Excluding paths and bicliques*

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Dedicated to the memory of Vera T. Sós.

Abstract

Classes of graphs excluding a path and a biclique as induced subgraphs are extensively studied in the literature. One of the key structural results for such graphs is a Ramsey-type result due to Galvin, Rival, and Sands (1982), establishing the existence of a function f bounding the maximum length of a path in terms of clique number ω. We improve the best known bound on f to a function that is a singly exponential in ω^c, for some constant c, which we show is best possible, up to optimizing c.

Our approach also has consequences for treedepth. In particular, we show that, for graphs excluding a path and a biclique as induced subgraphs, treedepth is bounded by a polynomial function of clique number. In turn, this result implies that every hereditary graph class that admits a function bounding treedepth of graphs in the class in terms of clique number, admits a polynomial such function. This gives a treedepth analogue of a recent result on pathwidth due to Hajebi (2025).

Keywords: induced path, induced biclique, longest path, clique number, treedepth, hereditary graph class, clique-polynomiality.

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1 Introduction

Galvin, Rival, and Sands [15] in 1982 and independently Atminas, Lozin, and Razgon [1] in 2012 proved that the only reason for a graph to have a long path is the presence of a long induced path, a large clique, or a large induced biclique. Denoting by $\omega(G)$ the clique number of G and by $\mathsf{pn}(G)$ the *path number* of G, that is, the maximum number of vertices in a (not necessarily induced) path in G, this result can be equivalently stated as follows.

Theorem 1 (Galvin, Rival, and Sands [15]; Atminas, Lozin, and Razgon [1])**.** *For every two positive integers s and t, there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every $\{P_s, K_{t,t}\}$-free graph G satisfies $\mathsf{pn}(G) \leq f(\omega(G))$.*

Questions related to Theorem 1 were the subject of many recent papers, see Duron, Esperet, and Raymond [13] for an overview. The best quantitative bound known up to now is a doubly exponential function of some power of the clique number. This bound follows from the main result of Hunter, Milojević, Sudakov, and Tomon [22] (see Section 3).

We improve this result by showing that for graphs excluding a path and a biclique as induced subgraphs, the path number is bounded above by a *singly exponential* function of some power of the clique number.

Theorem 2. *For every two positive integers s and t there exists a positive integer c such that every $\{P_s, K_{t,t}\}$-free graph G satisfies $\mathsf{pn}(G) \leq 2^{\omega(G)^c}$.*

Except for optimizing the dependency of c on the excluded path and biclique, this result is best possible. Indeed, we construct examples showing that the exponential dependency on the clique number in Theorem 2 cannot be avoided.

Our approach has consequences for treedepth, a graph parameter that plays a central role in the graph sparsity theory of Nešetřil and Ossona de Mendez [27]. We show that, for graphs excluding a path and a biclique as induced subgraphs, treedepth is bounded by a polynomial function of the clique number.

Theorem 3. *For every two positive integers s and t there exists a positive integer c such that every $\{P_s, K_{t,t}\}$-free graph G satisfies $\mathsf{td}(G) \leq \omega(G)^c$.*

Theorem 3 in turn implies the treedepth analogue of a result on pathwidth due to Hajebi [17]. To state this result, we need a couple of definitions. A *graph parameter* is a function mapping graphs to integers that does not distinguish between isomorphic graphs. A graph class is *hereditary* if it is closed under vertex deletion. A graph parameter ρ is said to be *clique-polynomial* if the following holds: for every hereditary graph class $\mathcal{G}$, if there exists a function f such that $\rho(G) \leq f(\omega(G))$ for every graph $G \in \mathcal{G}$, then there exists a polynomial function p such that $\rho(G) \leq p(\omega(G))$ for every graph $G \in \mathcal{G}$. Hajebi [17] showed that pathwidth is clique-polynomial. We show the same property for treedepth.

Theorem 4. *For every hereditary graph class $\mathcal{G}$, if there exists a function f such that $\mathsf{td}(G) \leq f(\omega(G))$ for every graph $G \in \mathcal{G}$, then there exists a polynomial function p such that $\mathsf{td}(G) \leq p(\omega(G))$ for every graph $G \in \mathcal{G}$.*

Let us mention that analogous results do not hold for treewidth and chromatic number, as shown by Chudnovsky and Trotignon [10] and by Briański, Davies, and Walczak [6], respectively, but do hold for degeneracy in some cases, due to a result of Bourneuf, Bucić, Cook, and Davies [5] and Girão and Hunter [16]. Indeed, they showed that for every graph H, there is a polynomial p such that every $K_{t,t}$-subgraph-free graph with no induced subdivision of H has average degree at most $p(t)$. By Ramsey's theorem [28], this implies that for every graph H, there is a polynomial p such that every $K_{t,t}$-free graph with no induced subdivision of H has average degree at most $p(\omega(G))$ and hence, degeneracy at most $p(\omega(G))$.

A sufficient condition for a graph parameter to be clique-polynomial is that it is *awesome*, which is a property based on the so-called *independence variants* of parameters (see Bešter Štorgel et al. [2]). While several parameters, including vertex cover number, feedback vertex set number, and odd cycle transversal number, are awesome, treedepth and pathwidth are not [2]. Nonetheless, Theorem 4 and the aforementioned result of Hajebi show that these two parameters satisfy a necessary condition for awesomeness, namely clique-polynomiality.

Our main results, Theorems 2 and 3, contribute to the list of results on classes of graphs excluding a path and a biclique as induced subgraphs. Such classes were extensively studied in the literature, from both structural and algorithmic points of view. For example, Kamiński and Pstrucha [24] showed that graphs excluding a path and a biclique as induced subgraphs admit finitely many subgraph-minimal obstructions to H-list-colorability (for any fixed graph H), which implies the existence of polynomial-time certifying algorithms for H-list-colorability. Dallard et al. [11] conjectured that such graphs have bounded *tree-independence number*, that is, they admit tree decompositions whose bags induce subgraphs of bounded independence number. Following a number of partial results (see [3, 7, 8, 20]), the conjecture was proved recently in a more general context by Hajebi and Spirkl [18]. This implies that all good algorithmic properties enjoyed by graph classes with bounded tree-independence number (see [12, 25, 30]) hold for graphs excluding a path and a biclique as induced subgraphs. This generalizes a number of previous results (see, e.g., [19, 23, 26]).

The paper is structured as follows. After collecting the necessary definitions and tools in Section 2, we explain in Section 3 how to obtain a quantitative version of Theorem 1 from the main result of Hunter et al. [22]. In Section 4, we derive two auxiliary results regarding bounds on treedepth and path number of P_s-free graphs in terms of pathwidth. In Section 5, we prove Theorem 2, while in Section 6, we prove Theorems 3 and 4.

2 Preliminaries

We denote by $\mathbb{N}$ the set of all nonnegative integers. All logarithms in this paper are binary. Throughout this paper, all graphs are finite, simple, and undirected. For a graph G, we denote its *order*, that is, the number of vertices, by $n(G)$.

For a positive integer n, we denote by P_n and K_n the n-vertex path graph and the n-vertex complete graph, respectively. In this paper, a *biclique* $K_{n,n}$ refers to a balanced complete bipartite graph of order $2n$.

For a subset of vertices $U \subseteq V(G)$ or a subgraph U of G, we write $G - U$ to denote the graph obtained from G by deleting all vertices in U and their incident edges. Given a family $\mathcal{F}$ of graphs, we say that a graph G is *$\mathcal{F}$-free* if no induced subgraph of G is isomorphic to a member of $\mathcal{F}$; furthermore, for a graph F, we say that G is *F-free* if it is $\{F\}$-free.

For integers k, ℓ, we denote by $[k, \ell]$ the set of integers greater than or equal to k not exceeding ℓ, and set $[k] = [1, k]$.

A *Hamiltonian path* in a graph G is a path that visits every vertex of G. A graph G is *traceable* if it admits a Hamiltonian path.

A *path decomposition* of a graph G is a sequence $(X_1, \ldots, X_r)$ of subsets of $V(G)$ called *bags* satisfying the following two properties:

(i) for every edge $uv \in E(G)$, there exists an index $i \in [r]$ such that $\{u, v\} \subseteq X_i$, and

(ii) for every $v \in V(G)$ there exist integers $k, \ell \in [r]$ with $k \leq \ell$ such that for all $j \in [r]$, we have $v \in X_j$ if and only if $j \in [k, \ell]$.

The *width* of a path decomposition is defined as $\max_{1 \leq i \leq r} |X_i| - 1$. The *pathwidth* of G, denoted by $\mathsf{pw}(G)$, is the minimum width over all possible path decompositions of G.

A *rooted tree* is a tree T together with a distinguished vertex $r \in V(T)$ called the *root*. A *rooted forest* is a disjoint union of rooted trees. The *transitive closure* of a rooted forest F is the graph with vertex set $V(F)$ where two vertices are adjacent if and only if one is an ancestor of the other in F. A *treedepth decomposition* of a graph G is a rooted forest F on the vertex set $V(G)$ such that G is a subgraph of the transitive closure of F. The *depth* of a rooted forest F is the maximum number of vertices on any root-to-leaf path in F. The *treedepth* of G, denoted by $\mathsf{td}(G)$, is the minimum depth over all treedepth decompositions of G.

2.1 Tools

For easier reference, we enumerate here several standard (folklore) facts that we will use, sometimes implicitly, throughout the paper (see, e.g., [4, 27]).

Fact 1. Let G be a graph and $C_1, \ldots, C_k$ its connected components. Then $\mathsf{pw}(G) = \max_{1 \leq i \leq k} \mathsf{pw}(C_i)$. Similarly, $\mathsf{td}(G) = \max_{1 \leq i \leq k} \mathsf{td}(C_i)$ and $\mathsf{pn}(G) = \max_{1 \leq i \leq k} \mathsf{pn}(C_i)$.

Fact 2. Path number, pathwidth, treedepth, and clique number are all monotone under subgraphs.

Fact 3. Let G be a graph containing an edge, and let I be the set of isolated vertices in G. Then $\mathsf{pn}(G) = \mathsf{pn}(G - I)$, $\mathsf{td}(G) = \mathsf{td}(G - I)$, and $\mathsf{pw}(G) = \mathsf{pw}(G - I)$.

Fact 4. Let G be a graph with a path decomposition $(X_1, \dots, X_r)$, and let H be a connected subgraph of G. Then there exist integers $k, \ell \in [r]$ with $\ell \le k$ such that for all $j \in [r]$, the bag X_j intersects H if and only if $j \in [\ell, k]$.

Fact 5. For every graph G, every path decomposition $(X_1, \dots, X_r)$ of G, and every clique K of G, there exists $i \in [r]$ such that $K \subseteq X_i$.

Fact 6. Let G be a graph. Then

$$\omega(G) \le \mathsf{pw}(G) + 1 \le \mathsf{td}(G) \le \mathsf{pn}(G) \le 2^{\mathsf{td}(G)} - 1\,.$$

The following theorem is an immediate consequence of [9, Theorem 2.1].

Theorem 5 (Chudnovsky, Hajebi, and Spirkl [9])**.** *For all positive integers r, s, and t, there exists an integer k such that every $\{P_r, K_s, K_{t,t}\}$-free graph has pathwidth at most k.*

Theorem 5 also follows immediately from [1, 15] and [29].

Theorem 6 (Hajebi [17])**.** *For every hereditary graph class $\mathcal{G}$, if there exists a function f such that $\mathsf{pw}(G) \le f(\omega(G))$ for every graph $G \in \mathcal{G}$, then there exists a polynomial function p such that $\mathsf{pw}(G) \le p(\omega(G))$ for every graph $G \in \mathcal{G}$.*

Together Theorems 5 and 6 imply:

Corollary 7. *For every two positive integers s and t there exists a positive integer c such that the following holds. Every $\{P_s, K_{t,t}\}$-free graph G satisfies $\mathsf{pw}(G) \le \omega(G)^c$.*

Proof. Fix positive integers s and t, and let $\mathcal{G}$ be $\{P_s, K_{t,t}\}$-free graphs. Theorem 5 implies that the pathwidth of every graph in $\mathcal{G}$ is bounded by some function of its clique number. By Theorem 6, the pathwidth of every graph in $\mathcal{G}$ is bounded by a polynomial function p of its clique number. Since a graph with $\omega(G) = 1$ satisfies $\mathsf{pw}(G) = 0$, it remains to justify that there exists some positive integer c such that $p(x) \le x^c$ for all $x \ge 2$. Indeed, after dropping the terms with negative coefficients, we observe that p is upper-bounded by a polynomial of some degree d in which all nonzero terms have the same positive coefficient a. Any such polynomial is upper bounded by a monomial of the form ax^{d+1}, which in turn is bounded from above by x^c for all $x \ge 2$, where $c = d + 1 + \lceil \log a \rceil$. □

2.2 Transitive closures of binary trees

For a positive integer k, denote by T_k^+ the transitive closure of a complete binary tree of depth k. See Fig. 1 for an example. The family $\{T_k^+\}_{k>0}$ will serve as a tightness example for several of our claims.

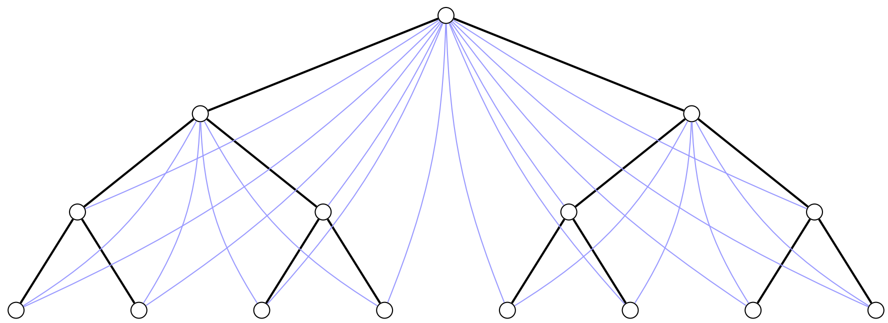

Figure 1: The graph T_k^+ is the transitive closure of a complete binary tree of depth k. The figure depicts T_4^+.

Proposition 8. *Let k be a positive integer. Then:*

(a) *For $k > 1$, the graph T_k^+ is isomorphic to the graph obtained from the disjoint union of two copies of T_{k-1}^+ by adding a universal vertex.*

(b) *T_k^+ is traceable, that is, $\mathsf{pn}(T_k^+) = |V(T_k^+)| = 2^k - 1$.*

(c) *T_k^+ is $\{P_4, K_{2,2}\}$-free.*

(d) *$\omega(T_k^+) = \mathsf{pw}(T_k^+) + 1 = \mathsf{td}(T_k^+) = k$.*

Proof. For completeness we offer short proofs in turn.

(a) Immediate from the definition and the fact that for $k > 1$, the complete binary tree of depth k is obtained from the disjoint union of two complete binary trees of depth $k-1$ by adding a new vertex as a root and making it adjacent to the roots of the two smaller trees.

(b) By induction on k. For $k > 1$, due to (a), the root r of T_k^+ is adjacent to all vertices in its left and right subtrees, both of which are isomorphic to T_{k-1}^+. Concatenating a Hamiltonian path of the left subtree, the root r, and a Hamiltonian path of the right subtree yields a Hamiltonian path of T_k^+.

(c) By induction on k, using (a) and the fact that the operations of disjoint union and addition of a universal vertex preserve $\{P_4, K_{2,2}\}$-freeness.

(d) By Fact 6, it suffices to show that $\mathsf{td}(T_k^+) \leq k \leq \omega(T_k^+)$. The former inequality follows from the fact that the underlying binary tree of depth k is a valid treedepth decomposition. For the latter one, observe that any root-to-leaf path in the underlying binary tree of depth k forms a clique of size k in T_k^+. □

3 Comparison with Hunter et al.

The main result of Hunter et al. [22] hides the dependency on the size of the excluded biclique in the $\Omega(\cdot)$ notation. The variant below can be obtained from the proof.

Theorem 9. *Let G be a $K_{r,r}$-subgraph-free graph containing a path on n vertices, where $n \geq r \geq 2$. Then G contains an induced path of length at least $\Omega(\frac{\log\log n}{r \log\log\log n})$.*

We now explain why Theorem 9 leads to a quantitative variant of Theorem 1 establishing a doubly exponential bound on the path number in terms of some power of the clique number.

Theorem 10. *For every two positive integers s and t there exists a positive integer c such that every $\{P_s, K_{t,t}\}$-free graph G satisfies $\mathsf{pn}(G) < 2^{2^{(\omega(G)+1)^c}}$.*

Proof. Theorem 9 implies that there exists a positive integer a such that for every $n \geq r \geq 2$, every graph G that does not contain $K_{r,r}$ as a subgraph, and which contains a path on n vertices, contains an induced path of length at least $a(\log\log n)^{0.99}/r$.

Fix s and t and let G be a $\{P_s, K_{t,t}\}$-free graph. Let $r = (\omega(G)+1)^t$. Since G is $K_{t,t}$-free, it does not contain $K_{r,r}$ as a subgraph. Indeed, if G contains a $K_{r,r}$, with parts A and B, then at least one of the two parts, say A, has independence number less than t, but then $G[A]$ would have to contain a clique of size $\omega(G)+1$, a contradiction. Here we are using the bound of Erdös and Szekeres [14] that every graph on at least $\binom{s+t-2}{t-1}$ vertices contains either a clique of size s or a stable set of size t, and the observation that $\binom{s+t-2}{t-1} \leq \binom{s+t-1}{t} \leq s^t$.

We claim that $\mathsf{pn}(G) < 2^{2^{(\omega(G)+1)^c}}$ where c is any integer satisfying

$$c \geq (t + \log s - \log a)/0.99\,.$$

Suppose for a contradiction that G contains a path on n vertices, where

$$n \geq 2^{2^{(\omega(G)+1)^c}}\,.$$

By assumption, G contains an induced path on at least $a(\log\log n)^{0.99}/r$ vertices. The inequality $n \geq 2^{2^{(\omega(G)+1)^c}}$ yields $\log\log n \geq (\omega(G)+1)^c$, and consequently

$$\begin{aligned} a\frac{(\log\log n)^{0.99}}{r} &\geq a\frac{(\omega(G)+1)^{0.99c}}{r} \\ &= a(\omega(G)+1)^{0.99c-t} \\ &\geq a2^{0.99c-t} \geq s\,. \end{aligned}$$

Hence, G contains an induced path on s vertices, a contradiction. □

4 Harnessing path decompositions

In this section, we bound treedepth and path number of P_s-free graphs in terms of pathwidth. We start by proving a linear relationship between treedepth and pathwidth.

Theorem 11. *Let $s \geq 3$ be an integer and let G be P_s-free graph with at least one edge. Then $\mathsf{td}(G) \leq (s-1) \cdot \mathsf{pw}(G)$.*

Proof. First observe that the inequality holds for graphs with $\mathsf{pw}(G) = 1$. Indeed, if $\mathsf{pw}(G) = 1$, then G is a forest, so $\mathsf{pn}(G) < s$ and Fact 6 implies

$$\mathsf{td}(G) \leq \mathsf{pn}(G) \leq (s-1)\,\mathsf{pw}(G)\,.$$

The proof for the general case is by induction on $n(G)$. If $n(G) = 2$, we are done since $\mathsf{pw}(G) = 1$.

For the rest of the proof let $n(G) \geq 3$ and $\mathsf{pw}(G) \geq 2$. Suppose first that G is disconnected. Denote by I the set of isolated vertices in G, and let $G_1, \ldots, G_k$ for some $k \geq 1$ be the non-trivial components of G. Then, the induction hypothesis yields $\mathsf{td}(G_i) \leq (s-1)\,\mathsf{pw}(G_i)$ for each $i \in [k]$. Hence, by Fact 1 and Fact 3, we obtain

$$\begin{aligned}
\mathsf{td}(G) = \mathsf{td}(G-I) &= \max_{1\leq i\leq k} \mathsf{td}(G_i) \\
&\leq (s-1)\left(\max_{1\leq i\leq k} \mathsf{pw}(G_i)\right) \\
&= (s-1)\,\mathsf{pw}(G-I) = (s-1)\,\mathsf{pw}(G)\,,
\end{aligned}$$

as desired.

We may therefore assume that G is connected. Let $(X_1, \ldots, X_m)$ be a path decomposition of G of width $\mathsf{pw}(G)$. Let $a_1 \in X_1$ and $a_m \in X_m$, and let P be a shortest path in G from a_1 to a_m. Since P is an induced path and G is P_s-free, $|V(P)| \leq s-1$. Furthermore, Fact 4 implies that $V(P) \cap X_i \neq \emptyset$ for every $i \in [m]$. It follows that $\mathsf{pw}(G-P) \leq \mathsf{pw}(G) - 1$. Let F be a treedepth decomposition of $G-P$ with depth equal to $\mathsf{td}(G-P)$ and let R be the set of roots of F. Let T be the tree obtained from the disjoint union of F and P by adding an edge between a_m and every vertex in R, and let a_1 be the root of T. Then, T is a treedepth decomposition of G with depth at most $\mathsf{td}(G-P)+(s-1)$, implying that $\mathsf{td}(G) \leq \mathsf{td}(G-P) + (s-1)$.

If $G-P$ has at least one edge, then the induction hypothesis implies that $\mathsf{td}(G-P) \leq (s-1)(\mathsf{pw}(G)-1)$, and thus

$$\mathsf{td}(G) \leq \mathsf{td}(G-P) + (s-1) \leq (s-1)\,\mathsf{pw}(G)\,,$$

as desired. We may thus assume that $G-P$ is edgeless. Then, $\mathsf{td}(G-P) \leq 1$ and, since $\mathsf{pw}(G) \geq 2$ and $s \geq 3$, we obtain that

$$\mathsf{td}(G) \leq \mathsf{td}(G-P) + (s-1) \leq s \leq (s-1)\,\mathsf{pw}(G)\,,$$

as desired. □

Theorem 11 and the inequality $\mathsf{pn}(G) \leq 2^{\mathsf{td}(G)} - 1$ (see Fact 6) imply that for every integer $s \geq 3$, every P_s-free graph with at least one edge satisfies

$$\mathsf{pn}(G) \leq 2^{(s-1)\,\mathsf{pw}(G)} - 1\,.$$

We recently learned of a better bound, obtained by a similar approach as the one used in the proof of Theorem 11, due to Hilaire and Raymond in [21, Theorem 6]: for every integer $s \geq 3$, every P_s-free graph with at least one edge satisfies

$$\mathsf{pn}(G) \leq (3s-3)^{\mathsf{pw}(G)+1}\,.$$

Using a similar strategy, we obtain a further improvement, as follows.

Theorem 12. *Let $s \geq 3$ be an integer and let G be P_s-free graph with at least one edge. Then*

$$\mathsf{pn}(G) \leq s^{\mathsf{pw}(G)} - 1\,.$$

Proof. We again start by verifying that, in the case $\mathsf{pw}(G) = 1$, we have a forest and the fact $\mathsf{pn}(G) < s$ implies our claim. We proceed by induction on $n(G)$ where the base case (i.e., $n(G) = 2$) is settled due to $\mathsf{pw}(K_2) = 1$.

Suppose now that $n(G) \geq 3$. Consider first the case where G is disconnected. Denote by I the set of isolated vertices in G and again recall Fact 3, in particular, $\mathsf{pw}(G) = \mathsf{pw}(G-I)$ and $\mathsf{pn}(G) = \mathsf{pn}(G-I)$. Denote by $G_1, \ldots, G_k$ the non-trivial components of G, and observe by the induction hypothesis that $\mathsf{pn}(G_i) \leq s^{\mathsf{pw}(G_i)} - 1$ for each $i \in [k]$. By Fact 1, we thus obtain

$$\begin{aligned} \mathsf{pn}(G) = \mathsf{pn}(G-I) &= \max_{1\leq i\leq k} \mathsf{pn}(G_i) \\ &\leq s^{\max_{1\leq i\leq k} \mathsf{pw}(G_i)} - 1 \\ &= s^{\mathsf{pw}(G-I)} - 1 = s^{\mathsf{pw}(G)} - 1\,, \end{aligned}$$

as desired.

Suppose now that G is connected, let $(X_1, \ldots, X_m)$ be a path decomposition of G realizing $\mathsf{pw}(G)$, and let P be a shortest path connecting some vertex of X_1 with some vertex of X_m. We again have and $P \cap X_i \neq \emptyset$ for every $i \in \{1, \ldots, m\}$, and also $|V(P)| < s$, since G is P_s-free. In particular, $\mathsf{pw}(G-P) \leq \mathsf{pw}(G) - 1$. Note that we may assume that $G-P$ has at least one vertex, since otherwise G is a path and has pathwidth 1.

Let $C_1, \ldots, C_\ell$ be the connected components of $G-P$. Let Q be a path in G. Consider how Q alternates between vertices of P and the connected components of $C_1, \ldots, C_l$. Since P contributes at most $s-1$ vertices to Q, the path Q can traverse at most s such components of $G-P$. Therefore, we can bound $\mathsf{pn}(G)$ by

$$\mathsf{pn}(G) \leq s \cdot \max_{i\in[\ell]} \mathsf{pn}(C_i) + (s-1) \leq s \cdot \mathsf{pn}(G-P) + s - 1\,.$$

If $G-P$ is edgeless, then $\mathsf{pn}(G-P) = 1$, which implies $\mathsf{pn}(G) \leq 2s-1 \leq s^{\mathsf{pw}(G)}-1$ due to $\mathsf{pw}(G) \geq 2$ and $s \geq 3$. Otherwise, the induction hypothesis on $G-P$

and the fact that $\mathsf{pw}(G-P) \leq \mathsf{pw}(G)-1$ imply that

$$\mathsf{pn}(G) \leq s\left(\left(s^{\mathsf{pw}(G)-1}-1\right)+1\right)-1 = s^{\mathsf{pw}(G)}-1,$$

as desired. □

We observe that exponential dependence on the pathwidth cannot be avoided due to, for instance, transitive closures of binary trees which are P_4-free (also see Fig. 1). Indeed, for any positive integer k, we have $\mathsf{pw}(T_k^+) = k$ while $\mathsf{pn}(T_k^+) = 2^{k+1}-1$ (see Proposition 8).

Remark. *The aforementioned result in [21, Theorem 6] is stated as:*

> *For every positive integer k, if G is a graph with pathwidth less than k that has a path of order n, then G has an induced path of order at least $\frac{1}{3}n^{1/k}$.*

Theorem 12 leads to the following improved bound:

> *For every integer $k \geq 2$, if G is a graph with pathwidth less than k that has a path of order n, then G has an induced path of order at least $(n+1)^{1/(k-1)}-1$.*

5 Improving a Ramsey-type result

Using the exponential relationship between the path number of a P_s-free graph and its pathwidth given by Theorem 12, we now derive one of our main results, the following quantitative improvement of Theorem 1.

Theorem 2. *For every two positive integers s and t there exists a positive integer c such that every $\{P_s, K_{t,t}\}$-free graph G satisfies $\mathsf{pn}(G) \leq 2^{\omega(G)^c}$.*

Proof. Fix two positive integers s and t. We may assume that $s \geq 2$. By Corollary 7, there exists a positive integer d such that every $\{P_s, K_{t,t}\}$-free graph G satisfies $\mathsf{pw}(G) \leq \omega(G)^d$. Let G be a $\{P_s, K_{t,t}\}$-free graph. We may assume that $\omega(G) \geq 2$, since otherwise $\mathsf{pn}(G) \leq 1 \leq 2^{\omega(G)^c}$ for every integer c. Note that $s \geq 3$, since G is P_s-free but has at least one edge. Hence, by Theorem 12, $\mathsf{pn}(G) \leq s^{\mathsf{pw}(G)}$. Let $c = d + \lceil \log\log s \rceil$. Then $\log s \leq 2^{c-d} \leq \omega(G)^{c-d}$ and hence, $(\log s)\cdot \omega(G)^d \leq \omega(G)^c$, which together with the inequality $\mathsf{pw}(G) \leq \omega(G)^d$ implies

$$\mathsf{pn}(G) \leq s^{\mathsf{pw}(G)} \leq s^{\omega(G)^d} \leq 2^{(\log s)\cdot\omega(G)^d} \leq 2^{\omega(G)^c},$$

as desired. □

Exponential dependence on the clique number cannot be avoided, e.g., due to the family of transitive closures of binary trees $\{T_k^+\}_{k>0}$ (also see Fig. 1), where $\omega(T_k^+) = k$ while $\mathsf{pn}(T_k^+) = 2^{k+1}-1$ by Proposition 8. Furthermore, the constant c must depend on s and t, as can be seen, for instance, by looking at the family of path graphs.

6 Treedepth is clique-polynomial

In what follows, we make use of the other main result from Section 4, the linear relationship between treedepth and pathwidth for P_s-free graphs given by Theorem 11, to prove Theorem 3.

Theorem 3. *For every two positive integers s and t there exists a positive integer c such that every $\{P_s, K_{t,t}\}$-free graph G satisfies $\mathsf{td}(G) \leq \omega(G)^c$.*

Proof. Fix s and t. By Corollary 7, there exists a positive integer d such that $\mathsf{pw}(G) \leq \omega(G)^d$, where G is an arbitrary $\{P_s, K_{t,t}\}$-free graph. We may assume that G has at least one edge, since otherwise $\mathsf{td}(G) \leq \omega(G)^d$. Note that $s \geq 3$, since G is P_s-free but has at least one edge. Hence, Theorem 11 implies that $\mathsf{td}(G) \leq (s-1) \cdot \mathsf{pw}(G)$. Since $\mathsf{pw}(G) \leq \omega(G)^d$, we obtain that $\mathsf{td}(G) \leq (s-1) \cdot \omega(G)^d$. Similarly as in the proof of Corollary 7, we conclude that there exists a positive integer c depending only on s and d (and hence, only on s and t) such that $\mathsf{td}(G) \leq \omega(G)^c$, as desired. □

Finally, we derive from Theorem 3 the analogue of Theorem 6 for treedepth, that is, we show that treedepth is clique-polynomial.

Theorem 4. *For every hereditary graph class $\mathcal{G}$, if there exists a function f such that $\mathsf{td}(G) \leq f(\omega(G))$ for every graph $G \in \mathcal{G}$, then there exists a polynomial function p such that $\mathsf{td}(G) \leq p(\omega(G))$ for every graph $G \in \mathcal{G}$.*

Proof. By Theorem 3 it is enough to prove that every graph in $\mathcal{G}$ is $\{P_s, K_{t,t}\}$-free. To this end let $\mathcal{G}$ be a (td, ω)-bounded graph class. Fix a function $f\colon \mathbb{N} \to \mathbb{N}$ such that $\mathsf{td}(G) \leq f(\omega(G))$ for all graphs $G \in \mathcal{G}$. We may assume that f is non-decreasing, since otherwise we could redefine $f(k)$ for $k \in \mathbb{N}$ by setting it to $\max_{i \in [k]} f(i)$. Let $s = t = 2^{f(2)}$. We claim that $\mathcal{G}$ excludes P_s. Suppose for a contradiction that $P_s \in \mathcal{G}$. Since $\omega(P_s) \leq 2$ and f is non-decreasing, we have that $\mathsf{td}(P_s) \leq f(2)$. By Fact 6, it holds that $s = \mathsf{pn}(P_s) \leq 2^{f(2)} - 1$, a contradiction. Furthermore, we claim that $\mathcal{G}$ excludes $K_{t,t}$. Suppose for a contradiction that $K_{t,t} \in \mathcal{G}$. Since $\omega(K_{t,t}) = 2$, we have that $\mathsf{td}(K_{t,t}) \leq f(2)$. Then, Fact 6 implies that $2^{f(2)+1} = 2t = \mathsf{pn}(K_{t,t}) \leq 2^{f(2)} - 1$, a contradiction. Since $\mathcal{G}$ is hereditary, every graph in $\mathcal{G}$ is $\{P_s, K_{t,t}\}$-free, as desired. □

References

[1] Aistis Atminas, Vadim V. Lozin, and Igor Razgon. Linear time algorithm for computing a small biclique in graphs without long induced paths. In *Algorithm theory—SWAT 2012*, volume 7357 of *Lecture Notes in Comput. Sci.*, pages 142–152. Springer, Heidelberg, 2012. doi:10.1007/978-3-642-31155-0_13.

[2] Kenny Bešter Štorgel, Clément Dallard, Vadim Lozin, Martin Milanič, and Viktor Zamaraev. Awesome graph parameters. Preprint available at https://arxiv.org/abs/2511.05285, 2025.

[3] Václav Blažej, J. Pascal Gollin, Tomáš Hons, Tomáš Masařík, Martin Milanič, Paweł Rzążewski, Ondřej Suchý, and Alexandra Wesolek. Tree-independence number of P_5-free graphs with no large bicliques. Preprint available at https://arxiv.org/abs/2605.03965, 2026.

[4] Hans L. Bodlaender. A partial k-arboretum of graphs with bounded treewidth. *Theoret. Comput. Sci.*, 209(1-2):1–45, 1998. doi:10.1016/S0304-3975(97)00228-4.

[5] Romain Bourneuf, Matija Bucić, Linda Cook, and James Davies. On polynomial degree-boundedness. *Adv. Comb.*, pages Paper No. 5, 16, 2024. doi:10.19086/aic.2024.5.

[6] Marcin Briański, James Davies, and Bartosz Walczak. Separating polynomial χ-boundedness from χ-boundedness. *Combinatorica*, 44(1):1–8, 2024. doi:10.1007/s00493-023-00054-3.

[7] Maria Chudnovsky, Julien Codsi, J. Pascal Gollin, Martin Milanič, and Varun Sivashankar. Tree-independence number and forbidden induced subgraphs: excluding a 6-vertex path and a $(2,t)$-biclique. Preprint available at https://arxiv.org/abs/2604.01999, 2026.

[8] Maria Chudnovsky, Julien Codsi, Daniel Lokshtanov, Martin Milanič, and Varun Sivashankar. Tree independence number V. Walls and claws. Preprint available at https://arxiv.org/abs/2501.14658, 2025.

[9] Maria Chudnovsky, Sepehr Hajebi, and Sophie Spirkl. Induced subgraphs and tree decompositions XVIII. Obstructions to bounded pathwidth. Preprint available at https://arxiv.org/abs/2412.17756, 2024.

[10] Maria Chudnovsky and Nicolas Trotignon. On treewidth and maximum cliques. *Innov. Graph Theory*, 2:223–243, 2025. doi:10.5802/igt.11.

[11] Clément Dallard, Matjaž Krnc, O-joung Kwon, Martin Milanič, Andrea Munaro, Kenny Štorgel, and Sebastian Wiederrecht. Treewidth versus clique number. IV. Tree-independence number of graphs excluding an induced star. Preprint available at https://arxiv.org/abs/2402.11222, 2024.

[12] Clément Dallard, Martin Milanič, and Kenny Štorgel. Treewidth versus clique number. II. Tree-independence number. *J. Combin. Theory Ser. B*, 164:404–442, 2024. doi:10.1016/j.jctb.2023.10.006.

[13] Julien Duron, Louis Esperet, and Jean-Florent Raymond. Long induced paths in sparse graphs and graphs with forbidden patterns. Preprint available at https://arxiv.org/abs/2411.08685, 2024.

[14] P. Erdös and G. Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935. URL: http://www.numdam.org/item?id=CM_1935__2__463_0.

[15] F. Galvin, I. Rival, and B. Sands. A Ramsey-type theorem for traceable graphs. *J. Combin. Theory Ser. B*, 33(1):7–16, 1982. doi:10.1016/0095-8956(82)90053-3.

[16] António Girão and Zach Hunter. Induced subdivisions in $K_{s,s}$-free graphs with polynomial average degree. *International Mathematics Research Notices*, 2025(4):rnaf025, 02 2025. doi:10.1093/imrn/rnaf025.

[17] Sepehr Hajebi. Polynomial bounds for pathwidth. Preprint available at https://arxiv.org/abs/2510.19120, 2025.

[18] Sepehr Hajebi and Sophie Spirkl. Tree-alpha and excluding finitely many graphs. Preprint available at https://arxiv.org/abs/2605.01223, 2026.

[19] Pavol Hell and Shenwei Huang. Complexity of coloring graphs without paths and cycles. *Discrete Appl. Math.*, 216:211–232, 2017. doi:10.1016/j.dam.2015.10.024.

[20] Claire Hilaire, Martin Milanič, and Đorđe Vasić. Treewidth versus clique number. V. Further connections with tree-independence number. *Journal of Graph Theory*, 112(3):337–351, 2026. doi:10.1002/jgt.70036.

[21] Claire Hilaire and Jean-Florent Raymond. Long induced paths in minor-closed graph classes and beyond. *Electron. J. Combin.*, 30(1):Paper No. 1.18, 26, 2023. doi:10.37236/11029.

[22] Zach Hunter, Aleksa Milojević, Benny Sudakov, and István Tomon. Long induced paths in $K_{s,s}$-free graphs. *Journal of Graph Theory*, 2026. To appear. doi:10.1002/jgt.70040.

[23] Edin Husić and Martin Milanič. A polynomial-time algorithm for the independent set problem in $\{P_{10}, C_4, C_6\}$-free graphs. In Ignasi Sau and Dimitrios M. Thilikos, editors, *Graph-Theoretic Concepts in Computer Science - 45th International Workshop, WG 2019, Vall de Núria, Spain, June 19-21, 2019, Revised Papers*, Lecture Notes in Computer Science, pages 271–284. Springer, 2019. doi:10.1007/978-3-030-30786-8_21.

[24] Marcin Kamiński and Anna Pstrucha. Certifying coloring algorithms for graphs without long induced paths. *Discrete Appl. Math.*, 261:258–267, 2019. doi:10.1016/j.dam.2018.09.031.

[25] Paloma T. Lima, Martin Milanič, Peter Muršič, Karolina Okrasa, Paweł Rzążewski, and Kenny Štorgel. Tree decompositions meet induced matchings: Beyond max weight independent set. *J. Comput. System Sci.*, 161:Paper No. 103819, 2026. doi:10.1016/j.jcss.2026.103819.

[26] Vadim Lozin and Dieter Rautenbach. Some results on graphs without long induced paths. *Inform. Process. Lett.*, 88(4):167–171, 2003. doi:10.1016/j.ipl.2003.07.004.

[27] Jaroslav Nešetřil and Patrice Ossona de Mendez. *Sparsity*, volume 28 of *Algorithms and Combinatorics*. Springer, Heidelberg, 2012. doi:10.1007/978-3-642-27875-4.

[28] F. P. Ramsey. On a Problem of Formal Logic. *Proc. London Math. Soc. (2)*, 30(4):264–286, 1929. doi:10.1112/plms/s2-30.1.264.

[29] Neil Robertson and P. D. Seymour. Graph minors. I. Excluding a forest. *J. Combin. Theory Ser. B*, 35(1):39–61, 1983. doi:10.1016/0095-8956(83)90079-5.

[30] Nikola Yolov. Minor-matching hypertree width. In Artur Czumaj, editor, *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018, New Orleans, LA, USA, January 7-10, 2018*, pages 219–233. SIAM, 2018. doi:10.1137/1.9781611975031.16.