INDUCED SUBGRAPHS AND TREE DECOMPOSITIONS XVIII. OBSTRUCTIONS TO BOUNDED PATHWIDTH

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Abstract. The *pathwidth* of a graph G is the smallest $w \in \mathbb{N}$ such that G can be constructed from a sequence of graphs, each on at most $w+1$ vertices, by gluing them together in a linear fashion. We provide a full classification of the unavoidable induced subgraphs of graphs with large pathwidth.

1. Introduction

The set of all positive integers is denoted by $\mathbb{N}$, and for every integer k, the set of all positive integers no greater than k is denoted by $\mathbb{N}_k$. Graphs in this paper have finite vertex sets, no loops and no parallel edges. For standard graph theoretic terminology, the reader is referred to [8].

For a graph $G = (V(G), E(G))$, the *treewidth* of G, denoted $\mathrm{tw}(G)$, is the smallest $w \in \mathbb{N}$ for which there is a tree T and an assignment of a subtree T_v of T to each vertex $v \in V(G)$ such that:

- for every edge $uv \in E(G)$, we have $V(T_u) \cap V(T_v) = \varnothing$;
- for every vertex $x \in V(T)$, we have $|\{v \in V(G) : x \in V(T_v)\}| \leq w + 1$.

The *pathwidth* of G, denoted $\mathrm{pw}(G)$, is defined analogously with T being a path (instead of a general tree).

Two central results in graph minor theory are complete descriptions of unavoidable minors in graphs of large treewidth and pathwidth. These are, respectively, planar graphs and forests:

Theorem 1.1 (Robertson and Seymour [14]). *For every planar graph H, there is a constant $f_{1.1} = f_{1.1}(H) \in \mathbb{N}$ such that every graph G with $\mathrm{tw}(G) > f_{1.2}$ has a minor isomorphic to H. Moreover, if H is not planar, then no such constant exists.*

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Theorem 1.2 (Robertson and Seymour [13])**.** *For every forest H, there is a constant $f_{1.2} = f_{1.2}(H) \in \mathbb{N}$ such that every graph G with $\mathrm{pw}(G) > f_{1.2}$ has a minor isomorphic to H. Moreover, if H is not a forest, then no such constant exists.*

The goal of this series of papers is to study the same questions for induced subgraphs (instead of minors). Here we prove an analogue of Theorem 1.2. Our main result, Theorem 2.1, identifies the following as the unavoidable induced subgraphs of graphs with large pathwidth (the exact statement and all necessary definitions will be given in Section 2):

- Complete graphs and complete bipartite graphs;
- Subdivided binary trees and their line graphs; and
- "Constellations" that are "interrupted" or "zigzagged."

We will derive Theorem 2.1 from another result, Theorem 1.3 below. In turn, Theorem 1.3 below is about "induced minors": a containment relation halfway between minors and induced subgraphs. Given a graph G, recall that a *minor* of G is a graph obtained from a subgraph of G by repeatedly contracting edges. Sometimes, when we want to define this operation on the class of simple graphs, loops and parallel edges arising in this process are removed. An *induced minor* of G is a graph obtained from an *induced* subgraph of G by repeatedly contracting edges, and removing all loops and parallel edges arising in this process.

Theorem 1.3. *For all $t \in \mathbb{N}$ and every forest H, there is a constant $f_{1.3} = f_{1.3}(t, H)$ such that every graph G with $\mathrm{pw}(G) > f_{1.3}$ has a subgraph isomorphic to K_{t+1}, an induced minor isomorphic to $K_{t,t}$ or an induced minor isomorphic to H.*

We find it more convenient to work with minors and induced minors in terms of "models," defined as follows. Let $G = (V(G), E(G))$ be a graph. For $X \subseteq V(G)$, we use both X and $G[X]$ to denote the induced subgraph of G with vertex set X (also called the *subgraph of G induced by X*). For $X, Y \subseteq V(G)$, we say that *X and Y are anticomplete in G* if $X \cap Y = \varnothing$ and there is no edge in G with an end in X and an end in Y. For $x \in V(G)$, we say that *x is anticomplete to Y in G* if $\{x\}$ and Y are anticomplete in G. For another graph H, an *H-model in G* is a $|V(H)|$-tuple $(A_v : v \in V(H))$ of pairwise disjoint connected induced subgraphs of G such that for all distinct and adjacent vertices $u, v \in V(H)$ in H, the sets A_u and A_v are not anticomplete in G. We call A_v *the branch set associated with v*. We also say that the H-model $(A_v : v \in V(H))$ in G is *induced* if for all distinct and non-adjacent vertices $u, v \in V(H)$ in H, the branch sets A_u and A_v are anticomplete in G. It is straightforward to observe that a graph G has a minor isomorphic to a graph H if and only if there is an H-model in G, and G has an induced minor isomorphic to a graph H if and only if there is an induced H-model in G.

We will state our main result, Theorem 2.1, in Section 2. There we also show how Theorem 2.1 follows from Theorem 1.3 combined with the main result of one of our earlier papers [5] in this series. The remainder of this paper will then be devoted to the proof of Theorem 1.3.

2. From induced minors to induced subgraphs

2.1. **Definitions.** The statement of our main result involves several definitions, some of which will also be used in later sections.

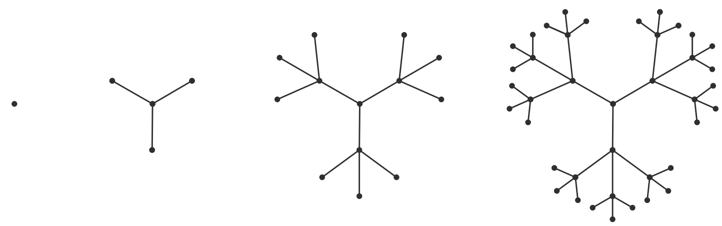

FIGURE 1. The trees $T_{3,r}$ for $r = 0, 1, 2, 3$.

FIGURE 2. A subdivision of $T_{2,3}$ (left) and its line graph (right).

Let G be a graph. A *stable set* in G is a set of pairwise non-adjacent vertices in G, and a *clique* in G is a set of pairwise adjacent vertices in G. Let X be a subset of $V(G)$. We denote by $N_G(X)$ the set of all vertices in $G \setminus X$ with at least one neighbor in X. If $X = \{x\}$, then we write $N_G(x)$ for $N_G(\{x\})$. For a set $\mathcal{X}$ of subsets of $V(G)$, we write $V(\mathcal{X}) = \bigcup_{X \in \mathcal{X}} X$. For a graph H, we say that G is H*-free* if G has no induced subgraph isomorphic to H. The *line graph* of G, denoted $L(G)$, is the graph with vertex set $E(G)$ such that $e, f \in E(G)$ are adjacent in $L(G)$ if and only if e and f share an end in G.

Let $d \in \mathbb{N}$ and let $r \in \mathbb{N} \cup \{0\}$. We denote by $T_{d,r}$ the unique (up to isomorphism) rooted tree of radius r such that, when $r \geq 1$, the root has degree d and every vertex that is neither the root nor a leaf has degree $d + 1$ (see Figure 1). For instance, $T_{2,r}$ is the full binary tree of radius r. It is well-known [13] that for every $r \in \mathbb{N}$, all subdivisions of $T_{2,2r}$ and their line graphs have pathwidth at least r (see Figure 2).

For an integer k, we denote by $\mathbb{N}_k$ the set of all positive integers no greater than k (so $\mathbb{N}_k = \varnothing$ if and only if $k \leq 0$). Let $k \in \mathbb{N}$ and let P be a k-vertex graph which is a path. Then we write $P = p_1\text{-}\cdots\text{-}p_k$ to mean that $V(P) = \{p_1, \ldots, p_k\}$ and $E(P) = \{p_i p_{i+1} : i \in \mathbb{N}_{k-1}\}$. We call the vertices p_1 and p_k the *ends of* P and refer to $P \setminus \{p_1, p_n\}$ as the *interior of* P, denoted P^*. For vertices $u, v \in V(P)$, we denote by u-P-v the subpath of P from u to v. The *length* of a path is the number of edges in it. It follows that a path P has distinct ends if and only if P has non-zero length, and P has non-empty interior if and only if P has length at least two. Given a graph G, a *path in* G is an induced subgraph of G which is a path.

A *constellation* is a graph $\mathfrak{c}$ in which there is a stable set $S_\mathfrak{c}$ such that every component of $\mathfrak{c} \setminus S_\mathfrak{c}$ is a path, and each vertex $x \in S_\mathfrak{c}$ has at least one neighbor in each component of $\mathfrak{c} \setminus S_\mathfrak{c}$. We denote by $\mathcal{L}_\mathfrak{c}$ the set of all components $\mathfrak{c} \setminus S_\mathfrak{c}$ (each of which is a path), and denote the constellation $\mathfrak{c}$ by the pair $(S_\mathfrak{c}, \mathcal{L}_\mathfrak{c})$. For $l, s \in \mathbb{N}$, by an (s, l)*-constellation* we mean a constellation $\mathfrak{c}$ with $|S_\mathfrak{c}| = s$ and $|\mathcal{L}_\mathfrak{c}| = l$. Given a graph G, by an (s, l)*-constellation in* G we mean an induced subgraph of G which is an (s, l)-constellation.

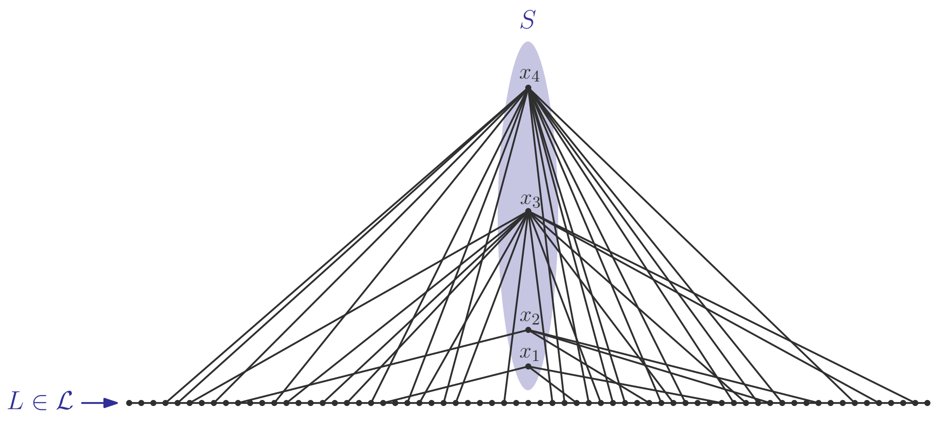

FIGURE 3. A $(4,1)$-constellation which is ample and interupted.

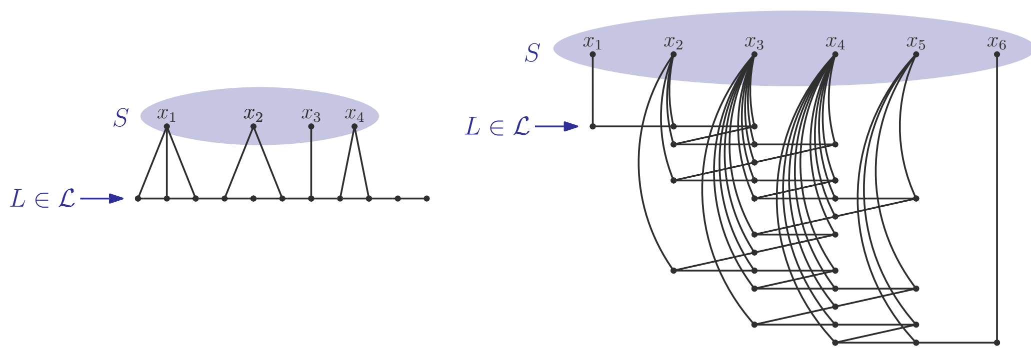

FIGURE 4. A $(4,1)$-constellation (left) and a $(6,1)$-constellation (right), both ample and 1-zigzagged.

We will need a few notions associated with a constellation $\mathfrak{c} = (S_\mathfrak{c}, \mathcal{L}_\mathfrak{c})$, which we define below:

- By a $\mathfrak{c}$-*route* we mean a path R in $\mathfrak{c}$ with ends in $S_\mathfrak{c}$ and with $R^* \subseteq V(\mathcal{L}_\mathfrak{c})$, or equivalently, with $R^* \subseteq L$ for some $L \in \mathcal{L}_\mathfrak{c}$.
- For $d \in \mathbb{N}$, we say that $\mathfrak{c}$ is d-*ample* if there is no $\mathfrak{c}$-route of length at most $d+1$. We also say that $\mathfrak{c}$ is *ample* if $\mathfrak{c}$ is 1-ample. It follows that $\mathfrak{c}$ is ample if and only if no two vertices in $S_\mathfrak{c}$ have a common neighbor in $V(\mathcal{L}_\mathfrak{c})$.
- We say that $\mathfrak{c}$ is *interrupted* if there is an enumeration $x_1, \ldots, x_s$ of all vertices in $S_\mathfrak{c}$ such that for all $i, j, k \in \mathbb{N}_s$ with $i < j < k$ and every $\mathfrak{c}$-route R from x_i to x_j, the vertex x_k has a neighbor in R (see Figure 3).
- For $q \in \mathbb{N}$, we say that $\mathfrak{c}$ is q-*zigzagged* if there is an enumeration $x_1, \ldots, x_s$ of all vertices in $S_\mathfrak{c}$ such that for all $i, k \in \mathbb{N}_s$ with $i < k$ and every $\mathfrak{c}$-route R from x_i to x_k, fewer than q vertices in $\{x_j : i < j < k\}$ are anticomplete to R in $\mathfrak{c}$ (see Figure 4).

Interrupted constellations form a slight extension of another construction from [2, 3], and zigzagged constellations are a fairly substantial generalization of a construction from [7, 11] (see [5] for further discussion).

2.2. **The main result.** With the above definitions in hand, we are now ready to state the main result of this paper:

Theorem 2.1. *For all $d, r, l, l', s, s' \in \mathbb{N}$, there is a constant $f_{2.1} = f_{2.1}(d, r, l, l', s, s')$ such that if G is a graph with $\mathrm{pw}(G) > f_{2.1}$, then one of the following holds.*

- (a) *There is an induced subgraph of G isomorphic to K_{r+1}, $K_{r,r}$, a subdivision of $T_{2,2r}$ or the line graph of a subdivision of $T_{2,2r}$.*
- (b) *There is a d-ample interrupted (s, l)-constellation in G.*
- (c) *There is a d-ample 2^{4r+1}-zigzagged (s', l')-constellation in G.*

Theorem 2.1 is "qualitatively" best possible, in the sense that:

- the outcomes of 2.1 themselves can have arbitrarily large pathwidth; and
- the statement of 2.1 will be false if any of the outcomes is omitted.

The first point is straightforward to check, and the second point is easily seen to be true for 2.1(a). For 2.1(b) and 2.1(c), the second point follows from the two results below that we proved in [5], and the fact that all constellations are K_4-free, and all ample constellations are $K_{3,3}$-free.

Theorem 2.2 (Chudnovsky, Hajebi, Spirkl [5])**.** *Let $\mathfrak{c}$ be an ample interrupted constellation. Then $\mathfrak{c}$ has no induced subgraph isomorphic to any of the following.*

- *An ample q-zigzagged $\left(3q + 6, 6\binom{q+2}{3}\right)$-constellation, where $q \in \mathbb{N}$.*
- *A subdivision of $T_{2,7}$ or the line graph of a subdivision of $T_{2,7}$.*

Theorem 2.3 (Chudnovsky, Hajebi, Spirkl [5])**.** *Let $q \in \mathbb{N}$ and let $\mathfrak{c}$ be an ample q-zigzagged constellation. Then $\mathfrak{c}$ has no induced subgraph isomorphic to any of the following.*

- *An ample interrupted $(2q + 6, 1)$-constellation.*
- *A subdivision of $T_{2,64q^2}$ or the line graph of a subdivision of $T_{2,64q^2}$.*

As mentioned earlier, Theorem 2.1 is a consequence of Theorem 1.3 combined with the main result of an earlier paper [5] in this series. For every $r \in \mathbb{N}$, we denote by $W_{r\times r}$ the r-by-r hexagonal grid, also known as the r-by-r *wall* (see Figure 5).

Theorem 2.4 (Chudnovsky, Hajebi, Spirkl [5])**.** *For all $d, l, l', r, s, s' \in \mathbb{N}$, there are constants $f_{2.4} = f_{2.4}(d, l, l', r, s, s') \in \mathbb{N}$ and $g_{2.4} = g_{2.4}(d, l, l', r, s, s') \in \mathbb{N}$ such that for every graph G with an induced minor isomorphic to $K_{f_{2.4},g_{2.4}}$ one of the following holds.*

- (a) *There is an induced subgraph of G isomorphic to $K_{r,r}$, a subdivision of $W_{r\times r}$, or the line graph of a subdivision of $W_{r\times r}$.*
- (b) *There is a d-ample interrupted (s, l)-constellation G.*
- (c) *There is a d-ample $2r^2$-zigzagged (s', l')-constellation in G.*

In addition to Theorem 2.4, we need the following observation about the presence of binary tree induced minors in walls (see Figure 5) and of general tree induced minors in binary trees (see Figure 6). The proofs are easy and we omit them.

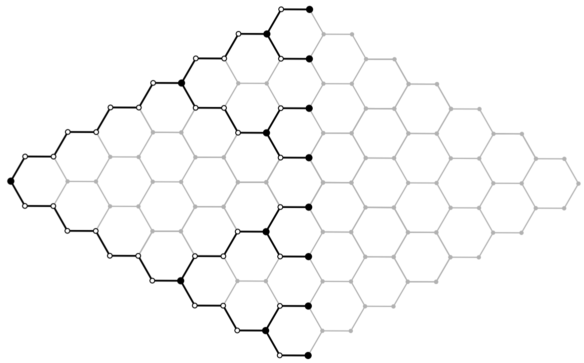

FIGURE 5. The graph $W_{8\times 8}$, and an induced subgraph of it isomorphic to a proper subdivision of $T_{2,3}$ (as in 2.5(a) for $r = 3$).

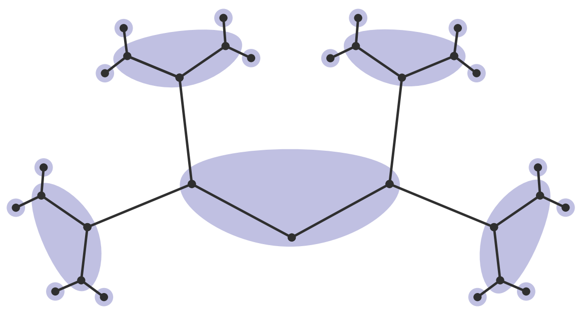

FIGURE 6. An induced $T_{4,2}$-model in $T_{2,4}$ (as in 2.5(b) for $d = r = 2$).

Observation 2.5. *The following hold for all $d, r \in \mathbb{N}$.*

(a) *There is an induced subgraph of $W_{2^r\times 2^r}$ isomorphic to a proper subdivision of $T_{2,r}$. Also, there is an induced subgraph of the line graph of $W_{2^r\times 2^r}$ isomorphic to the line graph of a proper subdivision of $T_{2,r}$. Consequently, if W is a subdivision of $W_{2^r\times 2^r}$, then both W and its line graph have an induced minor isomorphic to $T_{2,r}$.*

(b) *There is an induced minor of $T_{2,dr}$ isomorphic to $T_{2^d,r}$.*

We will also use the following from [10]:

Lemma 2.6 (Hickingbotham [10])**.** *For every $r \in \mathbb{N}$, if G is a graph with an induced minor isomorphic to $T_{2,8r}$, then G has an induced subgraph isomorphic to either a subdivision of $T_{2,r}$ or the line graph of a subdivision of $T_{2,r}$.*

Let us now deduce Theorem 2.1:

Proof of Theorem 2.1. Let

$$\phi = f_{2.4}(d, l, l', 2^{2r}, s, s')$$

and let

$$\gamma = g_{2.4}(d, l, l', 2^{2r}, s, s').$$

We claim that

$$f_{2.1} = f_{2.1}(d, l, l', r, s, s') = f_{1.3}(\max\{r, \phi, \gamma\}, T_{2,16r})$$

satisfies the theorem.

Let G be a graph of pathwidth larger than $f_{2.1}$. From Theorem 1.3, it follows that G has a subgraph isomorphic to K_{r+1}, an induced minor isomorphic to $K_{\phi,\gamma}$ or an induced minor isomorphic to $T_{2,16r}$. In the former case, 2.1(a) holds. Also, if G has an induced minor isomorphic to $T_{2,16r}$, then by Lemma 2.6, G has an induced subgraph isomorphic to either a subdivision of $T_{2,2r}$ or the line graph of a subdivision of $T_{2,2r}$, and again 2.1(a) holds. Therefore, we may assume that G has an induced minor isomorphic to $K_{\phi,\gamma}$. By the choice of ϕ, γ, we can apply Theorem 2.4 to G. Note that 2.4(a) along with Observation 2.5(a) (and the fact that $2^{2r} \geq r$) implies 2.1(a). Moreover, 2.4(b) directly implies 2.1(b), and 2.4(c) directly implies 2.1(c). This completes the proof of Theorem 2.1. ■

3. Seedlings and overview of the proof of Theorem 1.3

Here we give an overview of the steps in the proof of Theorem 1.3, beginning with some definitions.

Let G be a graph and let $X, Y \subseteq V(G)$. An *(X, Y)-path in G* is a path P in G that has an end in X and an end in Y, and subject to this property, $V(P)$ is minimal with respect to inclusion. Equivalently, an (X, Y)-path P in G is a path in G such that either

- P has length zero and $P \subseteq X \cap Y$; or
- P has non-zero length, one end of P belongs to $X \setminus Y$, the other end of P belongs to $Y \setminus X$, and we have $P^* \cap (X \cup Y) = \varnothing$.

In particular, every (X, Y)-path P in G has an end in X and an end in Y. We call the former the *X-end of P* and the latter the *Y-end of P*. So the unique vertex of a zero-length (X, Y)-path P is both the X-end and the Y-end of P.

Next, we define what we call a "seedling," a notion central to almost all of our proofs in this paper. Let G be a graph and let $\lambda \in \mathbb{N}$. A *λ-seedling in G* is a triple $(A, \mathcal{L}, Y)$ with the following specifications (see Figure 7):

- A is a path in G;
- $Y \subseteq V(G) \setminus A$; and
- $\mathcal{L}$ is a set of λ pairwise disjoint $(N_G(A), Y)$-paths in $G \setminus A$.

It follows in particular that $A \cap V(\mathcal{L}) = \varnothing$, and for every $L \in \mathcal{L}$, the $N(A)$-end of L is the only vertex in L with a neighbor in A.

By a *seedling in G* we mean a λ-seedling in G for some $\lambda \in \mathbb{N}$. Two seedlings $(A, \mathcal{L}, Y)$ and $(A', \mathcal{L}', Y')$ in a graph G are *disjoint* if $(A \cup V(\mathcal{L})) \cap (A' \cup V(\mathcal{L}')) = \varnothing$. We say that a seedling $(A, \mathcal{L}, Y)$ in G is *κ-rigid*, where $\kappa \in \mathbb{N}$, if there is no set $\mathcal{K}$ of κ pairwise anticomplete $(N(A), Y)$-paths in G such that $V(\mathcal{K}) \subseteq V(\mathcal{L})$.

For $t \in \mathbb{N}$, we say that a graph G is *t-tidy* if G is K_{t+1}-free and has no induced minor isomorphic to $K_{t,t}$. Then Theorem 1.3 says for all $t \in \mathbb{N}$ and every forest H, every t-tidy graph of sufficiently large pathwidth has an induced minor isomorphic to H. Roughly, the proof of Theorem 1.3 is in three steps. The first step is to prove the following. Note that $g_{3.1}$ does not depend on λ.

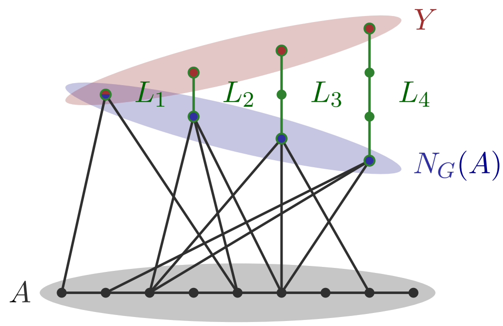

FIGURE 7. A 4-seedling $(A, \{L_1, L_2, L_3, L_4\}, Y)$.

Theorem 3.1. *For all $d, r, t, \lambda \in \mathbb{N}$, there are constants $f_{3.1} = f_{3.1}(d, r, t, \lambda) \in \mathbb{N}$ and $g_{3.1} = g_{3.1}(d, r, t) \in \mathbb{N}$ such that for every t-tidy graph G with $\mathrm{pw}(G) > f_{3.1}$ one of the following holds.*

(a) *G has an induced minor isomorphic to $T_{d,r}$.*
(b) *There is a λ-seedling in G which is $g_{3.1}$-rigid.*

The second step is to prove the following. Note, again, that $g_{3.2}$ does not depend on λ (nor on δ; but that does not matter much).

Theorem 3.2. *For all $t, \delta, \lambda, \kappa \in \mathbb{N}$, there are constants $f_{3.2} = f_{3.2}(t, \delta, \lambda, \kappa) \in \mathbb{N}$ and $g_{3.2} = g_{3.2}(t, \kappa) \in \mathbb{N}$ with the following property. Let G be a t-tidy graph and let $(A, \mathcal{L}, Y)$ be an $f_{3.2}$-seedling in G which is κ-rigid. Then there are δ pairwise disjoint λ-seedlings $(A_1, \mathcal{L}_1, Y_1), \dots, (A_\delta, \mathcal{L}_\delta, Y_\delta)$ in $G \setminus A$ with the following specifications.*

(a) *The paths $A_1, \dots, A_\delta$ are pairwise anticomplete in G.*
(b) *For every $i \in \mathbb{N}_\delta$, we have:*
   - *A and A_i are not anticomplete in G;*
   - *A and $V(\mathcal{L}_i)$ are anticomplete in G; and*
   - *$(A_i, \mathcal{L}_i, Y_i)$ is $g_{3.2}$-rigid.*

The third (and last) step is to use Theorem 3.2 to prove the following by induction on r:

Theorem 3.3. *For all $d, r, t, \kappa \in \mathbb{N}$, there is a constant $f_{3.3} = f_{3.3}(d, r, t, \kappa) \in \mathbb{N}$ with the following property. Let G be a t-tidy graph and let $(A, \mathcal{L}, Y)$ be an $f_{3.3}$-seedling in G which is κ-rigid. Then there is an induced $T_{d,r}$-model in $G[A \cup V(\mathcal{L})]$ where A is the branch set associated with the root of $T_{d,r}$.*

Now Theorem 1.3 is almost immediate from Theorems 3.1 and 3.3:

Proof of Theorem 1.3. Let $|V(H)| = h$. Let

$$\kappa = g_{3.1}(h, h, t);$$

$$\lambda = f_{3.3}(h, h, t, \kappa).$$

We claim that

$$f_{1.3} = f_{1.3}(t, H) = f_{3.1}(h, h, t, \lambda)$$

satisfies the theorem.

Let G be a t-tidy graph with $\mathrm{pw}(G) > f_{1.3}$. We will show that G has an induced minor isomorphic to H. Let H^+ be a tree obtained from H by adding a vertex with exactly one neighbor in each component of H. Then H is an induced subgraph of H^+. Also, H^+ has both maximum degree and radius at most $|V(H^+)|-1 = h$. It follows that H^+, and so H, is isomorphic to an induced subgraph of $T_{h,h}$. Thus, it suffices to prove that G has an induced minor isomorphic to $T_{h,h}$.

Now, since G has pathwidth more than $f_{3.1}(h, h, t, \lambda)$, it follows from Theorem 3.1 that either G has an induced minor isomorphic to $T_{h,h}$, or there is a λ-seedling in G which is κ-rigid (recall that $\kappa = g_{3.1}(h, h, t)$). In the former case, we are done. In the latter case, since $\lambda = f_{3.3}(h, h, t, \kappa)$, it follows from Theorem 3.3 that G has an induced minor isomorphic to $T_{h,h}$. This completes the proof of Theorem 1.3. ■

It remains to prove Theorems 3.1, 3.2 and 3.3, which we will do in Sections 4, 5 and 6.

4. Planting a seedling

In this section, we prove Theorem 3.1. We need a few results from the literature.

Theorem 4.1 (Ramsey [12]). *For all $s, t \in \mathbb{N}$, every graph on at least s^t vertices has either a stable set of cardinality s or a clique of cardinality $t + 1$.*

Theorem 4.2 (Hickingbotham [10]). *For all $r, s \in \mathbb{N}$, there is a constant $f_{4.2} = f_{4.2}(r, s)$ such that every graph G with $\mathrm{pw}(G) > f_{4.2}$ has either an induced minor isomorphic to $T_{2,r}$ or a minor isomorphic to K_s.*

We also need Theorem 4.3 below from [1], which we have also used in several earlier papers of this series.

Let X be a set. We denote the set of all subsets of X by 2^X and the set of all k-subsets of X, where $k \in \mathbb{N}$, by $\binom{X}{k}$. Let $k, l \in \mathbb{N}$ and let G be a graph. A (k, l)-*block* in G is a pair $(B, \mathcal{P})$ where $B \subseteq V(G)$ with $|B| \geq k$ and $\mathcal{P} : \binom{B}{2} \to 2^{V(G)}$ is map such that $\mathcal{P}_{\{x,y\}} = \mathcal{P}(\{x, y\})$, for each 2-subset $\{x, y\}$ of B, is a set of at least l pairwise internally disjoint paths in G from x to y. We say that $(B, \mathcal{P})$ is *strong* if for all distinct 2-subsets $\{x, y\}, \{x', y'\}$ of B, we have $V(\mathcal{P}_{\{x,y\}}) \cap V(\mathcal{P}_{\{x',y'\}}) = \{x, y\} \cap \{x', y'\}$; that is, each path $P \in \mathcal{P}_{\{x,y\}}$ is disjoint from each path $P' \in \mathcal{P}_{\{x',y'\}}$, except P and P' may share an end.

Let $t \in \mathbb{N}$. We say that a graph G is t-*clean* if G has no induced subgraph isomorphic to K_{t+1}, $K_{t,t}$, a subdivision of $W_{t\times t}$ or the line graph of a subdivision of $W_{t\times t}$.

Theorem 4.3 (Abrishami, Alecu, Chudnovsky, Hajebi, Spirkl [1]). *For all $k, l, t \in \mathbb{N}$, there is a constant $f_{4.3} = f_{4.3}(k, l, t) \in \mathbb{N}$ such that for every t-clean graph G with $\mathrm{tw}(G) > f_{4.3}$, there is a strong (k, l)-block in G.*

Finally, we need a lemma from [6]:

Lemma 4.4 (Chudnosvky, Hajebi, Spirkl [6]). *For all $s, t, \rho, \sigma \in \mathbb{N}$, there are constants $f_{4.4} = f_{4.4}(s, t, \rho, \sigma) \in \mathbb{N}$ and $g_{4.4} = g_{4.4}(s, \rho, \sigma) \in \mathbb{N}$ with the following property. Let G be a K_{t+1}-free graph and let $(B, \mathcal{Q})$ be a strong $(f_{4.4}, g_{4.4})$-block in G such that for every $\{x, y\} \subseteq B$, the paths $(Q^* : Q \in \mathcal{Q}_{\{x,y\}})$ are pairwise anticomplete in G. Then one of the following holds.*

(a) *There is an induced subgraph of G isomorphic to a proper subdivision of K_s.*
(b) *There is an induced minor of G isomorphic to $K_{\rho,\sigma}$.*

Now we can prove the main result of this section, which we restate:

Theorem 3.1. *For all $d, r, t, \lambda \in \mathbb{N}$, there are constants $f_{3.1} = f_{3.1}(d, r, t, \lambda) \in \mathbb{N}$ and $g_{3.1} = g_{3.1}(d, r, t) \in \mathbb{N}$ such that for every t-tidy graph G with $\mathrm{pw}(G) > f_{3.1}$ one of the following holds.*

(a) *G has an induced minor isomorphic to $T_{d,r}$.*
(b) *There is a λ-seedling in G which is $g_{3.1}$-rigid.*

Proof. Let

$$\phi = \phi(d, r, t) = f_{4.4}(rd^r + 1, t, t, t)$$

and let

$$\psi = \psi(d, r, t, \lambda) = f_{4.3}(\phi^t, \lambda, \max\{2^{dr}, t\}).$$

We claim that

$$f_{3.1} = f_{3.1}(d, r, t, \lambda) = f_{4.2}(dr, \psi + 2)$$

and

$$g_{3.1} = g_{3.1}(d, r, t) = g_{4.4}(rd^r + 1, t, t)$$

satisfy the theorem.

Let G be a t-tidy graph with $\mathrm{pw}(G) > f_{3.1}$. Suppose for a contradiction that neither 3.1(a) nor 3.1(b) holds; that is, G has no induced minor isomorphic to $T_{d,r}$, and there is no λ-seedling in G which is $g_{3.1}$-rigid.

(1) *G has no induced minor isomorphic to $T_{2,dr}$. Also, G is $\max\{2^{dr}, t\}$-clean.*

Since G has no induced minor isomorphic to $T_{d,r}$, it follows from Observation 2.5(b) (and the fact that $2^d > d$) that G has no induced minor isomorphic to $T_{2,dr}$. This, along with Observation 2.5(a), implies that G has no induced subgraph isomorphic to a subdivision of $W_{2^{dr}\times 2^{dr}}$ or the line graph of a subdivision of $W_{2^{dr}\times 2^{dr}}$. Also, recall that G is K_{t+1}-free and $K_{t,t}$-free. Therefore, G is $\max\{2^{dr}, t\}$-clean. This proves (1).

Since $\mathrm{pw}(G) > f_{3.1}$, it follows from Theorem 4.2, the choice of $f_{1.3}$ and the first bullet of (1) that G has a minor isomorphic to $K_{\psi+2}$; in particular, we have $\mathrm{tw}(G) \geq \mathrm{tw}(K_{\psi+2}) = \psi$ [8]. Moreover, by the second bullet of (1), G is $\max\{2^{dr}, t\}$-clean. Thus, by Theorem 4.3 and the choice of ψ, there is (ϕ^t, λ)-strong block $(B, \mathcal{P})$ in G.

Since G is K_{t+1}-free, it follows from Theorem 4.1 that there is a stable set $S \subseteq B$ in G with $|S| = \phi$. We further claim that:

(2) *For every $\{x,y\} \subseteq S$, there is a set $\mathcal{Q}_{\{x,y\}}$ of $g_{3.1}$ paths in G between x and y such that $V(\mathcal{Q}_{\{x,y\}}) \subseteq V(\mathcal{P}_{\{x,y\}})$ and the paths $(Q^* : Q \in \mathcal{Q}_{x,y})$ are pairwise anticomplete in G.*

Since S is a stable set, it follows that $N_G(y) \subseteq V(G) \setminus \{x\}$, and the paths in $\mathcal{P}_{\{x,y\}}$ have non-empty interiors; in particular, $\mathcal{L}_{\{x,y\}} = \{P^* : P \in \mathcal{P}_{\{x,y\}}\}$ is a set of λ pairwise disjoint $(N_G(x), N_G(y))$-paths in G. Thus, $(\{x\}, \mathcal{L}_{\{x,y\}}, N_G(y))$ is a λ-seedling in G. On the other hand, recall the assumption that there is no λ-seedling in G which is $g_{3.1}$-rigid. It follows that $(\{x\}, \mathcal{L}_{\{x,y\}}, N_G(y))$ is not $g_{3.1}$-rigid, and so there is a set $\mathcal{K}_{\{x,y\}}$ of $g_{3.1}$ pairwise disjoint and anticomplete $(N_G(x), N_G(y))$-paths in G with $V(\mathcal{K}_{\{x,y\}}) \subseteq V(\mathcal{L}_{\{x,y\}})$. But now $\mathcal{Q}_{\{x,y\}} = \{\{x,y\} \cup K : K \in \mathcal{K}_{\{x,y\}}\}$ is a set of $g_{3.1}$ paths in G between x and y such that $V(\mathcal{Q}_{\{x,y\}}) \subseteq V(\mathcal{P}_{\{x,y\}})$ and the paths $(Q^* : Q \in \mathcal{Q}_{x,y})$ are pairwise anticomplete in G. This proves (2).

Henceforth, for every 2-subset $\{x,y\}$ of S, let $\mathcal{Q}_{\{x,y\}}$ be as given by (2). Then $(S, \mathcal{Q})$ is a strong $(\phi, g_{3.1})$-block in G such that for every $\{x,y\} \subseteq S$, the paths $(Q^* : Q \in \mathcal{Q}_{x,y})$ are pairwise anticomplete in G. Since G is K_{t+1}-free with no induced minor isomorphic to $K_{t,t}$, it follows from Lemma 4.4 and the choice of ϕ and $g_{3.1}$ that G has an induced subgraph isomorphic to a proper subdivision of K_{rd^r+1}. In particular, since $|V(T_{d,r})| \leq rd^r + 1$, it follows that G has an induced subgraph isomorphic to a (proper) subdivision of $T_{d,r}$. But then G has an induced minor isomorphic to $T_{d,r}$, contrary to the assumption that 3.1(a) does not hold. This completes the proof of Theorem 3.1. ■

5. Growing a seedling

In this section, we prove Theorem 3.2. The main tool is Lemma 5.1 below about digraphs. So we start by clarifying our digraph terminology.

By a *digraph* we mean a pair $D = (V(D), E(D))$ where D is a finite set of *vertices* and $E(D) \subseteq (V(D) \times V(D)) \setminus \{(v,v) : v \in V(D)\}$ is the set of *edges*. In particular, our digraphs are loopless and allow at most one edge in each direction between every two vertices. Let D be a digraph. For $(u,v) \in E(D)$, we say that v is an *out-neighbor* of u and u is an *in-neighbor* of v. The *out-degree* (*in-degree*) of a vertex $v \in V(D)$ is the number of its out-neighbors (in-neighbors). The *underlying graph of* D is the graph G with $V(G) = V(D)$ and $E(G) = \{uv : (u,v) \in E(D) \text{ or } (u,v) \in E(D)\}$. A *stable set in* D is a stable set in the underlying graph of D. For $X \subseteq V(D)$, we denote by $D[X]$ the digraph with vertex set X and edge set $E(D) \cap (X \times X)$. It follows that the underlying graph of $D[X]$ is the subgraph of the underlying graph of D induced by X.

We need the following lemma; 5.1(a) is well-known, but we include a proof for the sake of completeness.

Lemma 5.1. *Let $q, r, s \in \mathbb{N}$ and let D be a digraph. Then the following hold.*

- (a) *If D has at least $2rs$ vertices of out-degree at most r, then there is a stable set of cardinality s in D.*
- (b) *If there are at least $2qrs$ vertices of out-degree at least qr in D, then there is an s-subset S of $V(D)$ with the following property: for every q-subset $\{v_1, \ldots, v_q\}$ of S, there are q pairwise disjoint r-subsets $R_1, \ldots, R_q$ of $V(D) \setminus S$ such that for each $i \in \mathbb{N}_q$, every vertex in R_i is an out-neighbor of v_i.*

Proof. We prove 5.1(a) by induction on s (for fixed r). The case $s = 1$ is trivial, so assume that $s \geq 2$. Choose a set X of exactly $2rs$ vertices, each with out-degree at most r in D. Let $D_1 = D[X]$. Then D_1 is a digraph in which every vertex has out-degree at most r. Let G_1 be the underlying graph of D_1. It follows that both D_1 and G_1 have at most $2r^2s$ edges. There are two cases to consider. First, assume that some vertex v has degree at most $2r-1$ in G_1. Let $D_2 = D[X \setminus (N_{G_1}(v) \cup \{v\})]$. Then we have $|V(D_2)| = |X| - |N_{G_1}(v) \cup \{v\}| \geq 2rs-2r \geq 2r(s-1)$, and every vertex of D_2 has out-degree at most r in D_2. By the inductive hypothesis applied to D_2, there is a stable set S in D_2 of cardinality $s-1$. But now since $S \subseteq V(D_2) = X \setminus (N_{G_1}(v) \cup \{v\})$, it follows that $S \cup \{v\}$ is stable set of cardinality s in D, as desired. Second, assume that every vertex in G_1 has degree at least $2r$. Then, since G_1 has $2rs$ vertices and at most $2r^2s$ edges, it follows G_1 is a $2r$-regular graph (on $2rs$ vertices), and so by Brook's theorem [4], G_1 admits a $2r$-coloring. Therefore, there is a stable set of cardinality $2rs/2r = s$ in G_1, and so in D. This proves 5.1(a).

Next, we prove 5.1(b), and for that we will use 5.1(a) which we just proved above. Let Y be the set of all vertices of out-degree at least qr in D; thus, $|Y| \geq 2qrs$. For each vertex $y \in Y$, choose a set Q_y of exactly qr out-neighbors of y in D. Let D' be the digraph with $V(D') = V(D)$ and $E(D') = \bigcup_{y \in Y}\{(y, z) : z \in Q_y\}$. Then $E(D') \subseteq E(D)$. Moreover, for every $y \in Y$, the set Q_y is exactly the set of all out-neighbors of y in D'. In particular, y has out-degree exactly qr in D', and so y has out-degree at most qr in $D'[Y]$. Since $|Y| \geq 2qrs$, it follows from 5.1(a) applied to $D'[Y]$ that there is a stable set S in $D'[Y]$ of cardinality s. In other words, S is a stable set in D' with $|S| = s$ and $S \subseteq Y$. From this and the definition of D', we deduce that for every $y \in S \subseteq Y$, we have $Q_y \subseteq V(D') \setminus S = V(D) \setminus S$.

Now, let $\{v_1, \dots, v_q\}$ be a q-subset of $S \subseteq Y$. Since $|Q_{v_1}| = \cdots = |Q_{v_q}| = qr$, it follows that for every $i \in \mathbb{N}_{q-1}$, we have $|Q_{v_{i+1}}| - ir \geq r$. In particular, there are r-subsets of $R_1, \dots, R_q$ of $Q_{v_1}, \dots, Q_{v_q}$, respectively, such that for every $i \in \mathbb{N}_{q-1}$, we have $R_{i+1} \subseteq Q_{v_{i+1}} \setminus (R_1 \cup \cdots \cup R_i)$. It follows that $R_1, \dots, R_q$ are pairwise disjoint. Moreover, for every $i \in \mathbb{N}_q$, since R_i is an r-subset Q_{v_i}, it follows that R_i is an r-subset of $V(D) \setminus S$ and every vertex in R_i is an out-neighbor of v_i. This completes the proof of Lemma 5.1. ■

The following lemma involves several applications of Lemma 5.1, and is the heart of the proof of Theorem 3.2 (see Figure 8).

Lemma 5.2. *Let $t, \delta, \lambda \in \mathbb{N}$, let G be a K_{t+1}-free graph and let $\mathcal{L}_0$ be a set of pairwise disjoint paths in G with*

$$|\mathcal{L}_0| = \left(10\delta^{t+3}\lambda^3\right)^t$$

such that no two paths $L, L' \in \mathcal{L}_0$ are anticomplete in G. For each $L \in \mathcal{L}_0$, let x_L, y_L be a labelling of the ends of L (where $x_L = y_L$ is possible). Then there are δ paths $L_1, \dots, L_\delta \in \mathcal{L}_0$ along with a vertex $z_{L_i} \in L_i$ for each $i \in \mathbb{N}_\delta$, as well as δ pairwise disjoint λ-subsets $\mathcal{L}_1, \dots, \mathcal{L}_\delta$ of $\mathcal{L}_0 \setminus \{L_1, \dots, L_\delta\}$, such that the following hold.

(a) *The paths $(x_{L_i}\text{-}L_i\text{-}z_{L_i} : i \in \mathbb{N}_\delta)$ are pairwise anticomplete in G.*
(b) *For each $i \in \mathbb{N}_\delta$, every path $L \in \mathcal{L}_i$ contains a vertex w_L distinct from x_L such that w_L is the only vertex in $w_L\text{-}L\text{-}y_L$ with a neighbor in $x_{L_i}\text{-}L_i\text{-}z_{L_i}$.*

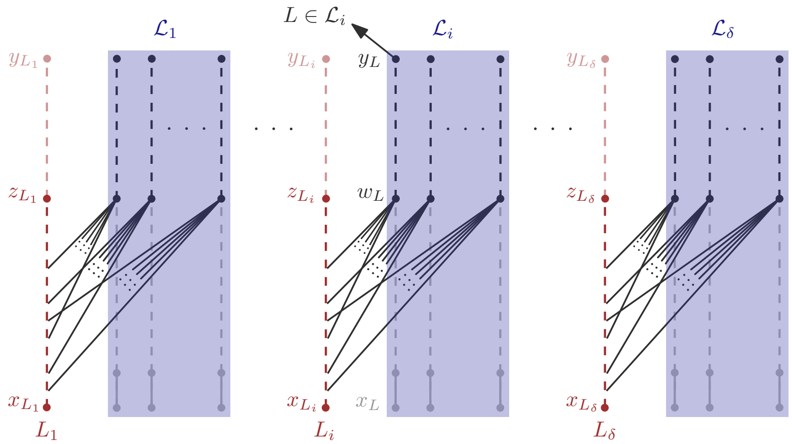

FIGURE 8. Lemma 5.2. Dashed lines represent paths of arbitrary length (possibly zero).

Proof. Since G is K_{t+1}-free, it follows from Theorem 4.1 that there exists $\mathcal{K}_1 \subseteq \mathcal{L}_0$ with

$$|\mathcal{K}_1| = 10\delta^{t+3}\lambda^3$$

such that $\{x_L : L \in \mathcal{L}_0\}$ is a stable set in G. Let D_1 be the digraph with $V(D_1) = \mathcal{K}_1$ such that for distinct $L, L' \in \mathcal{K}$, we have $(L, L') \in E(D_1)$ if and only if x_L has a neighbor in L'.

Suppose that D_1 has at least $2\delta^2\lambda$ vertices of out-degree at least $\delta\lambda$. Applying Lemma 5.1(b) to D_1 (with $r = \delta\lambda$ and $q = s = \delta$), we deduce that there is a δ-subset $\{L_1, \ldots, L_\delta\}$ of $\mathcal{K}_1 \subseteq \mathcal{L}_0$ as well as δ pairwise disjoint λ-subsets $\mathcal{L}_1, \ldots, \mathcal{L}_\delta$ of $\mathcal{K}_1 \setminus \{L_1, \ldots, L_\delta\} \subseteq \mathcal{L}_0 \setminus \{L_1, \ldots, L_\delta\}$ such that for each $i \in \mathbb{N}_\delta$, the vertex x_{L_i} has neighbors in every path in $\mathcal{L}_i$. Moreover, since $\{x_L : L \in \mathcal{K}_1\}$ is a stable set in G, it follows that:

- $\{x_{L_i} : i \in \mathbb{N}_\delta\}$ is a stable set in G; and
- for each $i \in \mathbb{N}_\delta$ and every $L \in \mathcal{L}_i$, traversing L from y_L to x_L, the first neighbor w_L of x_{L_i} in L is distinct from x_L. In particular, w_L is the only neighbor of x_{L_i} in w_L-L-y_L.

But now we are done by setting $z_{L_i} = x_{L_i}$ for every $i \in \mathbb{N}_\delta$.

Henceforth, assume that there are at most $2\delta^2\lambda$ vertices of out-degree at least $\delta\lambda$ in D_1. Since $2\delta^2\lambda \leq 2\delta^{t+3}\lambda^3$, it follows that there are at least $|V(D_1)| - 2\delta^{t+3}\lambda^3 = 8\delta^{t+3}\lambda^3$ vertices of out-degree at most $\delta\lambda$ in D_1. Thus, applying Lemma 5.1(a) to D_1 (with $r = \delta\lambda$ and $s = 4\delta^{t+2}\lambda^2$), it follows that there is a stable set $\mathcal{K}_2 \subseteq \mathcal{K}_1 = V(D_1)$ in D_1 with

$$|\mathcal{K}_2| = 4\delta^{t+2}\lambda^2.$$

Specifically, we have:

(3) *For all distinct $L, L' \in \mathcal{K}_2$, the end x_L of L is anticomplete to L' in G.*

Recall also that no two paths $L, L' \in \mathcal{K}_2 \subseteq \mathcal{L}_0$ are anticomplete in G. In particular, since $|\mathcal{K}_2| = 4\delta^{t+2}\lambda^2 > \delta\lambda$, it follows that for every $L \in \mathcal{K}_2$, there are at least $\delta\lambda$ paths $L' \in \mathcal{K}_2 \setminus \{L\}$ such that L, L' are not anticomplete in G. This, combined with (3), implies that:

(4) *For every $L \in \mathcal{K}_2$, there are distinct and adjacent vertices $z_L^-, z_L \in L$ such that L traverses x_L, z_L^-, z_L, y_L in order (where $x_L = z_L^-$ and $z_L = y_L$ are both possible), and the following hold.*

- *There are at least $\delta\lambda$ paths $L' \in \mathcal{K}_2 \setminus \{L\}$ for which x_L-L-z_L and L' are not anticomplete in G.*
- *There are at most $\delta\lambda$ paths $L' \in \mathcal{K}_2 \setminus \{L\}$ for which x_L-L-z_L^- and L' are not anticomplete in G.*

Next, let D_2 be the digraph with $V(D_2) = \mathcal{K}_2$ such that for all distinct $L, L' \in \mathcal{K}_2$, we have $(L, L') \in E(D_2)$ if and only if x_L-L-z_L and L' are not anticomplete in G.

By the first bullet of (4), every vertex has out-degree at least $\delta\lambda$ in D_2. Recall also that $|V(D_2)| = |\mathcal{K}_2| = 4\delta^{t+2}\lambda^2$. Thus, applying Lemma 5.1(b) to D_2 (with $q = \delta$, $r = \lambda$ and $s = 2\delta^{t+1}\lambda$), it follows that:

(5) *There is a $(2\delta^{t+1}\lambda)$-subset $\mathcal{K}_3$ of $\mathcal{K}_2 = V(D_2)$ with the following property: for every δ-subset $\{L_1, \dots, L_\delta\}$ of $\mathcal{K}_3$, there are δ pairwise disjoint λ-subsets $\mathcal{L}_1, \dots, \mathcal{L}_\delta$ of $\mathcal{K}_2 \setminus \mathcal{K}_3$ such that for all $i \in \mathbb{N}_\delta$ and $L \in \mathcal{L}_i$, the paths x_{L_i}-L_i-z_{L_i} and L are not anticomplete in G.*

From now on, let $\mathcal{K}_3$ be as given by (5). We claim that:

(6) *There is a δ-subset $\{L_1, \dots, L_\delta\}$ of $\mathcal{K}_3$ for which $(x_{L_i}$-L_i-$z_{L_i} : i \in \mathbb{N}_\delta)$ are pairwise anticomplete in G.*

To see this, let D_3 be the digraph with $V(D_3) = \mathcal{K}_3$ such that for all distinct $L, L' \in \mathcal{K}_3$, we have $(L, L') \in E(D_3)$ if and only if x_L-L-z_L^- and L' are not anticomplete in G. Since $\mathcal{K}_3 \subseteq \mathcal{K}_2$, it follows from the second bullet of (4) that every vertex in D_3 has out-degree at most $\delta\lambda$. Recall also that by (5), we have $|V(D_3)| = |\mathcal{K}_3| = 2\delta^{t+1}\lambda$. Thus, applying Lemma 5.1(a) to D_3 (with $r = \delta\lambda$ and $q = \delta^t$), we deduce that there is a stable set $\mathcal{K}_4 \subseteq \mathcal{K}_3 = V(D_3)$ of cardinality δ^t in D_3. It follows that for all distinct $L, L' \in \mathcal{K}_4$, the paths x_L-L-z_L^- and L' are anticomplete in G; in particular, x_L-L-z_L^- and $x_{L'}$-L'-$z_{L'}$ are anticomplete in G. Also, since $|\mathcal{K}_4| = \delta^t$ and since G is K_{t+1}-free, it follows from Theorem 4.1 that there are δ paths $L_1, \dots, L_\delta \in \mathcal{K}_4 \subseteq \mathcal{K}_3$ for which $\{z_{L_i} : i \in \mathbb{N}_\delta\}$ is a stable set in G. But now $(x_{L_i}$-L_i-$z_{L_i} : i \in \mathbb{N}_\delta)$ are pairwise anticomplete in G. This proves (6).

We can now finish the proof. Let $\{L_1, \dots, L_\delta\}$ be the δ-subset of $\mathcal{K}_3$ given by (6). By (5), there are δ pairwise disjoint λ-subsets $\mathcal{L}_1, \dots, \mathcal{L}_\delta$ of $\mathcal{K}_2 \setminus \mathcal{K}_3 \subseteq \mathcal{K}_2 \setminus \{L_1, \dots, L_\delta\} \subseteq \mathcal{L}_0 \setminus \{L_1, \dots, L_\delta\}$, such that for all $i \in \mathbb{N}_\delta$ and $L \in \mathcal{L}_i$, the paths x_{L_i}-L_i-z_{L_i} and L are not anticomplete in G. Moreover,

- By (6), the paths $(x_{L_i}$-L_i-$z_{L_i} : i \in \mathbb{N}_\delta)$ are pairwise anticomplete in G.
- For all $i \in \mathbb{N}_\delta$ and $L \in \mathcal{L}_i$, since $L_i, L \in \mathcal{K}_2 \subseteq \mathcal{K}_1$, it follows from (3) that traversing L from y_L to x_L, the first vertex w_L in L with a neighbor in x_{L_i}-L_i-z_{L_i} is distinct from x_L. In particular, w_L is the only vertex in w_L-L-y_L with a neighbor in x_{L_i}-L_i-z_{L_i}.

But now $(L_i, z_{L_i}, \mathcal{L}_i : i \in \mathbb{N}_\delta)$ satisfy 5.2(a) and 5.2(b). This completes the proof of Lemma 5.2. ■

We need one more lemma. The proof relies on the product version of Ramsey's theorem:

Theorem 5.3 (Graham, Rothschild, Spencer [9]). *For all $n, q, r \in \mathbb{N}$, there is a constant $f_{5.3} = f_{5.3}(n, q, r) \in \mathbb{N}$ with the following property. Let $U_1, \dots, U_n$ be n sets, each of cardinality at least $f_{5.3}$ and let W be a non-empty set of cardinality at most r. Let Φ be a map from the Cartesian product $U_1 \times \dots \times U_n$ to W. Then there exist $i \in W$ and a q-subset Z_j of U_j for each $j \in \mathbb{N}_n$, such that for every $z \in Z_1 \times \dots \times Z_n$, we have $\Phi(z) = i$.*

We will use the following both here and in the next section:

Lemma 5.4. *For all $r, s, t \in \mathbb{N}$, there is a constant $f_{5.4} = f_{5.4}(r, s, t) \in \mathbb{N}$ with the following property. Let G be a graph with no induced $K_{t,t}$-model and let $\mathcal{U}$ be a set of pairwise disjoint connected induced subgraphs of G. Assume that there is a $2rt$-subset $\mathcal{A}$ of $\mathcal{U}$ as well as $2rt$ pairwise disjoint $f_{5.4}$-subsets $(\mathcal{B}_U : U \in \mathcal{A})$ of $\mathcal{U} \setminus \mathcal{A}$ such that*

- *the sets in $\mathcal{A}$ are pairwise anticomplete in G; and*
- *for every $U \in \mathcal{A}$, the sets in $\mathcal{B}_U$ are pairwise anticomplete in G.*

Then there are $A_1, \dots, A_r \in \mathcal{A}$ along with an s-subset $\mathcal{B}_i$ of $\mathcal{B}_{A_i}$ for each $i \in \mathbb{N}_r$, such that $(A_i \cup V(\mathcal{B}_i) : i \in \mathbb{N}_r)$ are pairwise anticomplete in G. In particular, we have $f_{5.4} \geq s$.

Proof. Let

$$f_{5.4} = f_{5.4}(r, s, t) = f_{5.3}\left(2rt, \max\{s, t\}, 2^{4r^2t^2\binom{2rt}{2}}\right).$$

Fix an enumeration $\mathcal{A} = \{U_1, \dots, U_{2rt}\}$. For every $z = (B_1, \dots, B_{2rt}) \in \mathcal{B}_{U_1} \times \dots \times \mathcal{B}_{U_{2rt}}$,

- let E_z be the set of all ordered pairs $(i, j) \in \mathbb{N}_{2rt} \times \mathbb{N}_{2rt}$ with $i \neq j$ for which B_i and U_j are not anticomplete in G; and
- let $E'_z \in \mathcal{G}$ be the set of all 2-subsets $\{i, j\}$ of $\mathbb{N}_{2rt}$ for which B_i and B_j are not anticomplete in G.

It follows that the function $\Phi : \mathcal{B}_{U_1} \times \dots \times \mathcal{B}_{U_{2rt}} \to 2^{\mathbb{N}_{2rt} \times \mathbb{N}_{2rt}} \times 2^{\binom{\mathbb{N}_{2rt}}{2}}$ with $\Phi(z) = (E_z, E'_z)$ is well-defined. By Theorem 5.3 and the choice of $f_{5.4}$, we obtain $E \subseteq \mathbb{N}_{2rt} \times \mathbb{N}_{2rt}$ and $E' \subseteq \binom{\mathbb{N}_{2rt}}{2}$, as well as a $\max\{s, t\}$-subset $\mathcal{B}'_{U_i}$ of $\mathcal{B}_{U_i}$ for each $i \in \mathbb{N}_{2rt}$, such that for every $z \in \mathcal{B}'_{U_1} \times \dots \times \mathcal{B}'_{U_{2rt}}$, we have $\Phi(z) = (E, E')$.

Let D be the digraph with vertex set $\mathbb{N}_{2rt}$ and edge set E. We claim that:

(7) *Every vertex in D has out-degree less than t.*

Suppose not. Then there are $i, j_1, \dots, j_t \in \mathbb{N}_{2rt}$ such that $(i, j_1), \dots, (i, j_t) \in E$. Since $|\mathcal{B}'_{U_i}| = \max\{s, t\}$, we may choose t distinct sets $X_1, \dots, X_t \in \mathcal{B}'_{U_i}$. It follows that $X_1, \dots, X_t \in \mathcal{B}'_{U_i} \subseteq \mathcal{B}_{U_i}$ are pairwise anticomplete in G. Recall also that $U_{j_1}, \dots, U_{j_t} \in \mathcal{A}$ are pairwise anticomplete in G. Moreover, for every $z \in \mathcal{B}'_{U_1} \times \dots \times \mathcal{B}'_{U_{2rt}}$, since $\Phi(z) = (E, E')$, it follows that $E_z = E$, and so $(i, j_1), \dots, (i, j_t) \in E_z$. In particular, for all $k, l \in \mathbb{N}_t$, the sets X_k, U_{j_l} are not anticomplete in G. But now $(X_1, \dots, X_t, U_{j_1}, \dots, U_{j_t})$ is an induced $K_{t,t}$-model in G, a contradiction. This proves (7).

(8) $E' = \varnothing$.

Suppose that some 2-subset $\{i,j\}$ of $\mathbb{N}_{2rt}$ belongs to E'. Since $|\mathcal{B}'_{U_i}| = |\mathcal{B}'_{U_j}| = \max\{s,t\}$, we may choose t distinct sets $X_1,\dots,X_t \in \mathcal{B}'_{U_i}$ and t distinct sets $Y_1,\dots,Y_t \in \mathcal{B}'_{U_j}$. It follows that $X_1,\dots,X_t \in \mathcal{B}'_{U_i} \subseteq \mathcal{B}_{U_i}$ are pairwise anticomplete in G, and so are $Y_1,\dots,Y_t \in \mathcal{B}'_{U_j} \subseteq \mathcal{B}_{U_j}$. Also, for every $z \in \mathcal{B}'_{U_1} \times \dots \times \mathcal{B}'_{U_{2rt}}$, since $\Phi(z) = (E,E')$, it follows that $E'_z = E'$, and so $\{i,j\} \in E'_z$. In particular, for all $k,l \in \mathbb{N}_t$, the sets X_k, Y_l are not anticomplete in G. But now $(X_1,\dots,X_t,Y_1,\dots,Y_t)$ is an induced $K_{t,t}$-model in G, a contradiction. This proves (8).

Since $|V(D)| = 2rt$, it follows from (7) and Lemma 5.1(a) that D contains a stable set $\{k_1,\dots,k_r\} \subseteq \mathbb{N}_{2rt}$. For every $i \in \mathbb{N}_r$, let $A_i = U_{k_i}$ and choose an s-subset $\mathcal{B}_i$ of $\mathcal{B}'_{U_{k_i}} \subseteq \mathcal{B}_{U_{k_i}}$ (this is possible because $|\mathcal{B}'_{U_{k_i}}| = \max\{s,t\}$). Recall that $A_1,\dots,A_r \in \mathcal{A}$ are pairwise anticomplete in G. Since $\{k_1,\dots,k_r\}$ is a stable set in D, it follows that for all distinct $i,j \in \mathbb{N}_r$, we have $(i,j) \notin E$, and so A_i and $V(\mathcal{B}_j)$ are anticomplete in G. Also, by (8), the sets $V(\mathcal{B}_1),\dots,V(\mathcal{B}_r)$ are pairwise anticomplete in G. But now $(A_i \cup V(\mathcal{B}_i) : i \in \mathbb{N}_r)$ are pairwise anticomplete in G. This completes the proof of Lemma 5.4. ■

We can now prove the main result of this section, which we restate:

Theorem 3.2. *For all $t,\delta,\lambda,\kappa \in \mathbb{N}$, there are constants $f_{3.2} = f_{3.2}(t,\delta,\lambda,\kappa) \in \mathbb{N}$ and $g_{3.2} = g_{3.2}(t,\kappa) \in \mathbb{N}$ with the following property. Let G be a t-tidy graph and let $(A,\mathcal{L},Y)$ be an $f_{3.2}$-seedling in G which is κ-rigid. Then there are δ pairwise disjoint λ-seedlings $(A_1,\mathcal{L}_1,Y_1),\dots,(A_\delta,\mathcal{L}_\delta,Y_\delta)$ in $G \setminus A$ with the following specifications.*

(a) *The paths $A_1,\dots,A_\delta$ are pairwise anticomplete in G.*

(b) *For every $i \in \mathbb{N}_\delta$, we have:*
- *A and A_i are not anticomplete in G;*
- *A and $V(\mathcal{L}_i)$ are anticomplete in G; and*
- *$(A_i,\mathcal{L}_i,Y_i)$ is $g_{3.2}$-rigid.*

Proof. Let

$$f_{3.2} = f_{3.2}(t,\delta,\lambda,\kappa) = \kappa^{\left(10(\delta+3t\kappa)^{t+3}\lambda^3\right)^t};$$

and let

$$g_{3.2} = g_{3.2}(t,\kappa) = f_{5.4}(\kappa,1,t).$$

Let G be a t-tidy graph and let $(A,\mathcal{L},Y)$ be an $f_{3.2}$-seedling in G which is κ-rigid.

(9) *There is a $(10(\delta+3t\kappa)^{t+3}\lambda^3)^t$-subset $\mathcal{L}_0$ of $\mathcal{L}$ such that no two paths $L,L' \in \mathcal{L}_0$ are anticomplete in G.*

Let Γ be a graph with $V(\Gamma) = \mathcal{L}$ such that for distinct $L,L' \in \mathcal{L}$, we have $LL' \in E(\Gamma)$ if and only if L and L' are not anticomplete in G. Since $|V(\Gamma)| = |\mathcal{L}| = f_{3.2}$, it follows from Theorem 4.1 that Γ contains either a stable set of cardinality κ or a clique of cardinality $(30(\delta+3t\kappa)^{t+3}\lambda^3)^t$. In the former case, there is a set $\mathcal{K} \subseteq \mathcal{L}$ of κ pairwise anticomplete $(N(A),Y)$-paths in G, a contradiction to the assumption that $(A,\mathcal{L},Y)$ is κ-rigid. So the latter case holds; that is, there is a $(30(\delta+3t\kappa)^{t+3}\lambda^3)^t$-subset $\mathcal{L}_0$ of $\mathcal{L}$ such that no two paths $L,L' \in \mathcal{L}_0$ are anticomplete in G. This proves (9).

Henceforth, let $\mathcal{L}_0 \subseteq \mathcal{L}$ be as given by (9). In particular, $\mathcal{L}_0$ is a set of pairwise disjoint $(N(A), Y)$-paths in G, and so each path $L \in \mathcal{L}_0$ has an $N(A)$-end x_L and a Y-end y_L. We apply Lemma 5.2 to $\mathcal{L}_0$ along with the labelling x_L, y_L of the ends of each path $L \in \mathcal{L}$. Since G is K_{t+1}-free and $|\mathcal{L}_0| = (10(\delta + 3t\kappa)^{t+3}\lambda^3)^t$, we deduce that:

(10) *There is a $(\delta + 3t\kappa)$-subset $\mathcal{P}$ of $\mathcal{L}_0$, a vertex $z_P \in P$ for each $P \in \mathcal{P}$, and $\delta + 3t\kappa$ pairwise disjoint λ-subsets $(\mathcal{Q}_P : P \in \mathcal{P})$ of $\mathcal{L}_0 \setminus \mathcal{P}$, such that the following hold.*

- *The paths $(x_P\text{-}P\text{-}z_P : P \in \mathcal{P})$ are pairwise anticomplete in G.*
- *For each $P \in \mathcal{P}$, every $Q \in \mathcal{Q}_L$ contains a vertex w_Q distinct from x_Q such that w_Q is the only vertex in $w_Q\text{-}Q\text{-}y_Q$ with a neighbor in $x_P\text{-}P\text{-}z_P$.*

From now on, let $\mathcal{P}$ and $(z_P, \mathcal{Q}_P : P \in \mathcal{P})$ be as given by (10). For every $P \in \mathcal{P}$, let

$$A_P = x_P\text{-}P\text{-}z_P;$$

let

$$\mathcal{L}_P = \{w_Q\text{-}Q\text{-}y_Q : Q \in \mathcal{Q}_P\};$$

and let

$$Y_P = \{y_Q : Q \in \mathcal{Q}_P\}.$$

Then $A_P \subseteq P$ is a path in $G \setminus A$ and $Y_P \subseteq V(G \setminus A) \setminus A_P$. Moreover, by the second bullet of (10), for every $Q \in \mathcal{Q}_P$, the path $w_Q\text{-}Q\text{-}y_P$ is an $(N(A_P), Y_P)$-path in $G \setminus A_P$, and so $\mathcal{L}_P$ is a set of λ pairwise disjoint $(N(A_P), Y_P)$-paths in $G \setminus A_P$. Note also that by construction, $(A_P \cup V(\mathcal{L}_P) \cup Y_P : P \in \mathcal{P})$ are pairwise disjoint subsets of $V(\mathcal{L}_0) \subseteq G \setminus A$. Therefore,

(11) *The triples $((A_P, \mathcal{L}_P, Y_P) : P \in \mathcal{P})$ are pairwise disjoint λ-seedlings in $G \setminus A$.*

We also show that:

(12) *The following hold.*

- *The paths $(A_P : P \in \mathcal{P})$ are pairwise anticomplete in G.*
- *For every $P \in \mathcal{P}$, we have:*
  - *A and A_P are not anticomplete in G; and*
  - *A and $V(\mathcal{L}_P)$ are anticomplete in G.*

The first assertion is immediate from the first bullet of (10). We prove the second assertion. Let $P \in \mathcal{P}$. Then $x_P \in A_P$ has a neighbor in A (because x_P is the $N(A)$-end of P), and so A and A_P are not anticomplete in G. Moreover, by the second bullet of (10), we have

$$V(\mathcal{L}_P) \subseteq \bigcup_{Q \in \mathcal{Q}_P} Q \setminus \{x_Q\}.$$

Recall also that each path $Q \in \mathcal{Q}_P$ is an $(N(A), Y)$-path in $G \setminus A$ where x_Q is the $N(A)$-end of Q, and so A and $Q \setminus \{x_Q\}$ are anticomplete in G. It follows that A and $V(\mathcal{L}_P)$ are anticomplete in G. This proves (12).

(13) *There are* $P_1, \dots, P_\delta \in \mathcal{P}$ *such that* $(A_{P_i}, \mathcal{L}_{P_i}, Y_{P_i})$ *is* $g_{3.2}$*-rigid for every* $i \in \mathbb{N}_\delta$.

Suppose not. Recall that by (10), we have $|\mathcal{P}| = \delta + 3t\kappa$. So there is $3t\kappa$-subset $\mathcal{A}$ of $\mathcal{P}$ such that for every $P \in \mathcal{A}$, the λ-seedling $(A_P, \mathcal{L}_P, Y_P)$ in $G \setminus A$ is not $g_{3.2}$-rigid. By definition, this means for every $P \in \mathcal{K}$, there is a set $\mathcal{K}_P$ of $g_{3.2}$ pairwise anticomplete $(N(A_P), Y_P)$-paths in G with $V(\mathcal{K}_P) \subseteq V(\mathcal{L}_P)$. Also, by the first bullet of (12), the paths $(A_P : P \in \mathcal{A})$ are anticomplete. Since G has no induced $K_{t,t}$-model, and since $g_{3.2} = f_{5.4}(\kappa, 1, t)$, it follows from Lemma 5.4 that there are $P_1, \dots, P_\kappa \in \mathcal{A}$ as well as $K_i \in \mathcal{K}_{P_i}$ for each $i \in \mathbb{N}_\kappa$, such that $(A_{P_i} \cup K_i : i \in \mathbb{N}_\kappa)$ are pairwise anticomplete in G. Now, recall that for each $i \in \mathbb{N}_\kappa$, the vertex $x_{P_i} \in A_{P_i}$ has a neighbor in A, and $K_i \in \mathcal{K}_{P_i}$ is an $(N(A_{P_i}), Y_{P_i})$-path in G, which in turn implies that K_i has an end in $Y_{P_i} \subseteq Y$. Consequently, for every $i \in \mathbb{N}_\kappa$, there is an $(N(A), Y)$-path L_i in G with $L_i \subseteq A_{P_i} \cup K_i$. Moreover, we have

$$A_{P_i} \cup K_i \subseteq P_i \cup V(\mathcal{K}_{P_i}) \subseteq P_i \cup V(\mathcal{L}_{P_i}) \subseteq P_i \cup V(\mathcal{Q}_{P_i})$$

and so by (10), we have $A_{P_i} \cup K_i \subseteq V(\mathcal{L}_0) \subseteq V(\mathcal{L})$. But now $\mathcal{K} = \{L_1, \dots, L_\kappa\}$ is a set of κ pairwise anticomplete $(N(A), Y)$-paths in G with $V(\mathcal{K}) \subseteq V(\mathcal{L})$, violating the assumption that the seedling $(A, \mathcal{L}, Y)$ is κ-rigid. This proves (13).

Let $\{P_1, \dots, P_\delta\} \subseteq \mathcal{P}$ be as given by (13). For each $i \in \mathbb{N}_\delta$, let $A_i = A_{P_i}$, let $\mathcal{L}_i = \mathcal{L}_{P_i}$ and let $Y_i = Y_{P_i}$. From (11), (12) and (13), it follows that $(A_1, \mathcal{L}_1, Y_1), \dots, (A_\delta, \mathcal{L}_\delta, Y_\delta)$ are δ pairwise disjoint λ-seedlings in $G \setminus A$ satisfying both 3.2(a) and 3.2(b). This completes the proof of Theorem 3.2. ■

6. From a seedling to a tree

In this section, we prove Theorem 3.3. As shown at the end of Section 3, this will conclude the proof of Theorem 1.3.

Theorem 3.3. *For all* $d, r, t, \kappa \in \mathbb{N}$*, there is a constant* $f_{3.3} = f_{3.3}(d, r, t, \kappa) \in \mathbb{N}$ *with the following property. Let* G *be a* t*-tidy graph and let* $(A, \mathcal{L}, Y)$ *be an* $f_{3.3}$*-seedling in* G *which is* κ*-rigid. Then there is an induced* $T_{d,r}$*-model in* $G[A \cup V(\mathcal{L})]$ *where* A *is the branch set associated with the root of* $T_{d,r}$.

Proof. First, for each $r \in \mathbb{N}$, we define a function

$$\xi_r : \mathbb{N}^3 \to \mathbb{N}.$$

The definition is recursive in r, as follows. For $r = 1$, let

$$\xi_1(a, b, c) = b^a$$

for every $(a, b, c) \in \mathbb{N}^3$. For $r \geq 2$, assuming the function ξ_{r-1} is defined, let

$$\xi_r(a, b, c) = f_{3.2}(a, 2ab, \xi_{r-1}(a, f_{5.4}(b, b, a), g_{3.2}(a, c)), c)$$

for every $(a, b, c) \in \mathbb{N}^3$. This concludes the definition of the functions $(\xi_r : r \in \mathbb{N})$.

Back to the proof of 3.3, we will prove by induction on $r \in \mathbb{N}$, that for all $d, t, \kappa \in \mathbb{N}$,

$$f_{3.3} = f_{3.3}(d, r, t, \kappa) = \xi_r(t, d, \kappa)$$

satisfies the theorem.

Let G be a t-tidy graph and let $(A, \mathcal{L}, Y)$ be an $f_{3.3}$-seedling in G which is κ-rigid. Assume that $r = 1$. Then we have $|\mathcal{L}| = f_{3.3}(d, 1, t, \kappa) = \xi_1(t, d, \kappa) = d^t$. For each $L \in \mathcal{L}$, let x_L be the $N(A)$-end of L. Since G is K_{t+1}-free, it follows from Theorem 4.1 that there is a d-subset $\{L_1, \ldots, L_d\}$ of $\mathcal{L}$ for which $\{x_{L_1}, \ldots, x_{L_d}\}$ is a stable set in G. But now $(A, \{x_{L_1}\}, \ldots, \{x_{L_d}\})$ is an induced $T_{d,1}$-model in G where A is the branch set associated with the root of $T_{d,1}$.

Therefore, we may assume that $r \geq 2$. Let

$$\Delta = f_{5.4}(d, d, t).$$

In particular, we have $\Delta \geq d$. Let

$$\Lambda = \xi_{r-1}(t, \Delta, g_{3.2}(t, \kappa)).$$

Then we have

$$|\mathcal{L}| = f_{3.3}(d, r, t, \kappa) = \xi_r(t, d, \kappa) = f_{3.2}(t, 2dt, \Lambda, \kappa).$$

Applying Theorem 3.2 to $(A, \mathcal{L}, Y)$, we deduce that:

(14) *There are $2dt$ pairwise disjoint Λ-seedlings $(A_1, \mathcal{L}_1, Y_1), \ldots, (A_{2dt}, \mathcal{L}_{2dt}, Y_{2dt})$ in $G \setminus A$ with the following specifications.*

- *The paths $A_1, \ldots, A_{2dt}$ are pairwise anticomplete in G.*
- *For every $i \in \mathbb{N}_{2dt}$, we have:*
  - *A and A_i are not anticomplete in G;*
  - *A and $V(\mathcal{L}_i)$ are anticomplete in G; and*
  - *$(A_i, \mathcal{L}_i, Y_i)$ is $g_{3.2}(t, \kappa)$-rigid.*

Moreover, for every $i \in \mathbb{N}_{2dt}$, since $(A_i, \mathcal{L}_i, Y_i)$ is a $g_{3.2}(t, \kappa)$-rigid $\xi_{r-1}(t, \Delta, g_{3.2}(t, \kappa))$-seedling in G, it follows from the inductive hypothesis applied to $(A_i, \mathcal{L}_i, Y_i)$ that there is an induced $T_{\Delta,r-1}$-model $(A_{i,v} : v \in V(T_{\Delta,r-1}))$ in $G[A_i \cup V(\mathcal{L}_i)]$ where A_i is the branch set associated with the root of $T_{\Delta,r-1}$.

Let u_0 be the root of $T_{\Delta,r-1}$; thus, we have $A_i = A_{i,u_0}$ for every $i \in \mathbb{N}_{2dt}$. Let $u_1, \ldots, u_\Delta$ be the neighbors of u_0 in $T_{\Delta,r-1}$ and let $T_1, \ldots, T_\Delta$ be the components of $T_{\Delta,r-1} \setminus \{u_0\}$ containing $u_1, \ldots, u_\Delta$, respectively. It follows that T_i, for each $i \in \mathbb{N}_\Delta$, is isomorphic to $T_{\Delta,r-2}$ with root u_i (in particular, T_i is non-null because $r \geq 2$). Moreover, since $\Delta \geq d$, it follows that for every $i \in \mathbb{N}_\Delta$, there is an induced subgraph T^i of T_i isomorphic to $T_{d,r-2}$ with root u_i.

For each $i \in \mathbb{N}_{2dt}$ and every $j \in \mathbb{N}_\Delta$, let

$$B_i^j = \bigcup_{v \in V(T^j)} A_{i,v}.$$

Then $B_i^1, \ldots, B_i^\Delta$ are pairwise anticomplete in G for every $i \in \mathbb{N}_{2dt}$. Also, by the first bullet of (14), the sets $A_1, \ldots, A_{2dt}$ are pairwise anticomplete in G. Since $\Delta = f_{5.4}(d, d, t)$, it follows from Lemma 5.4 that there is a d-subset I of $\mathbb{N}_{2dt}$ as well as a d-subset $J_i \subseteq \mathbb{N}_\Delta$ for each $i \in I$, such that the d sets

$$\left(A_i \cup \left(\bigcup_{j \in J_i} B_i^j\right) : i \in I\right)$$

are pairwise anticomplete in G.

Now, for every $i \in I$, the subgraph of $T_{\Delta,r-1}$ induced by $\{u_0\}\cup(\bigcup_{j\in J_i} V(T^j))$ is isomorphic to $T_{d,r-1}$. Also, by the second bullet of (14), the sets A and $A_i = A_{i,u_0}$ are not anticomplete in G, whereas A and $\bigcup_{j\in J_i}\bigcup_{v\in V(T^j)} A_{i,v} = \bigcup_{j\in J_i} B_i^j \subseteq V(\mathcal{L}_i)$ are anticomplete in G. Hence,

$$\left(A; A_{i,v} : i \in I, v \in \{u_0\} \cup \left(\bigcup_{j\in J_i} V(T^j)\right)\right)$$

is an induced $T_{d,r}$-model in $G[A \cup V(\mathcal{L})]$ where A is the branch set associated with the root of $T_{d,r}$. This completes the proof of Theorem 3.3. ■

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