PIERCING AXIS-PARALLEL BOXES

MARIA CHUDNOVSKY, SOPHIE SPIRKL, AND SHIRA ZERBIB

ABSTRACT. Given a finite family $\mathcal{F}$ of axis-parallel boxes in $\mathbb{R}^d$ such that $\mathcal{F}$ contains no $k+1$ pairwise disjoint boxes, and such that for each two intersecting boxes in $\mathcal{F}$ a corner of one box is contained in the other box, we prove that $\mathcal{F}$ can be pierced by at most $ck \log\log(k)$ points, where c is a constant depending only on d. We further show that in some special cases the upper bound on the number of piercing points can be improved to ck. In particular, a linear bound in k holds if $\mathcal{F}$ consists of axis-parallel square boxes in $\mathbb{R}^d$, or if $\mathcal{F}$ is a family of axis-parallel rectangles in $\mathbb{R}^2$ in which the ratio between any of the side lengths of any rectangle is bounded by a constant.

1. INTRODUCTION

A *matching* in a hypergraph $H = (V, E)$ on vertex set V and edge set E is a subset of disjoint edges in E, and a *cover* of H is a subset of V that intersects all edges in E. The *matching number* $\nu(H)$ of H is the maximal size of a matching in H, and the *covering number* $\tau(H)$ of H is the minimal size of a cover. The fractional relaxations of these numbers are denoted as usual by $\nu^*(H)$ and $\tau^*(H)$. By LP duality we have that $\nu^*(H) = \tau^*(H)$.

Let $\mathcal{F}$ be a finite family of axis-parallel boxes in $\mathbb{R}^d$. We identify $\mathcal{F}$ with the hypergraph consisting with vertex set $\mathbb{R}^d$ and edge set $\mathcal{F}$. An old result due to Gallai is the following (see e.g. [**?**]):

Theorem 1.1 (Gallai)**.** *If $\mathcal{F}$ is a family of intervals in $\mathbb{R}$ (i.e., a family of boxes in $\mathbb{R}$) then $\tau(\mathcal{F}) = \nu(\mathcal{F})$.*

M. Chudnovsky: Princeton University, Princeton, NJ 08544. E-mail: mchudnov@math.princeton.edu. Supported by NSF grant DMS-1550991 and US Army Research Office Grant W911NF-16-1-0404.

S. Spirkl: Princeton University, Princeton, NJ 08544. E-mail: sspirkl@math.princeton.edu.

S. Zerbib: Department of Mathematics, University of Michigan, Ann Arbor. E-mail: zerbib@umich.edu. Partly supported by the New-England Fund, Technion.

A *rectangle* is an axis-parallel box in $\mathbb{R}^2$. In 1965, Wegner [**?**] conjectured that in a hypergraph of axis-parallel rectangles in $\mathbb{R}^2$, the ratio τ/ν is bounded by 2. Gýarfás and Lehel conjectured in [**?**] that the same ratio is bounded by a constant. The best known lower bound, $\tau = \lfloor 5\nu/3 \rfloor$, is attained by a construction due to Fon-Der-Flaass and Kostochka in [**?**]. Károlyi [**?**] proved that in families of axis-parallel boxes in $\mathbb{R}^d$ we have $\tau(\mathcal{F}) \le \nu(\mathcal{F})\left(1+\log\left(\nu(\mathcal{F})\right)\right)^{d-1}$, where $\log = \log_2$. Here is a short proof of Károlyi's bound.

Theorem 1.2 (Károlyi [**?**])**.** *If $\mathcal{F}$ is a finite family of axis-parallel boxes in $\mathbb{R}^d$, then $\tau(\mathcal{F}) \le \nu(\mathcal{F})\left(1+\log\left(\nu(\mathcal{F})\right)\right)^{d-1}$.*

Proof. We proceed by induction on d and $\nu(\mathcal{F})$. Note that if $\nu(\mathcal{F}) \in \{0,1\}$ then the result holds for all d. Now let $d, n \in \mathbb{N}$. Let $F_{d'} : \mathbb{R} \to \mathbb{R}$ be a function for which $\tau(\mathcal{T}) \le F_{d'}(\nu(\mathcal{T}))$ for every family $\mathcal{T}$ of axis-parallel boxes in $\mathbb{R}^{d'}$ with $d' < d$, or with $d = d'$ and $\nu(\mathcal{T}) < n$.

Let $\mathcal{F}$ be a family of axis-parallel boxes in $\mathbb{R}^d$ with $\nu(\mathcal{F}) = n$. For $a \in \mathbb{R}$, let H_a be the hyperplane $\{x = (x_1, \dots, x_d) : x_1 = a\}$. Write $L_a = \{x = (x_1, \dots, x_d) : x_1 \le a\}$, and let $\mathcal{F}_a = \{F \in \mathcal{F} : F \subseteq L_a\}$. Define $a^* = \min\{a : \nu(F_a) \ge \lceil \nu/2 \rceil\}$. The hyperplane H_{a^*} gives rise to a partition $\mathcal{F} = \bigcup_{i=1}^{3} \mathcal{F}_i$, where $\mathcal{F}_1 = \{F \in \mathcal{F} : F \subseteq L_{a^*} \setminus H_{a^*}\}$, $\mathcal{F}_2 = \{F \in \mathcal{F} : F \cap H_{a^*} \neq \emptyset\}$, and $\mathcal{F}_3 = \mathcal{F} \setminus (\mathcal{F}_1 \cup F_2)$. It follows from the choice of a^* that $\nu(\mathcal{F}_1) \le \lceil \nu(\mathcal{F})/2 \rceil - 1$, $\nu(\mathcal{F}_2) \le \nu(\mathcal{F})$, and $\nu(\mathcal{F}_3) \le \lfloor \nu(\mathcal{F})/2 \rfloor$.

Therefore,

$$
\begin{aligned}
F_d(\nu(\mathcal{F})) &\le \tau(\mathcal{F}_1) + \tau(\mathcal{F}_3) + \tau(\{F \cap H_{a^*} : F \in \mathcal{F}_2\}) \\
&\le F_d(\nu(\mathcal{F}_1)) + F_d(\nu(\mathcal{F}_3)) + F_{d-1}(\nu(\mathcal{F}_2)) \\
&\le F_d(\left\lceil \frac{\nu(\mathcal{F})}{2} \right\rceil - 1) + F_d(\left\lfloor \frac{\nu(\mathcal{F})}{2} \right\rfloor) + F_{d-1}(\nu(\mathcal{F})) \\
&\le 2\, \frac{\nu(\mathcal{F})}{2} \left(1 + \log\left(\frac{\nu(\mathcal{F})}{2}\right)\right)^{d-1} + \nu(\mathcal{F})\left(1+\log\left(\nu(\mathcal{F})\right)\right)^{d-2} \\
&\le \nu(\mathcal{F})\left(1+\log\left(\nu(\mathcal{F})\right)\right)^{d-1},
\end{aligned}
$$

implying the result. □

For $\nu(\mathcal{F}) = 1$, this implies the following well-known result (see e. g. [**?**]).

Observation 1.3. *Let $\mathcal{F}$ be a family of axis-parallel boxes with $\nu(\mathcal{F}) = 1$. Then $\tau(\mathcal{F}) = 1$.*

Note that for $\nu(\mathcal{F}) = 2$, we have that $\mathcal{F}_1 = \emptyset$, $\nu(\mathcal{F}_2) = 1$ and so $\tau(\mathcal{F}) \leq F_{d-1}(2) + 1$. Therefore, we have the following, which was also proved in [**?**].

Observation 1.4 (Fon-der-Flaass and Kostochka [**?**]). *Let $\mathcal{F}$ be a family of axis-parallel boxes in $\mathbb{R}^d$ with $\nu(\mathcal{F}) = 2$. Then $\tau(\mathcal{F}) \leq d + 1$.*

The bound from Theorem 1.2 was improved by Akopyan [**?**] to $\tau(\mathcal{F}) \leq (1.5 \log_3 2 + o(1))\nu(\mathcal{F}) \left(\log_2 (\nu(\mathcal{F}))\right)^{d-1}$.

A *corner* of a box F in $\mathbb{R}^d$ is a zero-dimensional face of F. We say that two boxes in $\mathbb{R}^d$ *intersect at a corner* if one of them contains a corner of the other.

A family $\mathcal{F}$ of connected subsets of $\mathbb{R}^2$ is a family of *pseudo-disks*, if for every pair of distinct subsets in $\mathcal{F}$, their boundaries intersect in at most two points. In [**?**], Chan and Har-Peled proved that families of pseudo-disks in $\mathbb{R}^2$ satisfy $\tau = O(\nu)$. It is easy to check that if $\mathcal{F}$ is a family of axis-parallel rectangles in $\mathbb{R}^2$ in which every two intersecting rectangles intersect at a corner, then $\mathcal{F}$ is a family of pseudo-disks. Thus we have:

Theorem 1.5 (Chan and Har-Peled [**?**]). *There exists a constant c such that for every family $\mathcal{F}$ of axis-parallel rectangles in $\mathbb{R}^2$ in which every two intersecting rectangles intersect at a corner, we have that $\tau(\mathcal{F}) \leq c\nu(\mathcal{F})$.*

Here we prove a few different generalizations of this theorem. In Theorem 1.6 we prove the bound $\tau(\mathcal{F}) \leq c\nu(\mathcal{F}) \log\log(\nu(\mathcal{F}))$ for families $\mathcal{F}$ of axis-parallel boxes in $\mathbb{R}^d$ in which every two intersecting boxes intersect at a corner, and in Theorem 1.7 we prove $\tau(\mathcal{F}) \leq c\nu(\mathcal{F})$ for families $\mathcal{F}$ of square axis-parallel boxes in $\mathbb{R}^d$, where in both cases c is a constant depending only on the dimension d. We further prove in Theorem 1.8 that in families $\mathcal{F}$ of axis-parallel boxes in $\mathbb{R}^d$ satisfying certain assumptions on their pairwise intersections, the bound on the covering number improves to $\tau(\mathcal{F}) \leq c\nu(\mathcal{F})$. For $d = 2$, these assumptions are equivalent to the assumption that there is a maximum matching $\mathcal{M}$ in $\mathcal{F}$ such that every intersection between a box in $\mathcal{M}$ and a box in $\mathcal{F} \setminus \mathcal{M}$ occurs at a corner. We use this result to prove our Theorem 1.10, asserting that for every r, if $\mathcal{F}$ is a family of axis-parallel rectangles in $\mathbb{R}^2$ with the property that the ratio between the side lengths of every rectangle in $\mathcal{F}$ is bounded by r, then $\tau(\mathcal{F}) \leq c\nu(\mathcal{F})$ for some constant c depending only on r.

Let us now describe our results in more detail. First, for general dimension d we have the following.

Theorem 1.6. *There exists a constant c depending only on d, such that for every family $\mathcal{F}$ of axis-parallel boxes in $\mathbb{R}^d$ in which every two intersecting boxes intersect at a corner we have $\tau(\mathcal{F}) \leq c\nu(\mathcal{F}) \log\log(\nu(\mathcal{F}))$.*

For the proof, we first prove the bound $\tau^*(\mathcal{F}) \leq 2^d\nu(\mathcal{F})$ on the fractional covering number of $\mathcal{F}$, and then use Theorem 1.11 below for the bound $\tau(\mathcal{F}) = O(\tau^*(\mathcal{F}) \log\log(\tau^*(\mathcal{F})))$.

An axis-parallel box is *square* if all its side lengths are equal. Note that if $\mathcal{F}$ consists of axis-parallel square boxes in $\mathbb{R}^d$, then every intersection in $\mathcal{F}$ occurs at a corner. Moreover, for axis-parallel square boxes we have $\tau(\mathcal{F}) = O(\tau^*(\mathcal{F}))$ by Theorem 1.11, and thus we conclude the following.

Theorem 1.7. *If $\mathcal{F}$ be a family of axis-parallel square boxes in $\mathbb{R}^d$, then $\tau(\mathcal{F}) \leq c\nu(\mathcal{F})$ for some constant c depending only on d.*

To get a constant bound on the ratio τ/ν in families of axis-parallel boxes in $\mathbb{R}^d$ which are not necessarily squares, we make a more restrictive assumption on the intersections in $\mathcal{F}$.

Theorem 1.8. *Let $\mathcal{F}$ be a family of axis-parallel boxes in $\mathbb{R}^d$. Suppose that there exists a maximum matching $\mathcal{M}$ in $\mathcal{F}$ such that for every $F \in \mathcal{F}$ and $M \in \mathcal{M}$, at least one of the following holds:*

(1) *F contains a corner of M;*
(2) *$F \cap M = \emptyset$; or*
(3) *M contains 2^{d-1} corners of F.*

Then $\tau(\mathcal{F}) \leq (2^d + (4+d)d)\nu(\mathcal{F})$.

For $d = 2$, this theorem implies the following corollary.

Corollary 1.9. *Let $\mathcal{F}$ be a family of axis-parallel rectangles in $\mathbb{R}^2$. Suppose that there exists a maximum matching $\mathcal{M}$ in $\mathcal{F}$ such that for every $F \in \mathcal{F}$ and $M \in \mathcal{M}$, if F and M intersect then they intersect at a corner. Then $\tau(\mathcal{F}) \leq 16\nu(\mathcal{F})$.*

Note that Corollary 1.9 is slightly stronger than Theorem 1.5. Here we only need that the intersections with rectangles in some fixed maximum matching $\mathcal{M}$ occur at corners, but we do not restrict the intersections of two rectangles $F, F' \notin \mathcal{M}$.

Given a constant $r > 0$, we say that a family $\mathcal{F}$ of axis-parallel boxes in $\mathbb{R}^d$ has an *r-bounded aspect ratio* if every box $F \in \mathcal{F}$ has $l_i(F)/l_j(F) \leq r$ for all $i, j \in \{1, \dots, d\}$, where $l_i(F)$ is the length of the orthogonal projection of F onto the ith coordinate.

For families of rectangles with bounded aspect ratio we prove the following.

Theorem 1.10. *Let $\mathcal{F}$ be a family of axis-parallel rectangles in $\mathbb{R}^2$ that has an r-bounded aspect ratio. Then $\tau(\mathcal{F}) \leq (14 + 2r^2)\nu(\mathcal{F})$.*

A result similar to Theorem 1.10 was announced in [**?**], but to the best of our knowledge the proof was not published.

An application of Theorem 1.10 is the existence of weak ε-nets of size $O\left(\frac{1}{\varepsilon}\right)$ for axis-parallel rectangles in $\mathbb{R}^2$ with bounded aspect ratio. More precisely, let P be a set of n points in $\mathbb{R}^d$ and let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$, each containing at least εn points of P. A *weak ε-net* for $\mathcal{F}$ is a cover of $\mathcal{F}$, and a *strong ε-net* for $\mathcal{F}$ is a cover of $\mathcal{F}$ with points of P. The existence of weak ε-nets of size $O\left(\frac{1}{\varepsilon}\right)$ for pseudo-disks in $\mathbb{R}^2$ was proved by Pyrga and Ray in [**?**]. Aronov, Ezra and Sharir in [**?**] showed the existence of strong ε-nets of size $O\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$ for axis-parallel boxes in $\mathbb{R}^2$ and $\mathbb{R}^3$, and the existence of weak ε-nets of size $O\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$ for all d was then proved by Ezra in [**?**]. Ezra also showed that for axis-parallel square boxes in $\mathbb{R}^d$ there exist an ε-net of size $O\left(\frac{1}{\varepsilon}\right)$. These results imply the following.

Theorem 1.11 (Aronov, Ezra and Sharir [**?**]; Ezra [**?**])**.** *If $\mathcal{F}$ is a family of axis-parallel boxes in $\mathbb{R}^d$ then $\tau(\mathcal{F}) \leq c\tau^*(\mathcal{F})\log\log(\tau^*(\mathcal{F}))$ for some constant c depending only on d. If $\mathcal{F}$ consists of square boxes, then this bound improves to $\tau(\mathcal{F}) \leq c\tau^*(\mathcal{F})$.*

An example where the smallest strong ε-net for axis-parallel rectangles in R^2 is of size $\Omega\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$ was constructed by Pach and Tardos in [**?**]. The question of whether weak ε-nets of size $O(\frac{1}{\varepsilon})$ for axis-parallel rectangles in $\mathbb{R}^2$ exist was raised both in [**?**] and in [**?**].

Theorem 1.10 implies a positive answer for the family of axis-parallel rectangles in $\mathbb{R}^2$ satisfying the r-bounded aspect ratio property:

Corollary 1.12. *For every fixed constant r, there exists a weak ε-net of size $O(\frac{1}{\varepsilon})$ for the family $\mathcal{F}$ of axis-parallel rectangles in $\mathbb{R}^2$ with aspect ratio bounded by r.*

Proof. Given a set P of n points, there cannot be $\frac{1}{\varepsilon}+1$ pairwise disjoint rectangles in $\mathcal{F}$, each containing at least εn points of P. Therefore $\nu(\mathcal{F}) \leq \frac{1}{\varepsilon}$. Theorem 1.10 implies that there is a cover of $\mathcal{F}$ of size $O(\frac{1}{\varepsilon})$. □

This paper is organized as follows. In Section 2 we prove Theorem 1.6. Section 3 contains definitions and tools. Theorem 1.8 is then proved in Section 4 and Theorem 1.10 is proved in Section 5.

2. Proof of Theorems 1.6 and 1.7

Let $\mathcal{F}$ be a finite family of axis-parallel boxes in $\mathbb{R}^d$, such that every intersection in $\mathcal{F}$ occurs at a corner. By performing small perturbations on the boxes, we may assume that no two corners of boxes of $\mathcal{F}$ coincide.

Proposition 2.1. *We have* $\tau^*(\mathcal{F}) \leq 2^d \nu(\mathcal{F})$.

Proof. Write $\nu(\mathcal{F}) = k$, and let $g : \mathcal{F} \to \mathbb{R}^+$ be a maximal fractional matching. By removing boxes $F \in \mathcal{F}$ for which $g(F) = 0$ and duplicating boxes if necessary, we may assume that $g(F) = \frac{1}{r}$ for all $F \in \mathcal{F}$, where r is the maximal size of a subset of boxes in $\mathcal{F}$ intersecting in a single point. Letting n be the number of boxes in $\mathcal{F}$ we have $\tau^*(H) = \frac{n}{r}$, and thus our aim is to show that $\frac{n}{r} \leq 2^d k$.

Since $\nu(\mathcal{F}) = k$, it follows from Turán's theorem that there are at least $n(n-k)/(2k)$ unordered intersecting pairs of boxes $\mathcal{F}$. Each such unordered pair contributes at least two pairs of the form (x, F), where x is a corner of a box $F' \in \mathcal{F}$, F is box in $\mathcal{F}$ different from F', and x pierces F. Therefore, since there are altogether $2^d n$ corners of boxes in $\mathcal{F}$, there must exist a corner x of a box $F \in \mathcal{F}$ that pierces at least $(n-k)/2^d k$ boxes in $\mathcal{F}$, all different from F. Together with F, x pierces at least $n/2^d k$ boxes, implying that $n/2^d k \leq r$. Thus $\frac{n}{r} \leq 2^d k$, as desired. □

Combining this bound with Theorem 1.11, we obtain the proofs of Theorems 1.6 and 1.7.

3. Definitions and tools

Let R be an axis-parallel box in $\mathbb{R}^d$ with $R = [x_1, y_1] \times \cdots \times [x_d, y_d]$. For $i \in \{1, \dots, d\}$, let $p_i(R) = [x_i, y_i]$ denote the orthogonal projection of R onto the i-th coordinate. Two intervals $[a, b], [c, d] \subseteq \mathbb{R}$, are *incomparable* if $[a, b] \not\subseteq [c, d]$ and $[c, d] \not\subseteq [a, b]$. We say that $[a, b] \prec [c, d]$ if $b < c$. For two axis-parallel boxes Q and R we say that $Q \prec_i R$ if $p_i(Q) \prec p_i(R)$.

Observation 3.1. *Let Q, R be disjoint axis-parallel boxes in $\mathbb{R}^d$. Then there exists $i \in \{1, \dots, d\}$ such that $Q \prec_i R$ or $R \prec_i Q$.*

Lemma 3.2. *Let Q, R be axis-parallel boxes in $\mathbb{R}^d$ such that Q contains a corner of R but R does not contain a corner of Q. Then, for all $i \in \{1, \dots, d\}$, either $p_i(R)$ and $p_i(Q)$ are incomparable, or $p_i(R) \subseteq p_i(Q)$, and there exists $i \in \{1, \dots, d\}$ such that $p_i(R) \subsetneq p_i(Q)$.*

Moreover, if $R \not\subseteq Q$, then there exists $j \in \{1, \dots, d\} \setminus \{i\}$ such that $p_i(R)$ and $p_i(Q)$ are incomparable.

Proof. Let $x = (x_1, \dots, x_d)$ be a corner of R contained in Q. By symmetry, we may assume that $x_i = \max(p_i(R))$ for all $i \in \{1, \dots, d\}$. Since $x_i \in p_i(Q)$ for all $i \in \{1, \dots, d\}$, it follows that $\max(p_i(Q)) \geq \max(p_i(R))$ for all $i \in \{1, \dots, d\}$. If $\min(p_i(Q)) \leq \min(p_i(R))$, then $p_i(R) \subseteq p_i(Q)$; otherwise, $p_i(Q)$ and $p_i(R)$ are incomparable. If $p_i(Q)$ and $p_i(R)$ are incomparable for all $i \in \{1, \dots, d\}$, then $y = (y_1, \dots, y_d)$ with $y_i = \min(p_i(Q))$ is a corner of Q and since $\min(p_i(Q)) > \min(p_i(R))$, it follows that $y \in R$, a contradiction. It follows that there exists an $i \in \{1, \dots, d\}$ such that $p_i(R) \subsetneq p_i(Q)$.

If $p_i(R) \subsetneq p_i(Q)$ for all $i \in \{1, \dots, d\}$, then $R \subseteq Q$; this implies the result. □

Lemma 3.3. *Let $\mathcal{F}$ be a family of axis-parallel boxes in $\mathbb{R}^d$. Let $\mathcal{F}'$ arise from $\mathcal{F}$ by removing every box in $\mathcal{F}$ that contains another box in $\mathcal{F}$. Then $\nu(\mathcal{F}) = \nu(\mathcal{F}')$ and $\tau(\mathcal{F}) = \tau(\mathcal{F}')$.*

Proof. Since $\mathcal{F}' \subseteq \mathcal{F}$, it follows that $\nu(\mathcal{F}') \leq \nu(\mathcal{F})$ and $\tau(\mathcal{F}') \leq \tau(\mathcal{F})$. Let $\mathcal{M}$ be a matching in $\mathcal{F}$ of size $\nu(\mathcal{F})$. Let $\mathcal{M}'$ arise from $\mathcal{M}$ by replacing each box R in $\mathcal{M} \setminus \mathcal{F}'$ with a box in $\mathcal{F}'$ contained in R. Then $\mathcal{M}'$ is a matching in $\mathcal{F}'$, and so $\nu(\mathcal{F}') = \nu(\mathcal{F})$. Moreover, let P be a cover of $\mathcal{F}'$. Since every box in $\mathcal{F}$ contains a box in F' (possibly itself) which, in turn, contains a point in P, we deduce that P is a cover of $\mathcal{F}$. It follows that $\tau(\mathcal{F}') = \tau(\mathcal{F})$. □

A family $\mathcal{F}$ of axis-parallel boxes is *clean* if no box in $\mathcal{F}$ contains another box in $\mathcal{F}$. By Lemma 3.3, we may restrict ourselves to clean families of boxes.

4. Proof of Theorem 1.8

Throughout this section, let $\mathcal{F}$ be a clean family of axis-parallel boxes in $\mathbb{R}^d$, and let $\mathcal{M}$ be a matching of maximum size in $\mathcal{F}$. We let $\mathcal{F}(\mathcal{M})$ denote the subfamily of $\mathcal{F}$ consisting of those boxes R in $\mathcal{F}$ for which for every $M \in \mathcal{M}$, either M is disjoint from R or M contains at least 2^{d-1} corners of R. Our goal is to bound $\tau(\mathcal{F}(\mathcal{M}))$.

Lemma 4.1. *Let $R \in \mathcal{F}(\mathcal{M})$. Then R intersects at least one and at most two boxes in $\mathcal{M}$. If R intersects two boxes $M_1, M_2 \in \mathcal{M}$, then there exists $j \in \{1, \dots, d\}$ such that $M_1 \prec_j M_2$ or $M_2 \prec_j M_1$, and for all $i \in \{1, \dots, d\} \setminus \{j\}$, we have that $p_i(R) \subseteq p_i(M_1)$ and $p_i(R) \subseteq p_i(M_2)$.*

Proof. If R is disjoint from every box in $\mathcal{M}$, then $\mathcal{M} \cup \{R\}$ is a larger matching, a contradiction. So R intersects at least one box in $\mathcal{M}$. Let M_1 be in $\mathcal{M}$ such that $R \cap M_1 \neq \emptyset$. We claim that there exists

$j \in \{1, \dots, d\}$ such that M_1 contains precisely the set of corners of R with the same jth coordinate.

By Lemma 3.2, there exists $j \in \{1, \dots, d\}$ such that $p_j(R) = [a, b]$ and $p_j(M_1)$ are incomparable. By symmetry, we may assume that $a \in p_j(M_1)$, $b \notin p_j(M_1)$. This proves that M_1 contains all 2^{d-1} corners of R with a as their jth coordinate, and our claim follows.

Consequently, $p_i(R) \subseteq p_i(M_1)$ for all $i \in \{1, \dots, d\} \setminus \{j\}$. Since R has exactly 2^d corners, and members of $\mathcal{M}$ are disjoint, it follows that there exist at most two boxes in $\mathcal{M}$ that intersect R. If M_1 is the only one such box, then the result follows. Let $M_2 \in \mathcal{M} \setminus \{M_1\}$ such that $R \cap M_1 \neq \emptyset$. By our claim, it follows that M_2 contains 2^{d-1} corners of R; and since M_1 is disjoint from M_2, it follows that M_2 contains precisely those corners of R with jth coordinate equal to b. Therefore, $p_i(R) \subseteq p_i(M_2)$ for all $i \in \{1, \dots, d\} \setminus \{j\}$. We conclude that $p_i(M_2)$ is not disjoint from $p_i(M_1)$ for all $i \in \{1, \dots, d\} \setminus \{j\}$, and since M_1, M_2 are disjoint, it follows from Observation 3.1 that either $M_1 \prec_j M_2$ or $M_2 \prec_j M_1$. □

For $i \in \{1, \dots, d\}$, we define a directed graph G_i as follows. We let $V(G_i) = \mathcal{M}$, and for $M_1, M_2 \in \mathcal{M}$ we let $M_1M_2 \in E(G_i)$ if and only if $M_1 \prec_i M_2$ and there exists $R \in \mathcal{F}(\mathcal{M})$ such that $R \cap M_1 \neq \emptyset$ and $R \cap M_2 \neq \emptyset$. In this case, we say that R *witnesses* the edge M_1M_2. For $i = \{1, \dots, d\}$, we say that R is *i-pendant* at $M_1 \in \mathcal{M}$ if M_1 is the only box of $\mathcal{M}$ intersecting R and $p_i(R)$ and $p_i(M_1)$ are incomparable. Note that by Lemma 4.1, every box R in $\mathcal{F}(\mathcal{M})$ satisfies exactly one of the following: R witnesses an edge in exactly one of the graphs G_i, $i \in \{1, \dots, d\}$; or R is i-pendant for exactly one $i \in \{1, \dots, d\}$.

Lemma 4.2. *Let $i \in \{1, \dots, d\}$. Let $Q, R \in \mathcal{F}(\mathcal{M})$ be such that Q witnesses an edge M_1M_2 in G_i, and R witnesses an edge M_3M_4 in G_i. If Q and R intersect, then either $M_1 = M_4$, or $M_2 = M_3$, or $M_1M_2 = M_3M_4$.*

Proof. By symmetry, we may assume that $i = 1$. Let $p_1(M_1) = [x_1, y_1]$ and $p_1(M_2) = [x_2, y_2]$. It follows that $p_1(Q) \subseteq [x_1, y_2]$. Let $a = (a_1, a_2, \dots, a_d) \in Q \cap R$. It follows that $a_j \in p_j(Q) \subseteq p_j(M_1) \cap p_j(M_2)$ and $a_j \in p_j(R) \subseteq p_j(M_3) \cap p_j(M_4)$ for all $j \in \{2, \dots, d\}$.

If $M_1 \in \{M_3, M_4\}$ and $M_2 \in \{M_3, M_4\}$, then $M_1M_2 = M_3M_4$, and the result follows. Therefore, we may assume that this does not happen. If $M_1 \in \{M_3, M_4\}$, we reflect every rectangle in $\mathcal{F}$ along the origin. When constructing G_1 for this family, we have $M_4M_3, M_2M_1 \in E(G_1)$, and $M_2 \notin \{M_3, M_4\}$. Thus, by symmetry, we may assume that M_1 is distinct from M_3 and M_4.

It follows that $a \notin M_1$, for otherwise R intersects three distinct members of $\mathcal{M}$, contrary to Lemma 4.1. Since R is disjoint from M_1, it follows that either $M_1 \prec_1 R$ or $R \prec_1 M_1$. But $p_1(Q) \subseteq [x_1, y_2]$, and since $Q \cap R \neq \emptyset$, it follows that $M_1 \prec_1 R$.

Since $M_3 \neq M_1$ and $p_j(M_3) \cap p_j(M_1) \ni a_j$ for all $j \in \{2, \ldots, d\}$, it follows that either $M_1 \prec_1 M_3$ or $M_1 \prec_1 M_3$. Since $M_1 \prec_1 R$ and $R \cap M_3 \neq \emptyset$, it follows that $M_1 \prec_1 M_3$.

Suppose that $a \in M_3$. Then $Q \cap M_3 \neq \emptyset$, and since $M_1 \prec_1 M_3$, we have that $M_3 = M_2$ as desired.

Therefore, we may assume that $a \notin M_3$, and thus $p_1(M_1) \prec p_1(M_3) \prec [a_1, a_1]$. Since $[y_1, a_1] \subseteq p_1(Q)$, it follows that $p_1(M_3) \cap p_1(Q) \neq \emptyset$. But $p_j(M_3) \cap p_j(Q) \ni a_j$ for all $j \in \{2, \ldots, d\}$, and hence $Q \cap M_3 \neq \emptyset$. But then $M_3 \in \{M_1, M_2\}$, and thus $M_3 = M_2$. This concludes the proof. □

The following is a well-known fact about directed graphs; we include a proof for completeness.

Lemma 4.3. *Let G be a directed graph. Then there exists an edge set $E \subseteq E(G)$ with $|E| \geq |E(G)|/4$ such that for every vertex $v \in V(G)$, either E contains no incoming edge at v, or E contains no outgoing edge at v.*

Proof. For $A, B \subseteq V(G)$, let $E(A, B)$ denote the set of edges of G with head in A and tail in B.

Let $X_0 = Y_0 = \emptyset$, $V(G) = \{v_1, \ldots, v_n\}$. For $i = 1, \ldots, n$ we will construct X_i, Y_i such that $X_i \cup Y_i = \{v_1, \ldots, v_i\}$, $X_i \cap Y_i = \emptyset$ and $|E(X_i, Y_i)| + |E(Y_i, X_i)| \geq |E(G|(X_i \cup Y_i))|/2$, where $G|(X_i \cup Y_i)$ denotes the induced subgraph of G on vertex set $X_i \cup Y_i$. This holds for X_0, Y_0. Suppose that we have constructed X_{i-1}, Y_{i-1} for some $i \in \{1, \ldots, n\}$. If $|E(X_{i-1}, \{v_i\})| + |E(\{v_i\}, X_{i-1})| \geq |E(Y_{i-1}, \{v_i\})| + |E(\{v_i\}, Y_{i-1})|$, we let $X_i = X_{i-1}, Y_i = Y_{i-1} \cup \{v_i\}$; otherwise, let $X_i = X_{i-1} \cup \{v_i\}, Y_i = Y_{i-1}$. It follows that X_i, Y_i still have the desired properties. Thus, $|E(X_n, Y_n)| + |E(Y_n, X_n)| \geq |E(G)|/2$. By symmetry, we may assume that $|E(X_n, Y_n)| \geq |E(G)|/4$. But then $E(X_n, Y_n)$ is the desired set E; it contains only incoming edges at vertices in X_n, and only outgoing edges at vertices in Y_n. This concludes the proof. □

Theorem 4.4. *For $i \in \{1, \ldots, d\}$, $|E(G_i)| \leq 4\nu(\mathcal{F})$.*

Proof. Let $E \subseteq E(G_i)$ as in Lemma 4.3. For each edge in E, we pick one box witnessing this edge; let $\mathcal{F}'$ denote the family of these boxes. We claim that $\mathcal{F}'$ is a matching. Indeed, suppose not, and let $Q, R \in \mathcal{F}'$ be distinct and intersecting. Let Q witness M_1M_2 and R

witness M_3M_4. By Lemma 4.2, it follows that either $M_1M_2 = M_3M_4$ (impossible since we picked exactly one witness per edge) or $M_1 = M_4$ (impossible because E does not contain both an incoming and an outgoing edge at $M_1 = M_4$) or $M_2 = M_3$ (impossible because E does not contain both an incoming and an outgoing edge at $M_2 = M_3$). This is a contradiction, and our claim follows. Now we have $\nu(\mathcal{F}) \geq |\mathcal{F}'| = |E| \geq |E(G)|/4$, which implies the result. □

A matching $\mathcal{M}$ of a clean family $\mathcal{F}$ of boxes is *extremal* if for every $M \in \mathcal{M}$ and $R \in \mathcal{F} \setminus \mathcal{M}$, either $(\mathcal{M} \setminus \{M\}) \cup \{R\}$ is not a matching or there exists an $i \in \{1, \ldots, d\}$ such that $\max(p_i(R)) \geq \max(p_i(M))$. Every family $\mathcal{F}$ of axis parallel boxes has an extremal maximum matching. For example, the maximum matching $\mathcal{M}$ minimizing $\sum_{M \in \mathcal{M}} \sum_{i=1}^{d} \max(p_i(M))$ is extremal.

Theorem 4.5. *For $i \in \{1, \ldots, d\}$, let $\mathcal{F}_i$ denote the set of boxes in $\mathcal{F}(\mathcal{M})$ that either are i-pendant or witness an edge in G_i. Then $\tau(\mathcal{F}_i) \leq (4 + d)\nu(\mathcal{F})$. If $\mathcal{M}$ is extremal, then $\tau(\mathcal{F}_i) \leq (3 + d)\nu(\mathcal{F})$.*

Proof. By symmetry, it is enough to prove the theorem for $i = 1$. For $M \in \mathcal{M}$, let $\mathcal{F}_M$ denote the set of boxes in $\mathcal{F}_1$ that either are 1-pendant at M, or witness an edge MM' of G_1. It follows that $\bigcup_{M \in \mathcal{M}} \mathcal{F}_M = \mathcal{F}_1$. For $M \in \mathcal{M}$, let $d^+(M)$ denote the out-degree of M in G_1. We will prove that $\tau(\mathcal{F}_M) \leq d^+(M) + d$ for all $M \in \mathcal{M}$.

Let $M \in \mathcal{M}$, and let $\mathcal{A}$ denote the set of boxes that are 1-pendant at M. Suppose that $\mathcal{A}$ contains two disjoint boxes M_1, M_2. Then $(\mathcal{M} \setminus \{M\}) \cup \{M_1, M_2\}$ is a larger matching than $\mathcal{M}$, a contradiction. So every two boxes in $\mathcal{A}$ pairwise intersect. By Observation 1.3, it follows that $\tau(A) = 1$.

Let $\mathcal{B} = \mathcal{F}_M \setminus \mathcal{A}$. Suppose that there is an edge $MM' \in E(G_1)$ such that the set $\mathcal{B}(M')$ of boxes in $\mathcal{B}$ that witness the edge MM' satisfies $\nu(\mathcal{B}(M')) \geq 3$. Then $\mathcal{M}$ is not a maximum matching, since removing M and M' from $\mathcal{M}$ and adding $\nu(\mathcal{B}(M'))$ disjoint rectangles in $\mathcal{B}(M')$ yields a larger matching. Moreover, for distinct $M', M'' \in \mathcal{M}$, every box in $\mathcal{B}(M')$ is disjoint from every box in $\mathcal{B}(M'')$ by Lemma 4.2. Thus, if there exist M', M'' such that $\nu(\mathcal{B}(M')) = \nu(\mathcal{B}(M'')) = 2$ and $M' \neq M''$, then removing M, M' and M'' and adding two disjoint rectangles from each of $\mathcal{B}(M')$ and $\mathcal{B}(M'')$ yields a bigger matching, a contradiction.

Let $p_1(M) = [a, b]$. Two boxes in $\mathcal{B}(M')$ intersect if and only if their intersections with the hyperplane $H = \{(x_1, \ldots, x_d) : x_1 = b\}$ intersect. If $\nu(\mathcal{B}(M')) = 1$, then $\tau(\mathcal{B}(M')) = 1$ by Observation 1.3. If $\nu(\mathcal{B}(M')) =$

2, then $\nu(\{F \cap H : F \in \mathcal{B}(M')\}) = 2$ and so

$$\tau(\mathcal{B}(M')) = \tau(\{F \cap H : F \in \mathcal{B}(M')\}) \leq d$$

by Observation 1.4.

Therefore,

$$\tau(\mathcal{B}) \leq \sum_{M' : MM' \in E(G_1)} \tau(\mathcal{B}(M')) \leq d^+(M) - 1 + d,$$

and since $\tau(\mathcal{A}) \leq 1$, it follows that $\tau(\mathcal{F}_M) \leq d^+(M) + d$ as claimed.

Summing over all rectangles in $\mathcal{M}$, we obtain

$$\begin{aligned}\tau(\mathcal{F}_i) &\leq \sum_{M \in \mathcal{M}} \tau(\mathcal{F}_M) \leq \sum_{M \in \mathcal{M}} (d^+(M) + d) \\ &= d|V(G_1)| + |E(G_1)| \leq d|\mathcal{M}| + 4|\mathcal{M}| = (4+d)\nu(\mathcal{F}),\end{aligned}$$

which proves the first part of the theorem.

If $\mathcal{M}$ is extremal, then every 1-pendant box at M also intersects H. Let M' be such that $\nu(\mathcal{B}(M'))$ is maximum. It follows that $\nu(\mathcal{A} \cup \mathcal{B}(M')) \leq 2$ and thus $\tau(\mathcal{A} \cup \mathcal{B}(M')) \leq d$, implying $\tau(\mathcal{F}_M) \leq d^+(M) + d - 1$. This concludes the proof of the second part of the theorem. □

Theorem 4.6. *Let $\mathcal{F}' \subseteq \mathcal{F}$ be the set of boxes $R \in \mathcal{F}$ such that for each $M \in \mathcal{M}$, either $M \cap R = \emptyset$, or M contains 2^{d-1} corners of R, or R contains a corner of M. Then $\tau(\mathcal{F}') \leq (2^d + (4+d)d)\nu(\mathcal{F})$. If $\mathcal{M}$ is extremal, then $\tau(\mathcal{F}') \leq (2^d + (3+d)d)\nu(\mathcal{F})$.*

Proof. We proved in Theorem 4.5 that $\tau(\mathcal{F}_i) \leq (4+d)\nu(\mathcal{F})$ for $i = 1, \ldots, d$. Let $\mathcal{F}'' = \mathcal{F}' \setminus \mathcal{F}(\mathcal{M})$. Then $\mathcal{F}''$ consists of boxes R such that R contains a corner of some box $M \in \mathcal{M}$. Let P be the set of all corners of boxes in $\mathcal{M}$. It follows that P covers $\mathcal{F}''$, and so $\tau(\mathcal{F}'') \leq 2^d\nu(\mathcal{F})$. Since $\mathcal{F}' = \mathcal{F}'' \cup \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_d$, it follows that $\tau(\mathcal{F}') \leq (2^d + (4+d)d)\nu(\mathcal{F})$. If $\mathcal{M}$ is extremal, the same argument yields that $\tau(\mathcal{F}') \leq (2^d + (3+d)d)\nu(\mathcal{F})$, since $\tau(\mathcal{F}_i) \leq (3+d)\nu(\mathcal{F})$ for $i = 1 \ldots, d$ by Theorem 4.5. □

We are now ready to prove our main theorems.

Proof of Theorem 1.8. Let $\mathcal{F}$ be a family of axis-parallel boxes in $\mathbb{R}^d$, and let $\mathcal{M}$ be a maximum matching in $\mathcal{F}$ such that for every $F \in \mathcal{F}$ and $M \in \mathcal{M}$, either $F \cap M = \emptyset$, or F contains a corner of M, or M contains 2^{d-1} corners of F. It follows that $\mathcal{F} = \mathcal{F}'$ in Theorem 4.6, and therefore, $\tau(\mathcal{F}) \leq (2^d + (4+d)d)\nu(\mathcal{F})$. □

5. Proof of Theorem 1.10

Let $\mathcal{M}$ be a maximum matching in $\mathcal{F}$, and let $\mathcal{M}$ be extremal. Observe that each rectangle $R \in \mathcal{F}$ satisfies one of the following:

- R contains a corner of some $M \in \mathcal{M}$;
- some $M \in \mathcal{M}$ contains two corners of R; or
- there exists $M \in \mathcal{M}$ such that $M \cap R \neq \emptyset$, and $p_i(R) \supseteq p_i(M)$ for some $i \in \{1, 2\}$.

By Theorem 4.6, $14\nu(\mathcal{F})$ points suffice to cover every rectangle satisfying at least one of the first two conditions. Now, due to the r-bounded aspect ratio, for each $M \in \mathcal{M}$ and for each $i \in \{1, 2\}$, at most r^2 disjoint rectangles $R \in \mathcal{F}$ can satisfy the third condition for M and i. Thus the family of projections of the rectangles satisfying the third condition for M and i onto the $(3-i)$th coordinate have a matching number at most r^2. Since all these rectangles intersect the boundary of M twice, by Theorem 1.1, we need at most r^2 additional points to cover them. We conclude that $\tau(\mathcal{F}) \leq (14 + 2r^2)\nu(\mathcal{F})$. □

Acknowledgments

We are thankful to Paul Seymour for many helpful discussions.