elsevier, New York (1975), 10.

[6] A. Gyárfás, E. Szemeredi and Zs. Tuza, "Induced subtrees in graphs of large chromatic number", *Discrete Mathematics* **30** (1980), 235–244.

[7] H. A. Kierstead and S.G. Penrice "Radius two trees specify -bounded classes", *Journal of Graph Theory* **18** (1994): 119-129.

[8] H.A. Kierstead and Y. Zhu, "Radius Three Trees in Graphs with Large Chromatic Number", *SIAM J. Discrete Math.* **17** (2004), 571–581.

[9] A. Scott, "Induced trees in graphs of large chromatic number", *Journal of Graph Theory* **24** (1997), 297-311.

[10] D.P. Sumner, "Subtrees of a graph and chromatic number", in *The Theory and Applications of Graphs*, (G. Chartrand, ed.), John Wiley & Sons, New York (1981), 557-576.