

Polynomial bounds for chromatic number VII. Disjoint holes

Maria Chudnovsky¹
Princeton University, Princeton, NJ 08544

Alex Scott²
Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK

Paul Seymour³
Princeton University, Princeton, NJ 08544

Sophie Spirkl⁴
University of Waterloo, Waterloo, Ontario N2L3G1, Canada

November 1, 2021; revised February 20, 2023

¹Supported by NSF grant DMS-2120644.

²Research supported by EPSRC grant EP/V007327/1.

³Supported by AFOSR grants A9550-19-1-0187 and FA9550-22-1-0234, and NSF grant DMS-2154169.

⁴We acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC), [funding reference number RGPIN-2020-03912]. Cette recherche a été financée par le Conseil de recherches en sciences naturelles et en génie du Canada (CRSNG), [numéro de référence RGPIN-2020-03912].

Abstract

A *hole* in a graph G is an induced cycle of length at least four, and a k -*multihole* in G is the union of k pairwise disjoint and non-neighbouring holes. It is well known that if G does not contain any holes then its chromatic number is equal to its clique number. In this paper we show that, for any integer $k \geq 1$, if G does not contain a k -multihole, then its chromatic number is at most a polynomial function of its clique number. We show that the same result holds if we ask for all the holes to be odd or of length four; and if we ask for the holes to be longer than any fixed constant or of length four. This is part of a broader study of graph classes that are polynomially χ -bounded.

1 Introduction

A function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ is a *binding function* for a graph G if $\chi(G) \leq \phi(\omega(G))$, where $\chi(G), \omega(G)$ denote the chromatic number of G and the size of the largest clique of G , respectively. A class \mathcal{C} of graphs is *hereditary* if for every $G \in \mathcal{C}$, every graph isomorphic to an induced subgraph of G also belongs to \mathcal{C} . A hereditary class \mathcal{C} is χ -*bounded* if there is a function ϕ that is a binding function for each $G \in \mathcal{C}$, and if so, we call ϕ a *binding function* for the class; if there exists a polynomial binding function, we say that \mathcal{C} is *poly- χ -bounded* (see [12] for a survey on χ -bounded classes, and [9] on poly- χ -bounded classes). While many classes are known to be χ -bounded, the proofs frequently give quite fast-growing functions, and it is natural to ask whether this is necessary. A remarkable conjecture of Louis Esperet [6] asserted that every χ -bounded hereditary class is poly- χ -bounded. But this was recently disproved by Brianiński, Davies and Walczak [2]. So the question now is: which hereditary classes are poly- χ -bounded?

A hereditary graph class is defined by excluding some induced subgraphs. A graph is *H-free* if it has no induced subgraph isomorphic to H , and $\{H_1, H_2\}$ -*free* means both H_1 -free and H_2 -free. There is a mass of results on χ -bounded classes where one of the excluded graphs is a forest, but in this paper we consider some classes where every excluded graph has a cycle. A *hole* is an induced cycle of length at least four, and *odd-hole-free* means containing no odd hole. A *four-hole* means a hole of length four. Let us say a *k-multihole* of a graph G is an induced subgraph with k components, each a cycle of length at least four. We denote the k -vertex path by P_k and the k -vertex cycle by C_k .

Graphs with no 1-multihole are chordal and hence perfect. The class of graphs with no k -multihole in which all the cycles have odd length, is shown in [10] to be χ -bounded, but it contains the class of $\{P_5, C_5\}$ -free graphs, and we cannot yet prove it is poly- χ -bounded (see [16] for the best current bounds). If we replace “odd” by “long”, the same applies: it is shown in [11] that for every $\ell \geq 0$, the class of graphs with no k -multihole in which all the cycles have length at least ℓ is χ -bounded (and we cannot yet prove it is poly- χ -bounded, for the same reason). But we can if we permit cycles of length four to be components of the multiholes we are excluding. We will show:

1.1 *For each integer $k \geq 1$, let \mathcal{C} be the class of all graphs G with no k -multihole in which every component either has length four or odd length. Then \mathcal{C} is poly- χ -bounded.*

Incidentally, there is a similar-looking theorem due to Z. Dvořák and J. Pekárek [5], the following:

1.2 *For each integer $k \geq 1$, let \mathcal{C} be the class of all graphs G with no induced subgraph that consists of k cycles of odd length. Then \mathcal{C} is poly- χ -bounded.*

But here the cycles of odd length may have length three, and this makes a huge difference.

If we change “odd” to “long”, there is a result parallel to 1.1:

1.3 *For all integers $k \geq 1$ and $\ell \geq 4$, let \mathcal{C} be the class of all graphs G with no k -multihole in which every component either has length four or length at least ℓ . Then \mathcal{C} is poly- χ -bounded.*

This second one we can make stronger (we could not prove the corresponding strengthening of the first):

1.4 *For all integers $k, s \geq 1$, and $\ell \geq 4$, let \mathcal{C} be the class of all graphs G such that no induced subgraph of G has exactly k components, each of which is either isomorphic to $K_{s,s}$ or a cycle of length at least ℓ . Then \mathcal{C} is poly- χ -bounded.*

(In general, $K_{s,t}$ denotes the complete bipartite graph with parts of cardinality s and t .) Both these results derive from a theorem about $K_{s,s}$, which we will explain in the next section.

2 Excluding a disjoint union, and self-isolation

If $A \subseteq V(G)$, $G[A]$ denotes the subgraph of G induced on A ; and we write $\chi(A)$ for $\chi(G[A])$ and $\omega(A)$ for $\omega(G[A])$. Two disjoint subsets of $V(G)$ are *anticomplete* if there are no edges between them, and *complete* if every vertex of the first subset is adjacent to every vertex of the second. A graph G *contains* a graph H if some induced subgraph of G is isomorphic to H , and such a subgraph is a *copy* of H . A function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ is *non-decreasing* if $\phi(x) \leq \phi(y)$ for all $x, y \in \mathbb{N}$ with $x \leq y$.

Let us say a graph H is *self-isolating* if for every non-decreasing polynomial $\psi : \mathbb{N} \rightarrow \mathbb{N}$, there is a polynomial $\phi : \mathbb{N} \rightarrow \mathbb{N}$ with the following property. For every graph G with $\chi(G) > \phi(\omega(G))$, there exists $A \subseteq V(G)$ with $\chi(A) > \psi(\omega(A))$, such that either

- $G[A]$ is H -free, or
- G contains a copy H' of H such that $V(H')$ is disjoint from and anticomplete to A .

Self-isolation is of interest in considering polynomial χ -boundedness for the class of H -free graphs, where H is a forest. Say a forest H is *good* if the class of H -free graphs is polynomially χ -bounded. It might be true that every forest is good (strengthening the Gyárfás-Sumner conjecture [7, 17] from χ -boundedness to polynomial χ -boundedness), but this has only been proved for a few simple kinds of tree H , and some (not all) of the forests that are disjoint unions of these trees. It is not known that if trees H_1, H_2 are good, then the disjoint union of H_1 and H_2 is good. For instance, trees of diameter three are good [15], but disjoint unions of them might not be as far as we know. But self-isolation helps here: if H_1 and H_2 are good forests, and one of them is self-isolating, then the disjoint union of H_1 and H_2 is good. Some good trees are known to be self-isolating (namely, stars and four-vertex paths), so we can happily take disjoint unions with them and preserve goodness.

Which graphs are self-isolating? We know very little at the moment: there are very few graphs that we know to have the property, and none that we know not to have the property. (Could it be that all graphs are self-isolating? Certainly, if we change the definition of self-isolating, replacing the polynomials ϕ, ψ by general functions, it is easy to show that all graphs have the property, by induction on $\omega(G)$.) A graph is self-isolating if all its components are self-isolating, but the only connected graphs that we know are self-isolating are complete graphs (proved below), paths of arbitrary length (proved in [4]), and complete bipartite graphs (proved in the next section). The main result of [14] was that stars are self-isolating, so our result that complete bipartite graphs are self-isolating generalizes this. The last takes up the main part of this paper, and is most of what we need to prove 1.1 and 1.4.

First, complete isolation:

2.1 Every complete graph is self-isolating.

Proof. (This proof was derived from a similar proof in [8].) Let $\psi : \mathbb{N} \rightarrow \mathbb{N}$ be a non-decreasing polynomial, and let H be a k -vertex complete graph. Let ϕ be the polynomial $\phi(x) = (x+1)^k \psi(x) + x$ for $x \in \mathbb{N}$. Now let G be a graph with chromatic number more than $\phi(\omega(G))$, and let K be a clique of G with cardinality $\omega(G)$. If $\omega(G) < k$, then the first bullet in the definition of self-isolating holds, so

we assume that $\omega(G) \geq k$. For each $X \subseteq K$ with $|X| = k$, let A_X be the set of vertices in $V(G) \setminus K$ that are nonadjacent to every vertex in X ; and for every $Y \subseteq K$ with $|Y| = k - 1$, let B_Y be the set of vertices in $V(G) \setminus K$ that are adjacent to every vertex in $K \setminus Y$. Thus $V(G) \setminus K$ is the union of the $\binom{\omega(G)}{k}$ sets A_X and the $\binom{\omega(G)}{k-1}$ sets B_Y ; and since

$$\binom{\omega(G)}{k} + \binom{\omega(G)}{k-1} = \binom{\omega(G)+1}{k} \leq (\omega(G)+1)^k,$$

and $\chi(G \setminus K) > (\omega(G)+1)^k \psi(\omega(G))$, one of the sets A_X or B_Y has chromatic number more than $\psi(\omega(G))$. If $\chi(A_X) > \psi(\omega(G))$ for some X , then $G[X]$ is a copy of H anticomplete to A_X , and since $\psi(\omega(G)) \geq \psi(\omega(A_X))$, the second bullet in the definition of self-isolating holds. If $\chi(B_Y) > \psi(\omega(G))$ for some Y , then since $|K \setminus Y| = \omega(G) - k + 1$ and B_Y is complete to $K \setminus Y$, it follows that $\omega(B_Y) < k$ and so $G[B_Y]$ is H -free, and the first bullet in the definition of self-isolating holds. This proves 2.1. ■

3 Complete bipartite isolation

We turn to the proof that

3.1 Every complete bipartite graph is self-isolating.

We will in fact prove something a little stronger. Let $\psi : \mathbb{N} \rightarrow \mathbb{N}$ be some non-decreasing function. An induced subgraph H of a graph G is ψ -*nondominating* if there exists a set $A \subseteq V(G)$ disjoint from and anticomplete to $V(H)$, with $\chi(A) \geq \psi(\omega(A))$. If $\psi : \mathbb{N} \rightarrow \mathbb{N}$ is a non-decreasing function and $q \geq 0$ is an integer, a (ψ, q) -*sprinkling* in a graph G is a pair (P, Q) of disjoint subsets of $V(G)$, such that

- $\chi(P) > \psi(\omega(P))$; and
- $\chi(Q) > \psi(\omega(Q)) + qr$, where r is the maximum over $v \in P$ of the chromatic number of the set of neighbours of v in Q .

(This is closely related to what was called a “ (ψ, q) -scattering” in [4].) We denote the number of vertices of a graph H by $|H|$. We will prove:

3.2 *Let $s \geq 1$ and $q \geq 0$ be integers, and let $\psi : \mathbb{N} \rightarrow \mathbb{N}$ be a non-decreasing polynomial. Then there is a polynomial $\phi : \mathbb{N} \rightarrow \mathbb{N}$ with the following property. For every graph G with $\chi(G) > \phi(\omega(G))$, either:*

- *there is a ψ -nondominating copy of $K_{s,s}$ in G , or*
- *there is a (ψ, q) -sprinkling in G .*

Proof of 3.1, assuming 3.2. Let $s, s' \geq 1$ be integers, where $s' \leq s$. We will show that $K_{s,s'}$ is self-isolating. (It is not enough to show this when $s = s'$, because we do not know that every induced subgraph of a self-isolating graph is self-isolating.) Let $\psi : \mathbb{N} \rightarrow \mathbb{N}$ be a non-decreasing polynomial, let $q = s + s'$, and let ϕ satisfy 3.2. Let G be a graph with $\chi(G) > \phi(\omega(G))$. We claim that either

there is a ψ -nondominating copy of $K_{s,s'}$ in G , or there exists $A \subseteq V(G)$ with $\chi(A) > \psi(\omega(A))$ such that $G[A]$ is $K_{s,s'}$ -free. If there is a ψ -nondominating copy of $K_{s,s}$ in G , then there is also one of $K_{s,s'}$, so by 3.2, we may assume that there is a (ψ, q) -sprinkling (P, Q) in G . If $G[P]$ is $K_{s,s'}$ -free, the claim holds, so we assume that there is a copy H of $K_{s,s'}$ in $G[P]$. Thus $|H| = q$. Let r be the maximum over $v \in P$ of the chromatic number of the set of neighbours of v in Q . The set of vertices in Q with a neighbour in $V(H)$ has chromatic number at most $|H|r = qr$; and $\chi(Q) > \psi(\omega(Q)) + qr$ from the definition of a (ψ, q) -sprinkling. Consequently H is ψ -nondominating, and hence $K_{s,s'}$ is self-isolating. \blacksquare

To prove 3.2 we will need the following lemma:

3.3 *For every graph G that is not a complete graph, there is a vertex v such that the set of vertices different from and nonadjacent to v has chromatic number at least $\chi(G)/\omega(G)$.*

Proof. Let X be a maximum clique of G , and for each $x \in X$, let D_x be the set of vertices of G different from and nonadjacent to x . Since G is nonnull, it follows that $X \neq \emptyset$. But $V(G)$ is the union of the sets $D_x \cup \{x\}$ over $x \in X$, because of the maximality of X ; and so there exists $v \in X$ such that $\chi(D_v \cup \{v\}) \geq \chi(G)/\omega(G)$. Choose such a vertex v with $D_v \neq \emptyset$ if possible. If $D_v \neq \emptyset$, then $\chi(D_v \cup \{v\}) = \chi(D_v)$, since there are no edges between v and D_v , and so the theorem holds. Thus we may assume (for a contradiction) that $D_v = \emptyset$, and so $1 = \chi(D_v \cup \{v\}) \geq \chi(G)/\omega(G)$. Since $\chi(G)/\omega(G) \geq 1$, equality holds, and so $\chi(D_x \cup \{x\}) \geq \chi(G)/\omega(G)$ for every $x \in X$; and so $D_x = \emptyset$ for all $x \in X$, from the choice of v . Consequently $V(G) = X$, and G is a complete graph, a contradiction. This proves 3.3. \blacksquare

The proof of 3.2 will be by examining the largest “template” in G . Let us say that, for all integers $t, k, s \geq 1$, a (t, k, s) -*template* in G is a sequence (A_1, \dots, A_k) of pairwise disjoint subsets of $V(G)$, each of cardinality t , such that for $1 \leq i < j \leq k$, and for every stable set $S \subseteq A_j$ with $|S| = s$, every vertex in A_i has a neighbour in S . (Thus, this last condition is trivially satisfied if $t < s$.) The next result will enable us to find a $(t, 2, s)$ -template. If $v \in V(G)$, we denote the set of neighbours of a vertex v by $N(v)$ or $N_G(v)$.

3.4 *Let $s, t \geq 1$ and $q \geq 0$ be integers, and let $\psi : \mathbb{N} \rightarrow \mathbb{N}$ be a non-decreasing polynomial. Let G be a graph with*

$$\begin{aligned} \chi(G) &> \omega(G)^s ((s + t^s) \psi(\omega(G)) + t) \quad \text{and} \\ \chi(G) &> q^s t + (2 + q + q^2 + \dots + q^{s-1}) \psi(\omega(G)) + 1. \end{aligned}$$

Then either

- there is a ψ -nondominating copy of $K_{s,s}$ in G , or
- there is a (ψ, q) -sprinkling in G , or
- G contains a $(t, 2, s)$ -template.

Proof. Define $p = \psi(\omega(G))$. For $0 \leq i \leq s$, define

$$\begin{aligned} m_i &= \omega(G)^{s-i} (t^s p + t) + (1 + \omega(G) + \dots + \omega(G)^{s-i-1}) p \\ n_i &= q^{s-i} t + (1 + q + q^2 + \dots + q^{s-i-1}) p. \end{aligned}$$

Thus $m_s = t^s p + t$, and $m_i = \omega(G)m_{i+1} + p$ for $0 \leq i < s$; and $n_s = t$ and $n_i = qn_{i+1} + p$ for $0 \leq i < s$. By hypothesis, $\chi(G) > m_0$ and $\chi(G) > n_0 + p + 1$. We claim we may assume that:

(1) *There is a vertex v_1 such that $\chi(N(v_1)) > n_1$ and $\chi(M(v_1)) > m_1$, where $M(v_1) = V(G) \setminus (N(v_1) \cup \{v_1\})$.*

Let S be the set of all vertices v with $\chi(N(v)) \leq n_1$. If $\chi(S) > p$, choose a subset $P \subseteq S$ with $\chi(P) = p + 1$, and let $Q = V(G) \setminus P$. Then

$$\chi(Q) \geq \chi(G) - (p + 1) > n_0 = p + qn_1,$$

and so (P, Q) is a (ψ, q) -sprinkling. We therefore assume that $\chi(S) \leq p$. Let $R = V(G) \setminus S$. Thus

$$\chi(R) \geq \chi(G) - p > m_0 - p = \omega(G)m_1 \geq \omega(G),$$

and so R is not a clique. By 3.3, there exists $v_1 \in R$ such that the set of vertices in R different from and nonadjacent to v_1 has chromatic number at least $\chi(R)/\omega(G) > m_1$, and so $\chi(M(v_1)) > m_1$. This proves we may assume that (1) holds.

Choose a stable set $S \subseteq V(G)$ with $|S| \leq s$, maximal such that $\chi(N(S)) > n_{|S|}$ and $\chi(M(S)) > m_{|S|}$, where $N(S)$ denotes the set of all vertices in $V(G) \setminus S$ that are adjacent to every vertex in S , and $M(S)$ denotes the set of all vertices in $V(G) \setminus S$ that are nonadjacent to every vertex in S . From (1), $|S| \geq 1$. Now there are two cases, $|S| < s$ and $|S| = s$.

Suppose first that $|S| < s$. Let A be the set of all vertices $v \in M(S)$ such that the set of neighbours of v in $N(S)$ has chromatic number at most $n_{|S|+1}$. Since $\chi(N(S)) > n_{|S|} = qn_{|S|+1} + p$, we may assume that $\chi(A) \leq p$, because otherwise $(A, N(S))$ is a (ψ, q) -sprinkling. Hence

$$\chi(B) \geq \chi(M(S)) - p > m_{|S|} - p = \omega(G)m_{|S|+1},$$

where $B = M(S) \setminus A$. Since $m_{|S|+1} \geq 1$ (because $t \geq 1$), it follows that B is not a clique, and so from 3.3, there is a vertex $v \in B$ such that the set of vertices in B , different from and nonadjacent to v , has chromatic number at least $\chi(B)/\omega(G) > m_{|S|+1}$. But then adding v to S contradicts the maximality of S .

Now suppose that $|S| = s$. Since $\chi(N(S)) > n_s = t$, we may choose $T \subseteq N(S)$ with $|T| = t$. Let A be the set of vertices in $M(S)$ that have s non-neighbours in T that are pairwise nonadjacent, and let $B = M(S) \setminus A$. For each stable set $S' \subseteq T$ with $|S'| = s$, we may assume that the set of vertices in $M(S)$ with no neighbour in S' has chromatic number at most p , because otherwise $G[S \cup S']$ is a ψ -nondominating copy of $K_{s,s}$. The number of such sets S' is at most t^s , and so $\chi(A) \leq t^s p$. Hence

$$\chi(B) \geq \chi(M(S)) - t^s p > m_s - t^s p = t,$$

and so there exists $M \subseteq B$ with $|M| = t$. But then (M, T) is a $(t, 2, s)$ -template. This proves 3.4. \blacksquare

We also need the following version of Ramsey's theorem (proved for instance in [14]).

3.5 *For all integers $s \geq 1$ and $r \geq 2$, if a graph G has no stable subset of size s and no clique of size more than r , then $|V(G)| < r^s$.*

Now we use 3.4 to prove 3.2, which we restate in a strengthened form:

3.6 *Let $s \geq 1$ and $q \geq 0$ be integers, and let $\psi : \mathbb{N} \rightarrow \mathbb{N}$ be a non-decreasing polynomial. Let $\phi, \phi' : \mathbb{N} \rightarrow \mathbb{N}$ be the polynomials defined by*

$$\begin{aligned}\phi'(x) &= x^s \left(s\psi(x) + (s+1)^s x^{s(s+1)}\psi(x) + (s+1)x^{s+1} \right) \\ &\quad + q^s (s+1)x^{s+1} + (2+q+q^2+\dots+q^{s-1})\psi(x) + 2, \text{ and} \\ \phi(x) &= (s+1)^{2s} x^{2+2s(s+1)}\psi(x) + (s+1)^s x^{1+s(s+1)}\phi'(x) + (x+1)(s+1)x^{s+1}.\end{aligned}$$

for all $x \in \mathbb{N}$. Let G be a graph with $\chi(G) > \phi(\omega(G))$. Then either:

- there is a ψ -nondominating copy of $K_{s,s}$ in G , or
- there is a (ψ, q) -sprinkling in G .

Proof. Suppose that neither of the two bullets of the theorem holds. Let $t = (s+1)\omega(G)^{s+1}$. Thus

$$\chi(G) > \omega(G)^2 t^{2s} \psi(\omega(G)) + \omega(G) t^s \phi'(\omega(G)) + (\omega(G) + 1)t.$$

We claim:

(1) *If $A \subseteq V(G)$ with $\chi(A) > \phi'(\omega(G))$ then $G[A]$ contains a $(t, 2, s)$ -template.*

Let $G' = G[A]$. Since $\chi(A) > \phi'(\omega(G))$ and ψ is nondecreasing, it follows that

$$\chi(G') > \omega(G')^s ((s+t^s)\psi(\omega(G'))) + t$$

and

$$\chi(G') > q^s t + (2+q+q^2+\dots+q^{s-1})\psi(\omega(G')) + 1.$$

By 3.4 applied to G' , either

- there is a ψ -nondominating copy of $K_{s,s}$ in G' (and hence in G), or
- there is a (ψ, q) -sprinkling in G' (and hence in G), or
- G' contains a $(t, 2, s)$ -template.

By assumption, neither of the first two bullets hold, so the third holds. This proves (1).

For $2 \leq k \leq \omega(G) + 1$, define $t_k = (s+1)\omega(G)^{s+1} - s(k-2)\omega(G)^s$. Thus $t_2 = t$, and $0 \leq t_k \leq t$ for $2 \leq k \leq \omega(G) + 1$. By (1) applied to G , there is a $(t_2, 2, s)$ -template in G . Choose an integer k with $2 \leq k \leq \omega(G) + 1$, maximum such that there is a (t_k, k, s) -template in G , and let (A_1, \dots, A_k) be such a template.

(2) $k \leq \omega(G)$.

Suppose that $k = \omega(G) + 1$. Inductively for $i = 1, \dots, k$, suppose that vertices a_1, \dots, a_{i-1} are

defined, and define a_i as follows. For $1 \leq h < i$, the non-neighbours of a_h in A_i do not include a stable set of cardinality s , from the definition of a (t_k, k, s) -template. Hence by 3.5 (taking $r = \omega(G)$), there are at most $\omega(G)^s$ vertices in A_i nonadjacent to a_h , and hence at most $\omega(G)^{s+1}$ vertices in A_i that are nonadjacent to at least one of a_1, \dots, a_{i-1} . Since

$$|A_i| = t_k \geq (s+1)\omega(G)^{s+1} - s(\omega(G)-1)\omega(G)^s > \omega(G)^{s+1},$$

some vertex $a_i \in A_i$ is adjacent to all of a_1, \dots, a_{i-1} . This completes the inductive definition. But then $\{a_1, \dots, a_{\omega(G)+1}\}$ is a clique in G , a contradiction. This proves (2).

Let $Z = V(G) \setminus (A_1 \cup \dots \cup A_k)$. For $1 \leq i \leq k$, let \mathcal{S}_i be the set of all stable sets contained in A_i with cardinality s . For each $S \in \mathcal{S}_i$, let D_S be the set of vertices in Z with no neighbour in S , and let Y_i be the union of the sets D_S over $S \in \mathcal{S}_i$.

$$(3) \quad |Z \setminus (Y_1 \cup \dots \cup Y_k)| < t_{k+1}.$$

Suppose not, and choose $A \subseteq Z \setminus (Y_1 \cup \dots \cup Y_k)$ with $|A| = t_{k+1}$. For $1 \leq i \leq k$, choose $B_i \subseteq A_i$ with $|B_i| = t_{k+1}$. Then $(A, B_1, B_2, \dots, B_k)$ is a $(t_{k+1}, k+1, s)$ -template, contrary to the maximality of k . This proves (3).

For each $v \in Y_1 \cup \dots \cup Y_k$, choose $i \in \{1, \dots, k\}$ minimum such that $v \in Y_i$, and choose $S \in \mathcal{S}_i$ such that $v \in D_S$. We call S the *home* of v .

(4) *Let $1 \leq i \leq k$, and let $S \in \mathcal{S}_i$. The set of vertices in D_S with home S has chromatic number at most $\omega(G)t^s\psi(\omega(G)) + \phi'(\omega(G))$.*

Let F be the set of vertices in D_S with home S . By 3.5, as in the proof of (2), for $i+1 \leq j \leq k$ there are at most $s\omega(G)^s$ vertices in A_j with a non-neighbour in S , and since $|A_j| = t_k = t_{k+1} + s\omega(G)^s$, there exists $B_j \subseteq A_j$ with $|B_j| = t_{k+1}$ complete to S . For $1 \leq h < i$, choose $B_h \subseteq A_h$ with $|B_h| = t_{k+1}$ arbitrarily. Let F' be the set of vertices $v \in F$ such that v has no neighbour in S' for some $j \in \{i+1, \dots, k\}$ and some $S' \in \mathcal{S}_j$ with $S' \subseteq B_j$. For $i+1 \leq j \leq k$, and each $S' \in \mathcal{S}_j$ with $S' \subseteq B_j$, the chromatic number of the set of vertices in F with no neighbour in S' is at most $\psi(\omega(G))$, since the copy of $K_{s,s}$ induced on $S \cup S'$ is not ψ -nondominating; and so $\chi(F') \leq \omega(G)t^s\psi(\omega(G))$, since there are at most $\omega(G)t^s$ choices for the pair (j, S') . Let $F'' = F \setminus F'$. If $G[F'']$ contains a $(t, 2, s)$ -template, then it contains a $(t_{k+1}, 2, s)$ -template (C_1, C_2) say; and then

$$(C_1, C_2, B_1, \dots, B_{i-1}, B_{i+1}, \dots, B_k)$$

is a $(t_{k+1}, k+1, s)$ -template in G , from the definition of a home, a contradiction. Thus $G[F'']$ contains no such template, and so $\chi(F'') \leq \phi'(\omega(G))$ by (1). Hence $\chi(F) \leq \omega(G)t^s\psi(\omega(G)) + \phi'(\omega(G))$. This proves (4).

Now every vertex in $Y_1 \cup \dots \cup Y_k$ has a home, and there are only at most $\omega(G)t^s$ choices of a home; so by (4),

$$\chi(Y_1 \cup \dots \cup Y_k) \leq \omega(G)^2 t^{2s} \psi(\omega(G)) + \omega(G)t^s \phi'(\omega(G)).$$

Hence by (3),

$$\begin{aligned}\chi(G) &\leq \omega(G)^2 t^{2s} \psi(\omega(G)) + \omega(G) t^s \phi'(\omega(G)) + |Z \setminus (Y_1 \cup \dots \cup Y_k)| + |A_1 \cup \dots \cup A_k| \\ &\leq \omega(G)^2 t^{2s} \psi(\omega(G)) + \omega(G) t^s \phi'(\omega(G)) + (\omega(G) + 1)t,\end{aligned}$$

a contradiction. This proves 3.6. ■

4 Odd holes

Now we deduce 1.1. Let us say a hole in G is *special* if its length is either four or odd. We need a result proved in [10], the following:

4.1 *Let $x \in \mathbb{N}$, and let G be a graph such that $\chi(N(v)) \leq x$ for every vertex $v \in V(G)$. If C is a shortest odd hole in G , the set of vertices of G that belong to or have a neighbour in $V(C)$ has chromatic number at most $21x$.*

We deduce:

4.2 *Let $\psi : \mathbb{N} \rightarrow \mathbb{N}$ be some non-decreasing polynomial, let $n \in \mathbb{N}$, and let G be a graph such that $\chi(N(v)) \leq n$ for every vertex $v \in V(G)$. If $\chi(G) > \max(\omega(G), 21n + \psi(\omega(G)))$ then G contains a ψ -nondominating special hole.*

Proof. Since $\chi(G) > \omega(G)$, G is not perfect, and so contains either a four-hole or an odd hole (by the strong perfect graph theorem [3], since odd antiholes of length at least seven contain four-holes). Let C be either a four-hole, or a shortest odd hole of G . Let A be the set of vertices in $V(G) \setminus V(C)$ that have no neighbour in $V(C)$, and $B = V(G) \setminus A$. If C has length four then $\chi(B) \leq 4n$, and if C is a shortest odd hole of G , then $\chi(B) \leq 21n$ by 4.1. Consequently $\chi(A) > \psi(\omega(G)) \geq \psi(\omega(A))$, and so C is a ψ -nondominating special hole. This proves 4.2. ■

We also need:

4.3 *Let G be a graph containing no four-hole, let $n \in \mathbb{N}$, and let $X \subseteq V(G)$ be the set of all $v \in V(G)$ with $\chi(N(v)) > n$. If $\chi(X) > \omega(G)$, then there exist disjoint sets $A, B \subseteq V(G)$, anticomplete, with $\chi(A), \chi(B) > n/2 - \omega(G)$.*

Proof. Let us say an edge xy of G is *rich* if $\chi(N(x) \setminus N(y)) > n/2 - \omega(G)$ and $\chi(N(y) \setminus N(x)) > n/2 - \omega(G)$. Since there is no four-hole, it is enough to prove that there is a rich edge.

Since $\chi(X) > \omega(G)$, the graph $G[X]$ is not perfect, and so contains a four-vertex induced path with vertices $v_1-v_2-v_3-v_4$ in order. Let

$$\begin{aligned}A_1 &= N(v_1) \setminus (N(v_3) \cup N(v_4)), \\ A_2 &= N(v_2) \setminus (N(v_4) \cup (N(v_1) \cap N(v_3))), \\ A_3 &= N(v_3) \setminus (N(v_1) \cup (N(v_2) \cap N(v_4))), \text{ and} \\ A_4 &= N(v_4) \setminus (N(v_2) \cup N(v_1)).\end{aligned}$$

Since there is no four-hole, $N(v_1) \cap N(v_3)$ is a clique, and so is $N(v_1) \cap N(v_4)$, and therefore $\chi(A_1) > n - 2\omega(G)$. Since $N(v_2) \cap N(v_4)$ and $N(v_1) \cap N(v_3)$ are cliques, it also follows that $\chi(A_2) > n - 2\omega(G)$, and similarly $\chi(A_i) > n - 2\omega(G)$ for $1 \leq i \leq 4$.

Now v_2 is anticomplete to $A_1 \setminus A_2$, and v_1 is anticomplete to $A_2 \setminus A_1$, so if $\chi(A_1 \cap A_2) \leq n/2 - \omega(G)$, then $\chi(A_1 \setminus A_2) > n/2 - \omega(G)$ and $\chi(A_2 \setminus A_1) > n/2 - \omega(G)$, and so the edge v_1v_2 is rich.

Thus we may assume that $\chi(A_1 \cap A_2) > n/2 - \omega(G)$, and similarly $\chi(A_3 \cap A_4) > n/2 - \omega(G)$. But $A_1 \cap A_2 \subseteq N(v_2) \setminus N(v_3)$, and $A_3 \cap A_4 \subseteq N(v_3) \setminus N(v_2)$, and so the edge v_2v_3 is rich. This proves 4.3. \blacksquare

We put 4.2 and 4.3 together to prove the following:

4.4 *Let $\psi : \mathbb{N} \rightarrow \mathbb{N}$ be some non-decreasing polynomial. If G is a C_4 -free graph with*

$$\chi(G) > 85\omega(G) + 43\psi(\omega(G))$$

then G contains a ψ -nondominating odd hole.

Proof. Let G be a C_4 -free graph with $\chi(G) > 85\omega(G) + 43\psi(\omega(G))$. Define $n = 4\omega(G) + 2\psi(\omega(G))$.

Let A be the set of all vertices v of G such that $\chi(N(v)) \leq n$, and $B = V(G) \setminus A$. By 4.2 applied to $G[A]$, we may assume that

$$\chi(A) \leq \max(\omega(A), 21n + \psi(\omega(A))) = 21n + \psi(\omega(A)) \leq 84\omega(G) + 43\psi(\omega(G))$$

and so $\chi(B) \geq \chi(G) - \chi(A) > \omega(G)$. By 4.3 there exist disjoint sets $X, Y \subseteq V(G)$, anticomplete, with $\chi(X), \chi(Y) > n/2 - \omega(G) \geq \omega(G) + \psi(\omega(G))$. Since $\chi(X) > \omega(G) \geq \omega(X)$, $G[X]$ is not perfect and so contains a special hole C , and hence an odd hole since G has no four-holes. Since $V(C)$ is anticomplete to Y , and $\chi(Y) > \psi(\omega(G)) \geq \psi(\omega(Y))$, C is ψ -nondominating. This proves 4.4. \blacksquare

This in turn is used to prove:

4.5 *Let $\psi : \mathbb{N} \rightarrow \mathbb{N}$ be some non-decreasing polynomial. Then there is a non-decreasing polynomial $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that if $\chi(G) > \phi(\omega(G))$ then G contains a ψ -nondominating special hole.*

Proof. Let $\psi'(x) = 85x + 43\psi(x)$ for $x \in \mathbb{N}$, and let ϕ satisfy 3.2 with ψ replaced by ψ' , taking $s = 2$ and $q = 4$. We will show that ϕ satisfies 4.5. Let G be a graph with $\chi(G) > \phi(\omega(G))$. By 3.2, either there is a ψ' -nondominating four-hole in G , or there is a $(\psi', 4)$ -sprinkling in G . In the first case, this four-hole is also ψ -nondominating, since $\psi(x) \leq \psi'(x)$ for $x \in \mathbb{N}$, so we assume the second case holds. Let (P, Q) be a $(\psi', 4)$ -sprinkling in G , and let r be the maximum chromatic number over $v \in P$ of the set of neighbours of v in Q . Thus $\chi(Q) > 4r + \psi'(\omega(Q))$, from the definition of a $(\psi', 4)$ -sprinkling. If $G[P]$ has a four-hole H , the set of vertices in Q with a neighbour in $V(H)$ has chromatic number at most $4r$, and so there is a subset of Q with chromatic number more than $\psi'(\omega(Q)) \geq \psi(\omega(Q))$ anticomplete to H , and so H is ψ -nondominating. Thus we may assume that $G[P]$ has no four-hole. By 4.4, $G[P]$, and hence G , contains a ψ -nondominating odd hole. This proves 4.5. \blacksquare

We deduce 1.1, which we restate:

4.6 *For each integer $k \geq 1$, let \mathcal{C} be the class of all graphs G with no k -multihole in which every component is special. Then \mathcal{C} is poly- χ -bounded.*

Proof. Let us say a k -multihole is *special* if each of its components is a special hole. We proceed by induction on k . The result is true when $k = 1$, because graphs containing no special hole are perfect; so we assume that $k \geq 2$, and there is a polynomial binding function $\psi : \mathbb{N} \rightarrow \mathbb{N}$ for the class of all graphs with no special $(k-1)$ -multihole \mathcal{C}_{k-1} (and we may assume ψ is non-decreasing). Let ϕ satisfy 4.5; we claim that ϕ is a binding function for the class of all graphs with no special k -multihole. Thus, let G be a graph with $\chi(G) > \phi(\omega(G))$; we must show that G contains a special k -multihole. By the choice of ϕ , G contains a ψ -nondominating special hole H say. Choose $A \subseteq V(G) \setminus V(H)$, anticomplete to $V(H)$, such that $\chi(A) > \psi(\omega(A))$. From the inductive hypothesis, $G[A]$ contains a special $(k-1)$ -multihole, and so G contains a special k -multihole. This proves 4.6. \blacksquare

5 Long holes

In this section we will prove 1.4. The proof is similar to that of 1.1. Fix an integer $\ell \geq 4$, and we say a hole is *long* if its length is at least ℓ . Let $\tau(G)$ denote the largest integer t such that G contains $K_{t,t}$ as a subgraph. We need a result proved in [1] (see also [13]), the following:

5.1 *There exists an integer $c > 0$ such that $\chi(G) \leq \tau(G)^c + 1$ for every graph G with no long hole.*

We deduce:

5.2 *Let $s \in \mathbb{N}$; then the class of $K_{s,s}$ -free graphs with no long hole is poly- χ -bounded.*

Proof. Let $c \geq 1$ be as in 5.1, and let ϕ be the polynomial $\phi(x) = x^{cs}$ for $x \in \mathbb{N}$. Let G be a $K_{s,s}$ -free graph with no long hole. We will show that ϕ is a binding function for G . Suppose that $\tau(G) \geq \omega(G)^s$, and let A, B be disjoint subsets of $V(G)$, both of cardinality at least $\omega(G)^s$ and complete to each other. By 3.5, there exist stable sets $A' \subseteq A$ and $B' \subseteq B$ both of cardinality s ; but then $G[A' \cup B']$ is a copy of $K_{s,s}$, a contradiction. So $\tau(G) < \omega(G)^s$. By 5.1,

$$\chi(G) \leq (\omega(G)^s - 1)^c + 1 \leq \omega(G)^{cs} = \phi(\omega(G)),$$

and so ϕ is a binding function for G , and hence for the class of $K_{s,s}$ -free graphs with no long hole. This proves 5.2. \blacksquare

Next we need an analogue of 4.1, the following:

5.3 *Let $n \in \mathbb{N}$, and let G be a graph such that $\chi(N(v)) \leq n$ for every vertex $v \in V(G)$. If C is a shortest long hole in G , the set of vertices of G that belong to or have a neighbour in $V(C)$ has chromatic number at most $(\ell + 1)n$.*

Proof. Let C have vertices $c_1-c_2-\dots-c_k-c_1$ in order. Let P be the path $c_1-c_2-\dots-c_{\ell-3}$, and let Q be the path $C \setminus V(P)$.

(1) If $v \in V(G) \setminus V(C)$ has no neighbour in $V(P)$, then all neighbours of v in $V(Q)$ belong to a three-vertex subpath of Q .

Suppose not, and choose i, j minimum and maximum respectively such that $c_i, c_j \in V(Q)$ are neighbours of v . Thus $j - i \geq 3$, and so

$$c_1-c_2-\dots-c_i-v-c_j-c_{j+1}-\dots-c_k-c_1$$

is a long hole (because $j \geq \ell - 2$) that is shorter than C , a contradiction. This proves (1).

For $1 \leq i \leq k$, let A_i be the set of vertices in $V(G) \setminus V(C)$ that are adjacent to c_i and to none of c_1, \dots, c_{i-1} .

(2) A_i is anticomplete to A_j for $\ell - 2 \leq i < j \leq k$ with $j - i \geq 4$.

Suppose that $u \in A_i$ and $v \in A_j$ are adjacent. Choose $j' \geq j$ maximum such that $c_{j'}$ is adjacent to v ; thus $j' \geq j \geq i + 4$, and so by (1), u is non-adjacent to $c_{j'}, \dots, c_k$. Hence

$$c_1-c_2-\dots-c_i-u-v-c_{j'}-c_{j'+1}-\dots-c_k-c_1$$

is a long hole shorter than C , a contradiction. This proves (2).

For $t = 1, 2, 3, 4$ let I_t be the set of all integers $i \in \{\ell - 2, \dots, k\}$ such that $i - t$ is divisible by four. Thus I_1, I_2, I_3, I_4 form a partition of $\{\ell - 2, \dots, k\}$. Moreover, for all $t \in \{1, \dots, 4\}$, and all distinct $i, j \in I_t$, there is no edge between $A_i \cup \{c_{i+1}\}$ and $A_j \cup \{c_{j+1}\}$, by (1) and (2); and so $\bigcup_{i \in I_t} A_i \cup \{c_{i+1}\}$ has chromatic number at most n . Hence the set of all vertices in $V(G)$ that belong to or have a neighbour in $V(C)$ has chromatic number at most $(\ell + 1)n$, since those that belong to or have a neighbour in P have chromatic number at most $(\ell - 3)n$, and the others have chromatic number at most $4n$. This proves 5.3. ■

Now we need an analogue of 4.3, the following:

5.4 Let $s \in \mathbb{N}$, let G be a $K_{s,s}$ -free graph, with no long hole of length at most $2s\ell$. Let $n \in \mathbb{N}$, and let $B \subseteq V(G)$ be the set of vertices v of G such that $\chi(N(v)) > n$. If $G[B]$ contains a long hole, then there exist disjoint sets $X, Y \subseteq B$, anticomplete, with $\chi(X), \chi(Y) > n - (2s\ell)^s \omega(G)^s$.

Proof. We may assume that $G[B]$ has a hole of length more than $2s\ell$, and so contains an induced path P with $r = 2s\ell - 2$ vertices. Let the vertices of P be $p_1-p_2-\dots-p_r$ in order. For each stable subset $S \subseteq V(P)$ with $|S| = s$, let D_S be the set of vertices in $V(G) \setminus V(P)$ that are adjacent to every vertex in S . Since G is $K_{s,s}$ -free, it follows from 3.5 that $|D_S| \leq \omega(G)^s$. Let D be the set of vertices in $V(G) \setminus V(P)$ that have s pairwise nonadjacent neighbours in $V(P)$. Since there are at most $(2s\ell)^s$ choices of S , it follows that $\chi(D) \leq (2s\ell)^s \omega(G)^s$. Let $F = V(G) \setminus (V(P) \cup D)$.

(1) For each $v \in F$, if i, j are minimum and maximum such that v is adjacent to p_i, p_j , then

$$j - i \leq (s - 2)(\ell - 2) + 1.$$

Let $v \in F$. Choose $t \geq 0$ maximum such that there exist $1 \leq i_1 < \dots < i_t \leq r$ satisfying:

- i_1 is the least i such that v is adjacent to p_i ;
- v is adjacent to p_{i_k} for $1 \leq k \leq t$;
- $i_{k+1} \geq i_k + 2$ for $1 \leq k \leq t - 1$;
- v is nonadjacent to p_j for $1 \leq k \leq t - 1$ and for each $j \in \{i_k + 2, \dots, i_{k+1} - 1\}$.

Since $\{p_{i_1}, p_{i_2}, \dots, p_{i_t}\}$ is a stable set, and $v \in F$, it follows that $t < s$. Moreover, for $1 \leq k < t$, v is nonadjacent to each p_j for each $j \in \{i_k + 2, \dots, i_{k+1} - 1\}$; so one of

$$v-p_{i_k}-p_{i_k+1}-\dots-p_{i_{k+1}}$$

$$v-p_{i_k+1}-p_{i_k+2}-\dots-p_{i_{k+1}}$$

is an induced cycle. This cycle has length at most $2s\ell$, since P has only $r = 2s\ell - 2$ vertices; and so the cycle has length less than ℓ , since G has no long hole of length at most $2s\ell$. Consequently $i_{k+1} - i_k \leq \ell - 2$, and so $i_t - i_1 \leq (s - 2)(\ell - 2)$. From the maximality of t , v is nonadjacent to p_j for all $j \geq i_t + 2$. This proves (1).

Let X be the set of neighbours of p_1 in $V(G) \setminus D$, and let Y be the set of neighbours of p_r in $V(G) \setminus D$.

(2) X is disjoint from and anticomplete to Y .

Since $r - 1 > (s - 2)(\ell - 2) + 1$, (1) implies that $X \cap Y = \emptyset$. Suppose that $u \in X$ and $v \in Y$ are adjacent. Choose $i \in \{1, \dots, r\}$ maximum such that u is adjacent to p_i , and choose $j \in \{1, \dots, r\}$ minimum such that v is adjacent to p_j . By (1), $i - 1 \leq (s - 2)(\ell - 2) + 1$, and $r - j \leq (s - 2)(\ell - 2) + 1$. Hence $i - 1 + r - j \leq 2((s - 2)(\ell - 2) + 1)$, and so

$$j - i \geq (r - 1) - 2((s - 2)(\ell - 2) + 1) = 4\ell + 4s - 13.$$

But then $u-p_i-p_{i+1}-\dots-p_j-v-u$ is a hole of length at least $4\ell + 4s - 10 \geq \ell$ and at most $2s\ell$, a contradiction. This proves (2).

But $\chi(N(p_1)) \geq n$, and so $\chi(X) \geq n - \chi(D) \geq n - (2s\ell)^s \omega(G)^s$, and the same for Y . This proves 5.4. ■

Next, combining 5.3 and 5.4, we have an analogue of 4.4:

5.5 *Let $s \in \mathbb{N}$, and let $\psi : \mathbb{N} \rightarrow \mathbb{N}$ be some non-decreasing polynomial. There is a non-decreasing polynomial $\phi : \mathbb{N} \rightarrow \mathbb{N}$ with the following property. If G is a $K_{s,s}$ -free graph with no long hole of length at most $2s\ell$, and no ψ -nondominating long hole, then $\chi(G) \leq \phi(\omega(G))$.*

Proof. By 5.2, there is a non-decreasing polynomial $\theta : \mathbb{N} \rightarrow \mathbb{N}$ that is a binding function for the class of $K_{s,s}$ -free graphs with no long hole. Define ϕ by

$$\phi(x) = 2\theta(x) + \psi(x) + (\ell + 1) ((2s\ell)^s x^s + \theta(x) + \psi(x)).$$

We claim that ϕ satisfies 5.5. Thus, let G be a $K_{s,s}$ -free graph with no long hole of length at most $2s\ell$, and no ψ -nondominating long hole. Let

$$n = (2s\ell)^s \omega(G)^s + \theta(\omega(G)) + \psi(\omega(G)).$$

Let A be the set of vertices $v \in V(G)$ such that $\chi(N(v)) \leq n$, and $B = V(G) \setminus A$.

$$(1) \chi(A) \leq \theta(\omega(G)) + \psi(\omega(G)) + (\ell + 1)n.$$

Suppose not. Then by 5.2, $G[A]$ has a long hole; let C be a shortest long hole of $G[A]$. By 5.3 applied to $G[A]$, the set of vertices of A that belong to or have a neighbour in $V(C)$ has chromatic number at most $(\ell + 1)n$, and so there is a subset of $A \setminus V(C)$ anticomplete to $V(C)$ with chromatic number more than $\chi(A) - (\ell + 1)n \geq \psi(\omega(G))$. Hence C is ψ -nondominating, a contradiction. This proves (1).

$$(2) \chi(B) \leq \theta(\omega(G)).$$

Suppose not. Then $G[B]$ has a long hole by 5.2. By 5.4, there exist disjoint sets $X, Y \subseteq B$, anticomplete, with $\chi(X), \chi(Y) > n - (2s\ell)^s \omega(G)^s$. Since $\chi(X) \geq \theta(\omega(G))$, $G[X]$ has a long hole, and it is ψ -nondominating since $\chi(Y) \geq \psi(\omega(G))$, a contradiction. This proves (2).

From (1) and (2), it follows that

$$\chi(G) \leq 2\theta(\omega(G)) + \psi(\omega(G)) + (\ell + 1)n.$$

This proves 5.5. ■

This implies:

5.6 *Let $s \in \mathbb{N}$, and let $\psi : \mathbb{N} \rightarrow \mathbb{N}$ be some non-decreasing polynomial. Then there is a non-decreasing polynomial $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that if $\chi(G) > \phi(\omega(G))$ then G contains either a ψ -nondominating copy of $K_{s,s}$, or a ψ -nondominating long hole.*

Proof. By 5.5, there is a non-decreasing polynomial $\psi' : \mathbb{N} \rightarrow \mathbb{N}$ with the following property. If G is a $K_{s,s}$ -free graph with no long hole of length at most $2s\ell$, and $\chi(G) > \psi'(\omega(G))$, then G contains a ψ -nondominating long hole.

Let ϕ satisfy 3.2 with ψ replaced by ψ' , taking $q = 2s\ell$. We claim that ϕ satisfies 5.6. Thus, let G be a graph with $\chi(G) > \phi(\omega(G))$. By 3.2, either there is a ψ' -nondominating copy of $K_{s,s}$ in G , or there is a $(\psi', 2s\ell)$ -sprinkling in G . In the first case, this copy of $K_{s,s}$ is also ψ -nondominating, since $\psi(x) \leq \psi'(x)$ for $x \in \mathbb{N}$, so we assume the second case holds. Let (P, Q) be a $(\psi', 2s\ell)$ -sprinkling in G , and let r be the maximum chromatic number over $v \in P$ of the set of neighbours of v in Q . Thus $\chi(Q) > 2s\ell r + \psi'(\omega(Q))$, from the definition of a $(\psi', 2s\ell)$ -sprinkling. If $G[P]$ contains H where H is

either a copy of $K_{s,s}$ or a long hole of length at most $2s\ell$, the set of vertices in Q with a neighbour in $V(H)$ has chromatic number at most $|H|r \leq 2slr$, and so there is a subset of Q with chromatic number more than $\psi'(\omega(Q)) \geq \psi(\omega(Q))$ anticomplete to H ; and therefore H is ψ -nondominating. Thus we may assume that $G[P]$ is $K_{s,s}$ -free and has no long hole of length at most $2s\ell$. By 5.5, $G[P]$, and hence G , contains a ψ -nondominating long hole. This proves 5.6. ■

Finally, we prove 1.4, which we restate:

5.7 *For all integers $k, s \geq 1$ and $\ell \geq 4$, let \mathcal{C} be the class of all graphs G such that no induced subgraph of G has exactly k components, each of which is either a copy of $K_{s,s}$ or a cycle of length at least ℓ . Then \mathcal{C} is poly- χ -bounded.*

Proof. (The proof is just like that of 4.6.) Let us say an induced subgraph H of a graph G is a k -object if it has exactly k components, and each is either a copy of $K_{s,s}$ or a cycle of length at least ℓ . Thus \mathcal{C}_k is the class of graphs with no k -object. We prove by induction on k that \mathcal{C}_k is poly- χ -bounded. The result is true when $k = 1$, by 5.2, so we assume that $k \geq 2$, and there is a polynomial binding function $\psi : \mathbb{N} \rightarrow \mathbb{N}$ for \mathcal{C}_{k-1} (and we may assume ψ is non-decreasing). Let ϕ satisfy 5.6; we claim that ϕ is a binding function for \mathcal{C}_k . Thus, let G be a graph with $\chi(G) > \phi(\omega(G))$; we must show that G contains a k -object. By the choice of c , G contains a ψ -nondominating induced subgraph H , where H is either a copy of $K_{s,s}$ or a long hole. Choose $A \subseteq V(G) \setminus V(H)$, anticomplete to $V(H)$, such that $\chi(A) > \psi(\omega(A))$. From the inductive hypothesis, $G[A]$ contains a $(k-1)$ -object, and so G contains a k -object. This proves 5.7. ■

References

- [1] M. Bonamy, N. Bousquet, M. Pilipczuk, P. Rzazewski, S. Thomassé and B. Walczak, “Degeneracy of P_t -free and $C_{\geq t}$ -free graphs with no large complete bipartite subgraphs”, *J. Combinatorial Theory, Ser. B*, **152** (2022), 353–378, [arXiv:2012.03686](https://arxiv.org/abs/2012.03686).
- [2] M. Briański, J. Davies and B. Walczak, “Separating polynomial χ -boundedness from χ -boundedness”, [arXiv:2201.08814](https://arxiv.org/abs/2201.08814).
- [3] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, “The strong perfect graph theorem”, *Annals of Math.* **164** (2006), 51–229.
- [4] M. Chudnovsky, A. Scott, P. Seymour and S. Spirkl, “Polynomial bounds for chromatic number. VI. Adding a four-vertex path”, *European J. Combinatorics*, to appear, [arXiv:2202.10412](https://arxiv.org/abs/2202.10412).
- [5] Z. Dvořák and J. Pekárek, “Induced odd cycle packing number, independent sets, and chromatic number”, *J. Graph Theory* (2023), 1–15, <https://doi.org/10.1002/jgt.22932>, [arXiv:2001.02411](https://arxiv.org/abs/2001.02411).
- [6] L. Esperet, *Graph Colorings, Flows and Perfect Matchings*, Habilitation thesis, Université Grenoble Alpes (2017), 24, <https://tel.archives-ouvertes.fr/tel-01850463/document>.
- [7] A. Gyárfás, “On Ramsey covering-numbers”, in *Infinite and Finite Sets, Vol. II* (Colloq., Keszthely, 1973), *Coll. Math. Soc. János Bolyai* **10**, 801–816.

- [8] I. Schiermeyer, “On the chromatic number of $(P_5, \text{windmill})$ -free graphs”, *Opuscula Math.* **37** (2017), 609–615.
- [9] I. Schiermeyer and B. Randerath, “Polynomial χ -binding functions and forbidden induced subgraphs: a survey”, *Graphs and Combinatorics* **35** (2019), 1–31.
- [10] A. Scott and P. Seymour, “Induced subgraphs of graphs with large chromatic number. VII. Gyárfás’ complementation conjecture” *J. Combinatorial Theory, Ser. B* **142** (2020), 43–55, [arXiv:1701.06301](#).
- [11] A. Scott and P. Seymour, “Induced subgraphs of graphs with large chromatic number. X. Holes of specific residue”, *Combinatorica* **39** (2019), 1105–1132, [arXiv:1705.04609](#).
- [12] A. Scott and P. Seymour, “A survey of χ -boundedness”, *J. Graph Theory* **95** (2020), 473–504, [arXiv:1812.07500](#).
- [13] A. Scott, P. Seymour and S. Spirkl, “Polynomial bounds for chromatic number. I. Excluding a biclique and an induced tree”, *J. Graph Theory* **102** (2023), 458–471, [arXiv:2104.07927](#).
- [14] A. Scott, P. Seymour and S. Spirkl, “Polynomial bounds for chromatic number. II. Excluding a star forest”, *J. Graph Theory*, **101** (2022), 318–322, [arXiv:2107.11780](#).
- [15] A. Scott, P. Seymour and S. Spirkl, “Polynomial bounds for chromatic number. III. Excluding a double star”, *J. Graph Theory*, **101** (2022), 323–340, [arXiv:2108.07066](#).
- [16] A. Scott, P. Seymour and S. Spirkl, “Polynomial bounds for chromatic number. IV. A near-polynomial bound for excluding the five-vertex path”, *Combinatorica*, to appear, [arXiv:2110.00278](#).
- [17] D. P. Sumner, “Subtrees of a graph and chromatic number”, in *The Theory and Applications of Graphs*, (G. Chartrand, ed.), John Wiley & Sons, New York (1981), 557–576.