

Attempting perfect hypergraphs

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Abstract

We study several extensions of the notion of perfect graphs to k -uniform hypergraphs. One main definition extends to hypergraphs the notion of perfect graphs based on coloring. Let G be a k -uniform hypergraph. A coloring of a k -uniform hypergraph G is proper if it is a coloring of the $(k-1)$ -tuples with elements in $V(G)$ in such a way that no edge of G is a monochromatic K_k^{k-1} .

A k -uniform hypergraph G is C_ω -perfect if for every induced subhypergraph G' of G we have:

- if $X \subseteq V(G')$ with $|X| < k-1$, then there is a proper $(\omega(G')-k+2)$ -coloring of G' (so $(k-1)$ -tuples are colored) that restricts to a proper $(\omega(G')-k+2)$ -coloring of $lk_{G'}(X)$ (so $(k-|X|-1)$ -tuples are colored).

Another main definition is the following: A k -uniform hypergraph G is hereditary perfect (or, briefly, H -perfect) if all links of sets of $(k-2)$ vertices are perfect graphs.

The notion of C_ω perfectness is not closed under complementation (for $k > 2$) and we define G to be doubly perfect if both G and its complement are C_ω perfect. We study doubly-perfect hypergraphs: In addition to perfect graphs nontrivial doubly-perfect graphs consist of a restricted interesting class of 3-uniform hypergraphs, and within this class we give a complete characterization of doubly-perfect H -perfect hypergraphs.

1 Introduction

The purpose of this paper is to study some extensions of the notion of perfect graphs to k -uniform hypergraphs, $k > 2$. We start with basic notation

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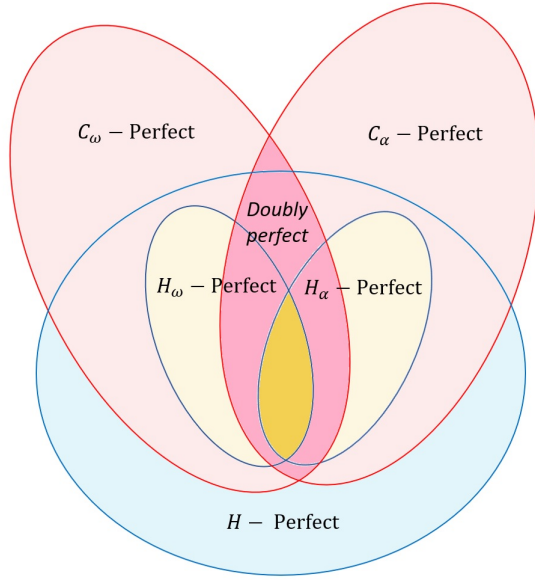


Figure 1: Classes of perfect hypergraphs considered in this paper. The figure describes (out of scale) the case of 3-uniform hypergraphs. For graphs all the classes coincide with the class of perfect graphs. For k -uniform hypergraphs, $k > 3$ the class of doubly-perfect hypergraphs contains only the empty hypergraph and complete hypergraph.

and terminology regarding hypergraphs followed by basic definitions and properties of perfect graphs. Denote by K_q^p the complete p -uniform hypergraph on q vertices. Let G be a k -uniform hypergraph. The complement G^c of G is the k -uniform hypergraph such that a k -subset X of $V(G)$ is a hyperedge of G^c if and only if X is not a hyperedge of G . For $X \subseteq V(G)$ we denote by $G[X]$ the k -uniform hypergraph induced by G on X , and by $lk_G(X)$ the $(k - |X|)$ -uniform hypergraph with vertex set $V(G) \setminus X$ and such that $Y \in E(G(X))$ if and only if $X \cup Y \in E(G)$ (this is the link of X). Observe that $lk_G^c(X) = lk_{G^c}(X)$. We denote by $\omega(G)$ the maximum size of $X \subseteq V(G)$ such that $G[X]$ is a complete k -uniform hypergraph, and by $\alpha(G)$ the maximum size of $X \subseteq V(G)$ such that $G^c[X]$ is a complete k -uniform hypergraph.

A graph G is perfect if for every induced subgraph H of G , $\chi(H) = \omega(H)$. The Weak Perfect Graph Theorem [16] asserts that G is perfect iff its complement G^c is perfect. The Strong Perfect Graph Theorem [5] asserts that G is perfect iff G is a Berge graph, namely neither G nor G^c contains an induced cycle of odd length greater than 3. The class of perfect graphs is a remarkable class of graphs with profound connections in mathematics, theoretical computer science and optimization. There is also a rich area of graph theory devoted to the study of graphs with forbidden induced subgraphs, and perfect graphs play a central role in

this study.

Our purpose here is to propose several extensions of the notion of perfect graphs to k -uniform hypergraphs, $k > 2$. Our main definition extends to hypergraphs the notion of perfect graphs based on coloring. Let G be a k -uniform hypergraph. A coloring of G is proper if it is a coloring of the $(k-1)$ -tuples with elements in $V(G)$ in such a way that no edge of G is a monochromatic K_k^{k-1} .

A k -uniform hypergraph is C_ω -perfect if for every induced subhypergraph G' of G we have:

- if $X \subseteq V(G')$ with $|X| < k-1$, then there is a proper $(\omega(G')-k+2)$ -coloring of G' (so $(k-1)$ -tuples are colored) that restricts to a proper $(\omega(G')-k+2)$ -coloring of $lk_{G'}(X)$ (so $(k-|X|-1)$ -tuples are colored).

In Section 2 we show that a weaker coloring property which resembles the Berge property for graphs suffices for C_ω -perfectness. Define next G to be C_α -perfect if its complement G^c is C_ω -perfect. Now, call G doubly perfect if G is both C_ω -perfect and C_α -perfect. We study the class of doubly-perfect hypergraphs in Section 3. A k -uniform hypergraph is clique friendly if every set of $k+1$ vertices contains either $k+1$ edges or at most two edges. It follows easily from the definition that C_ω -perfect hypergraphs are clique friendly and this implies that if G is a doubly-perfect hypergraph which is neither complete nor empty, then $k \leq 3$. Moreover, for $k=3$ to be doubly perfect G must be a cocycle, namely every 4 vertices span an even number of edges. An equivalent definition of cocycles which we rely on in Section 3 is the following: for a graph G we write $co(G)$ to be the 3-uniform hypergraph whose edges correspond to triples of vertices of G that span an odd number of edges. Every 3-uniform cocycle C can be written as $co(G)$ for some graph G .

A k -uniform hypergraph G is H -perfect (or hereditary perfect) if the link $lk_G(X)$ of every set X of $k-2$ vertices is a perfect graph. A k -uniform hypergraph is H_ω -perfect if it is H -perfect and clique friendly. In Section 2 we also prove that H_ω -perfect hypergraphs are C_ω -perfect. (The converse is not true.) Examples of H_ω -perfect (hence also C_ω -perfect) hypergraphs include k -partite hypergraphs, simple hypergraphs and hypergraphs of k -cliques of perfect graphs. In Section 3 we also give a full description of doubly-perfect hypergraphs which are also H -perfect.

Our notion of doubly-perfect hypergraphs is closely related (and yet not identical) to Simonyi's notion of "entropy splitting hypergraphs" [21]. This is also discussed in Section 3.

Our concluding Section 4 discusses other related notions of perfectness and possible connections. We note that a different notion of perfectness for hypergraphs was pioneered by Voloshin [22, 23] and further studied by Bujtás and Tuza [4]. Seeking hypergraph analogs of perfect graphs fits the 'high-dimensional combinatorics' programme of Linial [14, 15].

2 Perfect hypergraphs and proper colorings

2.1 Basic properties of C_ω perfect hypergraphs

2.1. Let G be a C_ω -perfect k -uniform hypergraph with $|V(G)| = k + 1$. Let $X \subseteq V(G)$ with $|X| < k - 1$, and suppose that $lk_G(X)$ is a clique. Then G is a clique. Consequently, every C_ω -perfect hypergraph is clique friendly.

Proof. Suppose G is not a clique, and so $\omega(G) = k$. Let $a, b, c \in V(G) \setminus X$. Since G is C_ω -perfect, there exists a proper 2-coloring of G (so the $(k-1)$ -tuples are colored) that restricts to a proper 2-coloring of the complete graph with vertex set $\{a, b, c\}$ (so the vertices are colored), a contradiction. This proves 2.1. \square

2.2. Let G be a clique-friendly k -uniform hypergraph with $|V(G)| = k + 1$. Let $X \subseteq V(G)$ with $|X| < k - 1$, and suppose that $lk_G(X)$ is a clique. Then G is a clique.

Proof. Suppose G is not a clique. Then $|E(G)| < k + 1$. Since $lk_G(X)$ is a clique, and $|V(G)| - |X| \geq 3$, we have that $|E(G)| \geq 3$, contrary to the fact that G is clique friendly. This proves 2.2. \square

2.3. If G is clique friendly then for every $X \subseteq V(G)$ with $|X| < k - 1$, and every clique K in $lk_G(X)$, we have that $K \cup X$ is a clique of G . Consequently, $\omega(lk_G(X)) \leq \omega(G) - |X|$.

Proof. First we prove the first statement. Induction on $|X|$. Let K be a clique in $lk_G(X)$. Assume first that $X = \{x\}$. By 2.2, $K \cup \{x\}$ is a clique of G , as required.

Now let $x \in X$; write $X' = X \setminus \{x\}$. Then K is a clique in $lk_G(x)(X')$, and so inductively $K \cup X'$ is a clique in $lk_G(x)$. But also, as we have seen in the base case of the induction, $(K \cup X') \cup \{x\}$ is a clique in G , as required. This proves the first statement.

To prove the second statement, let K be a clique in $lk_G(X)$ of size $\omega(lk_G(X))$. We have proved that $K \cup X$ is a clique of G , and consequently $\omega(G) \geq \omega(lk_G(X)) + |X|$.

This proves 2.3. \square

Now 2.1 and 2.3 imply:

2.4. If G is C_ω -perfect then for every $X \subseteq V(G)$ with $|X| < k - 1$, $\omega(lk_G(X)) \leq \omega(G) - |X|$.

2.2 Berge is perfect

A k -uniform hypergraph is Berge if for every induced subhypergraph G' of G we have:

- If $X \subseteq V(G')$ with $|X| = k - 2$, then there is a proper $(\omega(G') - k + 2)$ -coloring of G' (so $(k-1)$ -tuples are colored) that restricts to a proper $(\omega(G') - k + 2)$ -coloring of $lk_{G'}(X)$ (so $(k - |X| - 1)$ -tuples are colored).

2.5. A k -uniform hypergraph G is Berge iff it is C_ω -perfect.

Proof. Clearly if G is C_ω -perfect, then G is Berge. Thus we assume that G is Berge and show that G is C_ω -perfect. Since being Berge and being C_ω -perfect are both closed under taking induced subhypergraphs, it is enough to prove that:

- If $X \subseteq V(G)$ with $|X| \leq k-2$, then there is a proper $(\omega(G)-k+2)$ -coloring of G (so $(k-1)$ -tuples are colored) that restricts to a proper $(\omega(G)-k+2)$ -coloring of $lk_G(X)$ (so $(k-|X|-1)$ -tuples are colored).

Let $X \subseteq V(G)$. If $|X| = k-2$, the statement follows immediately from the first bullet in the definition of Bergeness. So we may assume that $|X| < k-2$.

Order the vertices of $V(G)$ as v_1, \dots, v_n so that $X = \{v_1, \dots, v_{|X|}\}$. Let $Y \subseteq V(G)$ with $|Y| = k-2$ and let i be maximum such that $v_i \in Y$. Let $R(Y) = lk_G(Y)[v_{i+1}, \dots, v_n]$. Then $R(Y)$ is a graph. Since $|Y| = k-2$, the first bullet of the definition of Bergeness implies that there is a proper $(\omega(G)-k+2)$ -coloring c_Y of G (so $(k-1)$ -tuples are colored) that restricts to a proper $(\omega(G)-k+2)$ -coloring of $lk_G(Y)$ (so vertices are colored). Since $R(Y)$ is a subgraph of $lk_G(Y)$, the coloring c_Y is also a coloring of $R(Y)$.

Let Z be a $(k-1)$ -tuple of vertices of G , and let Y be the $(k-2)$ -initial segment of Z . Define $c(Z) = c_Y(Z \setminus Y)$.

We show that c is proper. Let $F \in E(G)$, and let Y be the initial $(k-2)$ -segment of F . Then $F \setminus Y$ is an edge ab , say, of $R(Y)$ and so $c_Y(a) \neq c_Y(b)$. But now $c(Y \cup \{a\}) \neq c(Y \cup \{b\})$, and so F is not a monochromatic K_k^{k-1} in c .

Next we show that c restricts to a proper coloring of $lk_G(X)$. Let $F \in E(lk_G(X))$, let Z be the initial $(k-2-|X|)$ -segment of F , and let $Y = X \cup Z$. Then $|F \setminus Z| = 2$, say $F \setminus Z = \{a, b\}$. Since $X = \{v_1, \dots, v_{|X|}\}$, it follows that $c(Z \cup \{a\}) = c_Y(a)$ and $c(Z \cup \{b\}) = c_Y(b)$. But ab is an edge of $R(Y)$ and so $c_Y(a) \neq c_Y(b)$. Consequently, $c(Z \cup \{a\}) \neq c(Z \cup \{b\})$, and therefore F is not a monochromatic $K_{k-|X|}^{k-1-|X|}$ in c when viewed as a coloring of $lk_G(X)$. This proves that G is C_ω -perfect. \square

We now prove:

2.6. An H_ω -perfect hypergraph is C_ω -perfect.

Proof. Let G be an H_ω -perfect k -uniform hypergraph. By 2.5 it is enough to prove that G is Berge. We need to show that for every induced subhypergraph G' of G we have

- If $X \subseteq V(G')$ with $|X| = k-2$, then there is a proper $(\omega(G')-k+2)$ -coloring of G' (so $(k-1)$ -tuples are colored) that restricts to a proper $(\omega(G')-k+2)$ -coloring of $lk_{G'}(X)$ (so $(k-|X|-1)$ -tuples are colored).

Let G' be an induced subhypergraph of G , and let $X \subseteq V(G')$ with $|X| = k-2$. Order the vertices of $V(G)$ as v_1, \dots, v_n so that $X = \{v_1, \dots, v_{|X|}\}$. Let $Y \subseteq V(G')$ with $|Y| = k-2$ and let i be maximum such that $v_i \in Y$. Let $R(Y) = lk_{G'}(Y)[v_{i+1}, \dots, v_n]$. Then $R(Y)$ is a graph.

Moreover, $lk_{G'}(Y)$ is an induced subgraph of $lk_G(Y)$, and therefore $R(Y)$ is an induced subgraph of $lk_G(Y)[v_{i+1}, \dots, v_n]$. Since $|Y| = k - 2$ and G is H_ω -perfect (and therefore H -perfect), we have that $lk_G(Y)$ is a perfect graph. It follows that $R(Y)$ is a perfect graph. Consequently, $R(Y)$ has a proper $\omega(R(Y)) \leq \omega(lk_{G'}(Y))$ -coloring c_Y . Since G is H_ω -perfect (and therefore clique friendly) it follows that G' is clique-friendly, and so 2.3 implies that $\omega(lk_{G'}(Y)) \leq \omega(G') - k + 2$.

Let Z be a $(k-1)$ -tuple of vertices of G' , and let Y be the $(k-2)$ -initial segment of Z . Define $c(Z) = c_Y(Z \setminus Y)$.

We show that c is proper. Let $F \in E(G')$, and let Y be the initial $(k-2)$ -segment of F . Then $F \setminus Y$ is an edge ab , say, of $R(Y)$ and so $c_Y(a) \neq c_Y(b)$. But now $c(Y \cup \{a\}) \neq c(Y \cup \{b\})$, and so F is not a monochromatic K_k^{k-1} in c .

Since $lk_{G'}(X) = R(X)$, and the restriction of c to $R(X)$ is c_X , it follows that c restricts to a proper coloring of $lk_{G'}(X)$. This proves that G is Berge. Now 2.6 follows from 2.5. \square

To summarize we have:

2.7. H_ω -perfect \rightarrow Berge $\rightarrow C_\omega$ -perfect, and C_ω -perfect \rightarrow Berge.

2.3 Proper coloring and Ramsey numbers

A well-known equivalent definition [16] of perfect graphs (at the heart of the alternative proof of the Weak Perfect Graph Theorem given in [8]) is the following key fact:

2.8. A graph G is perfect if and only if for every induced subgraph G' of G we have that $\alpha(G')\omega(G') \geq |V(G')|$.

We mention yet another related notion of perfect hypergraphs. Denote by $R_s(k)$ the minimum integer such that for every $n \geq R_s(k)$ in every s -coloring of K_n^{k-1} there is a monochromatic K_k^{k-1} . Then $R_s(2) = s + 1$, and G is perfect if $|V(G')| < R_{\alpha(G')\omega(G')}(2)$ for every induced subgraph G' of G .

Definition of R -perfect: A k uniform hypergraph G is R -perfect if $|V(G)| < R_{(\alpha(G)-k+2)(\omega(G)-k+2)}(k)$, and this property holds for all induced subhypergraphs and links.

2.9. Let G be a k -uniform hypergraph. If

[PC] G admits a proper coloring with $\omega(G) - k + 2$ colors and its complement G^c admits a proper coloring with $\alpha(G) - k + 2$ coloring

then

$$(1) \quad |V(G)| < R_{(\alpha(G)-k+2)(\omega(G)-k+2)}(k).$$

Proof. Let c^G be the $(\omega(G) - k + 2)$ -coloring of $K_{[V(G)]}^{k-1}$ and let c^{G^c} be the $(\alpha(G) - k + 2)$ -coloring of $K_{[V(G)]}^{k-1}$. For every $(k-1)$ -tuple e let $c(e) = (c^G(e), c^{G^c}(e))$. Then c is a coloring of $K_{[V(G)]}^{k-1}$ with $(\alpha(G) - k + 2)(\omega(G) - k + 2)$ colors.

It remains to show that there is no monochromatic K_k^{k-1} in c . Suppose that $Y \subseteq V(G)$ is a monochromatic K_k^{k-1} in c . If $Y \in E(G)$, then

the coloring is not monochromatic in the first coordinate, and if $Y \notin E(G)$, then the coloring is not monochromatic in the second coordinate, a contradiction. This proves 2.9. \square

The class of k -uniform hypergraphs described by property [PC] for all induced subhypergraphs (but not necessarily links), and the class of R -perfect hypergraphs are both self-complementary classes that again, for $k = 2$ consist of the class of perfect graphs.

2.4 A few examples of H_ω -perfect hypergraphs

In this subsection we list a few constructions that yield H_ω -perfect k -uniform hypergraphs. First, a disjoint union of two H_ω -perfect hypergraphs is H_ω -perfect. Simple 3-uniform hypergraphs are H_ω -perfect, and so are tripartite 3-uniform hypergraphs. More generally k -uniform hypergraphs, in which no two edges share more than $k - 2$ vertices, are H_ω -perfect and so are k -partite k -uniform hypergraphs.

A simple induction gives another natural family:

2.10. Let G be a H_ω -perfect k -uniform hypergraph, and let H be the hypergraph of cliques of size r ($r > k$) in G . Then H is H_ω -perfect.

Remark: The class of graphs \mathcal{P}_k for which the hypergraphs of cliques of size k are H_ω -perfect seems an interesting extension of the class of perfect graphs.

2.5 H_ω -perfect triangulated spheres

Recall that k -partite k -uniform hypergraphs are H_ω -perfect. The converse is far from being true: indeed, simple hypergraphs are H_ω -perfect and they can have arbitrary large chromatic number [17]. Here is an especially nice class of H_ω -perfect hypergraphs.

2.11. Let G be an r -uniform hypergraph whose edges form a triangulation of an $(r - 1)$ -dimensional sphere other than K_{r+1}^r . Then if G is H_ω -perfect it is r -partite.

For $r = 3$ this is a reformulation of Ore's theorem for planar graphs. The general case follows from known high dimensional extensions of Ore's theorem that asserts that if all links of $(r - 3)$ -faces of an $(r - 1)$ -dimensional sphere S are even cycles then the graph of S is r -colorable. See, e.g., [9] (mainly for $r = 4$), [12] (for arbitrary dimensions), and references in these papers.

3 Perfect cocycles

3.1 H -perfect cocycles

Let G be a graph. An unordered triple $\{x, y, z\} \subseteq V(G)$ is G -odd if $|E(G[\{x, y, z\}])|$ is odd. We denote by $co(G)$ the 3-uniform hypergraph with vertex set $V(G)$ and such that $\{x, y, z\} \in E(co(G))$ if and only

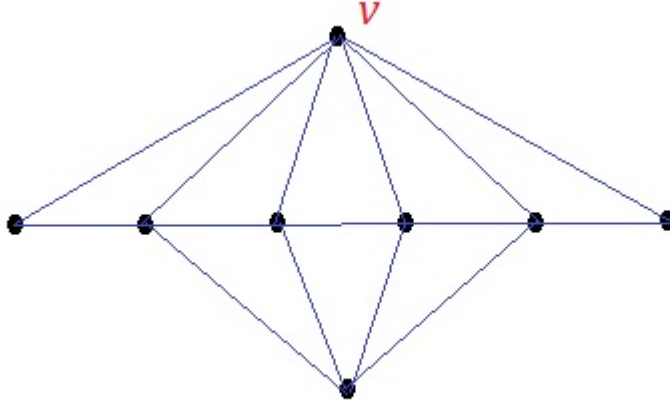


Figure 2: A graph whose cocycle is doubly perfect yet not H -perfect since $lk_{co(G)}(v) \sim C_7$.

$\{x, y, z\}$ is G -odd. Hypergraphs in $co(G)$ are called 3-cocycles or two-graphs ([20]). Note that every cocycle and every complement of a cocycle is clique friendly. Moreover, by (2.1) doubly perfect 3-uniform hypergraphs are cocycles.

We start with two observations.

3.1. Let G be a graph. Then

1. $co(G)^c = co(G^c)$.
2. For $X \subseteq V(G)$, $co(G)[X] = co(G[X])$.

3.2. A 3-cocycle is H -perfect if and only if it is both H_ω -perfect and H_α -perfect. In particular, if a 3-cocycle is H -perfect, then it is doubly perfect.

Proof. Let F be an H -perfect 3-cocycle. Then F^c is H -perfect. By 3.1 F^c is a cocycle. Then both F and F^c are clique friendly. It follows that F and F^c are both H_ω -perfect, and therefore F is both H_ω -perfect and H_α -perfect. Now by 2.7 both F and F^c are C_ω -perfect, and therefore F is doubly perfect. \square

The goal of this section is to describe graphs G for which $co(G)$ is H -perfect, and to study the larger class of graphs G for which $co(G)$ is doubly perfect.

For a graph G and a vertex $v \in V(G)$ we denote by $N_G(v)$ the set of neighbors of v , and by $M_G(v)$ the set of non-neighbors of v . Note that $v \notin N_G(v) \cup M_G(v)$. When there is no danger of confusion we omit the subscript G . Let $v \in V(G)$. We define the graph $G^+(v)$ as follows.

$V(G^+(v)) = V(G) \setminus \{v\}$ and $xy \in E(G^+(v))$ if and only if one of the following statements holds:

- $x, y \in N_G(v)$ and $xy \in E(G)$
- $x, y \in M_G(v)$ and $xy \in E(G)$
- $x \in N(v), y \in M(v)$ and $xy \notin E(G)$.

Note that $G^+(v) = lk_{co(G)}(v)$.

Next we define a family of graphs that we call “pre-odd-holes”. A pair (G, v) where G is a graph with an even number ≥ 6 of vertices, and $v \in V(G)$ is a pre-odd-hole centered at v if there exists an even integer k such that $V(G) \setminus \{v\}$ can be partitioned into k disjoint non-empty subsets P_1, \dots, P_k and the edges of G are as follows.

- $N_G(v) = \bigcup_{i \text{ even}} P_i$.
- For every i , $G[P_i]$ is a path with vertices $p_1^i, \dots, p_{n_i}^i$ in order.
- If $i \neq j$ and $i = j \pmod{2}$ then there are no edges between P_i and P_j .
- If $i \neq j \pmod{2}$ and $|i - j| \neq 1 \pmod{k}$, then every vertex of P_i is adjacent to every vertex of P_j .
- If $k > 2$, $j = i + 1$, or $i = k$ and $j = 1$ then $p_{n_i}^i$ is non-adjacent to p_1^j and all the other edges between P_i and P_j are present.
- If $k = 2$, then $p_{n_1}^1$ is non-adjacent to p_1^2 , p_1^1 is non-adjacent to $p_{n_2}^2$, and all the other edges between P_1 and P_2 are present.

A pair (G, v) is a pre-odd-antihole centered at v if (G^c, v) is a pre-odd-hole centered at v . We say that a graph G is a pre-odd-hole if (G, v) is a pre-odd-hole centered at v for some $v \in V(G)$; a pre-odd-antihole is defined similarly.

For a graph G and an induced subgraph H of G , we say that $v \notin V(H)$ is a center for H if v is complete to $V(H)$, and an anticenter for H if v is anticomplete to H .

We say that G is pure if

- No odd hole of G has a center.
- No odd antihole of G has a center.
- No odd hole of G has an anticenter.
- No odd antihole of G has an anticenter.
- No induced subgraph of G is a pre-odd-hole.
- No induced subgraph of G is a pre-odd-anti-hole.

We prove:

3.3. $co(G)$ is H -perfect if and only if G is pure.

Proof. By the Strong Perfect Graph theorem, it is enough to prove that G is pure if and only if for every $v \in V(G)$, the graph $G^+(v)$ is Berge.

Let us say that $v \in V(G)$ is pure if all of the following hold.

1. v is not a center for an odd hole of G .

2. v is not a center for an odd antihole of G .
3. v is not an anticenter for an odd hole of G .
4. v is not an anticenter for an odd antihole of G .
5. (H, v) is not a pre-odd-hole with center v for any induced subgraph H of G .
6. (H, v) is not a pre-odd-antihole with center v for any induced subgraph H of G .

Clearly G is pure if and only if every vertex of G is pure.

We show that for $v \in V(G)$ the graph $G^+(v)$ is Berge if and only if v is pure. Let $v \in V(G)$, write $N = N_G(v)$ and $M = M_G(v)$. Suppose C is an odd hole in $G^+(v)$ with vertices $c_1, \dots, c_{2t+1}, c_1$ in cyclic order. If $V(C) \subseteq N$, then v is a center for C , and if $V(C) \subseteq M$, then v is an anticenter for C . Now assume that C meets both N and M . Let $\{A, B\} = \{N, M\}$. Note that

- if x and y are two consecutive vertices of C , and they are both in A or both in B , then $xy \in E(G)$,
- if x and y are two consecutive vertices of C , and $x \in A$ and $y \in B$, then $xy \notin E(G)$
- if x, y are non-consecutive vertices of C , and they are both in A or both in B , then $xy \notin E(G)$,
- if x, y are non-consecutive vertices of C , and $x \in A$ and $y \in B$, then $xy \in E(G)$.

A sector of C is the vertex set of a maximal path of C that is contained in M or in N . Thus for some even integer k , the set $\{c_1, \dots, c_{2t+1}\}$ can be partitioned into sectors P_1, \dots, P_k where $\{c_1, \dots, c_{2t+1}\} \cap N = \bigcup_{i \text{ even}} P_i$. Now it is easy to check that $G[\{v, c_1, \dots, c_{2t+1}\}]$ is a pre-odd-hole centered at v .

Conversely, if (H, v) is a pre-odd-hole centered at v , then traversing the paths P_1, \dots, P_k (from the definition of a pre-odd-hole) in order we obtain an odd hole in $G^+(v)$.

This shows that $G^+(v)$ contains an odd hole if and only if v fails to satisfy one of the odd-numbered conditions for being pure. Applying this to G^c , we deduce that $G^+(v)$ contains an odd antihole if and only if v fails to satisfy one of the even-numbered conditions for being pure. Thus we showed that v is pure if and only if $G^+(v)$ is Berge. Since G is pure if and only if every vertex of G is pure, this proves 3.3. \square

In view of 3.2 we deduce

- 3.4. G is pure if and only if $co(G)$ is both H_ω -perfect and H_α -perfect. In particular, if G is pure, then $co(G)$ is doubly perfect.

3.2 Doubly-perfect cocycles

We present many examples of graphs G such that $co(G)$ is not doubly perfect. The class of graphs G for which $co(G)$ is doubly perfect is closed

under induced subgraphs and hence it can be described in terms of forbidden induced subgraphs. However, our examples suggest that such a description could be out of reach.

Start with a triangle-free graph H with chromatic number $\chi(H) > 3$. Next, add a new vertex v adjacent to an arbitrary subset $A \subset V(H)$. Consider now the graph G obtained from $H+v$ by the following operation: for every edge $e = (x, y)$ where $x \in A$ and $y \in V(H) \setminus A$, $e \in E(G)$ iff $e \notin E(H)$. For all other edges $e \in E(G)$ iff $e \in E(H)$. (In other words, we switch between edges and nonedges between A and $V(H) \setminus A$.)

3.5. $co(G)$ is not C_ω -perfect.

Proof. We need two claims.

3.6. Let $v \in V(G)$. Then $lk_{co(G)}(v) = H$

Proof. A triple $\{v, a, b\}$ belongs to $co(G)$ if it has an odd number of edges from G . If a and b are both in A or in $V(H) \setminus A$ this occurs iff ab is an edge in H . If $a \in A$ and $b \in V(H) \setminus A$ this occurs iff ab is not an edge of G and hence is an edge in H . \square

3.7. $\omega(co(G)) = 4$.

Proof. Since $lk_{co(G)}(v) = H$, there is not even K_4^3 in $co(G)$ that includes the vertex v . $co(G)$ and $co(H)$ restricted to all other vertices coincide and it is easy to verify that $co(H)$ contains no K_5^3 when H is triangle-free. \square

To conclude the proof of 3.5 we note that a proper coloring of pairs of vertices of $co(G)$ with t colors restricts to a proper vertex coloring of H with t colors, and since $\chi(H) > 3 = \omega(co(G)) - 1$, we deduce that $co(G)$ is not C_ω -perfect. \square

3.3 Simonyi's characterization of entropy splitting hypergraphs

Now we describe a notion of perfect hypergraphs which is closely related to (and yet interestingly different from) our notion of doubly-perfect hypergraphs. In [6] Csiszár, Körner, Lovász, Marton, and Simonyi gave a characterization of perfect graphs in terms of the equality case of certain subadditivity inequality (by Körner) involving graph entropy.

Gabor Simonyi [21] studied cases of equality for an extended entropic inequality for k -uniform hypergraphs. This led to the the class of entropy splitting hypergraphs giving an extension of the notion of perfectness to uniform hypergraphs. Both these papers are related to Körner's important notion of graph entropy [13] and subsequent works by Körner and Longo, and Körner and Marton.

As proved by Simonyi, for $k > 3$ only complete and edgeless hypergraphs have the entropy splitting property. For $k = 3$ entropy-splitting hypergraphs is a restricted class of 3-uniform hypergraphs and below are several equivalent characterizations of this class:

1. On every four vertices they have an even number of edges and, in addition, they do not contain a special hypergraph on 5 vertices as an induced subhypergraph. This special hypergraph is (in our language) $co(C_5)$, the cocycle defined by the five length cycle.
2. These 3-uniform hypergraphs can be obtained by starting with a single edge on three vertices and applying two operations (any number of times in any order):
 - a) taking the complementary 3-uniform hypergraph
 - b) duplicating a vertex.
 Here, duplicating a vertex v consists of introducing a new vertex v' that appears in an edge $v'xy$ iff vxy is also an edge. (The two vertices v and v' do not appear together in any edge.)
3. The class of cocycles of cographs. (We recall that cographs are graphs that can be obtained from a single vertex graph by repeated applications of disjoint union and taking complements. Equivalently they are the class of graphs with no induced path on four vertices.)
4. Entropy splitting 3-uniform hypergraphs have a “leaf-pattern” representation defined as follows: Given a tree with all its inner (non-leaf) vertices labeled with 0 or 1 (in an arbitrary manner), the leaf-pattern of this labeled tree is the following 3-uniform hypergraph. Its vertices are the leaves of the tree and three leaves, x, y, z form an edge iff the unique common point of the three paths $x - y, y - z, x - z$ is labeled with 1.

The equivalence of the classes given by the first and second items requires work and it turned out that this equivalence goes back to a 1984 paper by Gurvich [10].

It follows that entropy-splitting 3-uniform hypergraphs form a subclass of H -perfect 3-uniform cocycles (and hence also of doubly perfect 3-uniform hypergraphs). It is easy to see that cographs are pure. The first four obstruction to purity contain holes or antiholes, and therefore contain paths on four vertices. The last two require a little more analysis, but it is still true that they contain four-vertex paths. C_5 is pure simply because all obstruction to purity have at least 6 vertices. Therefore the 5-vertex hypergraph $co(C_5)$ is an H -perfect cocycle and it is not an entropy splitting hypergraph.

Remark: Gabor Simonyi pointed out to us also a direct inductive argument that entropy-splitting hypergraphs are doubly perfect.

4 Connections, problems, and other notions of perfectness

4.1 Vertex colorings and χ -boundedness

The families of hypergraphs described in this paper are closed under induced subhypergraphs and therefore can be described in terms of forbidden induced subhypergraphs. The study of families of graphs described in terms of forbidden induced subgraphs is a rich part of graph theory and

extensions for uniform hypergraphs are of interest. In our definitions we replaced vertex coloring for perfect graphs with proper colorings of sets of $(k - 1)$ vertices but it is of interest to also examine vertex colorings.

A family of k -uniform hypergraphs is χ -bounded if for some function f the vertices of every hypergraph G in the family can be covered by $f(\omega(G))$ independent sets. A family of k -uniform hypergraphs is χ^c -bounded if the family of its complements are χ -bounded or, in other words, if for some function f the vertices of every hypergraph G in the family can be covered by $f(\alpha(G))$ cliques. For graphs ($k = 2$) this notion was studied starting with works of Gyárfás and others, see [19] for a recent survey. For example, it is known that the class of graphs with no odd holes is χ -bounded.

Problem: Are H_ω -perfect (or even C_ω -perfect) k -uniform hypergraphs χ^c -bounded? (Equivalent formulation: Are H_α -perfect hypergraphs χ -bounded?)

As we already mentioned, H_ω -perfect hypergraphs are not χ -bounded since simple hypergraphs that can have arbitrary large chromatic number are H_ω -perfect.

It is easy to see (directly) that doubly-perfect 3-hypergraphs are χ -bounded with $f(m) = m - 1$. Indeed if $\omega(G) = t$ then for a vertex v $\omega(lk_G(v)) = t - 1$ and therefore the vertices of $lk_G(v)$ can be covered by $t - 1$ independent sets. Now, since G is a cocycle these independent sets remain independent in G and adding v to one of them describes $V(G)$ as the union of $t - 1$ independent set. The class of χ -bounded k -uniform hypergraphs with $f(m) = m - k + 2$ seems another interesting extension of perfect graphs. The strongest form of χ -boundedness that we can have for k -uniform hypergraphs is with $f(n) = \lceil n/(k - 1) \rceil$, and the restricted class of χ -bounded k -uniform hypergraphs with $f(n) = \lceil n/(k - 1) \rceil$ is yet another extension of perfect graphs worthy of study.

4.2 Analogs for chordal graphs and for interval graphs

A different avenue for perfect hypergraphs could start with various high dimensional extensions from the literature for chordal graphs. One definition is that a k -uniform hypergraph G is “chordal” if its clique complex $C(G)$ is $(k - 1)$ -Leray, meaning that $H_i(K) = 0$ for every $i \geq k - 1$ and every induced subcomplex K of $C(G)$. (Here, the clique complex $C(G)$ of a hypergraph G is the simplicial complex on the vertices of G whose faces correspond to sets of vertices that span a complete hypergraph.) This definition of chordality coincided with the definition of chordal graphs for $k = 2$, namely the homological condition simply asserts that the graph has no induced cycles with more than three edges. Certain refinements of this notion were considered in [2, 1]. A very different notion of “chordal hypergraphs” was defined by Voloshin in his book [24] Chapter 8.

It is known, [3] that if G is a k -uniform chordal hypergraph (according to the definition above) then it is χ^c -bounded, in other words if $\alpha = \alpha(G)$ then G can be covered by $f(\alpha, k)$ cliques.

An interesting larger class of hypergraphs extending the class of graphs with no odd holes would allow $(k - 1)$ -dimensional homological cycles provided they are k -partite. We do not know if χ^c -boundedness still holds.

(Imposing this condition both for G and its complement leads again to the class of perfect graphs for $k = 2$.)

A special case of chordal k -uniform hypergraphs (which motivated their definition) are those represented by collections of convex sets in R^{k-1} . Given such a collection we consider the hypergraph where vertices correspond to sets of the collection, and edges correspond to k sets with non-empty intersection. When $k = 2$ we obtain the class of interval graphs.

4.3 The Hadwiger–Debrunner property

Another notion worth considering is the following: Consider the class of k -uniform hypergraphs G with the following property (that we call the HD_r -property): If H is an induced subhypergraph of G and if p and q , $V(H) \geq p > q \geq k$, are integers satisfying

$$(2) \quad p(r-1) < (q-1)r,$$

and if every set of p vertices, contains a complete subhypergraph of size q then $V(G)$ can be covered by $p - q + 1$ cliques.

Wagner conjectured (see [7]) that every k -uniform hypergraph has the HD_k property, and we are mainly interested in the case that $r = k - 1$. Perfect graphs are graphs with the H_1 -property. A proof of the HD_{k-1} -property for k -uniform hypergraphs associated with families of convex sets in R^{k-1} goes back to Hadwiger and Debrunner [11].

4.4 Voloshin’s C-perfectness

A different notion of perfectness for hypergraphs was pioneered by Voloshin [22, 23] and further studied by Bujtás and Tuza [4]. Voloshin considered colorings of hypergraphs G where no edge is multicolored (or rainbow). He defined $\bar{\chi}(G)$ as the maximum number of colors in such a coloring. Clearly, $\bar{\chi}(G) \leq \alpha(G)$, and G is perfect according to Voloshin if for every induced subhypergraph H , $\bar{\chi}(H) = \alpha(H)$.

4.5 Is there a homological description of perfect graphs?

We mentioned that chordal graphs have a simple homological description: G is chordal if and only if for every induced subgraph H of G , $H_i(C(H)) = 0$, for every $i > 0$, where $C(H)$ is the clique complex of H . It is an interesting question if a similar homological definition exists for perfect graphs.

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