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LARGE RAINBOW MATCHINGS IN GENERAL GRAPHS

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Abstract. By a theorem of Drisko, any $2n-1$ matchings of size n in a bipartite graph have a rainbow matching of size n. Inspired by results and discussion of Barát, Gyárfás and Sárközy, we conjecture that if n is odd then the same is true also in general graphs, and that if n is even then $2n$ matchings of size n suffice. We prove that any $3n-2$ matchings of size n have a rainbow matching of size n.

1. Introduction

Given a system $\mathcal{C} = (C_1, \ldots, C_m)$ of sets of edges in a graph, a *rainbow matching* for $\mathcal{C}$ is a matching, each of whose edges is chosen from a different C_i. Note that we do not insist that the rainbow matching represents all C_is, so in fact our rainbow matchings are usually only partial. In this paper we consider the case in which the sets C_i are themselves matchings, and we are interested in questions of the form "how many matchings of size k are needed to guarantee the existence of a rainbow matching of size m." It is conjectured [2] that n matchings of size $n+1$ in a bipartite graph have a rainbow matching of size n, and the authors are not aware of any example refuting the possibility that n matchings of size $n+2$ in a general graph have a rainbow matching of size n. In the bipartite case the best current results are that n matchings of size $\lceil \frac{3}{2}n \rceil$ have a rainbow matching of size n [4], and that n matchings of size $n+o(n)$ have a rainbow matching of size n [15].

A surprising jump occurs when we insist that the matchings are of size n: we need to take $2n-1$ such matchings in a bipartite graph to guarantee a rainbow matching of size n. The following example shows that $2n-1$ is best possible:

Example 1.1. Let $M_i, 1 \leq i \leq n-1$ to be all equal to one of the two perfect matchings in the cycle C_{2n} and $M_i,\ n \leq i \leq 2n-2$ to be all equal to the other perfect matching. This is a system of $2n-2$ matchings of size n that does not have a rainbow matching of size n.

The fact that $2n-1$ matchings suffice is essentially due to Drisko [9], who proved the following special case:

Theorem 1.2. *Let A be an $m \times n$ matrix in which the entries of each row are all distinct. If $m \geq 2n-1$, then A has a transversal, namely a set of n distinct entries with no two in the same row or column.*

In [6] it was observed that this theorem follows also from Bárany's colorful extension of Caratheodory's theorem, and the fact that in bipartite graphs $\nu = \nu^*$, meaning that the notion of "matching" can be replaced by that of "fractional matching".

In [2] the theorem was formulated in the rainbow matchings setting, and given a short proof.

Theorem 1.3. *Any family $\mathcal{M} = (M_1, \ldots, M_{2n-1})$ of matchings of size n in a bipartite graph possesses a rainbow matching.*

In [4] it was shown that Example 1.1 is the only instance in which $2n-2$ matchings do not suffice. In [5] Theorem 1.3 was strengthened, using topological methods:

Theorem 1.4. *If $M_i,\ i = 1, \ldots, 2n-1$ are matchings in a bipartite graph satisfying $|M_i| = \min(i, n)$ for all $i \leq 2n-1$, then there exists a rainbow matching of size n.*

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Barát, Gyárfás and Sárközy considered the same problem in general graphs, and from their comments the following conjecture suggests itself:

Conjecture 1.5. [7] *For n even any $2n$ matchings of size n in any graph have a rainbow matching of size n, and for n odd any $2n-1$ matchings of size n have a rainbow matching of size n.*

In [6] a topological theorem of Kalai and Meshulam [13] was used to prove a fractional version of the conjecture, in an even stronger version, in which the given matchings are replaced by fractional matchings:

Theorem 1.6. *Let $F_1, \dots, F_{2n}$ be sets of edges in a general graph, satisfying $\nu^*(F_i) \geq n$. Then there exists a rainbow set $f_1 \in F_1, \dots, f_{2n} \in F_{2n}$ such that $\nu^*(\{f_1, \dots, f_{2n}\}) \geq n$.*

This is generalized in [6] to r-uniform r-partite hypergraphs, as follows: for any integer r and any real number m, any $\lceil rm - r + 1 \rceil$ fractional matchings of size m in an r-partite hypergraph have a rainbow fractional matching of size m.

There is a family of examples showing that for n even $2n-1$ matchings of size n do not necessarily have a rainbow matching of size n. Its construction is based on the following observation:

Observation 1.7. *Let $e = v_iv_j$ $(i < j)$be an edge connecting two vertices of $C_{2k} = v_1v_2 \dots v_{2k}$. Then e belongs to a perfect matching contained in $E(C_{2k}) \cup \{e\}$ if and only if $j - i$ (the length of e) is odd.*

The necessity of the condition follows from the fact that in order for e to participate in a perfect matching it has to enclose an even number of vertices on each of its sides, since the enclosed vertices have to be matched within themselves. The sufficiency follows from the fact that if the number of enclosed vertices is even, they can be matched within the cycle.

Example 1.8. To the system of matchings M_i in Example 1.1 add a matching C of size n, all of whose edges are of even length. The obtained family, consisting of $2n-1$ matchings, does not possess a rainbow matching of size n, since an even length edge cannot be completed by edges from the initial system of matchings to a perfect matching.

Note that a matching C as above exists if and only if n is even. The necessity of the evenness condition follows from the fact that for C as above, $\sum\{i + j \mid v_iv_j \in C\}$ is even, and on the other hand it is equal to $\sum\{i \mid 0 \leq i \leq 2n-1\} = \binom{2n}{2}$. The sufficiency is shown by a simple construction.

The aim of this paper is to prove a weaker version of Conjecture 1.5:

Theorem 1.9. *$3n-2$ matchings of size n in any graph have a rainbow matching of size n.*

2. Preliminaries and notation

For a set J of edges, $\bigcup J$ is the set of vertices participating in J.

Let F be a matching in a graph G, and let $K = E(G) \setminus F$. A path P is said to be $K-F$-*alternating* if every odd-numbered edge of P belongs to K and every even-numbered edge belongs to F.

An alternating path is *augmenting* if both its endpoints do not belong to $\bigcup F$. If P is augmenting then $E(P) \triangle F$ is a matching larger than F. The converse is also true, and standard:

Lemma 2.1. *If C, D are matchings and $|C| > |D|$ then $E(C) \cup E(D)$ contains a $C-D$ alternating path that augments D.*

Proof. Viewed as a multigraph, the connected components of $E(C) \cup E(D)$ are cycles (possibly digons) and paths that alternate between C and D edges. Since $|C| > |D|$, one of these paths contains more edges from C than from D, and is thus D-augmenting. □

Definition 2.2. Let F be a matching, let K be a set of edges disjoint from F. A vertex v is called *oddly (resp. evenly) reachable* if there exists a $K-F$-alternating path of odd (resp. even) length (number of edges), starting outside $\bigcup F$ and ending at v.

Let $OR(K, F)$ (resp. $ER(K, F)$) be the set of oddly (resp. evenly) reachable vertices.

Note that $V(G) \setminus \bigcup F \subseteq ER(K,F)$, since a vertex not matched by F has a zero length alternating path to itself. Note also that there exists a $K - F$ augmenting alternating path if and only if $OR(K,F) \not\subseteq \bigcup F$.

Definition 2.3. A graph G is called *factor-critical* if $G - v$ has a perfect matching for every $v \in V(G)$.

Lemma 2.4. *Let F be a matching in a graph G, let $K = E(G) \setminus F$, and suppose that $V(G) \setminus \bigcup F$ consists of a single vertex. Then a vertex x belongs to $ER(K,F)$ if and only if $G - x$ has a perfect matching.*

(Here and below $G - x$ is the graph obtained from G by removing the vertex x.)

Proof. Let a be the vertex unmatched by F. Suppose that there exists a perfect matching M of $G - x$. Then the $F-M$-alternating path starting at x with an edge of F must terminate at a with an edge of M. Reversing this path, we obtain a $K - F$ alternating path, starting at a and reaching x, with last edge belonging to F. This shows that $x \in ER(K,F)$. For the other direction, if $x \in ER(K,F)$ then let L be the even (namely ending with an F-edge) F-alternating path from a to x. Then $F \triangle L$ is a perfect matching of $G - x$. □

The following is obvious from the definitions:

Lemma 2.5. *A vertex $x \in \bigcup F$ is in $OR(K,F)$ if and only if it is matched by F to a vertex in $ER(K,F)$.*

Together with Lemma 2.4 this implies:

Corollary 2.6. *Let F be a matching in a graph G, covering all vertices apart from one vertex, a. Let $K = E(G) \setminus F$. Then G is factor-critical if and only if $V(G) \setminus \{a\} = DR(K,F)$.*

Combining the corollary with Lemma 2.4 gives:

Lemma 2.7. *Let H be a graph and J a matching in it that covers all vertices except for one vertex, a. Let $K = E(H) \setminus J$. Then the following are equivalent:*

(1) $V(H) \setminus \{a\} = DR(K,J)$.
(2) $V(H) \setminus \{a\} = ER(K,J)$.
(3) H is factor-critical.

3. Edmonds' blossoms algorithm

Definition 3.1. Let G be a graph, F a matching in G. A subgraph of $E(G)$ is called an F- *blossom tree* (or just blossom tree if the identity of F is clear) if it can be obtained from a rooted tree T with root r as follows. Subdivide every edge st of T by a vertex $m(st)$. Replace each vertex s of the original tree by a factor-critical graph $H(s)$, such that $F \restriction H(s)$ matches all vertices apart from a single vertex, which we call the *base* of $H(s)$ and denote by $base(H(s))$. For every child t of s choose some vertex $v \in H(s)$ and connect it to $m(st)$ by an edge of K. Connect $m(st)$ to $base(H(t))$ by an edge of F. The graphs $H(s)$ are called *blossoms*. We say that T *guides* the blossom tree.

A subgraph of G is called an F- *blossom forest* if each of its connected components is a blossom tree, and all non F-matched vertices are bases of blossom trees in it.

The vertices of a forest of the form $m(e)$ are called *inner*, and all other vertices of the forest are called *outer*. In particular, all bases of tree blossoms are outer. The vertices of $V(G) \setminus V(F)$ are neither inner nor outer. For a blossom forest Z we denote by $OUT(Z)$ the set of its outer vertices, by $INN(Z)$ the set of inner vertices, and by $BASE(Z)$ the set of bases of blossoms of Z.

Edmonds' algorithm starts from the empty matching, and generates a sequence $F_1, F_2, \ldots$ of matchings increasing in size. Once a matching $F = F_i$ has been reached, the algorithm starts a procedure of "blossom forming". An F-blossom forest Z is constructed, by adding an edge $e = uv$ of G at a time, satisfying the following condition:

(*) $u \in OUT(Z)$, and $v \in OUT(Z) \cup (V(G) \setminus V(Z))$. Furthermore, if u, v both belong to $OUT(Z)$, then they belong to distinct blossoms.

There are three cases to be distinguished:

(1) u and v belong to two distinct trees in Z. Then adding e to Z generates an augmenting F-alternating path, and $F = F_i$ can be enlarged, to obtain F_{i+1}. We then start constructing a blossom forest anew.
(2) If $v \in V(G) \setminus V(Z)$ then we extend Z by adding the edge e and the edge vw belonging to F.
(3) If v is in the same tree of Z as u and is in $OUT(Z)$, then adding e merges some blossoms into a larger blossom (the details can be found in [8]). This replaces Z by a forest with fewer blossoms.

The algorithm terminates when there are no edges of G satisfying (*).

Theorem 3.2. [10] *(see also* [8]*) At the terminating stage, F is maximum.*

Theorem 3.3. *Let F be a maximum matching in a graph G, let $K = E(G) \setminus F$, and let Z be the terminal blossom forest obtained by Edmonds' algorithm starting with F. Then*

(1) Let $x \in V(H(s))$, let Q be the path from $r(T)$ to s, where T is the tree guiding Z, that contains s. Then x is evenly reachable from $r(T)$ by a $K - F$-alternating path contained in $\bigcup_{t \in V(Q)} V(H(t))$, and is also oddly reachable by such a path unless it is equal to $base(H(s))$.
(2) $OR(K, F) = V(F) \setminus BASE(Z)$.
(3) $ER(K, F) = OUT(Z)$.

Proof. Let us first prove part (1). Follow the path from $r(T)$ to s on T, and use Corollary 2.6 to implant at each vertex t on this path, apart from s, an even $K - F$-alternating path from $base(H(t))$ to the vertex u in $H(t)$ such that the next edge of T emanates from u. At s itself, use the corollary to implant an odd $K - F$-alternating path from $base(H(s))$ to x if $x \neq base(H(s))$, and an even such path if x is any vertex in $H(s)$.

The same argument shows that every vertex in $INN(Z)$ is in $OR(K, F)$. By Lemma 2.5, it remains to be shown that $BASE(Z) \cap OR(Z) = \emptyset$. Assume to the contrary that $base(H(s))$ is reachable for some s by an odd $K - F$ alternating path P, contained in G. The path P starts at a non-F-matched vertex, which means at a base b_1 of some blossom $H(t_1)$. Since no edge of G satisfies (*), if the edge of P leaving $H(t_1)$ is uv_1, then $v_1 \in INN(Z)$. The next vertex on P is $base(H(t_2))$ for some t_2. When P leaves $H(t_2)$ it is again to $INN(Z)$, and so on. This shows that the last edge on P must be of the form $v_i base(H(s))$, namely ending in an F-edge, contrary to the assumption that P is odd. □

4. The main lemma

Throughout this section and the next, F is a matching in a graph G, and $K = E(G) \setminus F$. An edge e contained in $V(G)$ is called *enriching* if $OR(K \cup \{e\}, F) \supsetneq OR(K, F)$. In particular, an edge whose addition to $E(G)$ generates a matching larger than F is enriching. Note that if F is maximum and e is enriching then $e \notin E(G)$.

The key step in the proof of Theorem 1.9 is:

Lemma 4.1. *If F is a maximum matching in G, then any augmenting F-alternating path (of course, not contained in G) contains an enriching edge.*

Proof. Apply Edmonds' matching algorithm to F. Since F is maximum, this involves only constructing the blossom forest Z. As noted, the construction terminates when there are no edges in G satisfying (*).

Since adding $E(P)$ to $E(G)$ generates a matching larger than F, by Theorem 3.2 there exists $e = uv \in E(P)$ which does satisfy (*). We shall show that e is enriching.

There are two cases to be considered.

(1) $v \notin V(Z)$. Since $u \in OUT(Z)$, by Theorem 3.3 it belongs to $ER(K, F)$, and hence $v \in OR(K \cup \{e\}, F) \setminus OR(K, F)$.
(2) $u, v \in OUT(Z)$, $u \in V(H(p)) \setminus \{base(H(p))\}$, $v \in V(H(q)) \setminus \{base(H(q))\}$, for two distinct blossoms $H(p), H(q)$. Then one of p, q, say p, is not a descendant of the other in the forest guiding Z. By part (1) of Theorem 3.3 adding e to K makes $base(H(q))$ oddly reachable.

□

Remark 4.2. The proof given here is essentially the one given in one of the previous versions of the paper. We did not realize that we were repeating the proof of Edmonds' blossom algorithm, so we practically included also the proof of the validity of that algorithm, which made the proof longer than it is now. We are grateful to an anonymous referee for pointing out the possibility of using Edmonds' algorithm. Another proof, using the Edmonds'-Gallai decomposition theorem, can be found in [3].

5. Multicolored alternating paths and proof of Theorem 1.9

Given a family (namely, a multiset) $\mathcal{P}$ of F-alternating paths, an F-alternating path P is said to be $\mathcal{P}$-*multicolored* if $E(P) \setminus F$ is a rainbow set of the family $(E(Q),\ Q \in \mathcal{P})$, meaning that there is an injection $\phi :\ E(P) \setminus F \to \mathcal{P}$, such that $e \in E(\phi(e))$ for every $e \in E(P) \setminus F$.

Lemma 5.1. *If $\mathcal{P}$ is a family of augmenting F-alternating paths and $|\mathcal{P}| > 2|F|$ then there exists an augmenting $\mathcal{P}$-multicolored F-alternating path.*

Proof. Let $\mathcal{P} = (P_1, \dots P_m)$, where $m > 2|F|$. We define inductively sets K_i of edges, all disjoint from F. Let $K_0 = \emptyset$. By Lemma 4.1 the path P_1 contains an edge e_1, such that $OR(K_0 \cup \{e_1\}, F) \supsetneqq OR(K_0, F)$ (for clarity's sake - the last set is empty). Let $K_1 = K_0 \cup \{e_1\}$. Applying the lemma to the graph $F \cup K_1$ and the path P_2 yields an edge $e_2 \in E(P_2)$ such that $OR(K_1 \cup \{e_2\}, F) \supsetneqq OR(K_1, F)$. Let $K_2 = K_1 \cup \{e_2\}$. Continuing this way we inductively construct sets of edges K_i, and $K_i = K_{i-1} \cup \{e_i\}$ $(i \geq 1)$, $e_i \in E(P_i)$, and $OR(K_i, F) \supsetneqq OR(K_{i-1}, F)$. Since there are only $2|F|$ vertices in $\bigcup F$, for some $i \leq m$ the set $OR(K_i, F)$ will contain a vertex not in $\bigcup F$. The path witnessing it is an augmenting $K_i - F$-alternating path, which by the construction of K_i is $\mathcal{P}$-multicolored. □

Finally, we derive Theorem 1.9 from Lemma 5.1. We have to show that given $3n-2$ matchings $M_i,\ i \leq 3n-2$, each of size n, there exists a rainbow matching of size n. Let F be a rainbow matching of maximal size, and let $|F| = k$. We claim that $k \geq n$. Suppose to the contrary that $k < n$. Then there are at least $2k+1$ matchings M_i not represented in F. By Lemma 2.1 each of these generates an augmenting F-alternating path P_i, and by Lemma 5.1, there is an augmenting multicolored F-alternating path P using edges from the paths P_i. None of the colors appearing in P are used in F, and hence $F \triangle E(P)$ is a rainbow matching of size $k+1$, contradicting the maximality of k.

Remark 5.2. In [4] it was shown that in the bipartite case Corollary 5.1 only demands $|\mathcal{P}| > |F|$. In the case of general graphs Corollary 5.1 is sharp - $2|F|$ augmenting F-alternating paths do not suffice, as the following example shows. Let F be a matching $\{u_iv_i \mid i \leq k-1\} \cup \{xy\}$, let $P_1, \dots, P_k$ all be the same path $F \cup \{xu_1\} \cup \{v_{k-1}y\} \cup \{v_iu_{i+1} \mid i \leq k-2\}$ and let $P_{k+1}, \dots, P_{2k}$ all be equal to the same path $F \cup \{xv_1\} \cup \{u_{k-1}y\} \cup \{u_iv_{i+1} \mid i \leq k-2\}$.

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References

[1] R. Aharoni, N. Alon and E. Berger, Eigenvalues of $K_{1,k}$-free graphs and the connectivity of their independence complexes, *Jour. Graph Th.* **83** (2016), 384-391.
[2] R. Aharoni and E. Berger, Rainbow matchings and matchings in r-partite hypergraphs, *Electronic Jour. Combinatorics* (2009).
[3] R. Aharoni, E. Berger, M. Chudnovsky, D. Howard and P.D. Seymour, Large rainbow matchings in general graphs, arXiv:1611.03648v5.
[4] R. Aharoni, D. Kotlar and R. Ziv, Uniqueness of the extreme cases in theorems of Drisko and Erdős-Ginzburg-Ziv. *Eur. J. Comb.* **67** (2018), 222–229.
[5] R. Aharoni, D. Kotlar and R. Ziv, Degree Conditions for Matchability in 3-Partite Hypergraphs. *JGT* **87(1)** (2018), 61–71.
[6] R. Aharoni, R. Holzman, Z. Jiang, Fractional rainbow matchings. *submitted.*
[7] J. Barát, A. Gyárfás and J. Lehel, Rainbow matchings in bipartite multigraphs, *to appear in Periodica Mathematica Hungarica.*
[8] K. Bernhard and J. Vygen. *Combinatorial optimization: Theory and algorithms*, Springer, Fifth Edition, 2012, 258–263.
[9] A. A. Drisko, Transversals in row-Latin rectangles, *J. Combin. Theory Ser. A* **84** (1998), 181–195.
[10] J. Edmonds, Paths, trees, and flowers, Canad. J. Math. **17** (1965) 449-467.
[11] T. Gallai, Kritische Graphen II, Magyar Tud. Akad. Mat. Kutat. Int. Kőzl. **8** (1963), 373-395.
[12] M. X. Goemans, Advanced combinatorial optimization, Lecture 3, https://math.mit.edu/ goemans/18438F09/lec3.pdf.
[13] G. Kalai, R. Meshulam, A topological colorful Helly theorem, Adv. Math. **191** (2005), 305–311.
[14] L. Lovász and M. D. Plummer, *Matching Theory*, North-Holland, Mathematics Studies **121** (1986).

[15] A. Pokrovskyi, An approximate version of a conjecture of Aharoni and Berger, https://arxiv.org/abs/1609.06346.

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