INDUCED SUBGRAPHS AND TREE DECOMPOSITIONS IV. (EVEN HOLE, DIAMOND, PYRAMID)-FREE GRAPHS

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Abstract. A *hole* in a graph G is an induced cycle of length at least four, and an *even hole* is a hole of even length. The *diamond* is the graph obtained from the complete graph K_4 by removing an edge. A *pyramid* is a graph consisting of a vertex a called the *apex* and a triangle $\{b_1, b_2, b_3\}$ called the *base*, and three paths P_i from a to b_i for $1 \leq i \leq 3$, all of length at least one, such that for $i \neq j$, the only edge between $P_i \setminus \{a\}$ and $P_j \setminus \{a\}$ is b_ib_j, and at most one of P_1, P_2, and P_3 has length exactly one. For a family $\mathcal{H}$ of graphs, we say a graph G is *$\mathcal{H}$-free* if no induced subgraph of G is isomorphic to a member of $\mathcal{H}$. Cameron, da Silva, Huang, and Vušković proved that (even hole, triangle)-free graphs have treewidth at most five, which motivates studying the treewidth of even-hole-free graphs of larger clique number. Sintiari and Trotignon provided a construction of (even hole, pyramid, K_4)-free graphs of arbitrarily large treewidth. Here, we show that for every t, (even hole, pyramid, diamond, K_t)-free graphs have bounded treewidth. The graphs constructed by Sintiari and Trotignon contain diamonds, so our result is sharp in the sense that it is false if we do not exclude diamonds. Our main result is in fact more general, that treewidth is bounded in graphs excluding certain wheels and three-path-configurations, diamonds, and a fixed complete graph. The proof uses "non-crossing decompositions" methods similar to those in previous papers in this series. In previous papers, however, bounded degree was a necessary condition to prove bounded treewidth. The result of this paper is the first to use the method of "non-crossing decompositions" to prove bounded treewidth in a graph class of unbounded maximum degree.

1. Introduction

All graphs in this paper are simple. Let $G = (V(G), E(G))$ be a graph. An *induced subgraph* of G is a subgraph of G formed by deleting vertices. In this paper, we use induced subgraphs and their vertex sets interchangeably. A *tree decomposition* (T, χ) *of* G consists of a tree T and a map $\chi : V(T) \to 2^{V(G)}$ satisfying the following:

(i) For all $v \in V(G)$, there exists $t \in V(T)$ such that $v \in \chi(t)$;
(ii) For all $v_1v_2 \in E(G)$, there exists $t \in V(T)$ such that $v_1, v_2 \in \chi(t)$;
(iii) For all $v \in V(G)$, the subgraph of T induced by $\{t \in V(T)$ s.t. $v \in \chi(t)\}$ is connected.

The *width* of a tree decomposition (T, χ) of G is $\max_{t \in V(T)} |\chi(t)| - 1$. The *treewidth* of G, denoted $\text{tw}(G)$, is the minimum width of a tree decomposition of G. Treewidth was introduced by Robertson and Seymour in their study of graph minor theory.

Treewidth is roughly a measure of how "complicated" a graph is: forests have treewidth one, and in general, the smaller the treewidth, the more "tree-like" (and thus "uncomplicated") the graph. Graphs with bounded treewidth have nice structural properties, and many classic NP-hard problems can be solved in polynomial time in graphs with bounded treewidth (see [5] for more details). Understanding which graphs have bounded treewidth is an important question in the field of structural graph theory. This question is usually explored by considering substructures of graphs that, when present, cause the treewidth to be large, and when absent, guarantee that the treewidth is small. Robertson and Seymour famously gave a complete answer to this question in the case of subgraphs. By $W_{k\times k}$ we denote the $(k \times k)$-*wall*; see [2] for a full definition.

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Theorem 1.1 ([14]). *There is a function $f : \mathbb{N} \to \mathbb{N}$ such that every graph of treewidth at least $f(k)$ contains a subdivision of $W_{k\times k}$ as a subgraph.*

Recently, the study of which graphs have bounded treewidth has focused on *hereditary* graph classes; that is, classes of graphs defined by forbidden induced subgraphs. Most conjectures and theorems about the treewidth of hereditary graph classes fall into one of two categories: bounded treewidth in graph classes with bounded maximum degree, and logarithmic treewidth in graph classes with arbitrary maximum degree and bounded clique number. The main open question in the former category was the following:

Conjecture 1.2 ([1]). *For all $k, \Delta > 0$, there exists $c = c(k, \Delta)$ such that every graph with maximum degree at most Δ and treewidth more than c contains a subdivision of $W_{k\times k}$ or the line graph of a subdivision of $W_{k\times k}$ as an induced subgraph.*

In earlier papers in this series, several special cases of Conjecture 1.2 were resolved (see [2], [4]). Conjecture 1.2 was recently proven in [11], by a different method. In this paper, we use techniques similar to [2] and [4] to prove that a hereditary graph class of arbitrarily large maximum degree has bounded treewidth.

Before we state our main result, we define several types of graphs. If G is a path or a cycle, the *length* of G is $|E(G)|$. By P_k we denote the path on k vertices. If $P = p_1\text{-}p_2\text{-}\cdots\text{-}p_k$, then $P^* = P \setminus \{p_1, p_k\}$ denotes the *interior* of the path P. A *hole* of G is an induced cycle of length at least four. By C_4 we denote the hole of length four. Let G be a graph and let $v \in V(G)$. The *open neighborhood of v in G*, denoted $N_G(v)$, is the set of vertices of $V(G)$ adjacent to v. The *closed neighborhood of v in G*, denoted $N_G[v]$, is the union of v and $N_G(v)$. Let $X \subseteq V(G)$. The *open neighborhood of X in G*, denoted $N_G(X)$, is the set of vertices of $V(G) \setminus X$ with a neighbor in X. The *closed neighborhood of X in G*, denoted $N_G[X]$, is the union of X and $N_G(X)$. When the graph G is clear, we omit the subscript G from the open and closed neighborhoods. A *wheel* (H, w) is a hole H and a vertex $w \in V(G)$ such that w has at least three neighbors in H. We call w the *center* of the wheel (H, w). An *even wheel* is a wheel (H, w) such that $|N(w) \cap H|$ is even. A *line wheel* (H, v) is a hole H and a vertex $v \notin H$ such that $|H \cap N(v)|$ is the union of two disjoint edges. If $X, Y \subseteq V(G)$, we say X *is anticomplete to* Y if there are no edges with one endpoint in X and one endpoint in Y. We say that X *has a neighbor in* Y if X is not anticomplete to Y. We say that v is anticomplete to X if $\{v\}$ is anticomplete to X.

A *diamond* is the graph given by deleting an edge from K_4. A *theta* is a graph consisting of two non-adjacent vertices a and b and three paths P_1, P_2, P_3 from a to b of length at least two, such that P_1^*, P_2^*, P_3^* are pairwise disjoint and anticomplete to each other. We say this is a theta *between a and b through P_1, P_2, and P_3*. A *pyramid* is a graph consisting of a vertex a, a triangle b_1, b_2, b_3, and three paths P_1, P_2, P_3 from a to b_1, b_2, b_3, respectively, such that for $i \neq j$, the only edge between $P_i \setminus \{a\}$ and $P_j \setminus \{a\}$ is b_ib_j, and at most one of P_1, P_2, P_3 has length exactly one. We say a is the *apex* of the pyramid and $b_1b_2b_3$ is the *base* of the pyramid. A *prism* is a graph consisting of two disjoint triangles $a_1a_2a_3$ and $b_1b_2b_3$ and three paths P_1, P_2, P_3, with P_i from a_i to b_i, such that for all distinct $i, j \in \{1, 2, 3\}$, the only edges between P_i and P_j are a_ia_j and b_ib_j. Thetas, pyramids, and prisms are called *three-path configurations.*

If H is a graph, then by *H-free graphs* we mean the class of graphs which do not contain H as an induced subgraph. If $\mathcal{H}$ is a set of graphs, then by *$\mathcal{H}$-free graphs* we mean the class of graphs which are H-free for every $H \in \mathcal{H}$. Let $\mathcal{C}$ be the class of (C_4, diamond, theta, pyramid, prism, even wheel)-free graphs, and let $\mathcal{C}_t$ be the class of (C_4, diamond, theta, pyramid, prism, even wheel, K_t)-free graphs. The main result of this paper is the following theorem.

Theorem 1.3. *For all $t > 0$ there exists $c_t \geq 0$ such that $\mathrm{tw}(G) \leq c_t$ for every $G \in \mathcal{C}_t$.*

Note that thetas, prisms, and even wheels contain even holes, so (even hole, diamond, pyramid, K_t)-free graphs are a subclass of $\mathcal{C}_t$. Therefore, Theorem 1.3 implies the following:

Theorem 1.4. *For all $t > 0$ there exists $d_t \geq 0$ such that $\mathrm{tw}(G) \leq d_t$ for every (even hole, diamond, pyramid, K_t)-free graph G.*

Theorem 1.3 is the first result of this series that gives a constant bound on treewidth in a class of graphs with arbitrary maximum degree; the previous results have either obtained a constant bound on treewidth in graph classes with bounded degree ([2], [4]), or given a logarithmic bound on treewidth in graph classes with bounded clique number ([3]). Bounded treewidth results for similar graph classes were also proved in [7].

In [15], Sintiari and Trotignon construct (even hole, pyramid, K_4)-free graphs of arbitrarily large treewidth. These graphs contain diamonds; therefore, Theorem 1.4 is sharp in the sense that excluding diamonds is necessary to obtain bounded treewidth. Sintiari and Trotignon also made the following conjecture:

Conjecture 1.5 ([15])**.** *(Even hole, diamond, K_4)-free graphs have bounded treewidth.*

Theorem 1.4 is a special case of Conjecture 1.5. If Conjecture 1.5 can be proven using techniques similar to those used in this paper and in the previous papers of this series, then Theorem 1.4 is the base case to prove Conjecture 1.5. Indeed, we conjecture the following slight generalization of Conjecture 1.5.

Conjecture 1.6. *For all $t > 0$ there exists $c_t \geq 0$ such that $\text{tw}(G) \leq c_t$ for every (C_4, diamond, theta, prism, even wheel, K_t)-free graph G.*

In view of Conjecture 1.6, when possible, we prove the results of this paper for (C_4, diamond, theta, prism, even wheel, K_t)-free graphs instead of graphs in $\mathcal{C}_t$. To that end, we define $\mathcal{C}_t^*$ to be the class of (C_4, diamond, theta, prism, even wheel, K_t)-free graphs with no clique cutset. The ramifications of excluding a diamond on the treewidth of other classes of sparse graphs is still unclear, but it is of significant interest due to the profound impact it has on the behavior of maximal cliques.

1.1. **Proof outline.** Here, we give a brief outline of the ideas used in the proof of Theorem 1.3. Many of these ideas, or similar ones, appear in previous papers in this series. One major tool we use is that of balanced separators; a graph has a balanced separator for every normalized weight function on its vertices if and only if it has bounded treewidth.

Let $G \in \mathcal{C}_t$. We would like to apply decomposition techniques to G to obtain an induced subgraph β of G such that: (i) the treewidth of β is easy to compute, and (ii) there exists a function f such that if $\text{tw}(\beta) \leq c$, then $\text{tw}(G) \leq f(c)$. If we can find a decomposition that satisfies these two properties, then we can bound the treewidth of G.

To obtain property (i), we make use of the key result that if a graph $G \in \mathcal{C}_t$ is wheel-free, then G has bounded treewidth (by Theorem 3.12). In view of this result, we would like to guarantee that β is wheel-free. To do this, we consider star cutsets associated with wheel centers. A *star cutset* of a graph G is a set $C \subseteq V(G)$ such that $G \setminus C$ is not connected and there exists $v \in V(C)$ such that $C \subseteq N[v]$. We call v the *center* of the star cutset C. Let (H, v) be a wheel of G. Then, v is the center of a star cutset C of G, and H is not contained in the closed neighborhood of any connected component of $G \setminus C$ (by Lemma 2.7). Therefore, the star cutset with center v "breaks" the hole H. If β is contained in the closed neighborhood of a connected component of $G \setminus C$, then β does not contain the wheel (H, v). Therefore, star cutsets associated with wheel centers are a promising way to construct an induced subgraph β which is wheel-free, and thus whose treewidth is easy to compute.

To obtain property (ii), we make use of the relationship between treewidth and collections of decompositions with a property called "non-crossing." "Non-crossing decompositions" interact well with treewidth, and provide a way to obtain a function f such that if $\text{tw}(\beta) \leq c$, then $\text{tw}(G) \leq f(c)$. It turns out that there are natural decompositions corresponding to star cutsets, and in the case of graphs in $\mathcal{C}_t$, these decompositions are "nearly non-crossing" (a slight generalization of non-crossing that also cooperates with treewidth). Because of the way the decompositions corresponding to star cutsets are constructed, we obtain an induced subgraph β of G such that either β is wheel-free or β has a balanced separator. To prove that β has a balanced separator if it is not wheel-free, we use degeneracy to bound the degree of vertices which are wheel centers in β (a similar technique was used in [3]). In either case, β has bounded treewidth, and so, because of the function f obtained by the properties of non-crossing decompositions, G has bounded treewidth.

In previous papers in this series, decompositions corresponding to star cutsets were also used to reduce the problem of bounding the treewidth of G to bounding the treewidth of a "less complicated" induced subgraph β of G. But the collections of decompositions used in previous results were not nearly non-crossing; instead, we used that the graph classes had bounded degree to partition the decompositions into a bounded number of nearly non-crossing collections. In $\mathcal{C}_t$, we were able to slightly modify the star cutsets we consider in order to obtain a single collection of non-crossing decompositions. This eliminated the need for the bounded degree condition required for previous results of this series, and allowed us to prove for the first time that the treewidth of a graph class with unbounded maximum degree is bounded.

1.2. **Organization.** The remainder of the paper is organized as follows. In Section 2, we define several tools needed to prove that graphs $G \in \mathcal{C}_t$ have bounded treewidth. In Section 3, we construct a useful induced subgraph β of G and prove that β has bounded treewidth. Finally, in Section 4, we use that β has bounded treewidth to prove that G has bounded treewidth.

2. Tools

In this section, we describe the tools needed to prove that graphs in $\mathcal{C}_t$ have bounded treewidth. These tools fall into four categories: balanced separators, separations, central bags, and cutsets obtained from wheels.

2.1. **Balanced separators.** Let G be a graph. A *weight function on* G is a function $w : V(G) \to \mathbb{R}$. For $X \subseteq V(G)$, we let $w(X) = \sum_{x \in X} w(x)$. Let G be a graph, let $w : V(G) \to [0,1]$ be a weight function on G with $w(G) = 1$, and let $c \in [\frac{1}{2}, 1)$. A set $X \subseteq V(G)$ is a (w,c)*-balanced separator* if $w(D) \leq c$ for every component D of $G \setminus X$. The next two lemmas state how (w,c)-balanced separators relate to treewidth. The first result was originally proven by Harvey and Wood in [10] using different language, and was restated and proved in the language of (w,c)-balanced separators in [2].

Lemma 2.1 ([2], [10])**.** *Let G be a graph, let $c \in [\frac{1}{2}, 1)$, and let k be a positive integer. If G has a (w,c)-balanced separator of size at most k for every weight function $w : V(G) \to [0,1]$ with $w(G) = 1$, then* $\operatorname{tw}(G) \leq \frac{1}{1-c}k$.

Lemma 2.2 ([8])**.** *Let G be a graph and let k be a positive integer. If $\operatorname{tw}(G) \leq k$, then G has a (w,c)-balanced separator of size at most $k+1$ for every $c \in [\frac{1}{2}, 1)$ and for every weight function $w : V(G) \to [0,1]$ with $w(G) = 1$.*

2.2. **Separations.** A *separation* of a graph G is a triple (A, C, B), where $A, B, C \subseteq V(G)$, $A \cup C \cup B = V(G)$, A, B, and C are pairwise disjoint, and A is anticomplete to B. If $S = (A, C, B)$ is a separation, we let $A(S) = A$, $B(S) = B$, and $C(S) = C$. Two separations (A_1, C_1, B_1) and (A_2, C_2, B_2) are *nearly non-crossing* if every component of $A_1 \cup A_2$ is a component of A_1 or a component of A_2. A separation (A, C, B) is a *star separation* if there exists $v \in C$ such that $C \subseteq N[v]$. Let $S_1 = (A_1, C_1, B_1)$ and $S_2 = (A_2, C_2, B_2)$ be separations of G. We say S_1 is a *shield for* S_2 if $B_1 \cup C_1 \subseteq B_2 \cup C_2$.

Lemma 2.3. *Let G be a $(C_4$, diamond$)$-free graph with no clique cutset, let $v_1, v_2 \in V(G)$, and let $S_1 = (A_1, C_1, B_1)$ and $S_2 = (A_2, C_2, B_2)$ be star separations of G such that $v_i \subseteq C_i \subseteq N[v_i]$, B_i is connected, and $N(B_i) = C_i \setminus \{v_i\}$ for $i = 1, 2$. Suppose that $v_2 \in A_1$ and $B_2 \cap (B_1 \cup (C_1 \setminus \{v_1\})) \neq \emptyset$. Then, S_1 is a shield for S_2.*

Proof. Since $v_2 \in A_1$ and A_1 is anticomplete to B_1, it follows that $C_2 \subseteq A_1 \cup C_1$, and thus B_1 is contained in a connected component of $G \setminus C_2$. First, we show that $B_1 \subseteq B_2$. If there exists $x \in B_1 \cap B_2$, then it holds that $B_1 \subseteq B_2$, so we may assume that $B_1 \cap B_2 = \emptyset$. Consequently, $B_1 \subseteq A_2$, and so, since $B_2 \cap (B_1 \cup (C_1 \setminus \{v_1\})) = \emptyset$, it follows that there exists $x \in (C_1 \setminus \{v_1\}) \cap B_2$. But $N(B_1) = C_1 \setminus \{v_1\}$, $x \in B_2$, $B_1 \subseteq A_2$, and A_2 is anticomplete to B_2, a contradiction. This proves that $B_1 \subseteq B_2$.

Since every vertex of $C_1 \setminus \{v_1\}$ has a neighbor in B_1, and thus in B_2, it follows that $C_1 \setminus \{v_1\} \subseteq B_2 \cup C_2$. Now consider v_1. If there exists $x \in C_1 \setminus \{v_1\}$ such that $x \in B_2$, then v_1 has a neighbor in B_2 and so $v_1 \in B_2 \cup C_2$, as required. Thus we may assume that $C_1 \setminus \{v_1\} \subseteq C_2$, and so (since $v_2 \in A_1$) v_2 is complete to $C_1 \setminus \{v_1\}$.

Since G has no clique cutset and $N(B_1) = C_1$, there exist $x, y \in C_1 \setminus \{v_1\}$ such that x and y are non-adjacent. But now $\{x, y, v_1, v_2\}$ is either a diamond or a C_4, a contradiction. ■

Let G be a graph and let $w : V(G) \to [0,1]$ be a weight function on G with $w(G) = 1$. A vertex $v \in V(G)$ is called *balanced* if $w(D) \leq \frac{1}{2}$ for every component D of $G \setminus N[v]$, and *unbalanced* otherwise. Let U denote the set of unbalanced vertices of G. Let $v \in U$. The *canonical star separation for* v, denoted $S_v = (A_v, C_v, B_v)$, is defined as follows: B_v is the connected component of $G \setminus N[v]$ with largest weight, $C_v = \{v\} \cup (N(v) \cap N(B_v))$, and $A_v = V(G) \setminus (B_v \cup C_v)$. Note that B_v is well-defined since $v \in U$.

Let $\leq_A$ be the relation on U where for $x, y \in U$, $x \leq_A y$ if and only if $x = y$ or $y \in A_x$.

Lemma 2.4. *Let G be a (C_4, diamond)-free graph with no clique cutset, let $w : V(G) \to [0,1]$ be a weight function on G with $w(G) = 1$, let U be the set of unbalanced vertices of G, and let $\leq_A$ be the relation on U defined above. Then, $\leq_A$ is a partial order.*

Proof. We will show that $\leq_A$ is reflexive, antisymmetric, and transitive. The relation is reflexive by definition. Let $x, y \in U$ be such that $x \neq y$ and suppose that $x \leq_A y$. By Lemma 2.3, it holds that S_x is a shield for S_y, and so $B_x \cup C_x \subseteq B_y \cup C_y$. But $x \in C_x$, so $x \notin A_y$. Since $x \neq y$, it follows that $y \not\leq_A x$, and so the relation is antisymmetric.

Finally, suppose that $x, y, z \in U$ such that $x \leq_A y$ and $y \leq_A z$, so $y \in A_x$ and $z \in A_y$. By Lemma 2.3, it follows that S_x is a shield for S_y, so $B_x \cup C_x \subseteq B_y \cup C_y$. Since $z \in A_y$, it follows that $z \notin B_x \cup C_x$, so $z \in A_x$. Therefore, $x \leq_A z$, and the relation is transitive. ■

2.3. **Central bags.** Let G be a graph. We call a collection $\mathcal{S}$ of separations of G *smooth* if the following hold:

(i) S_1 and S_2 are nearly non-crossing for all distinct $S_1, S_2 \in \mathcal{S}$;
(ii) There is a set of unbalanced vertices $v(\mathcal{S}) \subseteq V(G)$ such that there is a bijection f from $v(\mathcal{S})$ to $\mathcal{S}$ with $v \in C(f(v)) \subseteq N[v]$ and $A(f(v)) \subseteq A_v$ for all $v \in \nu(\mathcal{S})$;
(iii) $v(\mathcal{S}) \cap A(S) = \emptyset$ for all $S \in \mathcal{S}$.

Let $\mathcal{S}$ be a smooth collection of separations of G. Then, the *central bag for* $\mathcal{S}$, denoted $\beta_{\mathcal{S}}$, is defined as follows:

$$\beta_{\mathcal{S}} = \bigcap_{S \in \mathcal{S}} (B(S) \cup C(S)).$$

Let G be a graph and let $w : V(G) \to [0,1]$ be a weight function on G with $w(G) = 1$. Let $\mathcal{S}$ be a smooth collection of separations of G, and let $\beta_{\mathcal{S}}$ be the central bag for $\mathcal{S}$. By property (iii) of smooth collections of separations, it holds that $v(\mathcal{S}) \subseteq \beta_{\mathcal{S}}$. Also note that $G \setminus \beta_{\mathcal{S}} = \bigcup_{S \in (\mathcal{S})} A(S)$. We now define the *inherited weight function* $w_{\mathcal{S}}$ *on* $\beta_{\mathcal{S}}$ as follows. Fix an ordering $\{v_1, \ldots, v_k\}$ of $v(\mathcal{S})$. For every $f(v_i) \in \mathcal{S}$, let $A^*(f(v_i))$ be the union of all connected components D of $\bigcup_{1 \leq j \leq k} A(f(v_j))$ such that $D \subseteq A(f(v_i))$ and $D \not\subseteq A(f(v_j))$ for every $j < i$. In particular, $(A^*(f(v_1)), \ldots, A^*(f(v_k)))$ is a partition of $\bigcup_{S \in \mathcal{S}} A(S)$. Now, $w_{\mathcal{S}}(v_i) = w(v_i) + w(A^*(f(v_i)))$ for all $v_i \in v(\mathcal{S})$, and $w_{\mathcal{S}}(v) = w(v)$ for all $v \notin v(\mathcal{S})$.

Lemma 2.5. *Let G be a graph, let $w : V(G) \to [0,1]$ be a weight function on G with $w(G) = 1$, and let $c \in [\frac{1}{2}, 1)$. Let $\mathcal{S}$ be a smooth collection of separations of G, let $\beta_{\mathcal{S}}$ be the central bag for $\mathcal{S}$, and let $w_{\mathcal{S}}$ be the inherited weight function on $\beta_{\mathcal{S}}$. Suppose that $X \subseteq \beta_{\mathcal{S}}$ is a $(w_{\mathcal{S}}, c)$-balanced separator of $\beta_{\mathcal{S}}$. Then, $Y = X \cup (N_G[X \cap v(\mathcal{S})] \cap \beta_{\mathcal{S}})$ is a (w, c)-balanced separator of G.*

Proof. Let $Q_1, \ldots, Q_m$ be the connected components of $\beta_{\mathcal{S}} \setminus X$. Let

$$A_i = \bigcup_{v_j \in Q_i \cap v(\mathcal{S})} A^*(f(v_j))$$

for $1 \leq i \leq m$. Since $N(A^*(f(v))) \subseteq C_v$ and $C_v \subseteq N[v]$ for all $v \in v(\mathcal{S})$, it follows that for every connected component D' of $G \setminus Y$, either $D' \subseteq Q_i \cup A_i$ or $D' \subseteq A^*(f(v))$ for some $v \in v(\mathcal{S}) \cap X$. Let D' be a component of $G \setminus Y$. Since $\mathcal{S}$ is smooth, it follows that $A^*(f(v)) \subseteq A_v$, and since v is unbalanced, it follows that $w(A_v) \leq \frac{1}{2}$. If $D' \subseteq A^*(f(v))$ for some $v \in v(\mathcal{S})$, then $w(D') \leq \frac{1}{2} \leq c$, so we may assume $D' \not\subseteq A^*(f(v))$. Therefore, $D' \subseteq Q_i \cup A_i$ for some $1 \leq i \leq m$. By the definition of $w_{\mathcal{S}}$, it holds that

$$\begin{aligned} w_{\mathcal{S}}(Q_i) &= w(Q_i) + \sum_{v \in Q_i \cap v(\mathcal{S})} w(A^*(f(v))) \\ &= w(Q_i) + w(A_i). \end{aligned}$$

Since X is a $(w_{\mathcal{S}}, c)$-balanced separator of $\beta_{\mathcal{S}}$, we have $w_{\mathcal{S}}(Q_i) \leq c$, and so $w(Q_i \cup A_i) \leq c$. Since $D' \subseteq Q_i \cup A_i$, it follows that $w(D') \leq c$ for every connected component D' of $G \setminus Y$. ■

2.4. **Wheels.** Recall that a *wheel* (H, w) is a hole H and a vertex $w \in V(G)$ such that w has at least three neighbors in H. If (H, w) is a wheel, a *sector of* (H, w) is a path $P \subseteq H$ of length at least one such that the ends of P are adjacent to w and P^* is anticomplete to w. A sector of (H, w) is *long* if it has length greater than one.

Lemma 2.6. *Let G be a (theta, even wheel)-free graph, let H be a hole of G, and let $v_1, v_2 \in V(G) \setminus V(H)$ be adjacent vertices each with at least two non-adjacent neighbors in H. Then, v_1 and v_2 have a common neighbor in H.*

Proof. Suppose that v_1 and v_2 have no common neighbors in H. Since $H \cup \{v_1\}$ is not a theta, it follows that (H, v_1) is a wheel. Similarly, (H, v_2) is a wheel. Let $Q \subseteq H$ be a long sector of (H, v_1). Then, $Q \cup \{v_1\}$ is a hole. Since v_2 is adjacent to v_1 and G has no even wheel and no theta, it follows that v_2 has an odd number of neighbors in $Q \cup \{v_1\}$, and thus v_2 has an even number of neighbors in Q. Since v_1 and v_2 have no common neighbors in H, every neighbor of v_2 in H is in the interior of a sector of (H, v_1). Therefore, v_2 has an even number of neighbors in H. Since (H, v_2) is a wheel, it follows that (H, v_2) is an even wheel, a contradiction. ■

A wheel (H, w) is a *twin wheel* if $N(w) \cap H$ is a path of length two. A wheel (H, w) is a *short pyramid* if $|N(w) \cap H| = 3$ and w has exactly two adjacent neighbors in H. A *proper wheel* is a wheel that is not a twin wheel or a short pyramid. A wheel (H, w) is a *universal wheel* if w is complete to H. We will use the following result about wheels and star cutsets. In [9], Lemma 2.7 is proven for a class of graphs called *C_4-free odd-signable graphs*. It is also shown in [9] that (C_4, even wheel, theta, prism)-free graphs are C_4-free odd-signable graphs. Since we are interested in (C_4, even wheel, theta, prism)-free graphs, we state Lemma 2.7 about (C_4, even wheel, theta, prism)-free graphs.

Lemma 2.7 ([9]). *Let G be a (C_4, even wheel, theta, prism)-free graph that contains a proper wheel (H, x) that is not a universal wheel. Let x_1 and x_2 be the endpoints of a long sector Q of (H, x). Let W be the set of all vertices $h \in H \cap N(x)$ such that the subpath of $H \setminus \{x_1\}$ from x_2 to h contains an even number of neighbors of x, and let $Z = H \setminus (Q \cup N(x))$. Let $N' = N(x) \setminus W$. Then, $N' \cup \{x\}$ is a cutset of G that separates Q^* from $W \cup Z$.*

In particular, Lemma 2.7 implies the following.

Lemma 2.8. *Let G be a (C_4, even wheel, theta, prism)-free graph and let (H, v) be a proper wheel of G that is not a universal wheel. Let (A, C, B) be a separation of G such that $v \in C \subseteq N[v]$, B is connected, and $N(B) = C \setminus \{v\}$. Then, $H \not\subseteq B \cup C$.*

Proof. By Lemma 2.7, there exist $x, y \in H$ such that there is no path from x to y with interior anticomplete to v. Suppose that $x, y \in B \cup C$. Since B is connected and every vertex of $C \setminus \{v\}$ has a neighbor in B, it follows that there is a path P from x to y with $P^* \subseteq B$. But $N(B) = C \setminus \{v\}$, and so v is anticomplete to P^*, a contradiction. It follows that $\{x, y\} \not\subseteq B \cup C$, and so $H \not\subseteq B \cup C$. ■

Finally, note that in a (diamond, pyramid)-free graph, every wheel is a proper wheel that is not a universal wheel.

3. Balanced separators and central bags

In this section, we construct a useful central bag for graphs in $\mathcal{C}_t$ and prove that the central bag has bounded treewidth. First, we state the observation that clique cutsets do not affect treewidth (this is a special case of Lemma 3.1 from [6]).

Lemma 3.1. *Let G be a graph. Then, the treewidth of G is equal to the maximum treewidth over all induced subgraphs of G with no clique cutset.*

Because of Lemma 3.1, we often assume that the graphs we work with do not have clique cutsets. The next lemma examines how three vertices can have neighbors in a connected subgraph.

Lemma 3.2 ([2]). *Let x_1, x_2, x_3 be three distinct vertices of a graph G. Assume that H is a connected induced subgraph of $G \setminus \{x_1, x_2, x_3\}$ such that H contains at least one neighbor of each of x_1, x_2, x_3, and that subject to these conditions $V(H)$ is minimal subject to inclusion. Then one of the following holds:*

(i) For distinct $i, j, k \in \{1, 2, 3\}$, there exists P that is either a path from x_i to x_j or a hole containing the edge $x_i x_j$ such that
- *$H = P \setminus \{x_i, x_j\}$, and*
- *either x_k has at least two non-adjacent neighbors in H or x_k has exactly two neighbors in H and its neighbors in H are adjacent.*

(ii) *There exists a vertex $a \in H$ and three paths P_1, P_2, P_3, where P_i is from a to x_i, such that*
- $H = (P_1 \cup P_2 \cup P_3) \setminus \{x_1, x_2, x_3\}$, *and*
- *the sets $P_1 \setminus \{a\}$, $P_2 \setminus \{a\}$ and $P_3 \setminus \{a\}$ are pairwise disjoint, and*
- *for distinct $i, j \in \{1, 2, 3\}$, there are no edges between $P_i \setminus \{a\}$ and $P_j \setminus \{a\}$, except possibly $x_i x_j$.*

(iii) *There exists a triangle $a_1a_2a_3$ in H and three paths P_1, P_2, P_3, where P_i is from a_i to x_i, such that*
- $H = (P_1 \cup P_2 \cup P_3) \setminus \{x_1, x_2, x_3\}$, *and*
- *the sets P_1, P_2, and P_3 are pairwise disjoint, and*
- *for distinct $i, j \in \{1, 2, 3\}$, there are no edges between P_i and P_j, except $a_i a_j$ and possibly $x_i x_j$.*

Using Lemma 3.2, we prove the following theorem, which can also be easily deduced from Lemma 1.1 of [13].

Theorem 3.3. *Let G be a (theta, triangle, wheel)-free graph. Then, $\mathrm{tw}(G) \leq 2$.*

Proof. By Lemma 3.1, we may assume G does not have a clique cutset (so in particular, G is connected).

(1) *G does not have a star cutset C with $v \in C \subseteq N[v]$ for some $v \in V(G)$.*

Suppose that $v \in V(G)$ and G has a star cutset $C \subseteq N[v]$ with $v \in C$. Since G does not have a clique cutset, there exist $x, y \in C \setminus \{v\}$ and two connected components D_1, D_2 of $G \setminus C$ such that $\{x, y\} \subseteq N(D_1) \cap N(D_2)$ (and since G is triangle-free, x and y are non-adjacent). Let P_1 be a path from x to y with $P_1^* \subseteq D_1$, let P_2 be a path from x to y with $P_2^* \subseteq D_2$, and let H be the hole given by $P_1 \cup P_2$. Since x and y are non-adjacent, v has at least two non-adjacent neighbors in H. If $N(v) \cap H = \{x, y\}$, then G contains a theta between x and y through x-v-y, x-P_1-y, and x-P_2-y, a contradiction, so v has at least three neighbors in H. Since G is triangle-free, the neighbors of v are pairwise non-adjacent. But now (H, v) is a wheel of G, a contradiction. This proves (1).

(2) $\deg(v) \leq 2$ *for all $v \in V(G)$.*

Let $v \in V(G)$ and let $B = G \setminus N[v]$. By (1), B is connected, and since G does not have a clique cutset, it follows that $N(B) = N(v)$. Suppose that $x, y, z \in N(v)$, and let $H \subseteq B$ be inclusion-wise minimal such that H contains a neighbor of x, y, and z. We apply Lemma 3.2. If case (ii) holds, then $H \cup \{v, x, y, z\}$ is a theta, a contradiction. Case (iii) does not hold because G is triangle-free. Therefore, case (i) holds. Then, up to symmetry between x, y, z, $H \cup \{x, z\}$ is a path from x to z such that y has two non-adjacent neighbors in H. But now $H' = \{v, x, z\} \cup H$ is a hole and y has three pairwise non-adjacent neighbors in H', so (H', v) is a wheel of G, a contradiction. This proves (2).

By (2), it follows that G is either a path or a cycle, and thus $\mathrm{tw}(G) \leq 2$. ■

Next, we give a simple characterization of the neighborhood of vertices in diamond-free graphs.

Lemma 3.4. *Let G be diamond-free and let $v \in V(G)$. Then, $N(v)$ is the union of disjoint, pairwise anticomplete cliques.*

Proof. If $N(v)$ contains P_3, then $v \cup N(v)$ contains a diamond, a contradiction. Therefore, $N(v)$ is P_3-free, and thus the union of disjoint, pairwise anticomplete cliques. ■

For $X \subseteq V(G)$, let $\mathrm{Hub}(X)$ denote the set of all vertices $x \in X$ for which there exists a proper wheel (H, x) with $H \subseteq X$.

Lemma 3.5. *Let G be a (theta, diamond)-free graph with no clique cutset and let $w : V(G) \to [0, 1]$ be a weight function on G with $w(G) = 1$. Let $\mathcal{S}$ be a smooth collection of separations of G, let $\beta_{\mathcal{S}}$ be the central bag for $\mathcal{S}$, and let $w_{\mathcal{S}}$ be the inherited weight function on $\beta_{\mathcal{S}}$. Let $v \in \beta_{\mathcal{S}}$ and (by Lemma 3.4) let $N_{\beta_{\mathcal{S}}}(v) \setminus \mathrm{Hub}(\beta_{\mathcal{S}}) = K_1 \cup \dots \cup K_t$, where $K_1, \dots, K_t$ are disjoint, pairwise anticomplete cliques. Assume that v is not a pyramid apex in $\beta_{\mathcal{S}}$. Let D be a component of $\beta_{\mathcal{S}} \setminus N[v]$. Then, at most two of $K_1, \dots, K_t$ have a neighbor in D.*

Proof. Suppose that $\{a, b, c\} \subseteq \{1, \dots, t\}$ such that K_a, K_b, K_c each have a neighbor in D. Let $x_1 \in K_a$, $x_2 \in K_b$, and $x_3 \in K_c$ be such that x_1, x_2, x_3 have neighbors in D. Note that $\{x_1, x_2, x_3\}$ is independent. Let $H \subseteq D$ be inclusion-wise minimal such that x_1, x_2, x_3 each have a neighbor in H. We apply Lemma 3.2. If case (ii) holds, then $\{v, x_1, x_2, x_3\} \cup H$ is a theta, a contradiction. If case (iii) holds, then $\{v, x_2, x_2, x_3\} \cup H$ is a pyramid of $\beta_{\mathcal{S}}$ with apex v, a contradiction. Therefore, up to symmetry between x_1, x_2, x_3, it holds that that $H \cup \{x_1, x_3\}$ is a path from x_1 to x_3. If x_2 has two non-adjacent neighbors in H, then x_2 is a proper wheel center for the hole given by $H \cup \{v\}$, a contradiction (since $x_2 \in N_{\beta_{\mathcal{S}}}(v) \setminus \mathrm{Hub}(\beta_{\mathcal{S}})$). Therefore, x_2 has exactly two adjacent neighbors in H. But now $\{v, x_2\} \cup H$ is a pyramid of $\beta_{\mathcal{S}}$ with apex v, a contradiction. ■

We next prove a result about balanced separators in central bags with balanced vertices. By $\omega(G)$ we denote the size of the largest clique of G.

Lemma 3.6. *Let G be a (C_4, theta, pyramid, prism, diamond)-free graph with no clique cutset, let $w : V(G) \to [0, 1]$ be a weight function on G with $w(G) = 1$, and let $c \in [\frac{1}{2}, 1)$. Let $\mathcal{S}$ be a smooth collection of separations of G, let $\beta_{\mathcal{S}}$ be the central bag for $\mathcal{S}$, and let $w_{\mathcal{S}}$ be the inherited weight function on $\beta_{\mathcal{S}}$. Suppose that there exists $v \in \beta_{\mathcal{S}}$ such that v is balanced in G, and assume that $|N(v) \cap \mathrm{Hub}(\beta_{\mathcal{S}})| < C$. Then $\beta_{\mathcal{S}}$ has a $(w_{\mathcal{S}}, c)$-balanced separator of size at most $6\omega(\beta_{\mathcal{S}}) + C$.*

Proof. Note that since $\mathcal{S}$ is smooth, it follows that $v \in \beta_{\mathcal{S}} \setminus \nu(\mathcal{S})$. Let $D_1, \dots, D_m$ be the components of $\beta_{\mathcal{S}} \setminus N[v]$.

(3) *$w_{\mathcal{S}}(D_i) \le \frac{1}{2}$ for all $1 \le i \le m$.*

Let D' be a component of $G \setminus N[v]$, and suppose $s \in v(\mathcal{S}) \cap D'$. Since $\mathcal{S}$ is smooth, it follows that $s \in \beta_{\mathcal{S}}$. Suppose that $u \in A(f(s)) \cap N(v)$. Then, since $v \in N(A(f(s)))$, it follows that $v \in A(f(s)) \cup C(f(s))$. But $v \in \beta_{\mathcal{S}}$, so $v \notin A(f(s))$ by the definition of $\beta_{\mathcal{S}}$, and thus $v \in C(f(s))$. But v is not adjacent to s, a contradiction. This proves that $A(f(s)) \cap N(v) = \emptyset$. Now, let $D'' \neq D'$ be a component of $G \setminus N[v]$, and suppose that $D'' \cap A(f(s)) \neq \emptyset$. Then, since $N(s) \cap D'' = \emptyset$, it follows that $D'' \subseteq A(f(s))$ and $N(D'') \subseteq N(v) \cap C(f(s))$. Since G has no clique cutset, it follows that there exist $x, y \in N(v) \cap C(f(s))$ with x and y non-adjacent. But now $\{s, v, x, y\}$ is a C_4, a contradiction. This proves that $D'' \subseteq B(f(s))$.

Let D be a component of $\beta_{\mathcal{S}} \setminus N[v]$ and let D' be the component of $G \setminus N[v]$ containing D. Now, for every $s \in v(\mathcal{S}) \cap D$, it holds that $A(f(s)) \subseteq D'$. It follows that $w_{\mathcal{S}}(D) \le w(D')$. Since v is balanced in G, it holds that $w(D') \le \frac{1}{2}$, and thus (3) follows.

By Lemma 3.4, let $N_{\beta_{\mathcal{S}}}(v) \setminus \mathrm{Hub}(\beta_{\mathcal{S}}) = K_1 \cup \dots \cup K_t$, where K_i is a clique for $1 \le i \le t$ and $K_1, \dots, K_t$ are disjoint and pairwise anticomplete to each other. Let H be a graph with vertex set $\{k_1, \dots, k_t, d_1, \dots, d_m\}$, where H contains an edge between k_i and d_j if and only if K_i has a neighbor in D_j.

(4) *H is (theta, wheel)-free and bipartite.*

By definition of H, $(\{k_1, \dots, k_t\}, \{d_1, \dots, d_m\})$ is a bipartition of H, so H is bipartite. By Lemma 3.5, $\deg(d_i) \le 2$ for $1 \le i \le m$, so H is wheel-free. Suppose that H contains a theta between x and y (with $x, y \in V(H)$). Since x and y have degree at least three, it follows that $x, y \in \{k_1, \dots, k_t\}$. Let P_1, P_2, P_3 be the three paths of a theta between x and y in H. Let Q_1, Q_2, Q_3 be paths in $\beta_{\mathcal{S}}$, where Q_i has ends x_i and y_i and is formed from P_i by replacing every path k_i-d_j-$k_\ell \subseteq P_i$ with a path from K_i to K_ℓ through D_j in $\beta_{\mathcal{S}}$. Note that since $x, y \in \{k_1, \dots, k_t\}$, it follows that $\{x_1, x_2, x_3\} \subseteq K_i$ and $\{y_1, y_2, y_3\} \subseteq K_j$ for some $1 \le i, j \le t$.

Let x'_i be the neighbor of x_i in Q_i for $i = 1, 2, 3$. Let H_1 be an inclusion-wise minimal connected subset of K_i such that H_1 contains neighbors of x'_1, x'_2, x'_3. We apply Lemma 3.2 to x'_1, x'_2, x'_3 and H_1. (Note that while x_1, x_2, x_3 are not necessarily distinct, x'_1, x'_2, x'_3 are distinct). Similarly, let y'_i be the neighbor of y_i in Q_i for $i = 1, 2, 3$. Let H_2 be an inclusion-wise minimal connected subset of K_j such that H_2 contains neighbors of y'_1, y'_2, y'_3. We apply Lemma 3.2 to y'_1, y'_2, y'_3, and H_2. Note that $\{x'_1, x'_2, x'_3\} \subseteq \bigcup_{1 \le i \le m} D_i$ and that x'_1, x'_2, x'_3 are each in distinct components D_i. Similarly, $\{y'_1, y'_2, y'_3\} \subseteq \bigcup_{1 \le i \le m} D_i$ and y'_1, y'_2, y'_3 are each in distinct components D_i. Therefore, $\{x'_1, x'_2, x'_3\}$ are pairwise non-adjacent and $\{y'_1, y'_2, y'_3\}$ are pairwise non-adjacent.

Suppose that $\{x'_1, x'_2, x'_3\} \cup H_1$ satisfies condition (i) of Lemma 3.2. If $\{y'_1, y'_2, y'_3\} \cup H_2$ satisfies condition (i), then $H_1 \cup H_2 \cup \{x'_1, x'_2, x'_3, y'_1, y'_2, y'_3\}$ is a prism or a line wheel, a contradiction. If $\{y'_1, y'_2, y'_3\} \cup H_2$ satisfies condition (ii), then $H_1 \cup H_2 \cup \{x'_1, x'_2, x'_3, y'_1, y'_2, y'_3\}$ is a pyramid, a contradiction. If $\{y'_1, y'_2, y'_3\} \cup$

H_2 satisfies condition (iii), then $H_1 \cup H_2 \cup \{x'_1, x'_2, x'_3, y'_1, y'_2, y'_3\}$ is a prism or a line wheel, a contradiction. Therefore, up to symmetry, neither $\{x'_1, x'_2, x'_3\} \cup H_1$ nor $\{y'_1, y'_2, y'_3\} \cup H_2$ satisfy condition (i).

Suppose that $\{x'_1, x'_2, x'_3\} \cup H_1$ satisfies condition (ii) of Lemma 3.2. If $\{y'_1, y'_2, y'_3\} \cup H_2$ satisfies condition (ii), then $H_1 \cup H_2 \cup \{x'_1, x'_2, x'_3, y'_1, y'_2, y'_3\}$ is a theta in $\beta_{\mathcal{S}}$, a contradiction. If $\{y'_1, y'_2, y'_3\} \cup H_2$ satisfies condition (iii), then $H_1 \cup H_2 \cup \{x'_1, x'_2, x'_3, y'_1, y'_2, y'_3\}$ is a pyramid, a contradiction. Therefore, up to symmetry, neither $\{x'_1, x'_2, x'_3\} \cup H_1$ nor $\{y'_1, y'_2, y'_3\} \cup H_2$ satisfy condition (ii).

It follows that $\{x'_1, x'_2, x'_3\} \cup H_1$ and $\{y'_1, y'_2, y'_3\} \cup H_2$ satisfy condition (iii). But now $H_1 \cup H_2 \cup \{x'_1, x'_2, x'_3, y'_1, y'_2, y'_3\}$ is a prism or a line wheel, a contradiction. This proves (4).

Let $w_H : V(H) \to [0,1]$ be a weight function on H defined as follows: $w_H(k_i) = w_{\mathcal{S}}(K_i)$ for $1 \leq i \leq t$, and $w_H(d_i) = w_{\mathcal{S}}(D_i)$ for $1 \leq i \leq m$. Let $\overline{w_H}$ be a weight function on H such that $\overline{w_H}(u) = \frac{w_H(u)}{w_H(H)}$ for every $u \in H$. Note that $w_H(H) \leq 1$, so for all $X \subseteq V(H)$, it holds that $\overline{w_H}(X) \geq w_H(X)$. Note also that $\overline{w_H}(H) = 1$.

(5) *$\beta_{\mathcal{S}}$ has a $(w_{\mathcal{S}}, c)$-balanced separator of size at most $6\omega(\beta_{\mathcal{S}}) + C$.*

Since H is (theta, triangle, wheel)-free, it follows from Theorem 3.3 and Lemma 2.2 that H has a $(\overline{w_H}, \frac{1}{2})$-balanced separator X of size at most 3. Let $Y = \{v\} \cup (N(v) \cap \text{Hub}(\beta_{\mathcal{S}})) \cup \{K_i \text{ s.t. } X \cap N_H[k_i] \neq \emptyset\}$. By assumption of the lemma, $N(v) \cap \text{Hub}(\beta_{\mathcal{S}}) < C$, and by Lemma 3.5, it holds that $|\{K_i \text{ s.t. } X \cap N_H[k_i] \neq \emptyset\}| \leq 2\omega(\beta_{\mathcal{S}})|X| \leq 6\omega(\beta_{\mathcal{S}})$. Therefore, $|Y| \leq 6\omega(\beta_{\mathcal{S}}) + C$.

We claim that Y is a $(w_{\mathcal{S}}, c)$-balanced separator of $\beta_{\mathcal{S}}$. Let F be a component of $\beta_{\mathcal{S}} \setminus Y$. Note that by construction of Y, it holds that $F \subseteq \bigcup_{1 \leq i \leq t} K_t \cup \bigcup_{1 \leq i \leq m} D_i$, that if $K_i \cap F \neq \emptyset$, then $K_i \subseteq F$, and if $D_i \cap F \neq \emptyset$, then $D_i \subseteq F$. Let $K_F = \{i \text{ s.t. } K_i \subseteq F\}$ and $D_F = \{i \text{ s.t. } D_i \subseteq F\}$. If $F = D_i$ for some $1 \leq i \leq m$, then it follows from (3) that $w_{\mathcal{S}}(F) \leq \frac{1}{2}$, so we may assume that $K_F \neq \emptyset$. Let $i \in D_F$ and suppose that $d_i \in X$. Then, $N(D_i) \subseteq Y$, so $K_F = \emptyset$, a contradiction. Therefore, $d_i \notin X$. Similarly, for $i \in K_F$, it holds that $k_i \notin X$, since $K_i \not\subseteq Y$. Let $Q = \{k_i \text{ s.t. } i \in K_F\} \cup \{d_i \text{ s.t. } i \in D_F\}$. Then, Q is contained in a connected component of $H \setminus X$, so $\overline{w_H}(Q) \leq \frac{1}{2}$. Finally,

$$w_{\mathcal{S}}(F) = \sum_{i \in K_F} w_{\mathcal{S}}(K_i) + \sum_{i \in D_F} w_{\mathcal{S}}(D_i) = w_H(Q) \leq \overline{w_H}(Q) \leq \frac{1}{2}.$$

This proves (5).

Now, Lemma 3.6 follows from (5). ■

Recall that $\mathcal{C}_t^*$ is the class of (C_4, diamond, theta, prism, even wheel, K_t)-free graphs with no clique cutset. Let $G \in \mathcal{C}_t^*$, let $w : V(G) \to [0,1]$ be a weight function on G with $w(G) = 1$, and let U be the set of unbalanced vertices of G. Let $X \subseteq U$. The *X-revised collection of separations*, denoted $\tilde{\mathcal{S}}_X$, is defined as follows. Let $u \in X$, and let $\tilde{S}_u = (\tilde{A}_u, \tilde{C}_u, \tilde{B}_u)$ be such that $\tilde{B}_u$ is the largest weight connected component of $G \setminus N[u]$, $\tilde{C}_u = C_u \cup \bigcup_{v \in (C_u \setminus \{u\}) \cap X} (N(u) \cap N(v))$, and $\tilde{A}_u = V(G) \setminus (\tilde{C}_u \cup \tilde{B}_u)$. Then, $\tilde{\mathcal{S}}_X = \{\tilde{S}_u : u \in X\}$. Note that the separations in $\tilde{\mathcal{S}}_X$ are closely related to canonical star separations. Specifically, for all $u \in X$, the following hold:

(i) $\tilde{B}_u = B_u$,
(ii) $C_u \subseteq \tilde{C}_u \subseteq N[u]$,
(iii) $\tilde{A}_u \subseteq A_u$,
(iv) $A_u \setminus N(u) \subseteq \tilde{A}_u$.

Lemma 3.7. *Let $G \in \mathcal{C}_t^*$, let $w : V(G) \to [0,1]$ be a weight function on G with $w(G) = 1$, let U be the set of unbalanced vertices of G, and let $X \subseteq U$ be the set of unbalanced vertices that are minimal under the relation $\leq_A$. Let $\tilde{\mathcal{S}} = \tilde{\mathcal{S}}_X$ be the X-revised collection of separations. Then, $\tilde{S}_u$ and $\tilde{S}_v$ are nearly non-crossing for all $\tilde{S}_u, \tilde{S}_v \in \tilde{\mathcal{S}}$.*

Proof. Since $\tilde{A}_x \subseteq A_x$ for all $x \in U$, it follows that either u and v are adjacent, $u \in \tilde{C}_v$, and $v \in \tilde{C}_u$, or u and v are non-adjacent, $u \in \tilde{B}_v$, and $v \in \tilde{B}_u$. Suppose first that u and v are non-adjacent. Then, $u \in \tilde{B}_v$, and $v \in \tilde{B}_u$. Since u is complete to $\tilde{C}_u \setminus \{u\}$, it follows that $\tilde{C}_u \subseteq \tilde{B}_v \cup \tilde{C}_v$, so $\tilde{A}_v \cap \tilde{C}_u = \emptyset$. By symmetry, $\tilde{A}_u \cap \tilde{C}_v = \emptyset$. It follows that every component of $\tilde{A}_u \cup \tilde{A}_v$ is a component of $\tilde{A}_u$ or a component of $\tilde{A}_v$, so $\tilde{S}_u$ and $\tilde{S}_v$ are nearly non-crossing.

Now suppose that u and v are adjacent. Then, $u \in \tilde{C}_v$, and $v \in \tilde{C}_u$. We may assume that there is a component of $\tilde{A}_u \cup \tilde{A}_v$ that is not a component of $\tilde{A}_u$ or a component of $\tilde{A}_v$. Therefore, there exists $u' \in \tilde{C}_u \cap \tilde{A}_v$, $v' \in \tilde{C}_v \cap \tilde{A}_u$, and a path P from u' to v' with $P^* \subseteq \tilde{A}_u \cap \tilde{A}_v$. We claim that $u' \in C_u$ and $v' \in C_v$. Suppose that $u' \in \tilde{C}_u \setminus C_u$. Then, there exists $x \in X \cap C_u$ such that u' is complete to $\{u, x\}$. If v is adjacent to u', then $u' \in \tilde{C}_v$ (since u' is a common neighbor of u and v), a contradiction, so v is not adjacent to u' and consequently $v \neq x$. Then, since $x \in X$ and x has a neighbor in $\tilde{A}_v \subseteq A_v$, it follows that $x \in C_v$, so x is adjacent to v. But now $uu'xv$ is a diamond, a contradiction. This proves that $u' \in C_u$ and similarly $v' \in C_v$.

Since $u, v \in U$ and $\tilde{B}_u = B_u$ and $\tilde{B}_v = B_v$, it follows that $w(\tilde{B}_u) \geq \frac{1}{2}$ and $w(\tilde{B}_v) \geq \frac{1}{2}$, and so $\tilde{B}_u \cap \tilde{B}_v \neq \emptyset$. Let $b \in \tilde{B}_u \cap \tilde{B}_v$. Since $u' \in C_u$, there exists a path Q_1 from u' to b with $Q_1 \setminus \{u'\} \subseteq \tilde{B}_u$. Similarly, there exists a path Q_2 from v' to b with $Q_2 \setminus \{v'\} \subseteq \tilde{B}_v$. Now, there exists a path Q from u' to v' with $Q \subseteq Q_1 \cup Q_2$, so in particular, $Q^* \subseteq \tilde{B}_u \cup \tilde{B}_v$. Note that $Q^* \cap (C_v \cap B_u) \neq \emptyset$ and $Q^* \cap (C_u \cap B_v) \neq \emptyset$, so in particular, Q^* contains a neighbor of u and a neighbor of v.

Let H be the hole given by $P \cup Q$. Since u is adjacent to u' and has a neighbor in Q^*, it follows that u has two non-adjacent neighbors in H. Similarly, v has two non-adjacent neighbors in H. Now, u and v are adjacent and each have two non-adjacent neighbors in H, so by Lemma 2.6, u and v have a common neighbor in H. However, since $u \in \tilde{C}_v$ and $v \in \tilde{C}_u$, it follows that $N(u) \cap N(v) \subseteq \tilde{C}_u \cap \tilde{C}_v$. Since $H \cap (\tilde{C}_u \cap \tilde{C}_v) = \emptyset$, we get a contradiction. ∎

Next, we show how to construct a useful collection of separations of G. We need a lemma which relies on the following result from [12].

Theorem 3.8 ([12])**.** *For every graph H and integer s there exists $d = d(H, s)$ such that every graph G of average degree at least d contains either a $K_{s,s}$ as a subgraph or an induced subdivision of H.*

Theorem 3.8 has the following corollary.

Lemma 3.9. *Let G be (theta, K_t)-free. Then, there is $\delta_t > 0$ such that G has degeneracy δ_t. In particular, there is an ordering $(v_1, \ldots, v_n)$ of $V(G)$ such that $|N(v_i) \cap \{v_{i+1}, \ldots, v_n\}| \leq \delta_t$ for all $1 \leq i \leq n-1$.*

We call δ_t the *hub constant for t*. Let $G \in \mathcal{C}_t$. We apply Lemma 3.9 to Hub(G). Let $v_1, \ldots, v_k$ be an ordering of Hub(G) such that for all $1 \leq i < j \leq k$, it holds that $|N(v_i) \cap \{v_{i+1}, \ldots, v_k\}| \leq \delta_t$ for all $1 \leq i \leq k-1$. Let U be the set of unbalanced vertices of G. Let m be defined as follows. If Hub$(G) \subseteq U$, then $m = k+1$. Otherwise, let m be such that v_m is the minimum element of Hub$(G) \setminus U$. Now, $\{v_1, \ldots, v_{m-1}\} \subseteq U$. Let M be the set of minimal vertices of $\{v_1, \ldots, v_{m-1}\}$ under the relation $\leq_A$, and let $\tilde{S}_M$ be the M-revised collection of separations. We call $(\{v_1, \ldots, v_k\}, m, M, \tilde{S}_M)$ the *hub division* of G. The next two lemmas describe properties of the hub division.

Lemma 3.10. *Let $G \in \mathcal{C}_t^*$, let $w : V(G) \to [0, 1]$ be a weight function on G with $w(G) = 1$, and let $(\{v_1, \ldots, v_k\}, m, M, \tilde{S}_M)$ be the hub division of G. Then, $\tilde{S}_M$ is a smooth collection of separations of G.*

Proof. By Lemma 3.7, it follows that S_1 and S_2 are nearly non-crossing for every distinct $S_1, S_2 \in \mathcal{S}_M$. By construction of $\tilde{S}_M$, there exists a set of vertices $v(\tilde{S}_M) = M$ such that there is a bijection f from $v(\tilde{S}_M)$ to $\tilde{S}_M$ with $v \in C(f(v)) \subseteq N[v]$, given by $f(x) = \tilde{S}_x$ for every $x \in M$. Finally, since M is minimal under the relation $\leq_A$ and $\tilde{A}_x \subseteq A_x$ for every $x \in M$, it holds that $M \cap A(\tilde{S}_x) = \emptyset$ for all $x \in M$. Therefore, $\tilde{S}_M$ is a smooth collection of separations of G. ∎

By Lemma 3.10, there is a central bag β_M for $\tilde{S}_M$ and an inherited weight function w_M on β_M.

Lemma 3.11. *Let $G \in \mathcal{C}_t^*$, let $w : V(G) \to [0, 1]$ be a weight function on G with $w(G) = 1$, and let $(\{v_1, \ldots, v_k\}, m, M, \tilde{S}_M)$ be the hub division of G. Let β_M be the central bag for $\tilde{S}_M$ and let w_M be the inherited weight function on β_M. Then, for all $1 \leq i \leq m-1$, v_i is not a proper wheel center of β_M.*

Proof. Let $1 \leq i \leq m-1$ and suppose that (H, v_i) is a wheel of β_M.

(6) *$H \cap A_{v_j} \neq \emptyset$ for some $v_j \in M$.*

By Lemma 2.8, $H \not\subseteq B_{v_i} \cup C_{v_i}$, so $H \cap A_{v_i} \neq \emptyset$. If $v_i \in M$, then $j = i$ satisfies the statement, so we may assume $v_i \notin M$. Then, there exists $v_j \in M$ such that $v_j \leq_A v_i$, so $v_i \in A_{v_j}$. Now, by Lemma 2.3, $A_{v_i} \subseteq A_{v_j}$, and so $H \cap A_{v_j} \neq \emptyset$. This proves (6).

By (6), there exists $v_j \in M$ such that $A_{v_j} \cap H \neq \emptyset$. Let $x \in A_{v_j} \cap H$. Since $v_j \in M$, it follows that $\beta_M \subseteq \tilde{B}_{v_j} \cup \tilde{C}_{v_j}$, and so $x \in \tilde{C}_{v_j}$. Let x' and x'' be the neighbors of x in H. Since A_{v_j} is anticomplete to B_{v_j} and thus $\tilde{B}_{v_j}$, it holds that $x', x'' \in \tilde{C}_{v_j}$. But v_j is complete to $\tilde{C}_{v_j}$, and so $\{v_j, x, x', x''\}$ is a diamond, a contradiction. ■

Let $R(t, s)$ denote the minimum integer such that every graph on at least $R(t, s)$ vertices contains either a clique of size t or an independent set of size s.

Theorem 3.12. *Let G be a (theta, pyramid, prism, wheel, K_t)-free graph and let $w : V(G) \to [0, 1]$ be a weight function on G. Then, G has a $(w, \frac{1}{2})$-balanced separator of size at most $R(t, 4)$.*

Proof. By Theorems 4.4 and 2.3 of [3], it follows that $\text{tw}(G) \leq R(t, 4) + |\text{Hub}(G)| - 1$. Since G is wheel-free, it follows that $|\text{Hub}(G)| = 0$. Therefore, $\text{tw}(G) \leq R(t, 4) - 1$. By Lemma 2.2, G has a $(w, \frac{1}{2})$-balanced separator of size at most $R(t, 4)$. ■

Finally, we prove the main result of this section: that if β_M is pyramid-free, then β_M has a balanced separator of bounded size.

Theorem 3.13. *Let $G \in \mathcal{C}_t^*$, let $w : V(G) \to [0, 1]$ be a weight function on G, and let $(\{v_1, \dots, v_k\}, m, M, \tilde{S}_M)$ be the hub division of G. Let β_M be the central bag for $\tilde{S}_M$ and let w_M be the inherited weight function on β_M. Assume that β_M is pyramid-free. Then, β_M has a $(w_M, \frac{1}{2})$-balanced separator of size at most $\max(R(t, 4) + 1, 6t + \delta_t)$.*

Proof. First, suppose that $m = k + 1$. Then, by Lemma 3.11, v is not a proper wheel center of β_M for all $v \in \text{Hub}(G)$. Since $\text{Hub}(\beta_M) \subseteq \text{Hub}(G)$, it follows that β_M is wheel-free. By Theorem 3.12, β_M has a $(w_M, \frac{1}{2})$-balanced separator of size at most $R(t, 4) + 1$.

Now, assume $m < k + 1$. We claim that $v_m \in \beta_M$. Suppose that $v_m \in A_{v_i}$ for some $v_i \in M$. Then, $N[v_m] \subseteq A_{v_i} \cup C_{v_i}$, so B_{v_i} is contained in a connected component D of $G \setminus N[v_m]$. Since $v_i \in M$, it follows that v_i is unbalanced, so $w(B_{v_i}) \geq \frac{1}{2}$. But now $w(D) \geq \frac{1}{2}$, so v_m is unbalanced, a contradiction. Therefore, $v_m \notin A_{v_i}$ for all $v_i \in M$. Since for all $v_i \in M$ it holds that $\tilde{A}_{v_i} \subseteq A_{v_i}$, it follows that $v_m \in \tilde{B}_{v_i} \cup \tilde{C}_{v_i}$, and so $v_m \in \beta_M$.

Next, consider $N(v_m) \cap \text{Hub}(\beta_M)$. By Lemma 3.11, $\text{Hub}(\beta_M) \subseteq \{v_m, v_{m+1}, \dots, v_k\}$. Therefore, $|N(v_m) \cap \text{Hub}(\beta_M)| \leq \delta_t$. Finally, by Lemma 3.10, $\tilde{S}_M$ is a smooth collection of separations of G. Now, by Lemma 3.6, β_M has a $(w_M, \frac{1}{2})$-balanced separator of size $6\omega(\beta_M) + \delta_t$. ■

4. Extending balanced separators

In this section, we prove that we can construct a bounded balanced separator of G given a bounded balanced separator of β_M. Together with the main result of the previous section, this is sufficient to prove Theorem 1.3. First, we need the following lemma.

Lemma 4.1. *Let $G \in \mathcal{C}_t^*$, let $w : V(G) \to [0, 1]$ be a weight function on G with $w(G) = 1$, and let $(\{v_1, \dots, v_k\}, m, M, \tilde{S}_M)$ be the hub division of G. Let β_M be the central bag for $\tilde{S}_M$. Let $v \in M$ be such that v is not a pyramid apex in β. Then, $|N_{\beta_M}(v) \setminus \text{Hub}(\beta_M)| \leq 2t$.*

Proof. By Lemma 3.4, let $N_{\beta_M}(v) \setminus \text{Hub}(\beta_M) = K_1 \cup \dots \cup K_\ell$, where $K_1, \dots, K_\ell$ are cliques and K_i and K_j are anticomplete to each other for $1 \leq i < j \leq \ell$. Recall that B_v is connected and $C_v \setminus \{v\} = N(B_v)$.

(7) *Let $k_1 \in K_1 \cap C_v$ and $k_2 \in K_2 \cap C_v$ be chosen such that the path P from k_1 to k_2 with interior in B_v is as short as possible among choices of $k_1 \in K_1$, $k_2 \in K_2$. Then, $P^* \subseteq \beta_M \setminus N[v]$.*

Suppose for a contradiction that $P^* \not\subseteq \beta_M$. Assume that P is chosen such that $P^* \setminus \beta_M$ is minimal. Then, there exists $v_i \in M$ such that $P^* \cap \tilde{A}_{v_i} \neq \emptyset$. Let a, b be the vertices of $\tilde{C}_{v_i} \cap P^*$ closest to k_1, k_2, respectively. Since there is a vertex of $\tilde{A}_{v_i}$ on the path from a to b through P^*, it follows that a is not adjacent to b. If v_i is not adjacent to v, then $P' = k_1$-P-a-v_i-b-P-k_2 is a path from k_1 to k_2 with $P'^* \subseteq B_v$. Since P is the shortest such path, it follows that the subpath from a to b in P is of length two. Let this subpath be a-u-b. But now $\{u, v_i, a, b\}$ is a C_4 or a diamond, a contradiction. Therefore, v_i is adjacent to v. Let H be the hole given by $\{v\} \cup P$. Now, v_i has at least three neighbors a, b, v in H. Suppose that v_i is adjacent to k_1. Then, $v_i \in K_1$, so v_i is not adjacent to k_2, and v_i-b-P-k_2 is a shorter path from a vertex of K_1 to a vertex of K_2, a contradiction. It follows that v_i is anticomplete to $\{k_1, k_2\}$.

By Lemma 2.7, $\{k_1, k_2\} \cap A_{v_i} \neq \emptyset$, and since v_i is complete to $\tilde{C}_{v_i}$ and anticomplete to $\{k_1, k_2\}$, it follows that $\{k_1, k_2\} \cap \tilde{A}_{v_i} \neq \emptyset$. But $\{k_1, k_2\} \subseteq \beta_M$, a contradiction. This proves (7).

(8) *There is a component D of $\beta_M \setminus N[v]$ such that for every $i \in \{1, \ldots, \ell\}$, some vertex of $K_i \cap C_v$ has a neighbor in D.*

Let D be a component of $\beta_M \setminus N[v]$ with neighbors in as many cliques K_i as possible. Let $K_1, \ldots, K_j$ be the cliques with a neighbor in D. By the definition of $\tilde{C}_v$ and since $N_{\beta_M}(v) \subseteq \tilde{C}_v$, for every clique K_i, it holds that $K_i \cap C_v \neq \emptyset$. Suppose for a contradiction that $j < \ell$. By (7), there is a path Q from K_1 to K_{j+1} with interior in $\beta_M \setminus N[v]$. Let D' be the component of $\beta_M \setminus N[v]$ containing Q^*, so $D \neq D'$. By the maximality of D, we may assume D' does not have a neighbor in K_2. By (7), there is also a path R from K_{j+1} to K_2 with interior in $\beta_M \setminus N[v]$. Let D'' be the component of $\beta_M \setminus N[v]$ with $R^* \subseteq D''$. Then $D'' \neq D, D'$. Let P be a path from K_1 to K_2 with interior in D. Since P, Q, and R are in distinct components of $\beta_M \setminus N[v]$, it follows that $P \cup Q \cup R$ is a hole of β_M. Now, $(P + Q + R, v)$ is a wheel in β_M, contrary to Lemma 3.11. This proves (8).

(9) *$K_i \cap C_v \neq \emptyset$ if and only if $K_i \cap \tilde{C}_v \neq \emptyset$.*

Since $C_v \subseteq \tilde{C}_v$, it holds that if $K_i \cap C_v \neq \emptyset$, then $K_i \cap \tilde{C}_v \neq \emptyset$. Suppose that $K_i \cap (\tilde{C}_v \setminus C_v) \neq \emptyset$ and let $x \in K_i \cap (\tilde{C}_v \setminus C_v)$. By the definition of $\tilde{C}_v$, it follows that there exists $u \in C_v \cap M$ such that $x \in N(u) \cap N(v)$. Recall that $N_{\beta_M}(v) \setminus \mathrm{Hub}(\beta_M) = K_1 \cup \ldots \cup K_\ell$, where $K_1, \ldots, K_\ell$ are cliques and K_i and K_j are anticomplete to each other for $1 \leq i < j \leq \ell$. Also, by Lemma 3.11, $M \subseteq \beta_M \setminus \mathrm{Hub}(\beta_M)$. Since u and x are adjacent and $\{u, x\} \subseteq N_{\beta_M}(v) \setminus \mathrm{Hub}(\beta_M)$, it follows that $u \in K_i$. But now $u \in C_v \cap K_i$, so $K_i \cap C_v \neq \emptyset$. This proves (9).

Let $I = \{1 \leq i \leq \ell : K_i \cap C_v \cap \beta_M \neq \emptyset\}$. By (8), there exists a component D of $\beta_M \setminus N[v]$ such that K_i has a neighbor in D for every $i \in I$. By Lemma 3.5, at most two of $K_1, \ldots, K_\ell$ have a neighbor in D. Therefore, $|I| \leq 2$. Since G is diamond-free and K_t-free, since $N(v) \cap \beta_M \subseteq \tilde{C}_v$, and by (9), it follows that $|N_{\beta_M}(v) \setminus \mathrm{Hub}(\beta_M)| \leq 2t$. This completes the proof. ∎

Now we prove the main result of this section.

Theorem 4.2. *Let $G \in \mathcal{C}_t^*$, let $w : V(G) \to [0, 1]$ be a weight function on G, and let $(\{v_1, \ldots, v_k\}, m, M, \tilde{S}_M)$ be the hub division of G. Let δ_t be the hub constant for t. Let β_M be the central bag for $\tilde{S}_M$ and let w_M be the inherited weight function on β_M. Assume that β_M has a $(w_M, \frac{1}{2})$-balanced separator of size C and that no vertex of M is a pyramid apex in β_M. Then, G has a $(w, \frac{1}{2})$-balanced separator of size $(2t + \delta_t + 1)C$.*

Proof. Let X be a $(w_M, \frac{1}{2})$-balanced separator of β_M with $|X| \leq C$. By Lemma 2.5, it follows that $X \cup (N[X \cap M] \cap \beta_M)$ is a $(w, \frac{1}{2})$-balanced separator of G. Let $u \in X \cap M$. By Lemma 4.1, it follows that $|N_{\beta_M}(u) \setminus \mathrm{Hub}(\beta_M)| \leq 2t$. By Lemma 3.11, $\mathrm{Hub}(\beta_M) \subseteq \{v_m, v_{m+1}, \ldots, v_k\}$, so it follows that $|N_{\beta_M}(u) \cap \mathrm{Hub}(\beta_M)| \leq \delta_t$. Therefore, $|X \cup (N[X \cap M] \cap \beta_M)| \leq (2t + \delta_t + 1)|X|$. ∎

Finally, we restate and prove Theorem 1.3.

Theorem 4.3. *Let $G \in \mathcal{C}_t$. Then, $\mathrm{tw}(G) \leq (4t + 2\delta_t + 2) \cdot \max(R(t, 4) + 1, 6t + \delta_t)$.*

Proof. By Lemma 3.1, we may assume that G has no clique cutset, so $G \in \mathcal{C}_t^*$. Let $w : V(G) \to [0, 1]$ be a weight function on G with $w(G) = 1$, and let $(\{v_1, \ldots, v_k\}, m, M, \tilde{S}_M)$ be the hub division of G. Let β_M be the central bag for $\tilde{S}_M$ and let w_M be the inherited weight function on β_M. By Theorem 3.13, β_M has a $(w_M, \frac{1}{2})$-balanced separator of size at most $\max(R(t, 4) + 1, 6t + \delta_t)$. Now, by Theorem 4.2, G has a $(w, \frac{1}{2})$-balanced separator of size $(2t + \delta_t + 1) \cdot \max(R(t, 4) + 1, 6t + \delta_t)$. Finally, by Lemma 2.1, $\mathrm{tw}(G) \leq (4t + 2\delta_t + 2) \cdot \max(R(t, 4) + 1, 6t + \delta_t)$. ∎

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