Proof of a conjecture of Plummer and Zha

Maria Chudnovsky[1]
Princeton University, Princeton, NJ 08544

Paul Seymour[2]
Princeton University, Princeton, NJ 08544

January 8, 2022; revised September 28, 2022

[1]Supported by NSF DMS-EPSRC grant DMS-2120644.
[2]Supported by AFOSR grant A9550-19-1-0187.

Abstract

Say a graph G is a *pentagraph* if every cycle has length at least five, and every induced cycle of odd length has length five. N. Robertson proposed the conjecture that the Petersen graph is the only internally 4-connected pentagraph, but this was disproved by M. Plummer and X. Zha in 2014. Plummer and Zha conjectured that every internally 4-connected pentagraph is three-colourable. We prove this: indeed, we will prove that every pentagraph is three-colourable.

1 Introduction

We say a graph G is *internally 4-connected* if it is 3-connected, and for every $X \subseteq V(G)$ with $|X| = 3$, if $G \setminus X$ has more than one component then one of them is a single vertex. (This is the definition used by Plummer and Zha [2], although it is non-standard.) Let us say a graph G is a *pentagraph* if every cycle has length at least five, and every induced cycle of odd length has length five. (All graphs in this paper are finite, and have no loops or parallel edges.) Such graphs seem to be richly structured; indeed N. Robertson [3] proposed the conjecture that the Petersen graph is the only internally 4-connected pentagraph, although this was disproved by M. Plummer and X. Zha [2] (see also [1]).

In the same paper, Plummer and Zha proposed the conjecture that all internally 4-connected pentagraphs have bounded chromatic number, and the stronger conjecture that they are all three-colourable. The first was proved by Xu, Yu, and Zha [5], who proved that all pentagraphs are four-colourable; and we will prove the second. Our main theorem is:

1.1 *Every pentagraph is three-colourable.*

(Some of the results of this paper also appear in a recent manuscript by Wu, Xu, and Xu [4], which a referee kindly brought to our attention.) The *girth* of G is the minimum length of a cycle in G; a *hole* in G is an induced cycle of length at least four, and an *odd hole* means a hole with odd length; and if $X \subseteq V(G)$, $G[X]$ denotes the subgraph induced on X. We first prove the result of Xu, Yu, and Zha [5], because the proof is short and pretty:

1.2 *Every pentagraph is four-colourable.*

Proof. It is enough to show that every connected pentagraph is four-colourable. Let G be a connected pentagraph, let $v_0 \in V(G)$, and for each $k \geq 0$ let L_k be the set of vertices with distance exactly k from v_0. Thus the sets $L_0, L_1, \ldots$ are pairwise disjoint and have union $V(G)$; and for each $k \geq 1$, each vertex in L_k has a neighbour in L_{k-1} and has no neighbour in $L_0 \cup \cdots \cup L_{k-2}$. Suppose that for some $k \geq 0$, $G[L_k]$ is not bipartite, and choose a minimum such value of k. Thus $k \geq 1$; and since $G[L_k]$ is not bipartite, it contains an odd cycle as a subgraph, and hence an induced odd cycle, and hence a hole C of length five (since G is a pentagraph). Let C have vertices c_1-c_2-c_3-c_4-c_5-c_1 in order. For $1 \leq i \leq 5$, let $d_i \in L_{k-1}$ be adjacent to c_i. It follows that each of $d_1, \ldots, d_5$ has only one neighbour in $V(C)$, since G has girth at least five, and hence $d_1, \ldots, d_5$ are all distinct. Thus $k \geq 2$. Since $G[L_0 \cup \cdots \cup L_{k-2}]$ is connected, and d_1, d_3 both have a neighbour in L_{k-2}, there is an induced path P between d_1, d_3 with interior in $L_0 \cup \cdots \cup L_{k-2}$. Since

$$d_1\text{-}P\text{-}d_3\text{-}c_3\text{-}c_4\text{-}c_5\text{-}c_1\text{-}d_1$$

is a hole of length at least six, it has even length and so P has odd length. Consequently

$$d_1\text{-}P\text{-}d_3\text{-}c_3\text{-}c_2\text{-}c_1\text{-}d_1$$

is an odd hole, and so it has length five, and therefore d_1d_3 is an edge. Similarly $d_3d_5, d_5d_2, d_2d_4, d_4d_1$ are edges, and so $G[L_{k-1}]$ has a cycle of length five, contradicting the choice of k.

Thus $G[L_k]$ is bipartite for each k, and so G is four-colourable. This proves 1.2. ∎

Now we turn to the proof of 1.1. This is a consequence of a stronger statement, that every non-bipartite pentagraph is either isomorphic to the Petersen graph, or has a vertex of degree at most two, or admits one of two kinds of decomposition, that a minimal non-three-colourable pentagraph cannot admit. Let us see these decompositions.

- A *path-3-cutset* means an induced three-vertex path P of G such that $G \setminus V(P)$ is disconnected.
- A *parity star-cutset* means a set $X \subseteq V(G)$ such that $G \setminus X$ is disconnected, and there is a vertex $x \in X$ such that x is adjacent to every other vertex in X, and there is a component A of $G \setminus X$ such that every two vertices in $X \setminus \{x\}$ are joined by an induced path of even length with interior in $V(A)$. We call this a *strong parity star-cutset* if A can be chosen such that in addition x has a neighbour in $V(A)$.

We will prove:

1.3 *Let G be a pentagraph. Then either*

- *G is bipartite; or*
- *G is isomorphic to the Petersen graph; or*
- *G has a vertex of degree at most two; or*
- *G admits a path-3-cutset or a strong parity star-cutset.*

Proof of 1.1, assuming 1.3. We prove by induction on $|V(G)|$ that every pentagraph is three-colourable. Let G be a pentagraph such that every pentagraph with fewer vertices is three-colourable. If G is isomorphic to the Petersen graph, or some vertex has degree at most two, then G is three-colourable; so by 1.3 we may assume that G admits a path-3-cutset or a parity star-cutset (indeed, a strong parity star-cutset, but we do not need "strong" here).

Suppose first that G admits a path-3-cutset, and let v_1-v_2-v_3 be an induced path such that $G \setminus \{v_1, v_2, v_3\}$ is disconnected. Let A_1 be the union of at least one and not all of the components of $G \setminus \{v_1, v_2, v_3\}$, and let A_2 be the union of all the other components. For $i = 1, 2$ let $G_i = G[A_i \cup \{v_1, v_2, v_3\}]$. From the inductive hypothesis, both G_1 and G_2 are three-colourable; let $\phi_i : V(G_i) \to \{1, 2, 3\}$ be a three-colouring, for $i = 1, 2$. We may assume that $\phi_i(v_1) = 1$ and $\phi_i(v_2) = 2$ for $i = 1, 2$. Thus $\phi_1(v_3), \phi_2(v_3) \in \{1, 3\}$, and if $\phi_1(v_3) = \phi_2(v_3)$ then G is three-colourable. Thus we may assume that $\phi_1(v_3) = 1$ and $\phi_2(v_3) = 3$. Let H_1 be the subgraph of G_1 induced on the set of vertices $v \in V(G_1)$ with $\phi_1(v) \in \{1, 3\}$. If v_1, v_3 belong to different components of H_1, then by exchanging colours in the component containing v_3, we obtain another three-colouring of G_1 that can be combined with ϕ_2 to show that G is three-colourable. So we may assume that v_1, v_3 belong to the same component of H_1, and so there is an induced path P_1 of H_1 between v_1, v_3. Consequently P_1 has even length, and length at least four since G has girth at least five. Define H_2 in G_2 similarly: then similarly we may assume that v_1, v_3 belong to the same component of H_2, and so there is an induced path P_2 of H_2 between v_1, v_3 with odd length, at least three. But then $P_1 \cup P_2$ is an induced cycle of G of odd length at least seven, a contradiction.

Now suppose that G admits a parity star-cutset, and let $X \subseteq V(G)$ and $v \in V(G) \setminus X$, such that v is adjacent to every vertex in X, and $G \setminus (X \cup \{v\})$ is disconnected, and there is a component

A of $G \setminus (X \cup \{v\})$ such that every two vertices in X are joined by an induced path of even length with interior in $V(A)$. Choose X minimal with this property, and let $A_1, \ldots, A_k$ be the components of $G \setminus (X \cup \{v\})$, where every two vertices in X are joined by an induced path of even length with interior in $V(A_1)$.

(1) *For $1 \le i \le k$ and for all distinct $x, x' \in X$, every induced path between x, x' with interior in A_i has a length that is even and at least four.*

For all distinct $x, x' \in X$, let $P_1(x, x')$ be an induced path of even length with interior in $V(A_1)$. It follows that $P_1(x, x')$ has length at least four, since v is adjacent to x, x' and G has girth at least five. For $2 \le i \le k$, if x, x' both have a neighbour in A_i, let $P_i(x, x')$ be an induced path between x, x' with interior in A_i. Thus $P_i(x, x')$ has length at least three, for the same reason. Since $P_1(x, x') \cup P_i(x, x')$ is an induced cycle of length at least seven, it follows that $P_i(x, x')$ has even length, and length at least four, for all choices of x, x' that both have a neighbour in A_i; and therefore every induced path between x, x' with interior in A_i has even length at least four. From the minimality of X, it follows that every vertex in X has a neighbour in A_i for $2 \le i \le k$; and so, by the same argument with A_1, A_2 exchanged, every induced path between x, x' with interior in A_1 has even length at least four. This proves (1).

For $1 \le i \le k$, let $G_i = G[V(A_i) \cup X \cup \{v\}]$.

(2) *For $1 \le i \le k$, there is a three-colouring $\phi_i : V(G_i) \to \{1, 2, 3\}$ with $\phi_i(v) = 1$ and $\phi_i(x) = 2$ for all $x \in X$.*

From the inductive hypothesis, G_i admits a three-colouring $\phi_i : V(G_i) \to \{1, 2, 3\}$ with $\phi_i(v) = 1$. Choose ϕ_i such that $\phi_i(x) = 3$ for as few vertices $x \in X$ as possible. We claim that $\phi_i(x) = 2$ for all $x \in X$. To see this, let X_2 be the set of $x \in X$ with $\phi_i(x) = 2$, and let X_3 be the set of $x \in X$ with $\phi_i(x) = 3$. Thus $X_2 \cup X_3 = X$. Let H be the subgraph of G_i induced on the set of vertices $u \in V(G_i)$ with $\phi_i(u) \in \{2, 3\}$. Suppose that $X_3 \neq \emptyset$, and let C be a component of H that contains a vertex of X_3. By exchanging colours in C, the choice of ϕ_i implies that some vertex of X_2 belong to C, and so there is a minimal induced path P of H between X_2, X_3, which therefore has odd length; but from the minimality of P, all internal vertices of P belong to A_i, contradicting (1). This proves (2).

From (2) it follows that G is three-colourable. This proves 1.1 (assuming 1.3). ∎

2 Pentagraphs that contain large parts of the Petersen graph

In this section we prove part of 1.3. A *clique cutset* of G is a clique X of G such that $G \setminus X$ is disconnected. In a pentagraph G, every clique has cardinality at most two, and so if G has an edge and G admits a clique cutset, then G admits a strong parity star-cutset. If P is an induced path, we denote the set of internal vertices of P by P^*. Two disjoint subsets X, Y of G are *anticomplete* if there are no edges between X, Y. Let us say two nonadjacent vertices s, t of a graph H are *linked* if there are induced paths Q_1, Q_2 of H both with ends s, t and both of length at least three, with lengths of different parity. We say that s, t are *odd-linked* if there is an induced path of H with ends s, t and with odd length at least five.

We begin with:

2.1 *Let G be a pentagraph that does not admit a clique cutset, and let H be an induced subgraph of G with $H \neq G$ and $|V(H)| \geq 3$. Then either*

- *there is a vertex $v \in V(G) \setminus V(H)$ with at least three neighbours in $V(H)$ (and therefore every two neighbours of v in $V(H)$ have distance at least three in H); or*
- *there exist nonadjacent $s, t \in V(H)$, and a vertex $v \in V(G) \setminus V(H)$ adjacent to s, t and with no other neighbours in $V(H)$ (and therefore s, t have distance at least three in H and are not odd-linked in H); or*
- *no vertex in $V(G) \setminus V(H)$ has more than one neighbour in $V(H)$, and there exist nonadjacent $s, t \in V(H)$, not linked in H, and an induced path P of G with length at least three, with ends s, t and with $P^* \subseteq V(G) \setminus V(H)$, such that every vertex of H with a neighbour in P^* is adjacent to both s, t.*

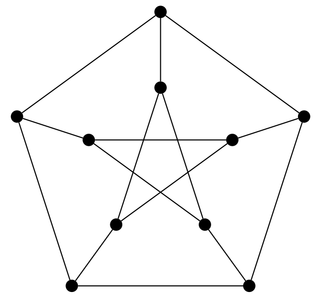
Proof. If some $v \in V(G) \setminus V(H)$ has at least three neighbours in $V(H)$ then the first bullet is satisfied, since G has girth at least five. If some $v \in V(G) \setminus V(H)$ has exactly two neighbours s, t in $V(H)$, and Q is an induced path of H with ends s, t and with odd length at least five, then adding v to Q makes a long odd hole G, a contradiction; so s, t are not odd-linked and the second bullet is satisfied. Thus we may assume that each vertex in $V(G) \setminus V(H)$ has at most one neighbour in $V(H)$.

Let C be a component of $G \setminus V(H)$. Since G does not admit a clique cutset, and $|V(H)| \geq 3$, it follows that there exist nonadjacent vertices in $V(H)$ both with neighbours in $V(C)$. Thus there is an induced path P with $P^* \subseteq V(C)$, with ends nonadjacent vertices of C. Choose P with P^* minimal, and let its ends be s, t. Since no vertex in $V(G) \setminus V(H)$ has more than one neighbour in $V(H)$, it follows that P has length at least three. If $v \in V(H)$ has a neighbour in P^*, and v is nonadjacent to s say, then from the minimality of P^*, it follows that v has only one neighbour in P^* and that neighbour is adjacent to t. Hence v is nonadjacent to t, since G has girth at least five; and this contradicts the minimality of P^*. So every vertex of H with a neighbour in P^* is adjacent to both s, t. Suppose that s, t are linked in H, and so there are induced paths Q_1, Q_2 of H between s, t, both of length at least three, and with lengths of different parity. Thus neither contains a vertex adjacent to both s, t, and so Q_1^*, Q_2^* are both anticomplete to P^*. Consequently both $P \cup Q_1, P \cup Q_2$ are long holes, and one has odd length, a contradiction; and so s, t are not linked, and the third bullet is satisfied. This proves 2.1. ∎

We deduce:

2.2 *Let G be a pentagraph that has an induced subgraph isomorphic to the Petersen graph. Then either G is isomorphic to the Petersen graph, or G admits a clique cutset.*

Proof. Let H be an induced subgraph of G isomorphic to the Petersen graph. No two vertices of H have distance at least three in H. Moreover, every two nonadjacent vertices of H are linked in H. The result follows from 2.1. This proves 2.2. ∎



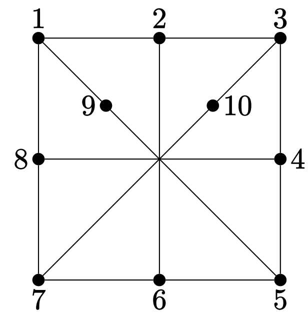
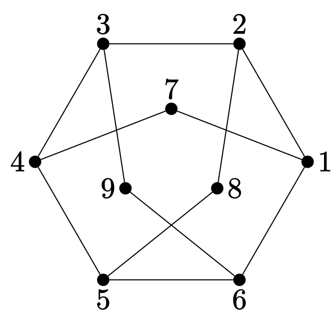


Figure 1: $\mathcal{P}$ and $\mathcal{P}^0$.

Let $\mathcal{P}$ denote the Petersen graph, and let $\mathcal{P}^0, \mathcal{P}^1, \mathcal{P}^2$ denote the graphs obtained from $\mathcal{P}$ by deleting one edge, one vertex, and two adjacent vertices respectively. (See figures 1 and 2.)

2.3 *Let G be a pentagraph that has an induced subgraph isomorphic to $\mathcal{P}^0$. Then either G is isomorphic to $\mathcal{P}^0$, or G admits a clique cutset.*

Proof. Let H be an induced subgraph of G isomorphic to $\mathcal{P}^0$, numbered as in figure 1. $(9, 10)$ is the only pair of vertices of H that have distance more than two in H. Consequently no three vertices of H pairwise have distance at least three; and every two vertices of H with distance at least three in H are odd-linked in H. Moreover, every two nonadjacent vertices are linked in H, and so the result follows from 2.1. This proves 2.3. ∎



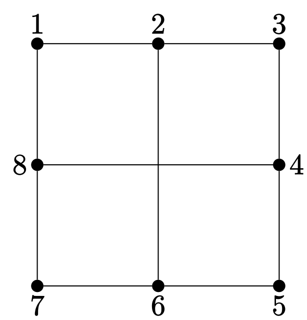


Figure 2: $\mathcal{P}^1$ and $\mathcal{P}^2$.

2.4 *Let G be a pentagraph that has an induced subgraph isomorphic to $\mathcal{P}^1$. Then either G is isomorphic to one of $\mathcal{P}, \mathcal{P}^0, \mathcal{P}^1$, or G admits a clique cutset.*

Proof. Let H be an induced subgraph of G isomorphic to $\mathcal{P}^1$, numbered as in figure 2. The only pairs of vertices that have distance at least three in H are the pairs of vertices in $\{7, 8, 9\}$. Thus if the first bullet of 2.1 holds then G contains $\mathcal{P}$ and the result follows from 2.2. If the second bullet of 2.1 holds, then G contains $\mathcal{P}_0$ and the result follows from 2.3. Every two nonadjacent vertices of H are linked, so the third bullet of 2.1 does not hold. This proves 2.4. ∎

2.5 *Let G be a pentagraph that has an induced subgraph isomorphic to $\mathcal{P}^2$. Then either G is isomorphic to one of $\mathcal{P}, \mathcal{P}^0, \mathcal{P}^1$, $\mathcal{P}^2$, or G admits a path-3-cutset or a strong parity star-cutset.*

Proof. Let H be an induced subgraph of G isomorphic to $\mathcal{P}^2$, numbered as in figure 2. The only pairs of vertices that have distance at least three in H are $(1,5)$ and $(3,7)$, so the first bullet of 2.1 does not hold, and if the second bullet of 2.1 holds then G contains $\mathcal{P}^1$ and the result follows from 2.4 (since if G has a clique cutset then it has a strong parity star-cutset). Thus we may assume that the third bullet of 2.1 holds, and in particular, no vertex in $V(G) \setminus V(H)$ has more than one neighbour in $V(H)$.

Let us say the four sets

$$\{1,2,3\}, \{3,4,5\}, \{5,6,7\}, \{7,8,1\}$$

are the *sides* of H. We may assume that $H \neq G$, and G does not admit a path-3-cutset, and so there is a connected subgraph F of $G \setminus V(H)$ such that $N(F)$ is not a subset of any side of H, where $N(F)$ denotes the set of vertices in H with a neighbour in $V(F)$. Choose F with $|V(F)|$ minimal. Since $N(F)$ is not a clique, there is an induced path P with ends $s, t \in N(F)$, nonadjacent; and as in 2.1, by choosing P with P^* minimal it follows that s, t are not linked in H, and no vertex of H has a neighbour in P^* except s, t and possibly a common neighbour of s and t. The only nonadjacent pairs of vertices of H that are not linked are $(1,3), (3,5), (5,7), (7,1)$, so from the symmetry we may assume that $s = 1$ and $t = 3$. Thus no vertex of $V(H)$ has a neighbour in P^* except $1, 3$ and possibly 2. Since $N(F) \not\subseteq \{1,2,3\}$, there is an induced path Q of G with interior in $V(F)$ with one end in P^* and the other in $\{4,5,6,7,8\}$. Thus Q has length at least two, and $F = P^* \cup Q^*$, from the minimality of $|V(F)|$. Let the vertices of Q be q_1-q_2-$\cdots$-q_k in order, where $k \geq 3$, and $q_1 \in P^*$, and $q_k \in \{4,5,6,7,8\}$. From the symmetry we may assume that $q_k \in \{4,5,6\}$. The vertices $1,2,3$ may have neighbours in Q^*, but since no vertex has more than one neighbour in $V(H)$, it follows from the minimality of $|V(F)|$ that q_k is the only vertex in $\{4,5,6,7,8\}$ with a neighbour in Q^*.

(1) $q_k = 4$.

There is an induced path R between $1, q_k$ with interior in $P^* \cup Q^*$, and it has length at least three since no vertex in $V(G) \setminus V(H)$ has more than one neighbour in $V(H)$. If $q_k = 5$, then one of

$$1\text{-}R\text{-}5\text{-}4\text{-}8\text{-}1, 1\text{-}R\text{-}5\text{-}6\text{-}7\text{-}8\text{-}1$$

is a long odd hole, a contradiction; and if $q_k = 6$, then one of

$$1\text{-}R\text{-}6\text{-}5\text{-}4\text{-}8\text{-}1, 1\text{-}R\text{-}6\text{-}7\text{-}8\text{-}1$$

is a long odd hole, a contradiction. Since $q_k \in \{4,5,6\}$, this proves (1).

(2) 2 *has no neighbour in $P^* \cup Q^*$, and P has length three.*

(See figure 3.) Suppose that 2 has a neighbour in $P^* \cup Q^*$; then there is an induced path R between $2, 4$ with interior in $P^* \cup Q^*$, and it has length at least three. But then one of

$$2\text{-}R\text{-}4\text{-}5\text{-}6\text{-}2, 2\text{-}R\text{-}4\text{-}8\text{-}7\text{-}6\text{-}2$$

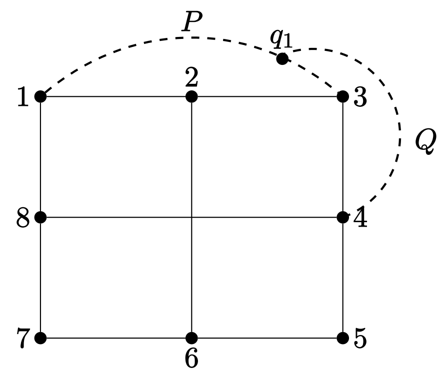


Figure 3: Step (2) of the proof of 2.5.

is a long odd hole, a contradiction. Thus 2 has no neighbour in $P^* \cup Q^*$. Consequently

$$1\text{-}P\text{-}3\text{-}2\text{-}1, 1\text{-}P\text{-}3\text{-}4\text{-}8\text{-}1$$

are both holes of length at least five, and one has odd length; so P has length three. This proves (2).

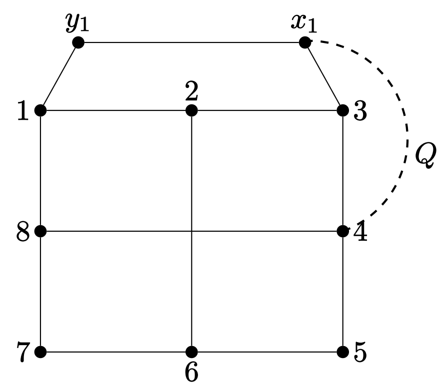


Figure 4: Step (3) of the proof of 2.5.

Let the vertices of P be 1-y_1-x_1-3 in order.

(3) 1 *has no neighbour in* $V(Q)$*, and so* $q_1 = x_1$*.*

Suppose that 1 has a neighbour in $V(Q)$; then there is an induced path R between $1, 4$ with $R^* \subseteq V(Q)$. Since $N(R^*)$ is not a subset of a side of H, the minimality of $|V(F)|$ implies that $R^* = V(F)$, and in particular $x_1 \in R^* \subseteq V(Q)$. Since $x_1 \notin Q^*$, it follows that $q_1 = x_1 \in R^*$, and so Q is a subpath of R, contradicting that 1 has a neighbour in $V(Q)$. This proves (3).

See figure 4. Let us say a 1-3 *handle* is an induced path R of G of length three between $1, 3$ such that $R^* \cap V(H) = \emptyset$ and no vertex in $V(H)$ has a neighbour in R^* except $1, 3$. Thus P is a 1-3 handle. Let X be the set of neighbours of 3 that belong to 1-3 handles, and let Y be the set of all neighbours of 1 that belong to 1-3 handles. Thus $x_1 \in X$ and $y_1 \in Y$, and $Y \cap V(Q) = \emptyset$ by (3). If some $y \in Y$ has a neighbour in Q^*, then $Q^* \cup \{y\}$ induces a connected subgraph of G and $1, 4$ both have neighbours in this subgraph, contrary to the minimality of $|V(F)|$. Thus Y is anticomplete to Q^*, and therefore $X \cap Q^* = \emptyset$. Let D be a connected induced subgraph of G, with $Q^* \subseteq V(D)$, maximal such that $V(D) \cap (X \cup \{3, 4\}) = \emptyset$ and no vertex in $Y \cup \{1, 2, 5, 6, 7, 8\}$ has a neighbour in $V(D)$. It follows that $x_1, 4$ both have a neighbour in D. (See figure 5.)

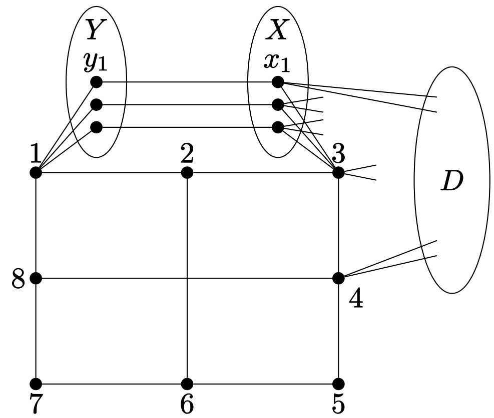


Figure 5: Handles in the proof of 2.5.

For every two vertices in $X \cup \{4\}$, there is a path between them of length four with middle vertex 1; and this path is induced since G has girth at least five. We may assume that $X \cup \{3, 4\}$ is not a strong parity star-cutset, and so D is not a component of $G \setminus (X \cup \{3, 4\})$. Consequently there is a vertex $v \in V(G) \setminus V(D)$ with a neighbour in $V(D)$ and with $v \notin X \cup \{3, 4\}$. From the maximality of D, it follows that v has a neighbour in $Y \cup \{1, 2, 5, 6, 7, 8\}$. Since there is a path between this neighbour and 3 of length at most three with vertex set in $V(H) \cup Y$, it follows that $3, v$ are nonadjacent.

(4) *v has a unique neighbour in* $Y \cup \{1, 2, 5, 6, 7, 8\}$.

Suppose that v has more than one neighbour in $Y \cup \{1, 2, 5, 6, 7, 8\}$. Every vertex in $V(G) \setminus V(H)$ has at most one neighbour in $V(H)$, as we saw earlier, so we may assume that v is adjacent to some $y \in Y$. Choose $x \in X$ adjacent to y. All neighbours of v in $V(H) \cup X \cup Y$ pairwise have distance at least three in $G[V(H) \cup X \cup Y]$, and so v has no more neighbours in Y, and none in $\{x, 1, 2, 3, 8\}$. Since v has two neighbours in $Y \cup \{1, 2, 5, 6, 7, 8\}$, it has a unique neighbour in $\{5, 6, 7\}$ (say u), and therefore is nonadjacent to 4. The paths

$$y\text{-}1\text{-}8\text{-}7\text{-}6\text{-}5,\ y\text{-}1\text{-}8\text{-}4\text{-}5\text{-}6,\ y\text{-}x\text{-}3\text{-}4\text{-}8\text{-}7$$

all have length five, and so v cannot be adjacent to both ends of any of them, and so $u \neq 5, 6, 7$, a contradiction. This proves (4).

Let u be the unique neighbour of v in $Y \cup \{1, 2, 5, 6, 7, 8\}$. Let R be an induced path with interior in $V(D) \cup \{v\}$ between x_1, u, and let S be a minimal path with interior in $V(D) \cup \{v\}$ between u and $\{3, 4\}$. Thus one of $3, 4$ is an end of S, and the other has no neighbour in $V(S)$.

- If $u = 5$, the union of R with one of x_1-y_1-1-8-7-6-5, x_1-y_1-1-2-6-5 is a long odd hole.
- If $u = 6$, the union of R with one of x_1-y_1-1-8-7-6, x_1-y_1-1-2-6 is a long odd hole.
- If $u = 7$, the union of R with one of x_1-y_1-1-8-7, x_1-y_1-1-2-6-7 is a long odd hole.
- If $u = 8$, let T be an induced path with interior in $V(D) \cup \{x_1, v\}$ between $3, u$; then T has length at least three and the union of T with one of 3-2-6-7-8, 3-2-1-8 is a long odd hole.
- If $u = 2$, let T be an induced path with interior in $V(D) \cup \{v\}$ between $4, u$; then T has length at least three and the union of T with one of 4-8-7-6-2, 4-5-6-2 is a long odd hole.
- If $u \in Y$ and 4 is an end of S, then the union of S with one of 4-5-6-2-1-u, 4-3-2-1-u is a long odd hole.
- If $u \in Y$ and 3 is an end of S, then S has length at least three, and the union of S with one of 3-2-1-u, 3-4-8-1-u is a long odd hole.
- If $u = 1$ and 4 is an end of S, then S has length at least three, and the union of S with one of 4-5-6-2-1, 4-3-2-1 is a long odd hole.
- If $u = 1$ and 3 is an end of S, then S has length at least three, and the union of S with one of 3-2-1, 3-4-8-1 is an odd hole. So S is a path of length three between 1,3 with interior in $V(D) \cup \{v\}$, and no vertex of H has a neighbour in S^* except $1, 3$. Consequently S is a 1-3 handle, and so $V(D) \cup \{v\}$ contains a vertex in X, a contradiction.

Thus in all cases we obtain a contradiction. This proves 2.5. ∎

3 Jumps across a pentagon

In view of 2.5, we turn our attention to pentagraphs that do not contain $\mathcal{P}^2$ as an induced subgraph. Let G be a pentagraph, and let C be a hole of length five in G. No vertex in $V(G) \setminus V(C)$ has more than one neighbour in $V(C)$, since G has girth five. Let P be an induced path with both ends in $V(C)$, nonadjacent, and with no other vertices in $V(C)$. We call P a *jump* over C. Let P have ends s, t; then we call P an *s-t jump*, and if c is the vertex of C adjacent to both s, t, we say P is a *jump across c*. If P is an s-t jump across c and no vertex of $V(C) \setminus \{c, s, t\}$ has a neighbour in P^*, we say P is a *local* jump. If P has length three we say that P is a *short* jump. Then, clearly,

- all jumps have length at least three;
- a jump P is short if and only if no vertex of $V(C) \setminus V(P)$ has a neighbour in P^*;
- short jumps are local;
- local jumps have odd length.

We need to analyze which pairs of vertices of C can be joined by short and local jumps. We begin with:

3.1 *Let G be a pentagraph not containing $\mathcal{P}^2$, and let C be a hole of length five in G. If P_1, P_2 are local jumps over C with exactly one common end c, then there is a short jump across c with interior in $P_1^* \cup P_2^*$, and consequently neither of them is short.*

Proof. (See figure 6.) Suppose that there are local jumps P_1, P_2 over C with exactly one common end c say, and there is no short jump across c with interior in $P_1^* \cup P_2^*$. Choose such P_1, P_2, c with $P_1^* \cup P_2^*$ minimal. (Note that $P_1^* \cap P_2^*$ may be nonempty.) Let C have vertices c_1-c_2-c_3-c_4-c_5-c_1 in

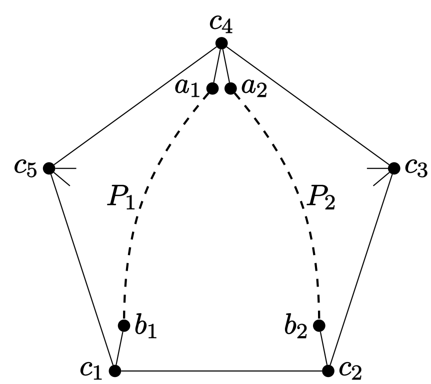


Figure 6: Local jumps with a common end.

order, where P_i is a c_i-c_4 jump for $i = 1, 2$. For $i = 1, 2$, let a_i, b_i be the vertices of P_i adjacent to c_i and c_4 respectively. For $i = 1, 2$, let $D_i = P_i^* \setminus \{a_i, b_i\}$.

(1) *$D_1 \cup \{b_1\}$ is disjoint from and anticomplete to $D_2 \cup \{b_2\}$.*

Suppose not. Since $b_2 \notin V(P_1)$ (because P_1 is local) and vice versa, and b_1, b_2 are not adjacent, it follows that either there is a path of $G[D_1 \cup D_2 \cup \{b_1\}]$ from b_1 to D_2 or a path of $G[D_1 \cup D_2 \cup \{b_2\}]$ from b_2 to D_1, and from the symmetry we may assume the first. Hence $D_2 \neq \emptyset$, so P_2 is not short, and so c_3 has a neighbour in D_2. Consequently there is a path between c_3 and $\{c_1, c_5\}$ with interior in $D_1 \cup D_2 \cup \{b_1\}$. Let Q be a minimal path from c_3 to one of c_1, c_5, with interior in $D_1 \cup D_2 \cup \{b_1\}$. It follows that one of c_1, c_5 is an end of Q and the other has no neighbour in Q^*. Moreover, neither of c_2, c_4 has a neighbour in Q^*, and so Q is a short jump. The choice of P_1, P_2 implies that Q is not a short jump across c_4, and so c_1 is an end of Q and Q is a short jump across c_2. There is no short jump across c_1 with interior in $P_1^* \cup Q^*$, since c_2 has no neighbour in $P_1^* \cup Q^*$; and since $P_1^* \cup Q^*$ is a proper subset of $P_1^* \cup P_2^*$, this contradicts the minimality of $P_1^* \cup P_2^*$. This proves (1).

If $a_1 = a_2$, then since the paths P_1, P_2 have odd length (because they are local), (1) implies that $G[V(P_1 \cup P_2)]$ is an odd hole, which therefore has length five; and so P_1, P_2 are both short. But then $G[V(C \cup P_1 \cup P_2)]$ is isomorphic to $\mathcal{P}^2$, a contradiction. Thus $a_1 \neq a_2$, and since $a_1 \notin V(P_2)$ and vice versa, (1) implies that P_1^*, P_2^* are disjoint, and every edge between them has an end in $\{a_1, a_2\}$. Since $G[V(P_1 \cup P_2)]$ is not a long odd hole, there is an edge between P_1^*, P_2^*; so from the symmetry we may assume that a_1 has a neighbour in $D_2 \cup \{b_2\}$, and therefore there is a c_2-c_4 jump with interior in $\{a_1, b_2\} \cup D_2$. Since c_1, c_5 have no neighbours in $\{a_1, b_2\} \cup D_2$, this c_2-c_4 jump is local, contrary to the minimality of $P_1^* \cup P_2^*$. This proves 3.1. ∎

To complete the proof of 1.3 we need:

3.2 *Let G be a pentagraph not containing $\mathcal{P}^2$, and let C be a hole of length five in G. Then either:*

- *some vertex of C has degree two; or*
- *G admits a path-3-cutset; or*
- *G admits a strong parity star-cutset.*

Proof. We claim first that we may number the vertices of C as c_1-c_2-c_3-c_4-c_5-c_1 in order, such that:

(1) *There are no short jumps across any of c_3, c_4, c_5, there are no local jumps across c_4, and every local jump across c_3 or c_5 contains a vertex that is in a short jump.*

Let S be the set of $c \in V(C)$ such that there is a short jump across c, and let L be the set of $c \in V(C)$ such that there is a local jump across c. Thus $S \subseteq L$. If L is a clique, then (1) holds; so we may assume that $c_2, c_5 \in L$, where the vertices of C are c_1-c_2-c_3-c_4-c_5-c_1 in order. 3.1 implies that if $c, c' \in L$ are nonadjacent, then neither of them is in S; so S is a clique, and every vertex of S is adjacent to every vertex of $L \setminus S$. In particular $S = \{c_1\}$, and $L = \{c_5, c_1, c_2\}$. By 3.1 again, either every local jump across c_2 contains a vertex in a short jump across c_1, or every local jump across c_5 contains such a vertex; and from the symmetry we may assume the second. This proves (1).

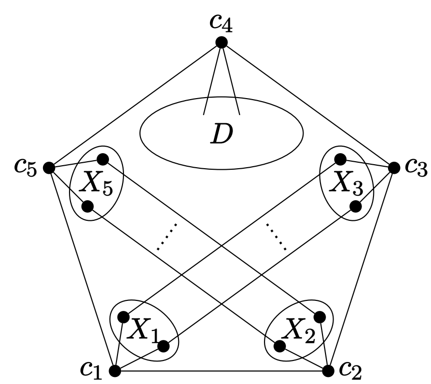


Figure 7: The numbering of C.

Let X_1, X_3 be the sets of vertices adjacent to c_1, c_3 respectively that are in short jumps across c_2 (thus, X_1, X_3 might be empty); and similarly let X_2, X_5 be the sets of vertices adjacent to c_2, c_5 respectively that are in short jumps across c_1. (See figure 7.) Let $X = X_1 \cup X_2 \cup X_3 \cup X_5$. Thus X is the set of all vertices that belong to the interior of short jumps.

Now, c_4 has no neighbour in $X \cup \{c_1, c_2\}$, but we may assume that c_4 has degree at least three, and so there is a connected induced subgraph D such that c_4 has a neighbour in $V(D)$ and $V(D) \cap (V(C) \cup X) = \emptyset$, and D is maximal with these properties. Let N be the set of vertices in $V(C) \cup X$ that have a neighbour in $V(D)$; so $c_4 \in N$.

(2) $c_1, c_2 \notin N$, *and* $N \cap (X_1 \cup X_2) = \emptyset$.

Suppose not; then from the symmetry we may assume that either c_1 or some member of X_1 belongs to N. Choose an induced path P between c_4, c_1 with interior in $D \cup X_1$. Since $P^* \setminus X_1 \subseteq D$, it follows that $P^* \cap X \subseteq X_1$; and so $|P^* \cap X| \leq 1$, since P is induced. Suppose that P is a local jump. From 3.1, there is no short jump across c_2, and in particular $X_1 = \emptyset$; and so $V(P) \cap X = \emptyset$, contrary to (1). Thus P is not local. Let Z be the set of vertices of P that are not equal or adjacent to c_1 or to c_4. Thus one of c_2, c_3 has a neighbour in Z, since P is not local and G has girth five. If also c_5 has a neighbour in P^*, this neighbour also belongs to Z, and so there is a minimal path Q from c_5 to one of c_2, c_3, with interior in Z. Then no vertex of C has a neighbour in Q^* except the ends of Q, and so Q is short, and therefore two vertices of Q^* belong to X; and since $Q^* \subseteq P^*$. But this contradicts that $Q^* \subseteq P^*$ and $|P^* \cap X| \leq 1$. So c_5 has no neighbour in P^*. If c_2 has a neighbour in P^*, choose a minimal path between c_2, c_4 with interior in P^*; then this is a local jump across c_3, containing no vertices in X, contrary to (1). Thus c_3 has a neighbour in P^*, and hence in Z, and none of c_2, c_4, c_5 have a neighbour in Z; and so there is a short jump across c_2 with interior in P^*, and therefore two vertices of P^* belong to X, a contradiction. This proves (2).

From (2) it follows that $N \subseteq X_3 \cup X_5 \cup \{c_3, c_4, c_5\}$. If $N \subseteq \{c_3, c_4, c_5\}$ then G admits a P_3-cut, so we may assume from the symmetry that some $x_3 \in X_3$ belongs to N. If also c_5 or some $x_5 \in X_5$ has a neighbour in D, then there is an induced path Q between c_3, c_5 with interior in $V(D) \cup X_3 \cup X_5$, and so neither of c_1, c_2 have a neighbour in it, and it is therefore a local jump across c_4, contrary to (1). Thus $N \subseteq X_3 \cup \{c_3, c_4\}$. But every two vertices in $X_3 \cup \{c_4\}$ are joined by an induced path of length four with interior in $X_1 \cup \{c_1, c_5\}$, and so $X_3 \cup \{c_3, c_4\}$ is a strong parity star-cutset. This proves 3.2. ∎

Finally we deduce 1.3, which we restate:

3.3 *Let G be a pentagraph. Then either*

- *G is bipartite; or*
- *G is isomorphic to the Petersen graph; or*
- *G has a vertex of degree at most two; or*
- *G admits a path-3-cutset or a strong parity star-cutset.*

Proof. Since $\mathcal{P}^0, \mathcal{P}^1, \mathcal{P}^2$ all have vertices of degree two, the result is true by 2.5 if G contains an induced subgraph isomorphic to $\mathcal{P}^2$; so we assume it does not. We may assume that G is not bipartite, and so it has a hole of length five. But then the result follows from 3.2. This proves 3.3. ∎

4 Construction?

Robertson's conjecture, that the Petersen graph is the only non-bipartite internally 4-connected pentagraph, is false, but perhaps something like it is true. For instance, in [1] the following is shown:

4.1 *The Petersen graph is the only cubic non-bipartite 3-connected pentagraph.*

A cubic 3-connected pentagraph cannot admit a path-3-cutset (because the middle vertex of the P_3 must have a neighbour on either side of the cutset, by 3-connectivity), and cannot admit a strong parity star-cutset (because the "strong" condition implies that such a cutset would be a path-3-cutset); and so in fact 4.1 follows easily from 1.3.

Is there some hope of extending 4.1 to larger classes of pentagraphs, or indeed to a construction for all pentagraphs? In [2], Plummer and Zha conjecture that every counterexample to Robertson's conjecture is "close to bipartite", and one might hope that this too would follow from 1.3, but we do not see how to show it. The problem is, let G be a pentagraph, with a parity star-cutset $X \cup \{x\}$, where x is adjacent to every vertex in X, and $A_1, \ldots, A_k$ are the components of $G \setminus (X \cup \{x\})$. As in the derivation of 1.1 from 1.3, we may assume that each vertex in X has a neighbour in each of $A_1, \ldots, A_k$, and every induced path with ends in X and no other vertex has even length. Let G_i be the subgraph induced on $V(A_i) \cup X \cup \{x\}$. We would like to apply an inductive hypothesis that says each G_i is "close to bipartite", but even if G is internally 4-connected, we do not know that $G_1, \ldots, G_k$ are 3-connected; for instance, they might have vertices of degree two.

On the other hand, a parity star-cutset is a "reversible" decomposition, that can be turned into something like a construction: in the notation above, if we do not know that G is a pentagraph, but we know that each of $G_1, \ldots, G_k$ is a pentagraph, it follows that G is indeed a pentagraph. So there is some hope here for a construction.

Acknowledgements

This research was conducted during an Oberwolfach workshop on graph theory in January 2022, and the authors are grateful to MFO for providing accommodations and facilities. We would like to thank Xingxing Yu for introducing us to this problem (at an open problem session during the workshop) and for supplying background information.

References

[1] D. Nelson, M. Plummer, N. Robertson and X. Zha, "On a conjecture concerning the Petersen graph", *Electronic J. Combinatorics* **18** (2011), #P20.

[2] M. Plummer and X. Zha, "On a conjecture concerning the Petersen graph: part II", *Electronic J. Combinatorics* **21** (2014), #P1.34.

[3] N. Robertson, personal communication to Xiaoya Zha.

[4] D. Wu, B. Xu and Y. Xu, "On coloring of graphs of girth $2\ell + 1$ without longer odd holes", `arXiv:2204.0628`.

[5] B. Xu, G. Yu, and X. Zha, "A note on chromatic number and induced odd cycles", *Electronic J. Combinatorics* **24** (4) (2017), #P4.32.