

# A note on simplicial cliques

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## Abstract

Motivated by an application in condensed matter physics and quantum information theory, we prove that every non-null even-hole-free claw-free graph has a simplicial clique, that is, a clique  $K$  such that for every vertex  $v \in K$ , the set of neighbours of  $v$  outside of  $K$  is a clique. In fact, we prove the existence of a simplicial clique in a more general class of graphs defined by forbidden induced subgraphs.

## 1 Introduction

All graphs in this paper are finite and simple. Let  $G$  be a graph. A *hole* in  $G$  is an induced cycle of length at least four. By a *path* we always mean an induced path. For  $v \in V(G)$  we denote by  $N(v)$  the set of neighbours of  $v$  (so  $v \notin N(v)$ ).  $G$  is *even-hole-free* if all holes in  $G$  have odd length, and  $G$  is *claw-free* if  $G$  has no induced subgraph isomorphic to  $K_{1,3}$ . A non-empty set  $K \subseteq V(G)$  is a *simplicial clique* if  $K$  is a clique, and for every  $v \in K$  we have

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that  $N(v) \setminus K$  is a clique. The unique element of a simplicial clique of size one is called a *simplicial vertex*.

This paper is motivated by a question from condensed matter physics and quantum information theory concerning so-called *spin models*, i.e. models of interacting qubits (two-level quantum systems). Each spin model is defined by a Hamiltonian operator, and to every such Hamiltonian one can associate a graph, called its *frustration graph*. In [4] a new method is given that allows us to “solve a model” (meaning in this case to find the spectrum and the eigenvectors of the Hamiltonian) whose frustration graph is even-hole-free, claw-free, and has a simplicial clique. This augments earlier results of [1] where it is shown that models whose frustration graphs are line-graphs are solvable using certain classical tools. The solution method of [4] uses only the structure of the frustration graph, and it is an extension of both [5] and [6]. The authors of [4] raised a question:

**1.1. Question:** *Does every non-null even-hole-free claw-free graph have a simplicial clique?*

In other words, does their new solvability result hold for all models whose frustration graphs are even-hole-free and claw-free? In this note we answer their question affirmatively (the “dome of an edge” is defined before the statement of (2.2)); in fact we prove a stronger result:

**1.2.** *Let  $G$  be a non-null even-hole-free claw-free graph.*

1. *If  $G$  is chordal, then  $G$  has a simplicial vertex.*
2. *For every hole  $H$  of  $G$  there is an edge  $ab$  of  $H$  such that the dome of  $ab$  is a simplicial clique.*

*In particular,  $G$  has a simplicial clique.*

We have an even stronger result, describing explicitly the structure of all such graphs, but the proof is harder, and we do not present it here. The main result of this paper is 2.2 which is a strengthening of 1.2, and we will explain it in Section 2.

We remark that the answer to 1.1 becomes negative if we omit either the assumption that the graph is even-hole-free or that the graph is claw-free. The complement of a cycle of odd length at least seven is a claw-free graph with no simplicial clique. Moreover, the square of a cycle of length at least nine is an example of a  $C_4$ -free claw-free graph with no simplicial clique. (The *square* of a graph  $G$  is the graph obtained from  $G$  by making every vertex adjacent to all its second neighbours.) And here is an example of an even-hole-free graph rather than just  $C_4$ -free. Let  $k$  be an odd positive integer. The following is a construction of an even-hole-free graph  $G_k$  with  $2k$  vertices and with no simplicial clique. Let the vertex set of  $G_k$  be the union of  $k$  disjoint pairs of adjacent vertices  $\{a_i, b_i\}$  where  $i \in \{1, \dots, k\}$ . For  $i = \{1, \dots, k-1\}$  add the edges  $a_i a_{i+1}, a_i b_{i+1}, b_i a_{i+1}$ ; add also the edges  $a_k a_1, a_k b_1, b_k a_1$ . There are

no more edges in  $G$ . It is easy to check that  $G_k$  is even-hole-free and has no simplicial clique.

In [2] an algorithm is presented that finds a simplicial clique in a claw-free graph if one exists. The authors of [4] also asked if that algorithm can be simplified when the input is known to be even-hole-free. An easy corollary of our main result 1.2 is such a simpler, but slower, algorithm 4.1. In fact 4.1 works under the more general assumptions of 2.2.

## 2 A strengthening

The goal of this section is to present our main result 2.2.

Let  $G$  be a graph. For  $X \subseteq V(G)$  we denote by  $G[X]$  the graph induced by  $G$  on  $X$ . For  $A \subseteq V(G)$  and  $x \in V(G) \setminus A$ , we say that  $x$  is *complete* to  $A$  if  $x$  is adjacent to every element of  $A$ , and that  $x$  is *anticomplete* to  $A$  if  $x$  is non-adjacent to every element of  $A$ . Two disjoint subsets  $A, B \subseteq V(G)$  are *complete* to each other if every vertex of  $B$  is complete to  $A$ , and *anticomplete* to each other if every vertex of  $B$  is anticomplete to  $A$ .

Next we define a few types of graphs. A graph is called *chordal* if it has no holes. A *jewel* is a graph consisting of a hole  $H = h_1 - \dots - h_k - h_1$  with  $k \geq 4$  and a vertex  $v \notin V(H)$  such that  $N(v) \cap V(H) = \{h_1, h_2, h_3, h_4\}$ . A *line wheel* is a graph consisting of a hole  $H = h_1 - \dots - h_k - h_1$  with  $k \geq 6$  and a vertex  $v \notin V(H)$  such that there exists  $i \in \{4, \dots, k-2\}$  with  $N(v) \cap V(H) = \{h_1, h_2, h_i, h_{i+1}\}$ . A *short prism* is a graph consisting of a hole  $h_1 - h_2 - h_3 - h_4 - h_1$  and a path  $p_1 - \dots - p_k$  such that  $\{p_1, \dots, p_k\} \cap \{h_1, h_2, h_3, h_4\} = \emptyset$ ,  $p_1$  is adjacent to  $h_1$  and to  $h_2$ , and  $p_k$  is adjacent to  $h_3$  and to  $h_4$ , and there are no other edges between  $\{p_1, \dots, p_k\}$  and  $\{h_1, h_2, h_3, h_4\}$ . Finally, the *seven-antihole* is the complement of a cycle of seven vertices. These graphs are depicted in Figure 1.

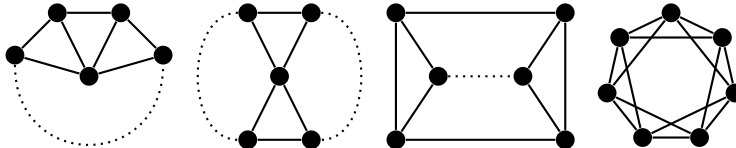


Figure 1: A jewel, a line wheel, a short prism and a seven-antihole (here dotted lines represent paths).

In what follows, whenever graph containment is mentioned, we will mean containment as an induced subgraph. We say that a graph  $G$  is *clean* if  $G$  is claw-free and contains no jewel, line wheel, short prism or seven-antihole. Note that clean graphs may contain even holes.

First we show:

**2.1.** *If  $G$  is claw-free and even-hole-free, then  $G$  is clean.*

*Proof.* Since  $G$  is even-hole free, and in particular  $C_4$ -free,  $G$  does not contain short prisms or seven-antiholes. Since a jewel contains a hole  $H$  of length  $k$ ,

and a hole  $h_1-v-h_4-h_5-\dots-h_k-h_1$  of length  $k-1$ ,  $G$  does not contain a jewel. Finally, at least one of the holes  $H$ ,  $h_2-\dots-h_i-v-h_2$ , and  $h_{i+1}-\dots-h_1-v-h_{i+1}$  is even, and so  $G$  does not contain a line wheel. This proves 2.1.  $\square$

We need one more definition. Let  $ab$  be an edge of a graph  $G$ . Let  $X(ab) = \{a, b\} \cup (N(a) \cap N(b))$ . The *dome* of  $ab$  is the set of all vertices  $y \in X(ab)$  such that  $N(y) \setminus X(ab)$  is a clique. We call the set  $X(ab) \setminus Y(ab)$  the *dome* of  $ab$ . We can now state our main result.

**2.2.** *Let  $G$  be a non-null clean graph.*

1. *If  $G$  is chordal then  $G$  has a simplicial vertex.*
2. *For every hole  $H$  of  $G$  there is an edge  $ab$  of  $H$  such that the dome of  $ab$  is a simplicial clique.*

*In particular,  $G$  has a simplicial clique.*

In view of 2.1 we immediately deduce 1.2.

### 3 The proof of the main theorem

In this section we prove 2.2. We start with a lemma.

**3.1.** *Let  $G$  be a clean graph, let  $H$  be a hole in  $G$ , and let  $v \in V(G) \setminus V(H)$ . Then one of the following holds:*

1.  *$v$  is anticomplete to  $V(H)$ .*
2.  *$|V(H)| = 5$  and  $v$  is complete to  $V(H)$ .*
3.  *$v$  has exactly two neighbours in  $H$  and they are consecutive.*
4.  *$v$  has exactly three neighbours in  $H$  and they form a path of  $H$ .*

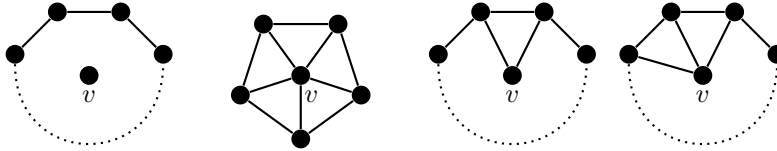


Figure 2: Outcomes of 3.1 (here dotted lines represent paths).

*Proof.* The outcomes of 3.1 are depicted in Figure 2. Write  $H = h_1-\dots-h_k-h_1$ . We may assume that  $v$  has a neighbour in  $V(H)$ , for otherwise 3.1.1 holds. If  $v$  is complete to  $V(H)$ , then, since  $G$  is claw-free and since  $G[V(H) \cup \{v\}]$  is not a jewel, it follows that  $k = 5$  and so 3.1.2 holds. Thus we may assume that  $v$  has a non-neighbour in  $V(H)$ , say  $v$  is adjacent to  $h_1$  and not to  $h_k$ . Since  $G$  is claw-free,  $v$  is adjacent to  $h_2$ , and  $|N(v) \cap V(H)| \leq 4$ . We may

assume that  $v$  has a neighbour in  $V(H) \setminus \{h_1, h_2, h_3\}$ , for otherwise 3.1.3 or 3.1.4 holds. Since  $G$  is claw-free,  $N(v) \cap (V(H) \setminus \{h_1, h_2, h_3\})$  is a clique, and therefore  $|N(v) \cap (V(H) \setminus \{h_1, h_2, h_3\})| \leq 2$ . Let  $i \in \{4, \dots, k-1\}$  be minimum such that  $v$  is adjacent to  $h_i$ . Since  $\{v, h_{i-1}, h_i, h_{i+1}\}$  is not a claw, it follows that either

- $i = 4$  and  $v$  is adjacent to  $h_3$ , or
- $v$  is adjacent to  $h_{i+1}$  and  $v$  has no other neighbours in  $V(H) \setminus \{h_1, h_2, h_i, h_{i+1}\}$ .

In the former case  $G[V(H) \cup \{v\}]$  is a jewel. Thus we may assume that the latter case holds. Since  $|N(v) \cap V(H)| \leq 4$ , it follows that  $N(v) \cap V(H) = \{h_1, h_2, h_i, h_{i+1}\}$ . But now  $G[V(H) \cup \{v\}]$  is a line wheel, a contradiction. This proves 3.1.  $\square$

Now we turn to the proof of 2.2.

*Proof.* If  $G$  has no hole, then  $G$  is chordal, and therefore has a simplicial vertex [3], and 2.2 holds. Thus we may assume that  $G$  has a hole. For an integer  $k \geq 4$  a subset  $W \subseteq V(G)$  is  $k$ -structured if (here the addition is mod  $k$ ):

- $W$  is the disjoint union of  $k$  non-empty cliques  $K_1, \dots, K_k$ ,
- for every  $i \in \{1, \dots, k\}$  every  $v \in K_i$  has a neighbour in  $K_{i-1}$  and a neighbour in  $K_{i+1}$ , and
- if  $i, j \in \{1, \dots, k\}$  and  $i \neq j \pm 1$  then  $K_i$  is anticomplete to  $K_j$ .

We call the partition  $(K_1, \dots, K_k)$  a  $k$ -structure of  $W$ .

Let  $H$  be a hole of  $G$ . Then  $H$  has length  $k \geq 4$ , and  $V(H)$  is a  $k$ -structured set. If possible, choose  $H$  with  $k \geq 5$ . Let  $W \subseteq V(G)$  be a  $k$ -structured set with  $k$ -structure  $(K_1, \dots, K_k)$ , where each  $K_i$  contains exactly one vertex of  $H$ , and such that  $W$  is inclusion-wise maximal with this property. In what follows addition and subtraction of indices of the  $k$ -structure is mod  $k$ .

- (1) *Let  $i \in \{1, \dots, k\}$ . If  $a, b \in K_i$  and  $N(b) \cap K_{i+1} \not\subseteq N(a) \cap K_{i+1}$ , then  $N(b) \cap K_{i-1} \subseteq N(a) \cap K_{i-1}$ .*

We may assume  $i = 1$ . If for each  $j \in \{2, k\}$  there exists  $a_j \in (N(b) \setminus N(a)) \cap K_j$ , then  $\{b, a_k, a, a_2\}$  is a claw in  $G$ , a contradiction. This proves (1).

- (2) *Let  $i \in \{1, \dots, k\}$ . For every  $a, b \in K_i$  either  $N(a) \cap K_{i+1} \subseteq N(b) \cap K_{i+1}$ , or  $N(b) \cap K_{i+1} \subseteq N(a) \cap K_{i+1}$ .*

We may assume  $i = 1$ . Suppose there exist  $a' \in (N(a) \setminus N(b)) \cap K_2$  and  $b' \in (N(b) \setminus N(a)) \cap K_2$ . Since  $K_1$  and  $K_2$  are cliques,  $a$  is adjacent to  $b$ , and  $a'$  is adjacent to  $b'$ . Now  $a-a'-b'-b-a$  is a hole of length four. Let  $C = N(a) \cap K_k$  and  $C' = N(a') \cap K_3$ . By (1)  $b$  is complete to  $C$ , and  $b'$  is complete to  $C'$ . Switching the roles of  $a$  and  $b$ , we deduce that  $b$  is anticomplete to  $K_k \setminus C$ , and, similarly,  $b'$  is anticomplete to  $K_3 \setminus C'$ . Since the graph  $G[\bigcup_{j=3}^k K_j]$  is connected,

there is a path  $P = p_1 - \dots - p_t$  in  $G[\bigcup_{j=3}^k K_j]$  with  $p_1 \in C$  and  $p_t \in C'$ ; we may assume  $P$  is chosen with  $t$  minimum. Then  $p_2, \dots, p_{t-1} \notin C \cup C'$ , and therefore  $V(P) \setminus \{p_1, p_t\}$  is anticomplete to  $\{a, b, a', b'\}$ . But now  $G[a, a', b, b', p_1, \dots, p_t]$  is a short prism in  $G$ , a contradiction. This proves (2).

- (3) *For every  $i \in \{1, \dots, k\}$ ,  $K_i$  is complete to at least one of  $K_{i-1}, K_{i+1}$ .*

We may assume  $i = 1$ . Suppose there exist  $u \in K_k, v, w \in K_1$  and  $z \in K_2$  (where possibly  $v = w$ ) such that  $u$  is not adjacent to  $w$ , and  $v$  is not adjacent to  $z$ .

First we claim that  $v, w$  can be chosen in such a way that  $uv$  and  $wz$  are edges. Suppose not; then we may assume that  $v$  is non-adjacent to both  $u$  and  $z$ . Since  $(K_1, \dots, K_k)$  is a  $k$ -structure, there exist  $n_u, n_z \in K_1$  such that  $u$  is adjacent to  $n_u$ , and  $z$  is adjacent to  $n_z$  (possibly  $n_z = w$ ). Since  $v$  is anticomplete to  $\{u, z\}$ , it follows from (1) that  $n_u$  is non-adjacent to  $z$ , and  $n_z$  is non-adjacent to  $u$ . But now we can choose  $v = n_u$  and  $w = n_z$ , and the claim holds.

In view of the claim in the previous paragraph we assume that  $uv$  and  $wz$  are adjacent (and in particular  $v \neq w$ ). Let  $u' \in K_k$  be a neighbour of  $w$ ; by (2)  $u'$  is adjacent to  $v$ . Now the set  $T = \bigcup_{i=3}^{k-1} K_i$  is non-empty and connected, and both  $u'$  and  $z$  have neighbours in  $T$ . Consequently, there is a path  $R$  from  $u'$  to  $z$  with  $V(R) \setminus \{u', z\} \subseteq T$ . Let  $x \in V(R)$  be the neighbour of  $u'$ . Then  $x \in K_{k-1}$ , and by (1)  $x$  is adjacent to  $u$ . It follows from the definition of a  $k$ -structure that  $V(R) \setminus \{z, x, u'\}$  is anticomplete to  $\{u, u', v, w\}$ . But now the hole  $z-w-v-u-x-R-z$  together with the vertex  $u'$  forms a jewel in  $G$ , a contradiction. This proves (3).

- Let  $j \in \{1, \dots, k\}$ . For every  $i \in \{2, \dots, k-2\}$ ,  $a_j \in K_j$  and*  
(4)  *$a_{j+i} \in K_{j+i}$ , there is a path  $P$  from  $a_j$  to  $a_{j+i}$  with  $V(P) \subseteq \bigcup_{t=j}^{j+i} K_t$  and using exactly one vertex from each of  $K_j, \dots, K_{j+i}$ .*

We may assume  $j = 1$ . The proof is by induction on  $i$ . Suppose first that  $i = 2$ . Since by (3)  $K_2$  is complete to at least one of  $K_1, K_3$ , it follows that  $a_1$  and  $a_3$  have a common neighbour  $a_2 \in K_2$ . Now  $a_1-a_2-a_3$  is the required path.

Now assume that  $i > 2$ , let  $a_{j+i-1} \in K_{j+i-1}$  be a neighbour of  $a_{j+i}$ . By the inductive hypothesis there is a path  $P$  from  $a_1$  to  $a_{j+i-1}$  with  $V(P) \subseteq \bigcup_{r=j}^{i+j-1} K_r$  using exactly one vertex from each of  $K_j, \dots, K_{i+j-1}$ . Now  $a_j-P-a_{j+i-1}-a_{j+i}$  is the required path. This proves (4).

*Let  $v \in V(G) \setminus W$ . For  $i \in \{1, \dots, k\}$  let  $N_i = K_i \cap N(v)$  and  $M_i = K_i \setminus N_i$ . The following hold for every  $i \in \{1, \dots, k\}$ :*

- (5) *1.  $N_i$  is anticomplete to at least one of  $M_{i-1}, M_{i+1}$ .*  
*2. If  $k > 4$ , then  $M_i$  is anticomplete to at least one of  $N_{i-1}, N_{i+1}$ .*

We may assume  $i = 1$ . By (3) we may assume that  $K_1$  is complete to  $K_2$ .

We first prove the first statement. We may assume that there exists  $m_2 \in M_2$ , for otherwise the claim holds ( $N_1$  is anticomplete to  $M_2$  because  $M_2 = \emptyset$ ). Now if  $n_1 \in N_1$  has a neighbour  $m_k \in M_k$ , then  $\{n_1, v, m_2, m_k\}$  is a claw, a contradiction. This proves that  $N_1$  is anticomplete to  $M_2$ , and (5).1 follows.

Next we prove the second statement. We may assume that there exist  $m_1 \in M_1$  and  $n_2 \in N_2$  such that  $m_1$  is adjacent to  $n_2$ . Let  $n_k \in N_k$ , then  $n_k \in N(m_1)$ . By (4) there exists a path  $P$  from  $n_2$  to  $n_k$  with  $V(P) \subseteq \bigcup_{i=2}^k K_i$  with  $|V(P) \cap K_i| = 1$  for every  $i \in \{2, \dots, k\}$ . But now we get a contradiction applying 3.1 to the hole  $m_1$ - $n_2$ - $P$ - $n_k$ - $m_1$  and the vertex  $v$ . This proves (5).2 and completes the proof of (5).

*Let  $v \in V(G) \setminus W$  and for  $i \in \{1, \dots, k\}$ , let  $N_i = N(v) \cap K_i$  and  $M_i = K_i \setminus N_i$ . Either*

- (6)
  - $N_i$  is non-empty for at most two consecutive values of  $i$  (mod  $k$ ) or
  - $k = 5$ ,  $v$  is complete to  $W$ , and  $K_i$  is complete to  $K_{i+1}$  for every  $i \in \{1, \dots, 4\}$ .

First we claim that we can choose  $j, l$  with  $N_j \neq \emptyset$  and  $N_l = \emptyset$ , and such that  $j = l \pm 2$ . If  $N_i \neq \emptyset$  for every  $i \in \{1, \dots, k\}$ , then the claim holds. If  $N_i = \emptyset$  for every  $i \in \{1, \dots, k\}$ , then the claim holds. Thus we may assume that some  $N_i$ s are empty, and some are not. By shifting the indices, we may assume  $N_1 \neq \emptyset$  and  $N_k = \emptyset$ . We may assume that  $N_t \neq \emptyset$  for some  $t \in \{3, \dots, k-1\}$  for otherwise (6) holds with  $i = 1$ . Let  $n_1 \in N_1$  and  $n_t \in N_t$ . By (4) there exists a path  $P$  from  $n_1$  to  $n_t$  such that  $V(P) \subseteq \bigcup_{j=1}^t K_j$  and  $P$  uses exactly one vertex from each of  $K_1, K_2, \dots, K_t$ . Also by (4) there exists a path  $Q$  from  $n_t$  to  $n_1$  such that  $V(Q) \subseteq K_1 \cup \bigcup_{j=t}^k K_j$  and  $Q$  uses exactly one vertex from each of  $K_t, K_{t+1}, \dots, K_k, K_1$ . Now  $F = n_1$ - $P$ - $n_t$ - $Q$ - $n_1$  is a hole, and  $n_1, n_t \in V(F)$ . Since  $t \geq 3$  and  $N_k = \emptyset$ , applying 3.1 to  $F$  and  $v$  we deduce that the fourth outcome of 3.1 holds and  $t = 3$ . Now we can set  $j = 1$  and  $l = t$ . This proves the claim.

By the claim of the previous paragraph (shifting the indices so that  $j = k$  and  $l = 2$ ) we may assume that  $N_k$  and  $N_2$  are both non-empty. For  $i \in \{2, k\}$  let  $a_i \in N_i$ . By (3)  $a_k$  and  $a_2$  have a common neighbour  $a_1 \in K_1$ .

Suppose  $\bigcup_{i=3}^{k-1} N_i = \emptyset$ . Since  $W$  is maximal,  $(K_1 \cup \{v\}, K_2, \dots, K_k)$  is not a  $k$ -structure for  $W \cup \{v\}$ , and therefore there exists  $a'_1 \in M_1$ . By symmetry we may assume that  $K_1$  is complete to  $K_2$ . Let  $a'_3 \in K_3$  be a neighbour of  $a_2$ ; then  $a'_3 \in M_3$ , contrary to (5).1. This proves that  $\bigcup_{i=3}^{k-1} N_i \neq \emptyset$ .

Suppose  $k = 4$ . Then there is symmetry between  $K_1$  and  $K_3$ , and we deduce that  $N_i \neq \emptyset$  for every  $i \in \{1, \dots, 4\}$ . By (3) we may assume that  $K_1$  is complete to  $K_2$ , and  $K_3$  to  $K_4$ . Now 3.1 implies that there is no hole  $n_1 n_2 n_3 n_4 n_1$  with  $n_i \in N_i$  for every  $i \in \{1, \dots, 4\}$ , and consequently either  $N_1$  is anticomplete to  $N_4$ , or  $N_3$  is anticomplete to  $N_2$ . By symmetry we may assume that  $N_1$  is anticomplete to  $N_4$ . Let  $n_1 \in N_1$  and  $n_4 \in N_4$ , and let  $m_1 \in K_1$  be adjacent

to  $n_4$  and  $m_4 \in K_4$  be adjacent to  $n_1$ . Then  $m_1 \in M_1$  and  $m_4 \in M_4$ . By (2) applied to  $m_1$  and  $n_1$ , we deduce that  $m_1$  is adjacent to  $m_4$ . If there exists  $m_2 \in M_2$ , then  $\{n_1, v, m_2, m_4\}$  is a claw, a contradiction. This proves (using symmetry) that  $M_2 \cup M_3 = \emptyset$ . Let  $n_2 \in K_2$ , and let  $n_3 \in K_3$  be adjacent to  $n_2$ . Then  $n_2 \in N_2$  and  $n_3 \in N_3$ . But now  $G[v, m_4, n_2, n_4, n_1, n_3, m_1]$  is a seven-antihole, a contradiction. This proves that  $k \geq 5$ .

By (5).2 it follows that  $a_1 \in N_1$ . We claim that  $k = 5$  and  $v$  is complete to  $W$ . Suppose  $v$  has a non-neighbour in  $W$ . Since  $\{v, a_k, a_2, x\}$  is not a claw for any  $x \in \bigcup_{i=4}^{k-2} K_i$ , it follows that  $\bigcup_{i=4}^{k-2} N_i = \emptyset$ .

We may assume that there is  $a_3 \in N_3$ . Since  $\{v, a_2, a_k, a_3\}$  is not a claw, it follows that  $a_2$  is adjacent to  $a_3$ . By (4) there is a path  $P$  from  $a_3$  to  $a_k$  with  $V(P) \subseteq \bigcup_{j=3}^k K_j$  and using exactly one vertex from each of  $K_3, \dots, K_k$ . Now  $F = a_k a_1 a_2 a_3 P a_k$  is a hole, and 3.1 implies that  $k = 5$  and  $v$  is complete to  $V(F)$ . We have proved that  $N_i \neq \emptyset$  for every  $i \in \{1, \dots, 5\}$  thus restoring the symmetry of the 5-structure. Since for  $n_1 \in N_1, n_2 \in N_2$  and  $n_4 \in N_4$ ,  $\{v, n_1, n_2, n_4\}$  is not a claw, we deduce (using symmetry) that  $N_i$  is complete to  $N_{i+1}$  for every  $i \in \{1, \dots, 5\}$ .

Next suppose that both  $M_5$  and  $M_2$  are non-empty. By (3) we may assume that  $K_1$  is complete to  $K_2$ . By (5).1,  $N_1$  is anticomplete to  $M_5$ . Since every vertex of  $M_5$  has a neighbour in  $K_1$ , it follows that  $M_1 \neq \emptyset$ . By (5).2  $M_1$  is anticomplete to  $N_5$ , but now  $m_1 \in M_1$  and  $n_1 \in N_1$  contradict (2). By symmetry we may assume  $M_i$  is non-empty for at most two consecutive values of  $i$ , and that  $M_1 \cup M_2 \cup M_3 = \emptyset$ . By (5).2  $N_4$  is anticomplete to  $M_5$ , and similarly  $M_4$  is anticomplete to  $N_5$ . By symmetry we may assume that  $M_4 \neq \emptyset$ . But now  $m_4 \in M_4$  and  $n_4 \in N_4$  contradicts (2). This proves that  $k = 5$  and  $v$  is complete to  $W$ . To complete the proof of (6) assume for a contradiction that there exist  $i \in \{1, \dots, 4\}$ ,  $k_i \in K_i$  and  $k_{i+1} \in K_{i+1}$  such that  $k_i$  is non-adjacent to  $k_{i+1}$ . We may assume  $i = 1$ . Then  $\{v, k_1, k_2, k_4\}$  is a claw in  $G$ , a contradiction. This proves (6).

For  $i \in \{1, \dots, k\}$  let  $K_{i,i+1}$  be the set of all vertices of  $V(G) \setminus W$  that have a neighbour in  $K_i$ , a neighbour in  $K_{i+1}$ , and no neighbour in  $W \setminus (K_i \cup K_{i+1})$ . The outcomes of (6) are summarized in Figure 3.

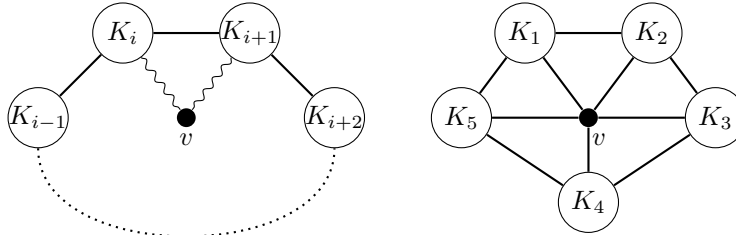


Figure 3: Outcomes of (6) (here wiggly lines represent possible adjacency, and the dotted arc represents the remainder of the  $k$ -structure).



Assume that  $K_{i,i+} \neq \emptyset$ . The following statements hold

1.  $K_i \cup K_{i+1} \cup K_{i,i+1}$  is a clique.
- (7) 2. If  $u \in V(G) \setminus W$  is complete to  $W$ , then  $u$  is anticomplete to  $K_{i,i+1}$ .
3.  $K_{i,i+1}$  is anticomplete to  $K_{i-1,i}$ .

Let  $v \in K_{1,2}$ . For  $i \in \{1, 2\}$  let  $N_i = K_i \cap N(v)$ , and let  $M_i = K_i \setminus N_i$ . By (5).1  $N_1$  is anticomplete to  $M_2$ , and  $N_2$  is anticomplete to  $M_1$ . If there exists  $m_1 \in M_1$ , then  $n_1, m_1$  contradict (2) for every  $n_1 \in N_1$ . Thus  $M_1 = \emptyset$ , and by symmetry  $M_2 = \emptyset$ . This proves that  $K_{1,2}$  is complete to  $K_1 \cup K_2$ .

Suppose  $k_1 \in K_1$  is non-adjacent to  $k_2 \in K_2$ . Let  $P$  be a path from  $k_2$  to a vertex  $k_k \in K_k$  as in (4), such that  $|V(P) \cap K_i| = 1$  for every  $i \in \{2, 3, \dots, k\}$ . By (3)  $K_1$  is complete to  $K_k$ , and so  $F = k_k - k_1 - v - k_2 - P - k_k$  is a hole. Let  $k'_1 \in K_1 \cap N(k_2)$ , then  $G[V(F) \cup \{k'_1\}]$  is a jewel in  $G$ , a contradiction. This proves that  $K_1$  is complete to  $K_2$ .

Since  $\{k_1, k_k, a, b\}$  is not a claw for any  $k_1 \in K_1, k_k \in K_k \cap N(k_1)$  and  $a, b \in K_{1,2}$ , it follows that  $K_{1,2}$  is a clique, and therefore  $K_1 \cup K_2 \cup K_{1,2}$  is a clique. By symmetry,  $K_1 \cup K_2 \cup K_{1,2}$  is a clique for every  $i$ . This proves (7).1.

Next suppose that  $u$  is complete to  $W$  and  $u$  is adjacent to  $w \in K_{1,2}$ . By (6)  $k = 5$ . Let  $k_3 \in K_3$  and  $k_5 \in K_5$ . Then  $\{u, w, k_3, k_5\}$  is a claw, a contradiction. This proves (7).2.

Finally, suppose that  $w_k \in K_{k,1}$  is adjacent to  $w_2 \in K_{1,2}$ . Let  $k_1 \in K_1, k_2 \in K_2$  and  $k_k \in K_k$ , and let  $P$  be a path from  $k_2$  to  $k_k$  as in (4) such that  $|V(P) \cap K_i| = 1$  for every  $i \in \{2, 3, \dots, k\}$ . Then  $F = k_2 - P - k_k - w_k - w_2 - k_2$  is a hole and  $G[V(F) \cup \{k_1\}]$  is a jewel, a contradiction. This proves (7).3 and completes the proof of (7).

For  $i \in \{1, \dots, k\}$ , let  $V(H) \cap K_i = \{h_i\}$ . Choose  $i \in \{1, \dots, k\}$  such that  $K_i$  is complete to  $K_{i+1}$  and, if possible, such that  $K_{i,i+1} \neq \emptyset$ ; we may assume  $i = 1$ . Set  $a = h_1$  and  $b = h_2$ , and let  $K = K_1 \cup K_2 \cup K_{1,2}$ .

By (6), every vertex of  $X(ab) = \{a, b\} \cup (N(a) \cap N(b))$  either belongs to  $K$  or is complete to  $W$  (and  $k = 5$ ). Since if  $y \in X(ab)$  is complete to  $W$ , then  $y$  has two non-adjacent neighbours in  $V(G) \setminus X(ab)$ , it follows that  $K$  contains the dome of  $ab$ . By (7).1  $K$  is a clique.

We prove that  $K$  is a simplicial clique, and therefore  $K$  equals the dome of  $ab$ . Suppose  $K$  is not a simplicial clique, and let  $v \in K$  and  $u, w \in V(G) \setminus K$  be such that  $u$  and  $w$  are adjacent to  $v$ , and  $u$  is non-adjacent to  $w$ . Suppose first that  $u$  is complete to  $W$ . By (6)  $k = 5$  and for every  $i$   $K_i$  is complete to  $K_{i+1}$ . By (7).2  $v \notin K_{1,2}$ . For  $i \in \{1, \dots, 5\}$  let  $k_i \in K_i$ . We may assume that  $v = k_1$ . Since  $u$  is non-adjacent to  $w$ , it follows that  $w \notin W$ . Since  $G[k_1, u, k_3, w, k_2]$  is not a jewel in  $G$ , it follows that  $w$  is not complete to  $W$ . By (5).1 (since  $k_1$  is complete to  $K_2 \cup K_5$ )  $w$  has a neighbour in at least one of  $K_2, K_5$ . By

(6)  $w \in K_{1,2} \cup K_{5,1}$ . Since  $w \notin K$ , it follows that  $w \in K_{5,1}$ . By (7).1  $K_5$  is complete to  $K_1$ . Since  $K_{1,5} \neq \emptyset$ , and since  $K_1, K_2$  were chosen with  $K_{1,2} \neq \emptyset$  if possible, it follows that there exists  $k \in K_{1,2}$ . By (7).2  $u$  is non-adjacent to  $k$ , and by (7).3  $w$  is non-adjacent to  $k$ . But now  $\{k_1, u, k, w\}$  is a claw in  $G$ , a contradiction. This proves that  $u$  is not complete to  $W$ . By symmetry,  $w$  is not complete to  $W$ .

Now suppose  $v \in K_1$ . Since  $u, w \notin K$ , it follows from (6) that  $u, w \in K_k \cup K_{k,1}$ . But then  $u$  is adjacent to  $w$  by (7).1, a contradiction. This proves that  $v \notin K_1$ , and, by symmetry,  $v \notin K_2$ .

It follows that  $v \in K_{1,2}$ . Moreover, applying the previous argument to an arbitrary vertex of  $K_1 \cup K_2$  in the role of  $v$ , we deduce that no vertex of  $K_1 \cup K_2$  is complete to  $\{u, w\}$ . Since  $\{v, u, w, p\}$  is not a claw for  $p \in K_1 \cup K_2$  it follows that every vertex of  $K_1 \cup K_2$  has a neighbour in  $\{u, w\}$ . Since  $u, w \notin K$ , (7).1 implies that each of  $u, w$  is anticomplete to at least one of  $K_1, K_2$ . By switching  $u$  and  $w$  if necessary, we may assume that  $u$  has a neighbour in  $K_1$  and is anticomplete to  $K_2$ . By (5).1  $u$  has a neighbour in  $K_k$ , but now  $u \in K_{k,1}$  is adjacent to  $v \in K_{1,2}$ , contrary to (7).3. Thus we have found an edge of  $H$  whose dome is a simplicial clique. This proves 2.2.  $\square$

## 4 The Algorithm

In this section we use 2.2 to design a simple algorithm that finds a simplicial clique in a clean graph.

**4.1.** *There is an algorithm with the following specifications.*

**Input:** A non-null clean graph  $G$ .

**Output:** A simplicial clique in  $G$ .

**Running time:**  $O(|V(G)|^5)$ .

*Proof.* Let  $|V(G)| = n$ . First, for every vertex  $v \in V(G)$  check if  $N(v)$  is a clique. If the answer is yes for some  $v$ , then output a simplicial clique  $\{v\}$ . This step can be done in time  $O(n^3)$ .

Now for every edge  $ab$  compute  $X(ab), Y(ab)$  and the dome of  $ab$ , and check if the dome of  $ab$  is a simplicial clique. This step can be done in time  $O(n^5)$ .

We now use 2.2 to prove that the algorithm will return a simplicial clique of  $G$ . If  $G$  is a chordal graph, then by the first statement of 2.2 the first step of the algorithm will return a simplicial clique; otherwise there is a hole in  $G$ , and so by the second statement of 2.2, the second step of the algorithm will return a simplicial clique. This proves 4.1.  $\square$

We remark that the algorithm of 4.1 can be used to produce, in time  $O(|V(G)|^2)$ , a list of at most  $|V(G)|^2$  sets one of which is guaranteed to be a simplicial clique. The rest of the running time is spent checking whether each of the sets is a simplicial clique.

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