

Polynomial bounds for chromatic number
VI. Adding a four-vertex path

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Abstract

A class of graphs is χ -bounded if there is a function f such that every graph G in the class has chromatic number at most $f(\omega(G))$, where $\omega(G)$ is the clique number of G ; the class is *polynomially χ -bounded* if f can be taken to be a polynomial. The Gyárfás-Sumner conjecture asserts that, for every forest H , the class of H -free graphs (graphs with no induced copy of H) is χ -bounded. Let us say a forest H is *good* if it satisfies the stronger property that the class of H -free graphs is polynomially χ -bounded.

Very few forests are known to be good: for example, it is open for the five-vertex path. Indeed, it is not even known that if every component of a forest H is good then H is good, and in particular, it was not known that the disjoint union of two four-vertex paths is good. Here we show the latter, and more generally, that if H is good then so is the disjoint union of H and a four-vertex path. We also prove a more general result: if every component of H_1 is good, and H_2 is any path (or broom) then the class of graphs that are both H_1 -free and H_2 -free is polynomially χ -bounded.

1 Introduction

A class of graphs is χ -bounded if there is a function f such that every graph G in the class has chromatic number at most $f(\omega(G))$, where $\omega(G)$ is the clique number of G ; the class is *polynomially* χ -bounded if f can be taken to be a polynomial. A graph is H -free if it has no induced subgraph isomorphic to H .

The Gyárfás-Sumner conjecture [4, 14] asserts:

1.1 Conjecture: *For every forest H , the class of H -free graphs is χ -bounded.*

There has been a great deal of recent progress on χ -bounded classes (see [9] for a survey), although the Gyárfás-Sumner conjecture remains open. In most cases, proofs of χ -boundedness give fairly fast-growing functions, so it is interesting to ask: when do we get the stronger property of polynomial χ -boundedness?

A provocative conjecture of Louis Esperet [3] asserted that every χ -bounded hereditary class is polynomially χ -bounded. But this was recently disproved by Briański, Davies and Walczak [1]. So the question now is: which hereditary classes are polynomially χ -bounded? In particular, can 1.1 be strengthened to polynomial χ -boundedness? Let us say a graph H is *good* if the class of H -free graphs is polynomially χ -bounded. Perhaps every forest is good, but the only trees currently known to be good are those not containing the five-vertex path P_5 [11]. It is not known whether P_5 is good (although see [12] for the best current bounds for $H = P_5$; and see [13] for the case when H is a general tree of radius two).

In the case of χ -boundedness, it is not hard to show that a forest H satisfies the Gyárfás-Sumner conjecture if and only if all its components do. But it has *not* been shown that if every component of a forest H is good then H is good. Indeed, only some very restricted forests are known to be good [8, 10]. One outstanding case was when H is the disjoint union of two copies of the four-vertex path P_4 ; and this was particularly annoying since the P_4 -free graphs are very well-understood and rather trivial.

We will prove the following:

1.2 *If H is a good forest, then the disjoint union of H and P_4 is also good.*

In particular, the disjoint union of two or more copies of P_4 is good. 1.2 is a consequence of the next result, about brooms. A (k, d) -broom is a tree obtained from a k -vertex path with one end v by adding d new vertices adjacent to v , and a broom is a tree that is a (k, d) -broom for some k, d . It is known that $(3, d)$ -brooms are good [6, 11], but this is not known for larger brooms (all of which contain P_5). We will show the following, which implies 1.2:

1.3 *Let H_1 be a forest such that every component of H_1 is good, and let H_2 be either a broom, or the disjoint union of a good forest and a number of paths. Then there is a polynomial ϕ such that $\chi(G) \leq \phi(\omega(G))$ for every $\{H_1, H_2\}$ -free graph G .*

($\{H_1, H_2\}$ -free means both H_1 -free and H_2 -free.) To deduce 1.2 from 1.3, let H be a good forest, let $H_1 = H_2$ be the disjoint union of H and P_4 , and apply 1.3.

Some notation and terminology: if G is a graph and $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced on X , and we sometimes write $\chi(X)$ for $\chi(G[X])$ and $\omega(X)$ for $\omega(G[X])$. Two disjoint subsets $A, B \subseteq V(G)$ are *complete* if every vertex in A is adjacent to every vertex of B , and *anti-complete* if there is no edge between A, B ; and we say a vertex v is *complete to* B if $\{v\}$ is complete

to B , and so on. A graph G *contains* a graph H if some induced subgraph of G is isomorphic to H , and such a subgraph is a *copy* of H . The *cone* of a graph H is obtained from H by adding a new vertex adjacent to every vertex of H .

Let us say a graph is *0-bad* if it is good; and a graph J is β -bad, where $\beta \geq 1$ is an integer, if either J is the disjoint union of two $(\beta - 1)$ -bad graphs, or J is the cone of a $(\beta - 1)$ -bad graph, or J is $(\beta - 1)$ -bad. In general, cones are not forests, so they are not good. Nevertheless, we will prove the following strengthening of 1.3:

1.4 *Let $\beta \geq 0$, let H_1 be a β -bad graph, and let H_2 be either a broom, or the disjoint union of a good forest and a number of paths. Then there is a polynomial ϕ such that $\chi(G) \leq \phi(\omega(G))$ for every $\{H_1, H_2\}$ -free graph G .*

This implies several results that were previously known. For instance, in [7] it is proved that:

1.5 *Let H_1 be either*

- *the disjoint union of a complete graph and a good graph, or*
- *the disjoint union of some complete graphs, or*
- *the cone of the disjoint union of some complete graphs.*

Let H_2 be a path. Then there is a polynomial ϕ such that $\chi(G) \leq \phi(\omega(G))$ for every $\{H_1, H_2\}$ -free graph G .

Some other results of [7, 8] are also special cases of 1.4.

2 Finding a disjoint union

Suppose that H is the disjoint union of good forests H_1, H_2 . Choose c_1, c_2 such that for $i = 1, 2$, every H_i -free graph G satisfies $\chi(G) \leq \omega(G)^{c_i}$. Thus, if G is H -free, we know that there do not exist disjoint, anticomplete subsets $P, Q \subseteq V(G)$ with $\chi(P) > \omega(P)^{c_1}$ and $\chi(Q) > \omega(Q)^{c_2}$; because then $G[P]$ is not H_1 -free, and $G[Q]$ is not H_2 -free, and the union of a copy of H_1 in $G[P]$ and a copy of H_2 in $G[Q]$ gives a copy of H , which is impossible.

But we do not really need P, Q to be anticomplete. It is enough that $\chi(P) > \omega(P)^{c_1}$, and $\chi(Q) > |H_1|r + \omega(Q)^{c_2}$, where r denotes the maximum over $v \in P$ of the chromatic number of the set of neighbours of v in Q ; because then if we choose a copy H'_1 of H in $G[P]$, the chromatic number of the set of vertices in Q with no neighbours in $V(H'_1)$ is at least $\chi(Q) - |H_1|r > \omega(Q)^{c_2}$, and so this set contains a copy of H_2 , a contradiction. In the proof to come later in the paper, this is the only way we will ever use that G is H -free; and so we might as well prove a stronger theorem, replacing the hypothesis that G is H -free with the weaker hypothesis that there is no suitable pair P, Q in G .

Thus we will be excluding pairs of disjoint sets P, Q where $\chi(P)$ is at least some power of $\omega(P)$, and for each vertex in P , its set of neighbours in Q has chromatic number at most some r that is small compared with the chromatic number of Q .

In our proof, it happens that when we find a suitable pair (P, Q) , it comes equipped with an extra vertex v that is complete to P and anticomplete to Q ; so we might as well prove that there is a “suitable triple” (v, P, Q) . Such a thing will also allow us to handle cones.

We denote the set of nonnegative integers by \mathbb{N} , and say a function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ is *non-decreasing* if $\phi(x) \leq \phi(x')$ for all $x, x' \in \mathbb{N}$ with $x \leq x'$.

Let $\psi : \mathbb{N} \rightarrow \mathbb{N}$ be non-decreasing, and let $q \geq 0$ be an integer. We say a (ψ, q) -*scattering* in a graph G is a triple (v, P, Q) where:

- P, Q are disjoint subsets of $V(G)$, and $v \in V(G) \setminus (P \cup Q)$;
- $\{v\}$ is complete to P and anticomplete to Q ;
- $\chi(P) > \psi(\omega(P))$; and
- $\chi(Q) > qr + \psi(\omega(Q))$, where r is the maximum, over $u \in P$, of the chromatic number of the set of neighbours of u in Q .

Thus we will replace the hypothesis in 1.4 that G is H_1 -free and H_1 is β -bad, with the hypothesis that G contains no (ψ, q) -scattering, for appropriate ψ, q . We will show:

2.1 *Let $\psi : \mathbb{N} \rightarrow \mathbb{N}$ be a non-decreasing polynomial and let $q \in \mathbb{N}$. Let H_2 be either a broom, or the disjoint union of a good forest and a number of paths. Then there is a polynomial $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that if $\chi(G) > \phi(\omega(G))$ and G contains no (ψ, q) -scattering, then G contains H_2 .*

Proof of 1.4, assuming 2.1. We proceed by induction on β . Let H_1 be β -bad, and let H_2 be either a broom, or the disjoint union of a good forest and a number of paths.

If H_1 is good, the result is true, so we assume that H_1 is not good, and therefore $\beta \geq 1$. Thus either H_1 is the disjoint union of two $(\beta - 1)$ -bad graphs J_1, J_2 , or the cone of a $(\beta - 1)$ -bad graph J_1 (and in this case let J_2 be the null graph). From the inductive hypothesis on β , for $i = 1, 2$ there is a non-decreasing polynomial ϕ_i such that if G is H_2 -free and J_i -free then $\chi(G) \leq \phi_i(\omega(G))$, and by replacing ϕ_1, ϕ_2 by $\phi_1 + \phi_2$ we may assume that $\phi_1 = \phi_2$.

Let $q = |J_1|$. By 2.1, there is a non-decreasing polynomial ϕ such that if $\chi(G) > \phi(\omega(G))$ and contains no (ϕ_1, q) -scattering, then G contains H_2 . We claim that ϕ satisfies 1.4.

Let G be $\{H_1, H_2\}$ -free, and suppose that $\chi(G) > \phi(\omega(G))$. Since G is H_2 -free, it follows from the choice of ϕ that G contains a (ϕ_1, q) -scattering (w, P, Q) say. Let r be the maximum, over $v \in P$, of the chromatic number of the set of neighbours of v in Q . Since $\chi(P) > \phi_1(\omega(P))$, there is an induced subgraph of $G[P]$ isomorphic to J_1 , say J'_1 . Hence G contains the cone of J_1 , so we may assume that H_1 is the disjoint union of J_1, J_2 . The set of vertices in Q with a neighbour in $V(J'_1)$ has chromatic number at most $r|J_1|$, and since

$$\chi(Q) > |J_1|r + \phi_2(\omega(Q)),$$

it follows that the set (say Q') of vertices in Q that are anticomplete to J'_1 has chromatic number more than $\phi_2(\omega(Q))$. From the choice of ϕ_2 , and since G is H_2 -free, it follows that $G[Q']$ is not J_2 -free; but then, combining this copy of J_2 with J'_1 , we find a copy of H_1 in G , a contradiction. This proves 1.4. ■

Let $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ be a non-decreasing function. We say a subgraph P of a graph G is σ -*nondominating* if there is a set $X \subseteq V(G) \setminus V(P)$, anticomplete to $V(P)$, with $\chi(X) > \sigma(\omega(X))$. Next we will show that to prove 2.1 it suffices to prove the following:

2.2 Let $\psi, \sigma : \mathbb{N} \rightarrow \mathbb{N}$ be non-decreasing polynomials, and let $q \geq 0$ an integer. Let H be a broom, and let J be a path. Then there is a non-decreasing polynomial $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that if G is a graph, and $\chi(G) > \phi(\omega(G))$, and G contains no (ψ, q) -scattering, then G contains H and a σ -nondominating copy of J .

Proof of 2.1, assuming 2.2. Let ψ, q, H_2 be as in 2.1. If H_2 is a broom, then 2.1 follows immediately from 2.2 (setting $H = H_2$ and setting J to be some path, for instance the one-vertex path). Thus we assume that H_2 is the disjoint union of a good forest J_1 and a forest J_2 that is a disjoint union of paths. Let $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ be a non-decreasing function such that every J_1 -free graph G has chromatic number at most $\sigma(\omega(G))$; and choose a path J such that J_2 is an induced subgraph of J . By 2.2 (setting H to be some broom, for instance with one vertex) there is a non-decreasing polynomial $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that if $\chi(G) > \phi(\omega(G))$ and G contains no (ψ, q) -scattering, then G contains a σ -nondominating copy J' of J .

We claim that ϕ satisfies 2.1. Thus we must show that if G is H_2 -free and contains no (ψ, q) -scattering then $\chi(G) \leq \phi(\omega(G))$. Suppose not. By the choice of f , and since G contains no (ψ, q) -scattering, it follows that G contains a copy J' of J , such that there is a set $X \subseteq V(G)$ with $\chi(X) > \sigma(\omega(X))$ anticomplete to $V(J'_1)$. But since $\chi(X) > \sigma(\omega(X))$, it follows that $G[X]$ contains J_1 , and since J contains J_2 , and $V(J)$ is anticomplete to X , it follows that G contains H_2 . This proves 2.1. ■

We remark that there is an appealing possible strengthening of 2.2, that we could not prove:

2.3 Conjecture: Let $\psi, \sigma : \mathbb{N} \rightarrow \mathbb{N}$ be non-decreasing polynomials, let $q \geq 0$ an integer, and let H be a broom. Then there is a non-decreasing polynomial $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that if G is a graph, and $\chi(G) > \phi(\omega(G))$, and G contains no (ψ, q) -scattering, then G a σ -nondominating copy of H .

Let us say a graph H is *self-isolating* if for every non-decreasing polynomial $\psi : \mathbb{N} \rightarrow \mathbb{N}$, there is a polynomial $\phi : \mathbb{N} \rightarrow \mathbb{N}$ with the following property: for every graph G with $\chi(G) > \phi(\omega(G))$, there exists $A \subseteq V(G)$ with $\chi(A) > \psi(\omega(A))$, such that either

- $G[A]$ is H -free, or
- G contains a copy H' of H such that $V(H')$ is disjoint from and anticomplete to A .

Which graphs are self-isolating? It is proved in [10] that stars are self-isolating, and we will show in [2] that complete graphs and complete bipartite graphs are self-isolating. Let us observe that 2.2 implies that:

2.4 Every path is self-isolating.

Proof. Let J be a path, and let $\psi : \mathbb{N} \rightarrow \mathbb{N}$ be a non-decreasing polynomial. Choose ϕ satisfying 2.2 with $H = J$ and $\sigma = \psi$ and $q = |J|$, and let G be a graph with $\chi(G) > \phi(\omega(G))$. We claim that either there is a ψ -nondominating copy of J in G , or there exists $A \subseteq V(G)$ with $\chi(A) > \psi(\omega(A))$ such that $G[A]$ is J -free. By 2.2 we may assume that there is a (ψ, q) -scattering (w, P, Q) in G . If $G[P]$ is J -free, the claim holds, so we assume that there is a copy J' of J in $G[P]$. Thus $|J'| = q$. Let r be the maximum over $v \in P$ of the chromatic number of the set of neighbours of v in Q . The set of vertices in Q with a neighbour in $V(J')$ has chromatic number at most $|J'|r = qr$; and $\chi(Q) > \psi(\omega(Q)) + qr$ from the definition of a (ψ, q) -scattering. Consequently J' is ψ -nondominating, and hence J is self-isolating. This proves 2.4. ■

3 Constructing a horn

Let $d \geq 0$ be an integer. If $A, B \subseteq V(G)$ are disjoint, we say that A is d -dense to B if for every vertex $v \in A$, the set of non-neighbours of v in B has chromatic number at most d . Let us say a (d, z) -horn in a graph G is a triple (v, A, B) where

- A, B are disjoint subsets of $V(G)$, and $v \in V(G) \setminus (A \cup B)$;
- v is complete to A and anticomplete to B ; and
- there is no $Z \subseteq A \cup B$ with $\chi(Z) \leq z$ such that $A \setminus Z$ is d -dense to $B \setminus Z$.

We will need a (d, z) -horn (v, A, B) where z is at least some large function of the clique number of $A \cup B$, and this section produces such a horn. We show in 3.5 that if G has sufficiently large chromatic number (and, for convenience, all its proper induced subgraphs have smaller chromatic number), then either G contains both a (k, s) -broom and a σ -nondominating k -vertex path, or G contains a (d, z) -horn. To complete the proof of 2.2, it therefore suffices to handle graphs G that contain (d, z) -horns, for suitably chosen values of d, z , and we will do so in the next section.

We will use the following well-known version of Ramsey's theorem, proved (for instance) in [10] ($|G|$ denotes the number of vertices of G):

3.1 *Let $x \geq 2$ and $y \geq 1$ be integers. For a graph G , if $|G| \geq x^y$, then G has either a clique of cardinality $x + 1$, or a stable set of cardinality y .*

If $v \in V(G)$, we denote by $N(v)$ or $N_G(v)$ the set of all neighbours of v in G . First, we need a result of Gyárfás [5] (we give the well-known proof, because it is so pretty.)

3.2 *Let $k \geq 1$ and $x \geq 0$ be integers. Let G be a connected graph such that $\chi(N(v)) \leq x$ for every vertex v . Let H be a connected induced subgraph of G , and let $v \in V(G) \setminus V(H)$ with a neighbour in $V(H)$. If $\chi(H) > (k - 2)x$, there is an induced k -vertex path of G with one end v and all other vertices in $V(H)$.*

Proof. We proceed by induction on k . The result is clear for $k \leq 2$, so we assume that $k \geq 3$. Let J be obtained from H by deleting all vertices in $N(v)$; thus $\chi(J) > (k - 3)x > 0$, and so there is a component H' of J with chromatic number more than $(k - 3)x$. Let $v' \in N(v) \cap V(H)$ with a neighbour in $V(H')$. From the inductive hypothesis applied to v', H' , there is an induced $(k - 1)$ -vertex path of G with one end v' and all other vertices in $V(H')$. Appending v to this path proves 3.2. ■

We deduce:

3.3 *Let $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ be non-decreasing, let $k, x \geq 1$ be integers, and let G be a graph. If $\chi(N(v)) \leq x$ for every $v \in V(G)$, and $\chi(G) > kx + \sigma(\omega(G))$, then there is a σ -nondominating k -vertex induced path P in G .*

Proof. We may assume that G is connected; choose $v \in V(G)$. Since $\chi(G \setminus v) > kx - 1 \geq (k - 2)x$, 3.2 (applied to v and to a component of $G \setminus v$ of maximum chromatic number) implies that G contains a k -vertex induced path P . The set of vertices of G with a neighbour in $V(P)$ has chromatic number at most kx , and the result follows. This proves 3.3. ■

The next result is also essentially due to Gyárfás (mentioned in [5]):

3.4 *Let H be a (k, s) -broom, and suppose that G is H -free, and $\chi(N(v)) \leq x$ for every $v \in V(G)$. Then*

$$\chi(G) \leq \max(\omega(G)^{2s}, (2s+1)(x+1) + (k-2)x).$$

Proof. Suppose that $\chi(G) > \max(\omega(G)^{2s}, (2s+1)(x+1) + (k-2)x)$. We may assume that G is connected. If every vertex of G has degree less than $\omega(G)^{2s}$ then $\chi(G) \leq \omega(G)^{2s}$, a contradiction, so some vertex v has at least $\omega(G)^{2s}$ neighbours. By 3.1 applied to $G[N(v)]$, there is a stable set S of neighbours of v , with $|S| = 2s$. Let M be the set of all vertices of G that do not belong to $S \cup \{v\}$ and have a neighbour in $S \cup \{v\}$. Thus $\chi(M) \leq (2s+1)x$. Let H be a component of $G \setminus (M \cup S \cup \{v\})$ of maximum chromatic number; then $\chi(H) \geq \chi(G) - (2s+1)(x+1) > (k-2)x$. Choose $u \in M \cup S \cup \{v\}$ with a neighbour in $V(H)$. Since no vertex of $S \cup \{v\}$ has a neighbour in $V(H)$, from the definition of M , it follows that $u \in M$. By 3.2 applied to u, H , there is an induced k -vertex path P of G with one end u and all other vertices in $V(H)$. Thus u is the only vertex of P with a neighbour in $S \cup \{v\}$. If u is adjacent to at least s vertices in S , then the subgraph induced on $V(P)$ and some s of these neighbours is a (k, s) -broom, a contradiction. Thus there exists $S' \subseteq S$ with $|S'| = s$, such that all vertices in S' are nonadjacent to u . If u is adjacent to v , the subgraph induced on $V(P) \cup S \cup \{v\}$ is a $(k+1, s)$ -broom, a contradiction. Thus u is adjacent to some $w \in S \setminus S'$, and nonadjacent to v . But then the subgraph induced on $V(P) \cup S' \cup \{v, w\}$ is a $(k+2, s)$ -broom, a contradiction. This proves 3.4. ■

3.5 *Let $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ be non-decreasing. Let $k, s, d, z \geq 0$ and $c \geq 2s$ be integers. Let G be a graph such that*

$$\begin{aligned} \chi(G) &> \omega(G)^c; \\ \chi(G') &\leq \omega(G')^c \text{ for every induced subgraph } G' \text{ of } G \text{ with } G' \neq G; \\ \omega(G)^c &\geq (\omega(G) - 1)^c + z + d\omega(G) + 2; \\ \omega(G)^c &\geq (2s+1)(z+1) + (k-2)z; \text{ and} \\ \omega(G)^c &\geq kz + \sigma(\omega(G)). \end{aligned}$$

Then either

- G contains a (d, z) -horn; or
- G contains a (k, s) -broom, and a σ -nondominating k -vertex path.

Proof. Suppose that $\chi(N(v)) \leq z$ for every vertex $v \in V(G)$. By 3.4, and since

$$\chi(G) > \omega(G)^c \geq \max(\omega(G)^{2s}, (2s+1)(x+1) + (k-2)z)$$

(because $c \geq 2s$), it follows that G contains a (k, s) -broom. By 3.3, since $\chi(G) - kz > \sigma(\omega(G))$, there is a σ -nondominating k -vertex induced path P in G , and so the second bullet holds.

Thus we assume that $\chi(N(v)) > z$ for some vertex v . Let A be the set of neighbours of v , and $B = V(G) \setminus (A \cup \{v\})$. We claim that (v, A, B) is a (d, z) -horn. Suppose not; then there exists $Z \subseteq A \cup B$ with $\chi(Z) \leq z$, such that $A \setminus Z$ is d -dense to $B \setminus Z$. Let $P \subseteq A \setminus Z$ be a clique with

cardinality $p = \omega(A \setminus Z)$. Then $p \geq 1$, since $\chi(Z) \leq z < \chi(A)$; and $p < \omega(G)$ since otherwise adding v would give a clique of cardinality $\omega(G) + 1$. For each $u \in P$, the set of vertices in $B \setminus Z$ nonadjacent to u has chromatic number at most d , since $A \setminus Z$ is d -dense to $B \setminus Z$; and so the set of vertices in B with a non-neighbour in P has chromatic number at most $pd \leq d\omega(G)$. The set of vertices in B complete to P has clique number at most $\omega(G) - p$ and so has chromatic number at most $(\omega - p)^c$. Hence $\chi(B \setminus Z) \leq pd + (\omega(G) - p)^c$, and so

$$\chi(G) \leq \chi(Z) + \chi(A \setminus Z) + \chi(B \setminus Z) + 1 \leq z + p^c + d\omega(G) + (\omega(G) - p)^c + 1.$$

Since $1 \leq p \leq \omega(G) - 1$, $p^c + (\omega(G) - p)^c \leq (\omega(G) - 1)^c + 1$, and so

$$\omega(G)^c < \chi(G) \leq z + d\omega(G) + (\omega(G) - 1)^c + 2,$$

a contradiction. This proves 3.5. ■

4 Making taller horns

In this section we prove 2.2, and hence complete the proofs of 2.1, 1.4, 1.3, and therefore 1.2. Because of 3.5, we may assume that G contains a (d, z) -horn, for some suitable values of d, z ; and now we will show that, provided that G does not contain the proscribed scattering, we can use this horn to make a “ k -tall” (d', z') -horn, which is a horn with a k -vertex path appended to its distinguished vertex. From such a horn, it is easy to obtain a (k, s) -broom and a σ -nondominating k -vertex path, to satisfy 2.2. The main step is therefore to convert an ℓ -tall horn to an $(\ell + 1)$ -tall horn, and for that we need the next result.

If $d, z, \omega \geq 0$ are integers, a graph G is (d, z, ω) -*unsplittable* if there is no partition (A, B, Z) of $V(G)$ such that $\chi(Z) \leq z$, and $\chi(A), \chi(B) > d\omega$, and A is d -dense to B . We begin with:

4.1 *If $d, z \geq 0$ are integers, every graph G admits a partition (D_0, D_1, \dots, D_k) of its vertex set with $k \leq \omega(G)$ such that $\chi(D_0) \leq z\omega(G)$ and $G[D_i]$ is $(d, z, \omega(G))$ -unsplittable for $1 \leq i \leq k$.*

Proof. We may assume that G is not $(d, z, \omega(G))$ -unsplittable, and so it admits a partition (D_0, D_1, D_2) such that $\chi(D_0) \leq z$, $\chi(D_1), \chi(D_2) > d\omega(G)$, and D_1 is d -dense to D_2 . Hence we may choose $k \geq 2$ maximum such that there is a sequence D_0, D_1, \dots, D_k of pairwise disjoint subsets of $V(G)$ with union $V(G)$, and with the following properties:

- $\chi(D_0) \leq (k - 1)z$
- D_i is d -dense to D_j for $1 \leq i < j \leq k$; and
- $\chi(D_i) > d\omega(G)$ for $1 \leq i \leq k$.

We claim:

(1) $k \leq \omega(G)$.

Suppose that $k > \omega(G)$, and define $d_i \in D_i$ for $1 \leq i \leq \omega(G) + 1$ inductively as follows. Let

$1 \leq i \leq \omega(G) + 1$, and suppose that d_1, \dots, d_{i-1} have been defined, all pairwise adjacent. The set of vertices in D_i that have a non-neighbour among d_1, \dots, d_{i-1} has chromatic number at most

$$(i-1)d \leq d\omega(G) < \chi(D_i),$$

and so some vertex $d_i \in D_i$ is adjacent to all of d_1, \dots, d_{i-1} . This completes the inductive definition. But then $\{d_1, \dots, d_{\omega(G)+1}\}$ is a clique of G , contradicting the definition of $\omega(G)$. This proves (1).

(2) For $1 \leq i \leq k$, $G[D_i]$ is $(d, z, \omega(G))$ -unsplittable.

Suppose that (A, B, Z) is a partition of D_i such that $\chi(Z) \leq z$, and $\chi(A), \chi(B) > d\omega(G)$, and A is d -dense to B . Then the sequence

$$(D_0 \cup Z, D_1, \dots, D_{i-1}, A, B, D_{i+1}, \dots, D_k)$$

contradicts the maximality of k . This proves (2).

From (1), (2), this proves 4.1. ▀

Let (v, A, B) be a (d, z) -horn in a graph G , and let $k \geq 1$ be an integer. We say that (v, A, B) is k -tall if there is an induced path P in G with k vertices, with one end v , such that $V(P) \setminus \{v\}$ is disjoint from and anticomplete to $A \cup B$. Thus every (d, z) -horn is 1-tall. We use 4.1 to prove a result which is the heart of the paper:

4.2 Let G be a graph, let $d, z, d', z', q \geq 0$ be integers, and let $\psi : \mathbb{N} \rightarrow \mathbb{N}$ be non-decreasing, satisfying:

$$\begin{aligned} z &\geq (2\psi(\omega(G)) + (1+q)z' + qd'\omega(G))\omega(G) \\ d &\geq (z' + d'\omega(G))\omega(G). \end{aligned}$$

Let (v, A, B) be an ℓ -tall (d, z) -horn in a graph G , for some $\ell \geq 1$. Then either

- there exist $P \subseteq A$ and $Q \subseteq B$ such that (v, P, Q) is a (ψ, q) -scattering; or
- there exist $v' \in A$ and disjoint subsets A', B' of B such that (v', A', B') is an $(\ell + 1)$ -tall (d', z') -horn.

Proof. Let $p = \psi(\omega(G))$. By 4.1, B admits a partition (D_0, D_1, \dots, D_k) with $k \leq \omega(G)$ such that $\chi(D_0) \leq z'\omega(G)$ and $G[D_i]$ is $(d', z', \omega(G))$ -unsplittable for $1 \leq i \leq k$. For $1 \leq i \leq k$, if $\chi(D_i) \leq q(z' + d'\omega(G)) + p$ let $P_i = \emptyset$, and if $\chi(D_i) > q(z' + d'\omega(G)) + p$ let P_i be the set of vertices $a \in A$ such that $\chi(U) \leq z' + d'\omega(G)$, where U is the set of neighbours of a in D_i . Let $P = P_1 \cup \dots \cup P_k$.

Suppose that $\chi(P_i) > p$, for some $i \in \{1, \dots, k\}$. Consequently $P_i \neq \emptyset$, and so

$$\chi(D_i) > q(z' + d'\omega(G)) + p \geq q(z' + d'\omega(G)) + \psi(\omega(P_i));$$

and for each $a \in P_i$, $\chi(U) \leq z' + d'\omega(G)$, where U is the set of neighbours of a in D_i . It follows that (v, P_i, D_i) is a (ψ, q) -scattering and the first bullet of the theorem holds. Thus we may assume that $\chi(P_i) \leq p$ for $1 \leq i \leq k$, and consequently $\chi(P) \leq p\omega(G)$.

Let Z be the union of P, D_0 , and all the sets D_i with $1 \leq i \leq k$ such that

$$\chi(D_i) \leq q(z' + d'\omega(G)) + p.$$

Consequently

$$\chi(Z) \leq 2p\omega(G) + z'\omega(G) + q(z' + d'\omega(G))\omega(G) \leq z.$$

Since (v, A, B) is a (d, z) -horn, it follows that $A \setminus Z$ is not d -dense to $B \setminus Z$; and so there exists $v' \in A \setminus P$ such that the set of vertices in $B \setminus Z$ that are nonadjacent to v' has chromatic number more than d . Since $B \setminus Z$ is the union of the sets D_i with $\chi(D_i) > q(z' + d'\omega(G)) + p$, there exists $i \in \{1, \dots, k\}$ with $\chi(D_i) \geq q(z' + d'\omega(G)) + p$ such that the set B' of vertices in D_i nonadjacent to v' has chromatic number more than $d/\omega(G)$. Since $v' \notin P$, the set A' of neighbours of v' in D_i has chromatic number more than $d'\omega(G) + z'$.

Let $Z' \subseteq D_i$ with $\chi(Z') \leq z'$. Thus $\chi(A' \setminus Z') \geq \chi(A') - \chi(Z') > d'\omega(G)$; and $\chi(B' \setminus Z') > d/\omega(G) - z' \geq d'\omega(G)$. Since $G[D_i]$ is $(d', z', \omega(G))$ -unsplittable, it follows that $A' \setminus Z'$ is not d' -dense to $B' \setminus Z'$. This proves that (v', A', B') is a (d', z') -horn.

Since (v, A, B) is ℓ -tall, there is an ℓ -vertex induced path P of G with one end v , such that $V(P) \setminus \{v\}$ is disjoint from and anticomplete to $A \cup B$. Then $P' = G[V(P) \cup \{v'\}]$ is an $(\ell + 1)$ -vertex path, and since $V(P)$ is anticomplete to B and hence to $A' \cup B'$, it follows that (v', A', B') is $(\ell + 1)$ -tall, and so the second bullet of the theorem holds. This proves 4.2. \blacksquare

Now we prove 2.2, which we restate:

4.3 *Let $k, s \geq 1$ and $q \geq 0$ be integers, and let $\psi, \sigma : \mathbb{N} \rightarrow \mathbb{N}$ be non-decreasing polynomials. Then there exists an integer $c \geq 0$ such that if G is a graph with $\chi(G) > \omega(G)^c$, and G contains no (ψ, q) -scattering, then G contains a (k, s) -broom and a σ -nondominating k -vertex path.*

Proof. Let $\zeta_k : \mathbb{N} \rightarrow \mathbb{N}$ be the polynomial defined by $\zeta_k(x) = \sigma(x) + x^s$, and let $\delta_k(x) = 0$. For $i = k - 1, \dots, 1$, define polynomials $\zeta_i, \delta_i : \mathbb{N} \rightarrow \mathbb{N}$ by

$$\begin{aligned} \zeta_i(x) &= 2x\psi(x) + (1 + q)x\zeta_{i+1}(x) + qx^2\delta_{i+1}(x) \\ \delta_i(x) &= x\zeta_{i+1}(x) + x^2\delta_{i+1}(x). \end{aligned}$$

Choose an integer $c \geq 2s$ such that

$$\begin{aligned} x^c &\geq (x - 1)^c + \zeta_1(x) + x\delta_1(x) + 2 \\ x^c &\geq (2s + 1)(\zeta_1(x) + 1) + (k - 2)\zeta_1(x), \text{ and} \\ x^c &\geq k\zeta_1(x) + \sigma(x) \end{aligned}$$

for all integers $x \geq 2$. We claim that c satisfies 4.3. To see this, let G be a graph with $\chi(G) > \omega(G)^c$, and suppose that G contains no (ψ, q) -scattering. We must show that G contains a (k, s) -broom and a σ -nondominating k -vertex path. We show this by induction on $|G|$. If there is an induced subgraph G' of G with $G' \neq G$ and $\chi(G') > \omega(G')^c$, then G' contains no (ψ, q) -scattering, and from the inductive hypothesis, G' contains a (k, s) -broom and a σ -nondominating k -vertex path, and hence so does G , as required. We may assume then that there is no such G' . Since $\chi(G) > \omega(G)^c$, it follows that $\omega(G) \geq 2$, and so the five displayed inequalities of 3.5 hold with z, d replaced by

$\zeta_1(\omega(G)), \delta_1(\omega(G))$ respectively. From 3.5, we may assume that G contains a $(\delta_1(\omega(G)), \zeta_1(\omega(G)))$ -horn, which is therefore 1-tall.

From 4.2, it follows that for $i = 2, \dots, k$, G contains an i -tall $(\delta_i(\omega(G)), \zeta_i(\omega(G)))$ -horn, and so contains a k -tall $(0, z)$ -horn (v, A, B) say, where $z = \zeta_k(\omega(G))$. Since this horn is k -tall, there is a k -vertex induced path P of G with one end v , such that $V(P) \setminus \{v\}$ is disjoint from and anticomplete to $A \cup B$. From the definition of a $(0, z)$ -horn, $\chi(A), \chi(B) > z$. Since $\chi(A) > z \geq \omega(A)^s$, 3.1 implies that there is a stable set $S \subseteq A$ with $|S| = s$, and so $G[V(P) \cup S]$ is a (k, s) -broom. Since $\chi(B) > z > \sigma(\omega(B))$, and $V(P)$ is anticomplete to B , P is σ -nondominating. This proves 4.3. ■

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