3-colorable subclasses of P_8-free graphs

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Abstract

In this paper, we study 3-colorable graphs having no induced 8-vertex path and no induced cycles of specific lengths. We prove a characterization by critical graphs in three particular cases.

1 Introduction

In this paper, graph are always finite and simple (no loops, parallel edges).

A *coloring* of a graph G assigns labels to vertices of G so that no two adjacent vertices receive the same label. For a fixed number of k colors, we speak of a *k-coloring*. Finding a coloring of a graph using smallest possible set of colors is a well-known hard problem. Even if we are promised that the graph can be colored using only 3 colors, the best polynomial-time algorithms can only guarantee to use n^ϵ colors in the coloring they produce [10]. The difficulty of this seems to stem from demanding the procedure to succeed on arbitrary graphs, with no tangible structure to take advantage of. This goes against experience with real-world graphs which often exhibit some type of structured behavior. Therefore from the theoretical perspective it is natural to focus on graphs where we impose certain structure. The most popular type of such a restriction is by forbidding either a subgraph, or an induced subgraph, a minor, or a similar fixed substructure. Here we focus on induced subgraphs.

We say that a graph G *contains* a graph H if G has an induced subgraph isomorphic to H. Otherwise, we say that G is H-free. The coloring problem on H-free graphs has been a focus of research over at least the past two decades. When the number of colors is to be minimized, the problem becomes easier only when H is very small, a subgraph of the 4-vertex path [11]. For a fixed number of $k \geq 3$ colors, the situation is different. The problem is still hard when H contains a cycle [9] or has a degree ≥ 3 vertex [8]. The remaining

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cases of interest are when H is fixed t-vertex path P_t or a disjoint union of paths. Therefore studying the structure and coloring of P_t-free graphs has been the focus of attention of many researchers in the area (see recent survey [7]). One approach in this direction is investigating *critical* graphs. These are graphs with no k-coloring, but minimal with this property, i.e., all their proper (induced) subgraphs admit a k-coloring. Characterizing critical graph for specific coloring problems is very useful for algorithmic purposes, in particular if the critical graphs have bounded size, or particular structure that is easy to test. In addition, this leads to *certifying* algorithms, i.e., the presence of critical graphs serves as a witness (certificate) of the "no" answer, rather than the algorithm merely returning "no" without a justification. It should be noted that finding all critical graphs is usually much harder than simply finding a polynomial-time algorithm. Thus it should come as no surprise that much less is known about critical graphs even in cases when efficient coloring algorithms exist.

In this paper, we study P_8-free graph, i.e., graphs with no induced 8-vertex path. In particular, we focus on 3-colorings of these graphs. Unlike k-coloring for $k \geq 4$ which is known to be hard on P_8-free graphs [7], the complexity of 3-coloring is wide open. For smaller path-lengths t, algorithms are known for $t \leq 7$ [2], and the structure of critical graphs is known for $t \leq 6$ [1, 3]. Moreover, the set of obstructions to 3-coloring is known to be infinite for graphs with no induced P_7 [3].

In view of this last fact it is of interest to consider 3-colorable P_8-free graphs with additional restrictions. In particular, we forbid specific cycle-lengths in these graphs. In [7], it has been shown that P_8-free graphs of girth ≥ 6 are 3-colorable. Put differently, P_8-free graphs with no induced l-cycles where $l \in \{3, 4, 5\}$ are 3-colorable. This is the starting point of our investigation. We extend this result in three specific ways. Namely, we study P_8-free graphs with no induced l-cycles, where l is either from $\{3, 4\}$ or from $\{3, 5\}$ or from $\{4, 5\}$. For the first two, we show that all such graphs are 3-colorable, while for the last one we provide a complete list of (5) critical graphs. Our method is based on describing the structure resulting from forbidding specific cycles. The colorability is then a consequence of this structure. While this paper was being prepared for publication, it was shown in [5] that there are only finitely many minimal (under vertex deletion) non-3-colorable graphs that are P_8-free and have no induced cycles of length 4. However, the proof of [5] uses a computer, while our proofs are all done by hand.

Our results are summarized as the following theorems.

Theorem 1. *Let G be a connected graph of girth at least 5 with no induced P_8. Then one of the following holds:*

(i) G is the Petersen graph,
(ii) G is the Heawood graph,
(iii) G is the graph we obtain by contracting one edge in the Heawood graph,
(iv) G contains a vertex of degree ≤ 2.

In particular, G is 3-colorable.

Theorem 2. *Let G be a connected P_8-free graph with no triangle and no induced 5-cycle. Then at least one of the following holds:*

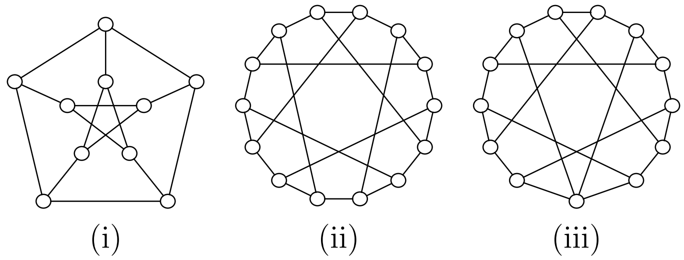

Figure 1: Outcomes of Theorem 1

(i) G contains distinct vertices u, v where $N(u) \subseteq N(v)$ (we say that v dominates u),
(ii) G admits a homomorphism to a 7-cycle.
In particular, G is 3-colorable.

Theorem 3. *Let G be a P_8-free graph with no induced 4- and 5-cycle. Then the following are equivalent:*
(i) G is 3-colorable.
(ii) G contains none of the graphs in Figure 2 as a subgraph (not necessarily induced).

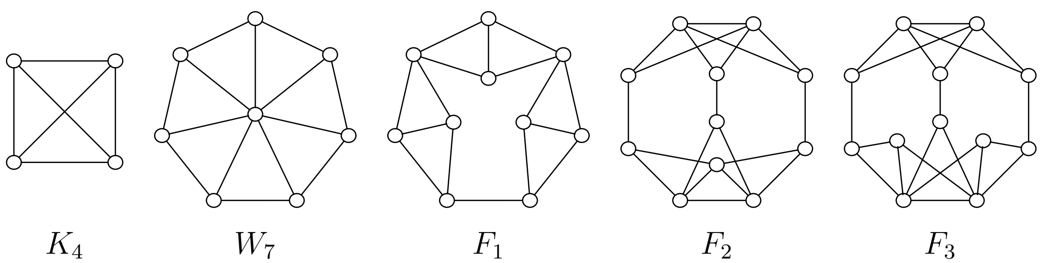

Figure 2: Forbidden subgraphs for 3-coloring P_8-free graphs with no induced 4- and 5- cycles.

In the rest of the paper, we prove these three theorems in individual sections.

2 Notation and Definitions

All graphs considered here are simple (no loops or parallel edges). The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The edge set $E(G)$ consists of unordered pairs $\{u, v\}$. For brevity, we write uv to denote the edge $\{u, v\}$.

We write $N(v)$ to denote the neighborhood of v, i.e., the set of vertices $u \neq v$ where $uv \in E(G)$. The degree of a vertex v is the size of its neighborhood. For a set $X \subseteq V(G)$, we write $N(X)$ to denote the set $\bigcup_{v \in X} (N(v) \setminus X)$.

For graphs G, H, we say that G is *H-free* if G does not contain H. For $X \subseteq V(G)$, we write $G[X]$ to denote the subgraph of G induced by X, i.e., the graph whose vertex set is X where two vertices are adjacent if and only if they are adjacent in G. We write $G - X$ to denote the subgraph of G induced by $V(G) \setminus X$. We write $G - x$ in place of $G - \{x\}$. A connected component of G is maximal connected subgraph of G.

A set $X \subseteq V(G)$ is a *clique* if all vertices in X are pairwise adjacent. A set $X \subseteq V(G)$ is a *stable set* or an *independent set* if no two vertices in X are adjacent. A set $X \subseteq V(G)$ is *complete* to $Y \subseteq V(G)$ if every $x \in X$ is adjacent to every $y \in Y$ (in particular, $X \cap Y = \emptyset$). A set $X \subseteq V(G)$ is *anticomplete* to $Y \subseteq V(G)$ if $X \cap Y = \emptyset$, and there are no edges in G with one endpoint in X and the other in Y. We say that x is complete to, or anticomplete to, Y if $\{x\}$ is complete to Y, or anticomplete to Y, respectively.

We write $x_1 - x_2 - \ldots - x_t$ to denote a path in G going through vertices $x_1, x_2, \ldots, x_t$ in this order. The path may not be induced. The *length* of a path is defined to be the number of edges in it. Similarly, we write $x_1 - x_2 - \ldots - x_t - x_1$ to denote a (not necessarily induced) cycle in G going through $x_1, x_2, \ldots, x_t$ and back to x_1. We write P_k to denote the k-vertex path, and write C_k to denote the k-vertex cycle. We write K_k to denote the complete graph on k vertices.

The girth of a graph G, denoted by $g(G)$, is the length of a shortest cycle in G. If G has no cycles, the girth of G is defined to be infinity. A k-*coloring* of G is a mapping $c : V(G) \to \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ for all $uv \in E(G)$. The chromatic number of G is the smallest k for which G has a k-coloring. In a *partial* coloring some vertices may not have a value assigned. These vertices are *uncolored*.

3 No 3- and 4-cycles

In this section, we prove Theorem 1.

Since the Petersen graph is 3-colorable and the Heawood graph is 2-colorable, and since a vertex-minimal non-3-colorable graph cannot contain a vertex of degree at most 2, it is enough to prove the first assertion of the theorem.

Let G be a connected graph of girth at least 5 with no induced 8-vertex path P_8. For contradiction, suppose that G is a counterexample to the claim of Theorem 1. Namely, G is not one of the graphs in Figure 1 and every vertex in G has degree at least 3.

To simplify the discussion, we need to state a few useful facts as the following claim.

($\star$) *If u is adjacent to v, then u is anticomplete to $N(v) \setminus \{u\}$ and $N(N(v) \setminus \{u\})$. In particular, if G contains a 5-cycle C, then every vertex $u \notin V(C)$ has at most one neighbor in $V(C)$.*

If u is adjacent to v and also to $w \in N(v) \setminus \{u\}$, then $u - v - w - u$ is a cycle in G. If u is adjacent to v and $w \in N(N(v) \setminus \{u\})$, then $u - v - z - w$ for some $z \in N(v) \setminus \{u\}$ is a cycle in G. Since the girth of G is at least 5, neither of the two is possible. If G contains a 5-cycle C, then no vertex $u \notin V(C)$ can have two neighbors in $V(C)$, since the diameter of C is 2. This proves ($\star$).

In the following sequence of claims, we prove that G cannot contain any of the graphs in Figures 3-6. This will provide a contradiction.

(0) *Suppose that G contains a copy of H_1, labeled as in Figure 3. Let $u, v \notin V(H_1)$ be vertices such that $u \in N(x)$ and $v \in N(y)$. If u adjacent to $v_i \in V(H_1)$, then v is adjacent to one of v_i, v_{i-2}, v_{i+2} (indices modulo 5) and has no other neighbor in $V(H_1) \cup \{u\} \setminus \{y\}$.*

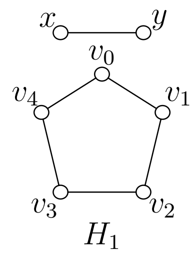

Figure 3: Case 1 of the proof of Theorem 1

For contradiction, assume that v is anticomplete to $\{v_i, v_{i-2}, v_{i+2}\}$. Then v is also non-adjacent to at least one of v_{i-1}, v_{i+1} by ($\star$). By symmetry, we may assume that v is non-adjacent to v_{i+1}. By ($\star$), we further have that v is anticomplete to $\{u, x\}$, and u is anticomplete to $\{y, v_{i+1}, v_{i+2}, v_{i-2}\}$. Thus $v–y–x–u–v_i–v_{i+1}–v_{i+2}–v_{i-2}$ is an induced P_8 in G, a contradiction. So v necessarily has at least one neighbor in $\{v_i, v_{i-2}, v_{i+2}\}$ and by ($\star$) that is its only neighbor in $V(H_1) \cup \{u\}$ besides y. This proves (0).

(1) *G does not contain H_1.*

Suppose that G contains an induced copy of H_1, labeled as shown in Figure 3. Choose the edge xy to be such that the distance in G from $\{x, y\}$ to $\{v_0, v_1, \ldots, v_4\}$ is smallest possible. Since G is connected, this implies that one of x, y has a common neighbor with one of $v_0, \ldots, v_4$. By symmetry, we may assume that there exists a vertex u_1 adjacent to both v_0 and x. Since every vertex in G has degree at least 3, let u_2 be another neighbor of x, and let u_3, u_4 be other neighbors of y.

By (0), both u_3 and u_4 are each adjacent to one of v_0, v_2, v_3. By ($\star$) they are not both adjacent to v_0. Thus by symmetry, we may assume that u_4 is adjacent to v_2. This implies by ($\star$) that u_3 is non-adjacent to v_2, and by (0) and ($\star$) that u_2 is adjacent to one of v_2, v_4. If u_2 were adjacent to v_4, then we would conclude by (0) that u_3 is adjacent to one of v_1, v_4 but then u_3 would have two neighbors in $\{v_0, \ldots, v_4\}$ contradicting ($\star$). Therefore u_2 is adjacent to v_2 and a symmetric argument gives that u_3 is adjacent to v_0.

Now, since every vertex in G has degree at least 3, there exists a neighbor $z \notin V(H_1)$ of v_3. By ($\star$), we see that z is anticomplete to $\{v_0, v_1, v_2, v_4, u_2, u_4\}$ and at least one of u_1, u_3. Using (0), we deduce that z is non-adjacent to both x and y. By symmetry, we may assume that z is non-adjacent to u_1. This implies that $z–v_3–v_4–v_0–u_1–x–y–u_4$ is an induced P_8 in G, a contradiction. This proves (1).

(2) *G does not contain H_2.*

Suppose that G contains the Petersen graph H_2, labeled as in Figure 4. Since G itself is not the Petersen graph and since G is connected, there exists a vertex x adjacent to some vertex of H_2. By symmetry, we may assume that x is adjacent to v_0. Then ($\star$) implies that x has no other neighbors in $V(H_2)$ because H_2 has diameter 2. Since every vertex in G has degree at least 3, we deduce that x has another neighbor $y \notin V(H_2)$. Similarly to x, the vertex y has at most one neighbor in $V(H_2)$.

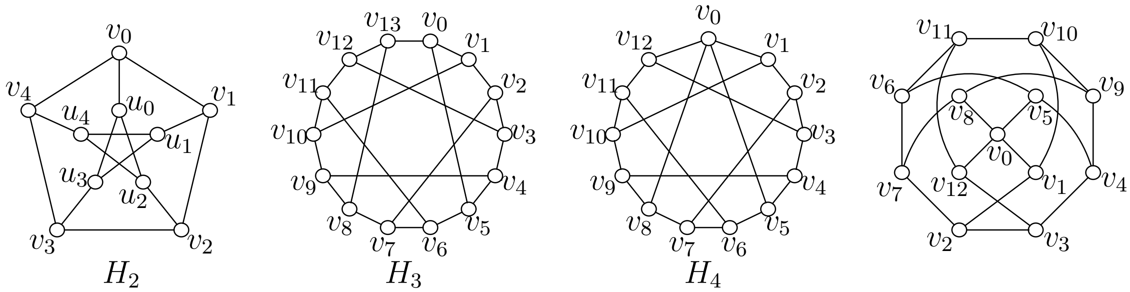

Figure 4: Cases 2-4 of the proof of Theorem 1

We focus our attention now on the edge xy. By (1), we see that y has at least one neighbor in each of the following three 5-cycles: $v_1–v_2–v_3–u_3–u_1–v_1$, $v_2–v_3–v_4–u_4–u_2–v_2$, $u_0–u_2–u_4–u_1–u_3–u_0$. However, the three 5-cycles do not have a vertex in common and so in order for y to hit all three 5-cycles it has to have at least 2 neighbors in $V(H_2)$, a contradiction. This proves (2).

(3) *G does not contain the Heawood graph H_3.*

Suppose that G contains an induced copy of the Heawood graph H_3, labeled as in Figure 4. Since G is not the Heawood graph and since G is connected, there exists a vertex x with a neighbor in $V(H_3)$. By symmetry, we may assume that x is adjacent to v_0. This implies by ($\star$) that x is anticomplete to $V(H_3) \setminus \{v_0, v_3, v_7, v_9, v_{11}\}$. If x is adjacent to v_3, then x is non-adjacent to v_{11} by ($\star$), in which case $\{x, v_0, v_1, v_2, v_3, v_6, v_{11}\}$ induces a copy of H_1 in G, contradicting (1). Thus x is non-adjacent to v_3, and if x is adjacent to v_9, then $\{x, v_0, v_1, v_{10}, v_9, v_3, v_{12}\}$ induces a copy of H_1 in G. Therefore x is also non-adjacent to v_9. But now $x–v_0–v_1–v_2–v_3–v_4–v_9–v_8$ is an induced P_8 in G, a contradiction. This proves (3).

(4) *G does not contain H_4.*

Suppose that G contains H_4, labeled as in Figure 4. Since G is not H_4 and since G is connected, there exists a vertex $x \notin V(H_4)$ with a neighbor in $V(H_4)$. By symmetry (see the alternative drawing of H_4 in Figure 4), we may assume that x is adjacent to one of v_0, v_1, v_2.

Suppose first that x is adjacent to v_1. Then ($\star$) implies that x is anticomplete to $V(H_4) \setminus \{v_1, v_4, v_6\}$. If x is adjacent to v_6, then it is non-adjacent to v_4 by ($\star$), in which case $\{x, v_1, v_2, v_7, v_6, v_4, v_9\}$ induces a copy of H_1 in G, contradicting (1). Thus x is non-adjacent to v_6, and so $x–v_1–v_2–v_3–v_{12}–v_{11}–v_6–v_5$ is an induced P_8 in G, a contradiction. We therefore conclude that x is not adjacent to v_1.

Next suppose that x is adjacent to v_2. Then x is anticomplete to $V(H_4) \setminus \{v_2, v_5, v_9, v_{11}\}$ and has at most one neighbor among v_5, v_9, v_{11} by ($\star$). Looking at the edge xv_2 and using (1), we find that x must have a neighbor in the 5-cycle $v_0–v_5–v_4–v_9–v_8–v_0$ and also in the 5-cycle $v_0–v_5–v_6–v_{11}–v_{12}–v_0$. This implies that x is adjacent to v_5 and has no further neighbors in $V(H_4)$. But this leads us to conclude that $\{x, v_5, v_6, v_7, v_2, v_9, v_{10}\}$ induces a copy of H_1 in G, contradicting (1).

It remains to consider the case when x is adjacent to v_0. In this case x has no other neighbor in $V(H_4)$ by ($\star$). Since every vertex in G has degree at least 3, there exist a neighbor $y \notin V(H_4)$ of x. By repeating the above argument (applied to y in place of x), we conclude that y is anticomplete to $V(H_4) \setminus \{v_0\}$ and is also non-adjacent to v_0 by $(\star)$. But now $y - x - v_0 - v_1 - v_2 - v_7 - v_6 - v_{11}$ is an induced P_8 in G, a contradiction.

This proves (4).

(5) *G does not contain an induced 8-cycle.*

Suppose that G contains an induced 8-cycle $C = v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 - v_0$. Since every vertex in G has degree at least 3, the vertex v_0 has a neighbor $u_1 \notin V(C)$. It follows by $(\star)$ that u_1 is anticomplete to $\{v_1, v_2, v_6, v_7\}$. If u_1 is adjacent to v_3, then it is anticomplete to $\{v_4, v_5\}$ by $(\star)$, in which case $\{u_1, v_0, v_1, v_2, v_3, v_5, v_6\}$ induces H_1 in G, contradicting (1). Thus u_1 is non-adjacent to v_3, and by symmetry, also to v_5. Now since $u_1 - v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6$ is not an induced P_8, we deduce that u_1 is adjacent to v_4 and has no other neighbors in $V(C)$. Applying the same argument to other vertices in C, we find new vertices u_2, u_3, u_4 where u_2 is adjacent in C to v_1, v_5, and u_3 is adjacent to v_2, v_6 while u_4 is adjacent to v_3, v_7. By $(\star)$, we have that u_1, u_2, u_3, u_4 are pairwise distinct and u_1u_2, u_2u_3, u_3u_4, u_1u_4 are non-edges.

Suppose that u_1 is non-adjacent to u_3. Since every vertex in G has degree at least 3, there exists $x \notin V(C) \cup \{u_1, u_2, u_3, u_4\}$ adjacent to u_1. By $(\star)$, we see that x is anticomplete to $V(C) \setminus \{v_2, v_6\}$. If x is adjacent to v_2, then it is non-adjacent to v_6 by $(\star)$, in which case $\{u_1, v_0, v_1, v_2, x, v_5, v_6\}$ induces H_1 in G, contradicting (1). Thus x is non-adjacent to v_2, and by symmetry, also non-adjacent to v_6. If x is also non-adjacent to u_3, then $x - u_1 - v_4 - v_5 - v_6 - u_3 - v_2 - v_1$ is an induced P_8 in G, a contradiction. This proves that x is adjacent to u_3 and has no neighbor in $V(C)$. If x is also adjacent to both u_2, u_4, then u_2 is non-adjacent to u_4 and $V(C) \cup \{u_1, u_2, u_3, u_4, x\}$ induces a copy of H_4 in G, contradicting (4). Thus, by symmetry, x is non-adjacent to u_2. If x is adjacent to u_4 and also u_2 is adjacent to u_4, then $V(C) \cup \{u_1, u_2, u_3, u_4, x\}$ induces H_4 in G. If x is adjacent to u_4 but u_2 is not, then $v_1 - u_2 - v_5 - v_4 - v_3 - u_4 - x - u_3$ is an induced P_8 in G, a contradiction. This shows that x is also non-adjacent to u_4. Moreover, if u_2 is adjacent to u_4, then $v_0 - u_1 - x - u_3 - v_6 - v_5 - u_2 - u_4$ is an induced P_8 in G. Therefore u_2 is non-adjacent to u_4. This allows us to repeat the above argument for u_2 and u_4 to find a vertex y complete to $\{u_2, u_4\}$ and anticomplete to $V(C) \cup \{u_1, u_3\}$. In particular, y is distinct from x. If x is adjacent to y, then $V(C) \cup \{u_1, u_2, u_3, u_4, x, y\}$ induces a copy of the Heawood graph H_3 in G, contradicting (3). Thus x is non-adjacent to y, but then $v_0 - u_1 - x - u_3 - v_6 - v_5 - u_2 - y$ is an induced P_8 in G, a contradiction.

This shows that u_1 is adjacent to u_3, and by symmetry, u_2 is adjacent to u_4. But now we find that $v_0 - u_1 - u_3 - v_2 - v_3 - u_4 - u_2 - v_5$ is an induced P_8 in G, a contradiction. This proves (5).

(6) *G does not contain H_6.*

Suppose that G contains an induced H_6, labeled as in Figure 5. Since every vertex in G has degree at least 3, the vertex v_0 has a neighbor $x \notin V(H_6)$. By $(\star)$, the vertex x is also anticomplete to $\{v_1, v_2, u_0, u_1, u_2, u_3\}$.

Suppose first that x is adjacent to v_4. Then by $(\star)$, we see that x has no other neighbors in $V(H_6)$, and so it has a neighbor $y \notin V(H_6)$, since every vertex in G has degree at least 3.

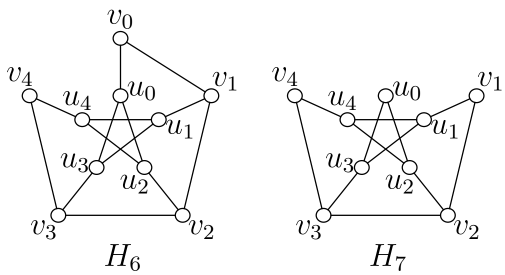

Figure 5: Cases 6-7 of the proof of Theorem 1

By (⋆), we conclude that y is anticomplete to $\{v_0, u_0, v_1, v_3, v_4, u_4\}$ and so it has at most one neighbor in $V(H_6)$. Looking at the edge xy and using (1), we find that y has a neighbor in each of the 5-cycles: $v_1–v_2–v_3–u_3–u_1–v_1$, $v_1–v_2–u_2–u_4–u_1–v_1$, $v_2–v_3–u_3–u_0–u_2–v_2$, and $u_0–u_2–u_4–u_1–u_3–u_0$. But these 5-cycles have no vertex in common, so in order for y to hit each of them it must have at least 2 neighbors in $V(H_6)$, a contradiction.

This shows that x is non-adjacent to v_4. If it is also anticomplete to $\{v_3, u_4\}$, then $\{u_1, u_3, v_3, v_4, u_4, x, v_0\}$ induces H_1 in G, contradicting (1). Therefore we conclude x is adjacent in $V(H_6)$ only to v_0 and exactly one of v_3, u_4 by (⋆). By a symmetric argument (applied to v_4), we find a new vertex z adjacent to v_4, one of v_1, u_0, and no other vertex in $V(H_6)$. If x is adjacent to u_4, and z is adjacent to u_0, then xz is not an edge by (⋆) in which case $x–v_0–v_1–u_1–u_3–v_3–v_4–z$ is an induced P_8 in G, a contradiction. Similarly, if x is adjacent to u_4, and z is adjacent to v_1, then $x–v_0–u_0–u_2–v_2–v_3–v_4–z$ is an induced P_8 in G, a contradiction. By symmetry, only one possibility remains: x is adjacent to v_3, and z is adjacent to v_1. In this case $x–v_0–u_0–u_3–u_1–u_4–v_4–z$ is an induced P_8 in G, a contradiction. This proves (6).

(7) *G does not contain H_7.*

Suppose that G contains an induced H_7, labeled as in Figure 5. Since every vertex in G has degree at least 3, the vertex u_0 has a neighbor $x \notin V(H_7)$. By (⋆), the vertex x is anticomplete to $V(H_7) \setminus \{u_0, v_1, v_4\}$. If x is adjacent to both v_1, v_4, then $V(H_7) \cup \{x\}$ induces in G a copy of the Petersen graph H_2, contradicting (2). If x is adjacent to exactly one of v_1, v_4, then $V(H_7) \cup \{x\}$ induces in G a copy of H_6, contradicting (6). Thus x has no neighbor in $V(H_7) \setminus \{u_0\}$. This implies that x has another neighbor $y \notin V(H_7)$, since every vertex in G has degree at least 3. It follows by (⋆) that y is anticomplete to $\{u_0, u_2, u_3\}$. If y is adjacent to u_4, then y has no other neighbors in $V(H_7)$ by (⋆), in which case $\{v_1, v_2, v_3, u_3, u_1, x, y\}$ induces a copy of H_1 in G, contradicting (1). Thus y is non-adjacent to u_4, and by symmetry, also non-adjacent to u_1. Similarly, if y is adjacent to v_3, then y has no other neighbors in $V(H_7)$ by (⋆) and so $\{v_1, v_2, u_2, u_4, u_1, x, y\}$ induces H_1 in G. Thus y is non-adjacent to v_3 and by symmetry, also non-adjacent to v_2. If y is also non-adjacent to v_4, then $\{v_2, v_3, v_4, u_4, u_2, x, y\}$ induces H_1 in G. This shows that y is adjacent to v_4, and by symmetry, also adjacent to v_1. But now $V(H_7) \cup \{y\}$ induces a copy of H_6 in G, contradicting (6). This proves (7).

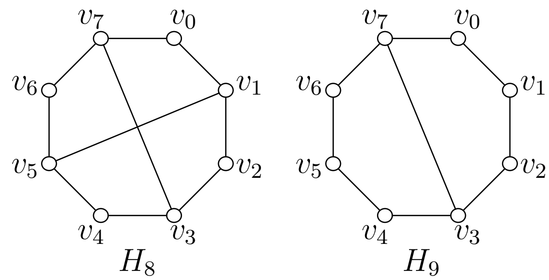

Figure 6: Cases 8-9 of the proof of Theorem 1

(8) *G does not contain H_8.*

Suppose that G contains an induced H_8, labeled as in Figure 6.

First we show that $N(v_0) \setminus V(H_8)$ is anticomplete to $N(v_2) \setminus V(H_8)$. For contradiction, let $x \in N(v_0) \setminus V(H_8)$ be adjacent to $y \in N(v_2) \setminus V(H_8)$. By $(\star)$, we see that x is anticomplete to $V(H_8) \setminus \{v_0, v_4\}$, and if x is adjacent to v_4, then $V(H_8) \cup \{x\}$ induces a copy of H_7 in G, contradicting (7). Thus the only neighbor of x in $V(H_8)$ is v_0, and by symmetry, the only neighbor of y in $V(H_8)$ is v_2. Since every vertex in G has degree at least 3, we find $z, w \notin V(H_8)$ where $z \in N(u_4)$ and $w \in N(u_6)$ which likewise have no other neighbors in $V(H_8)$. If x is also adjacent to z, then y is non-adjacent to z by $(\star)$, in which case z–x–y–v_2–v_1–v_5–v_6–v_7 is an induced P_8 in G, a contradiction. Thus x is non-adjacent to z which proves that y is adjacent to z, because if not, then $\{x, y, v_2, v_1, v_0, z, v_4\}$ induces a copy of H_1 in G, contradicting (1). Repeating this argument for y, z, we conclude that z is adjacent to w, and thus w is adjacent to x by the same token. But now x–y–z–w–x is a cycle in G, contradicting $(\star)$.

This shows that $N(v_0) \setminus V(H_8)$ is anticomplete to $N(v_2) \setminus V(H_8)$, and by symmetry, it is also anticomplete to $N(v_6) \setminus V(H_8)$. Suppose that $N(v_0) \setminus V(H_8)$ is also anticomplete to $N(v_4) \setminus V(H_8)$. Since every vertex in G has degree at least 3, there exists $x \in N(v_0) \setminus V(H_8)$ and $z \in N(v_4) \setminus V(H_8)$. By our assumption, x is not adjacent to z, and as shown in the previous paragraph, both x and z have no other neighbors in $V(H_8)$. Thus there is a neighbor $u \notin V(H_8)$ of x, since every vertex in G has degree at least 3. By $(\star)$, we conclude that u is anticomplete to $\{v_0, v_1, v_7\}$. Since $N(v_0) \setminus V(H_8)$ is anticomplete to $(N(v_2) \cup N(v_4) \cup N(v_6)) \setminus V(H_8)$, the vertex u is also anticomplete to $\{v_2, v_4, v_6\}$. So $(\star)$ implies that u is adjacent to at most one of v_3, v_5. By symmetry, we may assume that u is non-adjacent to v_3. If u is non-adjacent to z, we conclude that u–x–v_0–v_1–v_2–v_3–v_4–z is an induced P_8 in G, a contradiction. So u is adjacent to z and therefore u–x–v_0–v_1–v_2–v_3–v_4–z–u an induced the 8-cycle in G, contradicting (5).

This proves that $N(v_0) \setminus V(H_8)$ is not anticomplete to $N(v_4) \setminus V(H_8)$, and by symmetry, $N(v_2) \setminus N(V_8)$ is not anticomplete to $N(v_6) \setminus N(V_8)$. Thus there exist vertices $x, y, z, w \notin V(H_8)$ where $x \in N(v_0)$, $y \in N(v_2)$, $z \in N(v_4)$ and $w \in N(v_6)$ such that x is adjacent to z, and y is adjacent to w. We again conclude that x, y, z, w have no other neighbors in $V(H_8) \cup \{x, y, z, w\}$. But now v_0–x–z–v_4–v_3–v_2–y–w is an induced P_8 in G, a contradiction. This proves (8).

(9) *G does not contain H_9.*

Suppose that G contains an induced H_9, labeled as in Figure 6. Since every vertex in G has degree at least 3, we have that v_0 has a neighbor $x \notin V(H_9)$. From ($\star$), we deduce that x is anticomplete to $V(H_9) \setminus \{v_0, v_4, v_5\}$. If x is also anticomplete to $\{v_4, v_5\}$, then $x{-}v_0{-}v_1{-}v_2{-}v_3{-}v_4{-}v_5{-}v_6$ is an induced P_8 in G, a contradiction. Thus x is adjacent to exactly one of v_4, v_5 by ($\star$).

Suppose that x is adjacent to v_5. Then x is non-adjacent to v_4 and so it has another neighbor $y \notin V(H_9)$, since every vertex in G has degree at least 3. By ($\star$), we see that y is anticomplete to $V(H_9) \setminus \{v_2, v_3\}$. If y is also anticomplete to $\{v_2, v_3\}$, then $y{-}x{-}v_5{-}v_6{-}v_7{-}v_3{-}v_2{-}v_1$ is an induced P_8 in G, a contradiction. If y is adjacent to v_2, then $y{-}v_2{-}v_1{-}v_0{-}v_7{-}v_6{-}v_5{-}v_4$ is an induced P_8. Thus y is adjacent to v_3 and non-adjacent to v_2, and so $\{x, y, v_3, v_4, v_5, v_6, v_7, v_0\}$ induces a copy of H_8 in G, contradicting (8).

This shows that x is non-adjacent to v_5 and thus adjacent to v_4. Repeating the argument for v_6 instead of v_0, we find a new vertex z adjacent in $V(H_9)$ only to v_2 and v_6. If z is non-adjacent to x, then $v_0{-}v_1{-}v_2{-}z{-}v_6{-}v_5{-}v_4{-}x{-}v_0$ is an induced 8-cycle in G, contradicting (5). Therefore z is adjacent to x, which implies that $\{v_0, x, v_4, v_3, v_2, z, v_6, v_7\}$ induces a copy of H_8 in G, contradicting (8). This proves (9).

(10) *G does not contain an induced 5-cycle.*

Suppose that G contains an induced 5-cycle $C = v_0{-}v_1{-}v_2{-}v_3{-}v_4{-}v_0$.

First, we prove that $V(C)$ dominates G. For contradiction, let x be a vertex with no neighbor in $V(C)$. Since every vertex in G has degree at least 3, let u_1, u_2, u_3 be three distinct neighbors of x. If u_1 has no neighbors in $V(C)$, then $V(C) \cup \{x, u_1\}$ induces a copy of H_1 in G, contradicting (1). Thus u_1 has a neighbor in $V(C)$ and by symmetry, both u_2 and u_3 have neighbors in $V(C)$. By ($\star$), the vertices u_1, u_2, u_3 have disjoint sets of neighbors in $V(C)$. Therefore neighbors of two of them, say u_1 and u_2, are consecutive on the cycle C; say u_1 is adjacent to v_1, and u_2 is adjacent to v_2. By ($\star$), the vertices u_1 and u_2 are non-adjacent and have no other neighbors in $V(C)$. But now we find that $V(C) \cup \{u_1, u_2, x\}$ induces a copy of H_9 in G, contradicting (9).

This shows that every vertex in $V(G) \setminus V(C)$ has a neighbor in $V(C)$. Since every vertex in G has degree at least 3, let $u_0 \notin V(C)$ be a neighbor of v_0. By ($\star$), we see that u_0 has no other neighbors in $V(C)$. Thus u_0 has other neighbors $y, z \notin V(C)$, since every vertex in G has degree at least 3. By our previous argument, both y and z have neighbors in $V(C)$. Thus by ($\star$), the set $\{y, z\}$ is anticomplete to $\{v_0, v_1, v_4\}$, and y, z do not have a common neighbor in $V(C)$. Thus, by symmetry, we may assume that y is adjacent to v_2, and z is adjacent to v_3. But then $V(C) \cup \{u_0, y, z\}$ induces a copy of H_8 in G, contradicting (9).

This proves (10).

(11) *G does not contain an induced 6-cycle.*

Suppose that G contains an induced 6-cycle $C = v_0{-}v_1{-}v_2{-}v_3{-}v_4{-}v_5{-}v_0$. Since every vertex in G has degree at least 3, there are vertices $u_1, u_2 \notin V(C)$ where u_1 is adjacent to v_0, and u_2 is adjacent to v_2. From ($\star$) and (10), we conclude that u_1, u_2 are distinct non-adjacent vertices that have no other neighbors in $V(C)$. Thus u_1 has another neighbor $u_3 \notin V(C)$, since every vertex in G has degree at least 3. We deduce by ($\star$) and (10) that u_3 is anticomplete to $V(C) \setminus \{v_3\}$. If u_3 is also non-adjacent to v_3, then either $u_3-u_1-v_0-v_5-v_4-v_3-v_2-u_2$ induces a P_8 in G, or $u_3-u_1-v_0-v_5-v_4-v_3-v_2-u_2-u_3$ is an induced 8-cycle, contradicting (5). This shows that u_3 is adjacent to v_3. Repeating the argument for u_2 instead of u_1, we find a vertex $u_4 \notin V(C)$ adjacent to u_2, v_5 and no other vertex in $V(C)$. Moreover, u_4 is anticomplete to $\{u_1, u_3\}$, and u_3 is non-adjacent to u_2 by ($\star$) and (10). But this means that $v_0-u_1-u_3-v_3-v_2-u_2-u_4-v_5-v_0$ is an induced 8-cycle in G, contradicting (5). This proves (11).

(12) *G is a tree.*

Suppose that G contains an induced k-cycle $C = v_0-v_1-\cdots-v_{k-1}-v_0$. If $k \geq 9$, then $v_0-v_1-\cdots-v_7$ is an induced P_8 in G, a contradiction. Thus $k \leq 8$ which by ($\star$), (10), (11), and (5) implies that $k = 7$. Now since every vertex in G has degree at least 3, let $u_1 \notin V(C)$ be a neighbor of v_0, and let $u_2 \notin V(C)$ be a neighbor of v_5. By ($\star$), (10), and since $k = 7$, we see that u_1 and u_2 are distinct non-adjacent vertices with no other neighbors in $V(C)$. This implies that $u_1-v_0-v_1-v_2-v_3-v_4-v_5-u_2$ is an induced P_8 in G, a contradiction. This proves (12).

By (12), we find that G is a tree and so contains a vertex of degree 1, a contradiction to our assumption that every vertex in G has degree at least 3. This completes the proof. $\square$

4 No 3- and 5-cycles

In this section, we prove Theorem 2.

Since a vertex-minimal non-3-colorable graph cannot contain two vertices one of which dominates the other, and since C_7 is 3-colorable, it is enough to prove the first assertion of the theorem.

Let G be a connected P_8-free graph with no triangles and no induced 5-cycles. Observe that since G has no triangles, every 5-cycle in G is induced, and so in fact G has no C_5-subgraph. We may assume that no vertex of G dominates another vertex of G, i.e. there are no vertices $u, v \in V(G)$ such that $N(u) \subseteq N(v)$. Moreover, we may assume that G is not bipartite, since any bipartite graph admits a homomorphism to K_2 which is a subgraph of the 7-cycle.

(13) *G does not contain a 7-cycle anticomplete to two adjacent vertices (see Figure 7).*

Suppose that G contains an induced copy of the graph shown in Figure 7. Namely, let $C = v_0-v_1-\ldots-v_6-v_0$ be a 7-cycle and x, y be adjacent vertices anticomplete to it. Since G is connected, there exists a path from a vertex in $V(C)$ to one of x, y. Let $P = u_1-u_2\ldots-u_k$ be a shortest such path, where $u_1 \in V(C)$ and $u_k \in \{x, y\}$. Note that $k \geq 3$, since the cycle is anticomplete to $\{x, y\}$. Thus u_2 exists and has at least one neighbor in $V(C)$. Let us examine the neighbors of u_2 in $V(C)$. If u_2 is adjacent to v_i, then u_2 is anticomplete to $\{v_{i-1}, v_{i+1}, v_{i-3}, v_{i+3}\}$, since G contains no triangle and no 5-cycle. Therefore u_2 is additionally only adjacent to exactly one of v_{i-2}, v_{i+2}. This shows that

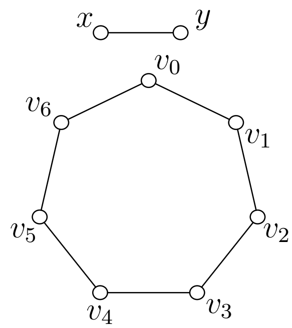

Figure 7: Case 1 of the proof of Theorem 2

there is an index i such that u_2 is adjacent to v_i and anticomplete to $\{v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\}$. Similarly, u_{k-1} exists and is adjacent to one of x, y. Since G contains no triangles, u_{k-1} is neither complete nor anticomplete to $\{x, y\}$. By symmetry, we may assume that u_{k-1} is adjacent to x and not to y. Put together, since the path P is shortest possible, we conclude that $v_{i+4}-v_{i+3}-v_{i+2}-v_{i+1}-v_i-u_2-\ldots-u_{k-1}-x-y$ is an induced path of length ≥ 7 which therefore contains P_8, a contradiction. This proves (13).

A *template* $W \subseteq V(G)$ consists of non-empty sets $X_0, X_1, \ldots, X_6 \subseteq V(G)$ with the following properties (index arithmetic modulo 7):

(W1) each X_i is a stable set,

(W2) X_i is anticomplete to X_j whenever $2 \leq |i-j| \leq 5$,

(W3) each $v \in X_i$ has both a neighbor in X_{i+1} and a neighbor in X_{i-1}.

Observe that if W is a template in G, then $G[W]$ admits a homomorphism to a 7-cycle $v_0-v_1-\ldots-v_6-v_0$ by simply mapping each $v \in X_i$ to v_i, for all i.

Thus if we show that $W = V(G)$ is a template in G, then the theorem will be proved. Since G is not bipartite, it contains an odd cycle. Since G is P_8-free and contains no triangle and no 5-cycle, it follows that G contains a 7-cycle. In particular, G contains a template. Thus, for contradiction, consider a largest template W in G, and assume $W \neq V(G)$.

(14) *If $x \in V(G) \setminus W$ has a neighbor in X_i, then x is anticomplete to X_{i+1}.*

For contradiction and by symmetry, assume that $x \in V(G) \setminus W$ has a neighbor v_0 in X_0, and a neighbor $u_1 \in X_1$. Note that v_0 is not adjacent to u_1 or otherwise $\{x, v_0, u_1\}$ is a triangle. Thus by (W3), v_0 has a neighbor $v_1 \in X_1 \setminus \{u_1\}$, and u_1 has a neighbor $u_0 \in X_0 \setminus \{v_0\}$. We conclude that x is anticomplete to $\{u_0, v_1\}$, since G has no triangles. Moreover, u_0 is not adjacent to v_1, or otherwise $x-v_0-v_1-u_0-u_1-x$ is a 5-cycle. By (W3), let $v_2 \in X_2$ be any neighbor of v_1. Note that v_2 is non-adjacent to u_1, since otherwise $x-u_1-v_2-v_1-v_0-x$ is a 5-cycle. Next let $v_6 \in X_6$ be a neighbor of v_0; we conclude that v_6 is non-adjacent to u_0, or otherwise $x-v_0-v_6-u_0-u_1-x$ is a 5-cycle.

Now, by (W3), let $v_3 \in X_3$ be a neighbor of v_2, let $v_5 \in X_5$ be a neighbor of v_6, let $v_4 \in X_4$ be a neighbor of v_3, and let $w_4 \in X_4$ be a neighbor of v_5. If possible, choose $v_4 = w_4$. Thus if $v_4 \neq w_4$, we conclude that v_4 is non-adjacent to v_5, and w_4 is non-adjacent to v_3. So $v_4-v_3-v_2-v_1-v_0-v_6-v_5-w_4$ is an induced P_8. Therefore $v_4 = w_4$ and $v_0-v_1-\ldots-v_6-v_0$ is a

7-cycle anticomplete to $\{u_0, u_1\}$ where u_0u_1 is an edge, contradicting (13). This proves (14).

(15) *If $x \in V(G) \setminus W$ has a neighbor in X_i, then x is anticomplete to X_{i+3}.*

For contradiction and by symmetry, suppose that $x \in V(G) \setminus W$ has a neighbor in $v_0 \in X_0$ and a neighbor $u_3 \in X_3$.

By (W3), let $u_2 \in X_2$ be any neighbor of u_3, let $u_1 \in X_1$ be any neighbor of u_2, and let $u_0 \in X_0$ be any neighbor of u_1. Observe that $u_0 \neq v_0$ and u_1 is non-adjacent to v_0, since otherwise $x–u_3–u_2–u_1–v_0–x$ is a 5-cycle. Similarly, note that x is non-adjacent to u_0, since otherwise $x–u_0–\ldots u_3–x$ is a 5-cycle. Moreover, x is anticomplete to $\{u_1, u_2\}$ by (14). If there are vertices $y, z \in X_6$ such that $y \in N(u_0) \setminus N(v_0)$ and $z \in N(v_0) \setminus N(u_0)$, then we conclude that x is anticomplete to $\{y, z\}$ by (14), and so $y–u_0–u_1–u_2–u_3–x–v_0–z$ is an induced P_8. Therefore such vertices do not exist which proves that u_0, v_0 have a common neighbor $w_6 \in X_6$. By (14), w_6 is not adjacent to x.

Now by renaming the template (X_0 switches place with X_3) and repeating the above paragraph, we conclude that there exists $v_3 \in X_3 \setminus \{u_3\}$ and $w_4 \in X_4$, where x is anticomplete to $\{v_3, w_4\}$, and w_4 is a common neighbor of u_3 and v_3. This yields an induced path $v_3–w_4–u_3–x–v_0–w_6–u_0–u_1$, a contradiction. This proves (15).

We are now ready to derive a contradiction. Consider any $x \in V(G) \setminus W$. Choose x so that it has neighbors in largest number of sets X_i possible. Since G is connected, x has at least one neighbor in W. By symmetry, we may assume that x has a neighbor in X_0. Then by (14), x is anticomplete to $X_1 \cup X_6$. Moreover, by (15), x is anticomplete to $X_3 \cup X_4$, and at least one of X_2, X_5. By symmetry, we may assume that x is anticomplete to X_2. If x has a neighbor in X_5, then x can be safely added to X_6, since it is anticomplete to $X_1 \cup \cdots \cup X_4 \cup X_6$, has at least one neighbor in X_0 and one neighbor in X_5. Then $W \cup \{x\}$ is a larger template, impossible.

We therefore conclude that x is also anticomplete to X_5. Recall that x has a neighbor $v_0 \in X_0$. By (W3), v_0 has a neighbor $v_1 \in X_1$. Since v_1 does not dominate x, there exists $w \in N(x) \setminus N(v_1)$. Note that w is not in $X_1 \cup \cdots \cup X_6$, since x is anticomplete to these sets; w may possibly be in X_0. We claim that w is anticomplete to $X_3 \cup X_4$. If $w \in X_0$, this follows from the definition of a template, so we may assume that $w \notin W$. Now w has neighbors in at most one set X_i, by the choice of x (since x only has neighbors in X_0). If w has neighbors in X_3, we can safely enlarge W by putting x in X_1 and w in X_2. Similarly, if w has neighbors in X_4, we put x in X_6 and w in X_5. Therefore w is anticomplete to $X_3 \cup X_4$, and the claim follows.

Now, note that $w \neq v_0$, since w is non-adjacent to v_1, while v_0 is. By (W3), let $v_2 \in X_2$ be any neighbor of v_1, let $v_3 \in X_3$ be any neighbor of v_2, let $v_4 \in X_4$ be any neighbor of v_3, and finally, let $v_5 \in X_5$ be any neighbor of v_4. Note that x is anticomplete to $\{v_1, \ldots, v_5\}$. Consider the path $x–v_0–\ldots–v_5$. Since w is adjacent to x, it follows that w is anticomplete to $\{v_0, v_2\}$, since G has no triangles and no 5-cycles. Moreover, w is anticomplete to $\{v_3, v_4\}$, since w is anticomplete to $X_3 \cup X_4$. Thus, since $w–x–v_0–\ldots–v_5$ is not an induced P_8, we conclude that w is adjacent to v_5. In particular, w has neighbors in X_5 and so it has no neighbors in any other set X_i, by the choice of x. But now consider a different path. By

(W3), let $u_6 \in X_6$ be any neighbor of v_0, let $u_5 \in X_5$ be any neighbor of u_6, let $u_4 \in X_4$ be any neighbor of u_5, let $u_3 \in X_3$ be any neighbor of u_4, and finally let $u_2 \in X_2$ be any neighbor of u_3. Note that w is anticomplete to $\{u_2, u_3, u_4, u_6\}$, since it is anticomplete to $W \setminus X_5$. Moreover, w is non-adjacent to u_5, since otherwise w–x–v_0–u_6–u_5–w is a 5-cycle. But now w–x–v_0–u_6–u_5–u_4–u_3–u_2 is an induced P_8, a contradiction.

We therefore conclude that no such a vertex x exists and thus $W = V(G)$ is indeed a template. This proves that G is homomorphic to a 7-cycle, as promised. □

5 No 4- and 5-cycles

In this section, we prove Theorem 3.

The forward direction (i)⇒(ii) of Theorem 3 follows immediately by verifying that none of the graphs in Figure 2 is 3-colorable. So it remains to prove (ii)⇒(i).

Consider a smallest counterexample, namely a smallest 4-chromatic graph G with no induced P_8, no induced 4- and 5-cycles, and containing none of the graphs in Figure 2 as a subgraph. In the following series of claims we will prove that such a graph G does not exist. This will prove (ii)⇒(i).

We start by noting that G is a connected graph and has minimum degree 3.

(16) *G is connected and every vertex of G has degree at least 3.*

Clearly, if G is disconnected, then all its connected components are smaller than G so they are 3-colorable by the minimality of G, and so is G. If a vertex $v \in V(G)$ has degree 2 or less, then there is by the minimality of G a 3-coloring of $G - v$ which can be completed to a 3-coloring of G, since there is always an available color for v, a contradiction.

This proves (16).

The following lemma will be used many times to analyze the structure of G.

(17) *Let a–b–c–d be a path in G such that $ac, bd \notin E(G)$. Then also $ad \notin E(G)$. Moreover, if $x \in N(d) \setminus \{c\}$ is adjacent to a or b, then x is complete to $\{b, c\}$.*

If $ad \in E(G)$, then $\{a, b, c, d\}$ induces a 4-cycle in G. Thus $ad \notin E(G)$. Now consider $x \in N(d) \setminus \{c\}$ such that x is not complete to $\{b, c\}$. If x is adjacent to b, then x is non-adjacent to c and hence x–b–c–d–x is an induced 4-cycle in G. Thus x is non-adjacent to b. If x is adjacent to both a and c, then x–a–b–c–x is an induced 4-cycle in G. If x is adjacent to a and non-adjacent to c, then a–b–c–d–x–a is an induced 5-cycle in G, a contradiction. Therefore x is also non-adjacent to a. This proves (17).

A set $S \subseteq V(G)$ is *connected* if $G[S]$ is a connected graph. A *star cutset* of G is a cutset S where $S \subseteq N(v) \cup \{v\}$ for some $v \in S$. We say that (S, v) is *2-connected* if $S \setminus \{v\}$ is connected (please note that this is slightly different from the usual definition of 2-connectivity).

(18) *G has no 2-connected star cutset.*

Let S be a star cutset of G where $v \in S$ is such that $S \subseteq N(v) \cup \{v\}$ and (S, v) is 2-connected. Let K be a connected component of $G - S$. The minimality of G implies that $G[V(K) \cup S]$

is 3-colorable. Likewise $G - K$ is 3-colorable. (Both subgraphs are strictly smaller than G.) The colorings of the two subgraphs agree on $S \setminus \{v\}$ (up to exchanging colors), since only two colors appear there (the third color is assigned to v), and $S \setminus \{v\}$ is connected. Thus by possibly permuting the colors we can match the two colorings on S to produce a 3-coloring of G, contradicting our choice of G. This proves (18).

A set $S \subseteq V(G)$ is *homogeneous* if $V(G) \setminus S$ splits into two sets A, B where S is complete to A and anticomplete to B. A homogeneous set is non-trivial if $1 < |S| < |V(G)|$.

(19) *Every non-trivial homogeneous set of G consists of two adjacent vertices.*

Let S be a non-trivial homogeneous set of G. Let A, B be the partition of $V(G) \setminus S$ where S is complete to A and anticomplete to B. Since S is non-trivial and G is connected, we conclude that A is non-empty. Let a be in A. Since G has no K_4, no induced C_5, W_7, and C_9 or larger induced cycle, it follows that $G[S]$ is bipartite. If S has no edges, then for any v in S, we can 3-color $G - v$ by the minimality of G, and then assign v a color of any vertex in $S \setminus \{v\}$ to produce a 3-coloring of G. Thus S has has adjacent vertices x, y. This implies that A is stable, since G contains no K_4. If $|S| > 2$, then we 3-color $G - (S \setminus \{x, y\})$ by the minimality of G. In this coloring, all vertices of A have the same color; we assign the remaining two colors to the bipartition of S to complete the coloring. Thus S consists of two adjacent vertices, as claimed. This proves (19).

(20) *Let $C = v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_0$ be an induced 7-cycle in G. Let X denote the set of all vertices in $V(G) \setminus V(C)$ with at least one neighbor in $V(C)$. Let Y be the remaining vertices, i.e. $Y = V(G) \setminus (X \cup V(C))$. Then the following is true.*

- (20a) *Each $x \in X$ has at most 4 neighbors in $V(C)$ and the neighbors are consecutive along the cycle.*
- (20b) *Each $x \in X$ with a neighbor $y \in Y$ has 3 or 4 consecutive neighbors in $V(C)$.*
- (20c) *$Y = \emptyset$.*

Let x be in X. Since G has no induced W_7 and since $x \in X$, it follows that x has both a neighbor and a non-neighbor in $V(C)$. We may assume that x is adjacent to v_0 and non-adjacent to v_1. Thus x is non-adjacent to v_2, and to v_3, since G has no induced C_4, or C_5. Thus x has at most 4 neighbors in $V(C)$, among v_4, v_5, v_6, v_0. The neighbors must be consecutive, since G has no induced C_4, C_5. This proves (20a).

Suppose further that x has a neighbor y in Y. Since $y–x–v_0–v_1–v_2–v_3–v_4–v_5$ is not an induced P_8, it follows that x must be adjacent to at least one of v_4, v_5. But since the neighbors of x are consecutive, x is adjacent to v_5, v_6 and possibly to v_4. This proves (20b).

For (20c), consider a connected component K of $G[Y]$. Let $S = N(K)$, the neighborhood of K. By the definition of X and K, we conclude that S is a subset of X. By (18), S is not a clique (note that S separates K from C). Let x, w be non-adjacent vertices in S. Since $S = N(K)$, there exists an xw-path P whose all internal vertices are in $V(K)$. Consider shortest such path $x = a_0 – a_1 – \ldots – a_k = w$. Note that P is induced, by the minimality. Also, $k \geq 2$ since x, w are non-adjacent. By (20b), we may assume that x is adjacent to v_5, v_6, v_0, possibly to v_4, and anticomplete to v_1, v_2, v_3. If $k \geq 4$, then $v_3–v_2–v_1–v_0–x–a_1–a_2–a_3$ is an induced P_8 (since a_1, a_2, a_3 are in K in that case).

Suppose that $k = 3$. Then x is adjacent to v_4, since otherwise $v_4 - v_3 - v_2 - v_1 - v_0 - x - a_1 - a_2$ is an induced P_8. Since G has no induced C_5, we conclude that w is anticomplete to $\{v_4, v_5, v_6, v_0\}$, or else $v_i - x - a_1 - a_2 - w - v_i$ is an induced C_5 for some i in $\{0, 4, 5, 6\}$. By (20b), it follows that w is complete to $\{v_1, v_2, v_3\}$ and has no other neighbors in $V(C)$. But now $a_1 - a_2 - w - v_3 - v_4 - v_5 - v_6 - v_0$ is an induced P_8.

It follows that $k = 2$. Since G has no induced C_4, C_5, we conclude that w is anticomplete to $\{v_4, v_5, v_6, v_0, v_1\}$ and possibly also to v_3 if x is adjacent to v_4. But then w can only have 1 or 2 neighbors in $V(C)$, contradicting (20b). Therefore we must conclude that P cannot exist, and neither could x, w. This proves (20).

(21) *Let $C = v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 - v_0$ be an induced 8-cycle in G. Let $x \in V(G) \setminus V(C)$. Then either*

- *(i) x is complete or anticomplete to $V(C)$, or*
- *(ii) x has exactly 3, exactly 4, or exactly 5 neighbors in $V(C)$ and they are consecutive along the cycle, or*
- *(iii) $N(x) \cap V(C) = \{v_i, v_{i+4}\}$ for some i (indices modulo 8).*

Let $x \in V(G) \setminus V(C)$. If x is complete or anticomplete to $V(C)$, we obtain outcome (i). Thus by symmetry we may assume that x is adjacent to v_0 and non-adjacent to v_1. By (17) applied to $v_3 - v_2 - v_1 - v_0$ we deduce that x is also anticomplete to $\{v_2, v_3\}$. Suppose x is non-adjacent to v_6, Since $G[V(C) \setminus \{v_7\} \cup \{x\}]$ is not P_8, it follows that x has a neighbor in $\{v_4, v_5\}$. By (17) applied to x and $v_5 - v_6 - v_7 - v_0$, we conclude that x is non-adjacent to v_5. So x is adjacent to v_4, and by (17) applied to $v_7 - v_6 - v_5 - v_4$ and x, we deduce that x is non-adjacent to v_7; consequently outcome (iii) holds. This proves that x is adjacent to v_6. Since G contains no induced C_4, it follows that x is adjacent to v_7, and that x is adjacent to v_4 only if x is adjacent to v_5, and outcome (ii) holds. This proves (21).

(22) *Let $C = v_0 - \dots v_k - v_0$ be an induced cycle where $k \in \{6, 7, 8\}$ and let $u, v \in V(G) \setminus V(C)$. Then there is no i such that $\{u, v\}$ is complete to $\{v_i, v_{i+2}\}$ (indices modulo k).*

Suppose that such i exists. Since there is no induced C_4 in G, it follows that u, v, v_{i+1} are pairwise adjacent. But now $\{u, v, v_i, v_{i+1}\}$ is a K_4, a contradiction. This proves (22).

(23) *Let $C = v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$ be an induced 6-cycle in G. Let q be a vertex complete to $V(C)$. Then*

- (23a) *each $x \in V(G) \setminus (V(C) \cup \{q\})$ has at most 2 neighbors in $V(C)$ and they are consecutive along the cycle.*
- (23b) *$G[N(q)]$ is connected, and $G[N(q) \setminus \{v_i\}]$ is connected for every $i \in \{0, \dots, 5\}$ with $N(v_i) \subseteq N(q) \cup \{q\}$.*
- (23c) *$G - (N(q) \cup \{q\})$ is connected and has at least one vertex.*
- (23d) *$N(v_i) \not\subseteq N(q) \cup \{q\}$ for all $i \in \{0, 1, \dots, 5\}$.*

To prove (23a), consider a vertex $x \notin V(C) \cup \{q\}$ with a neighbor in $V(C)$. By symmetry, assume that x is adjacent to v_0. By (22), x is anticomplete to $\{v_2, v_4\}$. Thus x is non-adjacent

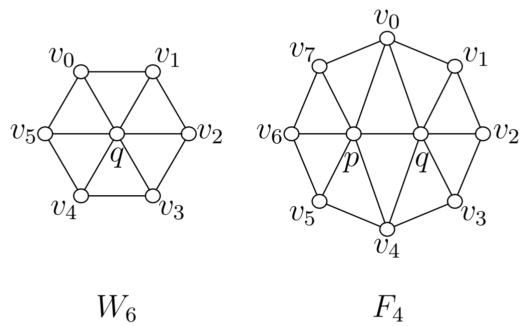

Figure 8: Excluded subgraphs.

to v_3 by (17) applied to $v_3-v_2-v_1-v_0$. So $N(x) \cap V(C) \subseteq \{v_5, v_0, v_1\}$ and furthermore, x is non-adjacent to at least one of v_1, v_5 by (22). This proves (23a).

To prove (23b), suppose for a contradiction that either $G[N(q)]$ is not connected, or that there is i such that $N(v_i) \subseteq N(q) \cup \{q\}$ and $G[N(q) \setminus \{v_i\}]$ is not connected. In the former case, let $B = \{q\}$ and in the latter let $B = \{v_i, q\}$. Note that $V(C) \setminus B$ is contained in one connected component of $G[N(q) \setminus B]$; let K denote this component. Observe that $G - B$ is connected for otherwise (B, q) is a 2-connected star cutset in G, contradicting (18). Consider a shortest path $P = x_1-x_2-\ldots-x_l$ in $G \setminus B$ from a vertex of $V(C) \setminus B$ to a vertex of $G[N(q)] - V(K)$. Note that P is an induced path and q is not complete to $V(P)$. Since q is complete to the ends of P and G contains no induced C_4, C_5, it follows that $l \geq 5$. The minimality of P implies that $x_j \notin V(C)$ for all $j \geq 2$ and moreover, $V(C) \setminus B$ is anticomplete to $\{x_3, x_4, \ldots, x_l\}$. Since $x_1 \in V(C)$, the vertex x_2 has a neighbor in $V(C)$, and therefore by (23a), we may assume by symmetry that x_2 is adjacent to $v_0 = x_1$, possibly to v_5, and otherwise has no other neighbors in $V(C)$. This implies $i \neq 0$ (if v_i exists). If v_i does not exist or if $i \in \{3, 4, 5\}$, choose $Q = x_2-v_0-v_1-v_2$, while if $i \in \{1, 2\}$ choose $Q = x_2-v_0-v_5-v_4$ if $x_2v_5 \notin E(G)$ and otherwise choose $Q = x_2-v_5-v_4-v_3$. Observe that Q does not contain v_i (if i exists). If $l \geq 6$, it follows that $x_6-x_5-\ldots x_2-Q$ is an induced P_8. Thus we conclude that $l = 5$ and so q is anticomplete to $\{x_2, x_3, x_4\}$. If v_i does not exist or if v_i is anticomplete to $\{x_2, x_3, x_4\}$, then $x_4-x_3-x_2-v_0-v_1-v_2-v_3-v_4$ is an induced P_8. Thus v_i must exist and must have a neighbor in $\{x_2, x_3, x_4\}$, contrary to the fact that $N(v_i) \subseteq N(q) \cup \{q\}$. This proves (23b).

To prove (23c), let $G' = G - (N(q) \cup \{q\})$. By (23b) $N(q)$ is connected. Since $|N(q)| \geq 6$, (19) implies that $N(q)$ is not a homogeneous set, and so G' has at least one vertex. Moreover, by (18) $(N(q) \cup \{q\}, q)$ is not a 2-connected star cutset, and so G' is connected. This proves (23c).

Finally, to prove (23d), suppose for a contradiction (and by symmetry) that $N(v_0) \subseteq N(q) \cup \{q\}$. Note that $V(G) \setminus (N(q) \cup \{q\})$ is non-empty by (23c). By (23b) v_0 is not a cutvertex of $G[N(q)]$. But now $(N(q) \cup \{q\} \setminus \{v_0\}, q)$ is a 2-connected star cutset separating v_0 from $V(G) \setminus (N(q) \cup \{q\})$, contrary to (18). This proves (23d). This proves (23).

(24) *G contains no induced double-6-wheel F_4 (see Figure 8).*

Suppose that G contains an induced copy of F_4, labeled as in Figure 8. By (23d) applied to the 6-cycle $v_0–v_1–v_2–v_3–v_4–p–v_0$, the vertex v_2 has a neighbor x non-adjacent to q. By (23a) it follows that x is non-adjacent to p, v_0, v_4. By (17) applied to x and $x–v_2–q–p–v_6$, we deduce that x is non-adjacent to v_6. Since (22) implies that x is not complete to $\{v_1, v_3\}$, we get a contradiction to (21) applied to $v_0–v_1–v_2–v_3–v_4–v_5–v_6–v_7–v_0$ and x. This proves (24).

(25) *Let $C = v_0 – v_1 – v_2 – v_3 – v_4 – v_5 – v_0$ be an induced 6-cycle in G. Let q be a vertex complete to $V(C)$. Then there is no index i such that the vertices v_i, v_{i+2} (indices modulo 6) are connected by an induced path of length 4 whose internal vertices are anticomplete to q.*

For a contradiction, we may assume by symmetry that there is an induced path $P = v_0–a_0–a_1–a_2–v_2$ between v_0 and v_2 where $\{a_0, a_1, a_2\}$ is anticomplete to q. By (17) applied to a_1 and $v_i–q–v_0–a_0$ where $i \in \{2, 3, 4\}$, we deduce that $\{a_0, a_1\}$ is anticomplete to $\{v_2, v_3, v_4\}$. By symmetry, $\{a_1, a_2\}$ is anticomplete to $\{v_0, v_4, v_5\}$.

Suppose first that v_1 is adjacent to a_0. By (23a) a_0 is anticomplete to $\{v_2, v_3, v_4, v_5\}$. Then (17) applied to v_1 and $a_0–a_1–a_2–v_2$ yields that v_1 is complete to $\{a_0, a_1, a_2\}$. Again by (23a) a_2 is anticomplete to $\{v_3, v_4, v_5\}$. Since there is no K_4 in G and by (23a), it follows that $N(a_1) \cap V(C) = \{v_1\}$. But now the vertices $\{v_0, v_1, v_2, v_3, v_4, v_5, a_0, a_1, a_2, q\}$ induce a copy of F_4 in G, contradicting (24). So v_1 is not adjacent to a_0, and by (17) applied to $a_2–a_1–a_0–v_0$, we deduce that v_1 is anticomplete to $\{a_0, a_1, a_2\}$.

Now suppose that $v_5 \notin N(a_0)$ and $v_3 \notin N(a_2)$. By (23d), v_3 has a neighbor $y \notin N(q)$. By (22), y is anticomplete to $\{v_1, v_5\}$, and not complete to $\{v_2, v_4\}$. By (17) applied to y and $a_0–v_0–q–v_3$, we deduce that y is non-adjacent to a_0. Note that $C' = v_0–a_0–a_1–a_2–v_2–v_3–v_4–v_5–v_0$ is an induced 8-cycle. It follows by (21) that y has exactly 3, 4, or 5 consecutive neighbors on C'. Since y is adjacent to at most one of v_2, v_4 and is non-adjacent to v_5, we conclude that y is complete to $\{v_2, v_3, a_2\}$ and anticomplete to $\{v_0, v_4, v_5, a_0\}$. Moreover, y is also adjacent to a_1, since otherwise $v_1–v_0–a_0–a_1–a_2–y–v_3–v_4$ is an induced P_8 in G. Now by (23d), the vertex v_4 has a neighbor z non-adjacent to q. By (23a), z is anticomplete to $\{v_0, v_1, v_2\}$ and one of v_3, v_5. Thus (21) applied to C' and z implies that z is adjacent to a_1 and has no other neighbor in $V(C')$. In particular, z is non-adjacent to v_3. But then we get a contradiction applying (17) to z and the path $a_1–y–v_3–v_4$.

This shows that either $v_5 \in N(a_0)$ or $v_3 \in N(a_2)$ or both. By symmetry we may assume that $v_5 \in N(a_0)$. By (23d), v_4 has a neighbor z non-adjacent to q. By (17) applied to $a_0–v_0–q–v_4$, we deduce that z is anticomplete to $\{a_0, v_0\}$. By the same token, z is anticomplete to $\{v_1, v_2, a_2\}$. Moreover, z is non-adjacent to a_1 by (17) applied to $a_1–a_0–v_5–v_4$. It follows that $N(z) \cap \{v_0, \dots, v_5, a_0, a_1, a_2\} \subseteq \{v_3, v_4, v_5\}$, and z is not complete to $\{v_3, v_5\}$ by (22). Now if z is non-adjacent to v_5, then $z–v_4–v_5–a_0–a_1–a_2–v_2–v_1$ is an induced P_8 in G, a contradiction. This shows that z is adjacent to v_5 and so z is non-adjacent to v_3. Reversing the roles of a_0 and a_2, we deduce that a_2 is non-adjacent to v_3, and $z–v_4–v_3–v_2–a_2–a_1–a_0–v_0$ is an induced P_8 in G, a contradiction. This proves (25).

(26) *G contains no induced 6-wheel W_6 (see Figure 8).*

Suppose that G contains an induced copy of W_6, labeled as in Figure 8, where $C = v_0–v_1–v_2–v_3–v_4–v_5–v_0$ is the 6-cycle that is complete to q. By (23d), each vertex on C

has a neighbor in $G - (N(q) \cup \{q\})$ which is a connected graph by (23c). Thus every two vertices in C are connected by path whose internal vertices are anticomplete to q. This allows us to consider a shortest path $P = x_1 - x_2 - \ldots - x_l$ whose endpoints are v_i and v_{i+2} (modulo 6) for some value of i. We may assume that $x_1 = v_0$ and $x_l = v_2$. Since q is complete to the ends of P and anticomplete to the interior vertices of P, it follows that $l \geq 5$ (for otherwise G contains C_4 or C_5). By (25), it follows that $l \geq 6$. By the minimality of l, v_4 is anticomplete to $V(P)$. If v_5 is anticomplete to $\{x_2, \ldots, x_{l-1}\}$, then $v_4 - v_5 - x_1 - x_2 - \ldots - x_{l-1} - v_2$ is an induced path of length at least 8 in G, a contradiction. By symmetry it follows that both v_5 and v_3 have neighbors in $\{x_2, \ldots, x_{l-1}\}$. By (23a) and the minimality of l, it follows that v_5 is adjacent to x_2, v_3 is adjacent to x_{l-1} and there are no other edges between $\{v_3, v_5\}$ and $\{x_2, \ldots, x_{l-1}\}$. Since $v_4 - v_5 - x_2 - \ldots - x_{l-1} - v_2$ is not an induced path of length at least 8 in G, it follows that $l = 6$, and therefore $D = v_3 - v_4 - v_5 - x_2 - x_3 - x_4 - x_5 - v_3$ is an induced 7-cycle in G. It follows from the minimality of l that v_1 is anticomplete to $\{x_2, \ldots, x_{l-1}\}$, and so v_1 is anticomplete to $V(D)$, contrary to (20c). This proves (26).

(27) *Let $C = v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_0$ be an induced 6-cycle in G. Then every vertex $x \in V(G) \setminus V(C)$ has at most 3 neighbors in $V(C)$ and the neighbors are consecutive.*

Let $x \in V(G) \setminus V(C)$. We may assume that x has at least two neighbors in $V(C)$. If x is complete to $V(C)$, then $V(C) \cup \{x\}$ induces a 6-wheel W_6 contrary to (26). Thus x is adjacent to v_i and non-adjacent to v_{i+1} for some i. Then by (17) applied to $v_i - v_{i+1} - v_{i+2} - v_{i+3}$, we conclude that x is anticomplete to $\{v_{i+2}, v_{i+3}\}$. If x is adjacent to v_{i-2} but not to v_{i-1}, then $x - v_i - v_{i-1} - v_{i-2} - x$ is an induced 4-cycle in G. In all other cases, x has at most 3 consecutive neighbors in $V(C)$. This proves (27).

Let $C = v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_0$ be an induced 7-cycle in G. For $i \in \{0, 1, \ldots 6\}$, let L_i denote the set of *leaves* at v_i, i.e., vertices x with $N(x) \cap V(C) = \{v_i\}$. Let H_i be the set of *hats* opposite v_i, i.e., vertices x with $N(x) \cap V(C) = \{v_{i+3}, v_{i-3}\}$. A *clone* at v_i is a vertex x with $N(x) \cap V(C) = \{v_{i-1}, v_i, v_{i+1}\}$.

Next we prove a few properties of leaves and hats.

(28) *Let $C = v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_0$ be an induced 7-cycle in G. Let $a \in L_0 \cup H_4$ and $b \in L_1 \cup L_2 \cup H_5$. Then*

- *If $b \in L_1$, then $ab \notin E(G)$.*
- *If $b \in L_2 \cup H_5$, then $ab \in E(G)$, $a \in H_4$, and $b \in H_5$.*

For the first claim, suppose that $b \in L_1$ and $ab \in E(G)$. Then $b - a - v_0 - v_6 - v_5 - v_4 - v_3 - v_2$ is an induced P_8. This proves first claim. For the second claim, suppose that $b \in L_2 \cup H_5$. If $ab \notin E(G)$, then $b - v_2 - v_3 - v_4 - v_5 - v_6 - v_0 - a$ is an induced P_8. Thus $ab \in E(G)$. This implies $a \in H_4$ and $b \in H_5$ or else we have an induced C_4 or C_5. This proves (28).

(29) *Let $C = v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_0$ be an induced 7-cycle in G, let $h \in H_5$, and let $y \in V(G) \setminus V(C)$ be adjacent to h. Then one of the following holds:*

- $\{v_2, v_3\} \subseteq N(y) \cap V(C) \subseteq \{v_2, v_3, v_4, v_5\}$
- $\{v_0, v_1\} \subseteq N(y) \cap V(C) \subseteq \{v_0, v_1, v_5, v_6\}$

– $y \in L_5$

Since G has no K_4, it follows that y is not complete to $\{v_1, v_2\}$. If y is adjacent to v_2, then y has up to 4 consecutive neighbors among $\{v_2, v_3, v_4, v_5\}$ by (20a). From (28), we conclude that y must be adjacent to v_3 and so the first outcome of (29) holds. Thus we may assume that y is non-adjacent to v_2. By symmetry, if y is adjacent to v_1, we obtain the second outcome. So y is also non-adjacent to v_1. Applying (17) to $h{-}v_2{-}v_3{-}v_4$ and to $h{-}v_1{-}v_0{-}v_6$, we deduce that y is anticomplete to $\{v_3, v_4, v_6, v_0\}$. Thus y is adjacent to v_5 by (20c) which yields the third outcome. This proves (29).

(30) *Let $C = v_0 - v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_0$ be an induced 7-cycle in G, let $l \in L_1$, and let $y \in V(G) \setminus V(C)$ be adjacent to l. If y is non-adjacent to v_1, then $y \in H_1 \cup L_4 \cup L_5$.*

By (20c) y has a neighbor in $V(C)$. Applying (17) to the path $l{-}v_1{-}v_2{-}v_3$, we deduce that y is non-adjacent to v_2, v_3. By symmetry y is non-adjacent to v_6, v_0, and the claim follows. This proves (30).

(31) *Let $C = v_0 - \ldots - v_6 - v_0$ be an induced 7-cycle in G. Let u_0 be a clone at v_0, let l_3 be a leaf at v_3, and let l_0 be adjacent to l_3, v_0, u_0 and have no other neighbor in $V(C)$. Then there is no clone at v_4 and no clone at v_5.*

Let $C' = C \setminus \{v_0\} \cup \{u_0\}$. Suppose for a contradiction there is either a clone at v_4 or a clone at v_5. Since l_3 has degree at least 3, we conclude that l_3 has a neighbor $w \notin \{l_0, v_3\}$. Suppose first that w is non-adjacent to v_3. By (30) applied to C and C' (with l_3 and w) either w is adjacent to v_6, or w is complete to $\{v_0, u_0\}$. By (17) applied to w and the paths $l_3{-}l_0{-}x{-}v_6$ with $x \in \{u_0, v_0\}$, we deduce that w is complete to $\{l_0, v_0, u_0\}$. But now $\{u_0, v_0, l_0, w\}$ induces a K_4, a contradiction. This proves that w is adjacent to v_3.

We claim that $N(w) \cap (V(C) \cup \{u_0, l_0\}) = \{v_3\}$. Suppose first that $N(w) \cap (V(C) \cup \{u_0\}) \neq \{v_3\}$. Then by (20a) w is adjacent to at least one of v_2, v_4. By (27) applied to $v_3{-}l_3{-}l_0{-}v_0{-}v_1{-}v_2{-}v_3$, and w, it follows that w is non-adjacent to v_1. Suppose w is adjacent to v_4. By (28) applied to C, w and l_3, it follows that w is adjacent to at least one of v_2, v_5. If there is a clone t at v_4, then (since there is no K_4) w is non-adjacent to t, and therefore by (22) not to v_5. So w is adjacent to v_2 and we get a contradiction to (28) applied to $C \setminus \{v_4\} \cup \{t\}$, l_3 and w. This proves that there is no clone at v_4, and so there is a clone t at v_5. Again since there is no K_4, w is not complete to $\{v_5, t\}$, and we may assume that w is non-adjacent to v_5, and so w is adjacent to v_2, and $G[V(C) \cup \{w, t, u_0\}]$ contains F_1 as a subgraph, a contradiction. This proves that w is non-adjacent to v_4. But now we get a contradiction to (28) applied to C, l_3, and w. This shows that w is non-adjacent to v_4 and also to v_2, again by applying (28) to C, w, l_3. This proves that $N(w) \cap (V(C) \cup \{u_0\}) = \{v_3\}$. If w is adjacent to l_0, then $G[\{v_0, l_0, l_3, v_3, v_4, v_5, v_6, u_0, t, w\}]$ (where t is a clone at v_4 or at v_5) contains F_1 as a subgraph, a contradiction. So w is non-adjacent to l_0. Therefore $N(w) \cap (V(C) \cup \{u_0, l_0\}) = \{v_3\}$ as claimed.

By symmetry (swap l_0, l_3 for v_1, v_2), there is a vertex z with $N(z) \cap (V(C) \cup \{l_0, l_3\}) = \{v_2, v_3\}$. By (28) applied to C, z and w, it follows that z is non-adjacent to w. Since the degree of z is at least 3, there is $y \notin \{v_2, v_3\}$ adjacent to z. Suppose y is non-adjacent to v_3. By (29) applied to C, z and y it follows that either y is complete to $\{v_1, v_2\}$, or $N(y) \cap V(C) = \{v_6\}$. Suppose that y is complete to $\{v_1, v_2\}$. Since G has no K_4, we may assume that y is non-adjacent to v_0. Now we get a contradiction to (21) applied to v_3–z–y–v_1–v_0–v_6–v_5–v_4–v_3 and l_3. Thus y is not complete to $\{v_1, v_2\}$, and so $N(y) \cap V(C) = \{v_6\}$. By (17) applied to y–z–v_3–w, and y with z–v_3–l_3–l_0 it follows that y is anticomplete to $\{l_0, l_3, w\}$. But now y–z–v_2–v_1–v_0–l_0–l_3–w is an induced P_8, a contradiction. This proves that y is adjacent to v_3, and by (29) y is adjacent to v_4, and possibly v_5, v_6, and has no other neighbors in $V(C) \cup V(C')$. Since there is either a clone at v_4 or a clone at v_5, it follows from (20a) and (22) that y is non-adjacent to v_6. If y is adjacent to l_3, then applying the argument at of the previous paragraph to y instead of w, we get a contradiction since y is adjacent to z. Therefore we conclude that y is non-adjacent to l_3. Now by (27) applied to w and v_3–v_2–v_1–v_0–l_0–l_3–v_3 and since y–z–v_2–v_1–v_0–l_0–l_3–w is not a P_8, we deduce that y is adjacent to w. Now we get a contradiction by applying (21) to v_6 and v_0–l_0–l_3–w–y–z–v_2–v_1–v_0. This proves (31).

(32) *G contains no vertex of degree 3 with exactly one pair of neighbors that are adjacent.*

Let $x \in V(G)$ be a vertex where $N(x) = \{v_1, u_1, v_6\}$ and $v_1u_1 \in E(G)$ while $v_1v_6, u_1v_6 \notin E(G)$. Since G is minimal non-3-colorable, $G - x$ is 3-colorable. Consider a 3-coloring c of $G - x$. Since G is not 3-colorable, it follows that all 3 colors appear on the vertices v_1, u_1, v_6. By symmetry $c(v_1) = 1$, $c(u_1) = 2$, and $c(v_6) = 3$.

Let V_{13} denote the set of vertices u with $c(u) \in \{1, 3\}$. Similarly, V_{23} are the set of vertices u with $c(u) \in \{2, 3\}$. If v_1 and v_6 are in different connected components of $G[V_{13}]$, then we can switch colors 1 and 3 in the connected component of $G[V_{13}]$ that contains v_1 and then color x by 1 to produce a 3-coloring of G. Since this is not possible, consider a shortest path Q from v_1 to v_6 in $G[V_{13}]$. Note that x has no neighbors on this path, since the only neighbors of x are v_1, u_1, v_6 by the assumption of the claim. Also note that Q has odd length at least 3, since the colors 1 and 3 alternate on Q, and v_1 is non-adjacent to v_6. If Q has length 3, then x–Q–x is an induced 5-cycle. If Q has length 7 or greater, then Q contains an induced P_8. Therefore $Q = v_1$–v_2–v_3–v_4–v_5–v_6 where $c(v_2) = c(v_4) = 3$ and $c(v_3) = c(v_5) = 1$. Applying the same argument to $u_1 \in V_{23}$ in place of $v_1 \in V_{13}$, we deduce that there is also an induced path $P = u_1$–u_2–u_3–u_4–u_5–v_6 where $c(u_3) = c(u_5) = 2$ and $c(u_2) = c(u_4) = 3$. Note that the vertices $v_1, \dots, v_5$ are not necessarily distinct from $u_1, \dots, u_5$. In particular, it is possible that one of u_2, u_4 is one of v_2, v_4 (but all other vertices are distinct). Also the vertices u_2, v_2, u_4, v_4, v_6 are pairwise non-adjacent (or identical), since they all have color 3. Let $C = x$–v_1–v_2–v_3–v_4–v_5–v_6–x and $D = x$–u_1–u_2–u_3–u_4–u_5–v_6–x. By (20a) applied to C and u_1 we deduce that u_1 is non-adjacent to v_4, and so $v_4 \neq u_2$. Similarly, $u_4 \neq v_2$. We may assume that P and Q are chosen with $V(P) \cup V(Q)$ minimal.

Assume first that $u_2 \neq v_2$. If u_2 is non-adjacent to v_1, then v_1 is a hat for D, and v_2 is anticomplete to $\{x, u_2\}$, and so it follows from (29) that v_2 is adjacent to u_4, a contradiction. Thus u_2 is adjacent to v_1, and by symmetry v_2 is adjacent to u_1. Since $N(u_2) \cap V(C) \subseteq \{v_1, v_3, v_5\}$ it follows that u_2 is a leaf for C, and so $N(u_2) \cap V(C) = \{v_1\}$. Since $u_2 \neq v_2$, the minimality of $V(P) \cup V(Q)$ implies that v_2 is non-adjacent to u_3, and u_2 is non-adjacent to v_3.

Suppose that u_3 is adjacent to v_1. By (17), applied to the path v_3–v_2–v_1–u_3 and u_4, it follows that u_4 is non-adjacent to v_3, and in particular $u_4 \neq v_4$. Since u_3 is non-adjacent to x, v_2, it follows that u_3 is a leaf for C. Since u_3 is anticomplete to $V(C) \setminus \{v_1\} \cup \{u_1\}$, (20c) implies that u_1 is not a clone for C, and so by (20a), $N(u_1) \cap V(C) = \{x, v_1, v_2, v_3\}$. By (20a), v_1 is non-adjacent to u_4, and so by (20c), we deduce that u_4 is adjacent to v_5. By (20), v_5 is adjacent to u_5. Since v_3 is adjacent to u_1, by symmetry v_4 is adjacent to u_5 and $G|[V(P) \cup V(Q) \cup \{x\}]$ contains F_3 as a subgraph, a contradiction. This proves that u_3 is non-adjacent to v_1, and similarly v_3 is non-adjacent to u_1.

By (30), this implies that v_3 is adjacent to at least one of u_4, u_5 and has no other neighbors in $V(D)$. Likewise, u_3 is adjacent to at least one of v_4, v_5 and has no other neighbors in $V(C)$.

Assume first that $u_4 = v_4$. This implies by (20a) that u_5 is adjacent to v_5. If u_3 is non-adjacent to v_5, then $D' = G[V(D) \setminus \{u_5\} \cup \{v_5\}]$ is an induced 7-cycle and we contradict (31) for D' with v_2 playing the role of l_0 and v_3 playing the role of l_3, since u_5 is a clone at v_5 in D'. Therefore we conclude that u_3 is adjacent to v_5, and by symmetry v_3 is adjacent to u_5. But now $G[V(P) \cup V(Q) \cup \{x\}]$ contains F_2 as a subgraph, a contradiction.

We may therefore assume that $u_4 \neq v_4$. Next, suppose that v_3 is adjacent to u_4. Since v_4 is anticomplete to u_2, v_6, (29) and (30) applied to $G[V(D) \setminus \{u_1\} \cup \{v_1\}]$, v_3 and v_4 imply that v_4 adjacent to one of v_1, x, a contradiction. Therefore v_3 is adjacent to u_5 and non-adjacent to u_4. Now by (20a) applied to C and u_5, it follows that u_5 is adjacent to v_4, v_5, and by symmetry v_5 is complete to $\{u_3, u_4, u_5, v_6\}$. So $G[V(P) \cup V(Q) \cup \{x\}]$ contains F_3 as a subgraph, a contradiction. This proves that $u_2 = v_2$.

Now, if $u_4 = v_4$ then by (20a) $G[x, v_1 \ldots, v_6, u_1, u_3, u_5]$ contains F_1 as a subgraph, a contradiction, so $u_4 \neq v_4$. Since $N(u_4) \cap V(C) \subseteq \{v_3, v_5\}$, it follows from the minimality of $V(P) \cup V(Q)$ and the fact that $u_4 \neq v_4$, that u_4 is a leaf for C. Since if $N(v_4) \cap V(D) = \{u_3\}$ and $N(u_4) \cap V(C) = \{v_5\}$, then u_4–v_5–v_4–u_3–u_4 is a C_4 in G, it follows that either $N(u_4) \cap V(C) = \{v_3\}$ and $N(v_4) \cap V(D) = \{u_3\}$, or $N(u_4) \cap V(C) = \{v_5\}$ and $N(v_4) \cap V(D) = \{u_5\}$. Since G contains no C_4,C_5, in the former case u_3, u_4 are leaves for C and u_1, u_5 are clones for C, and in the latter case u_4, u_5 are leaves for C, and u_1, u_3 are clones for C. In both cases we get a contradiction to (31). This proves (32).

(33) *G contains no adjacent vertices u, v such that $N(u) \subseteq N(v) \cup \{v\}$.*

Consider adjacent u, v such that $N(u) \subseteq N(v) \cup \{v\}$. Among possible candidates, choose u, v so that $N(u)$ is smallest possible

Observe that $N(u) \setminus \{v\}$ is a stable set, since G contains no K_4. Since G is a minimal counterexample, $G - u$ is 3-colorable. Consider a fixed 3-coloring c of $G - u$. Since G is not 3-colorable, all 3 colors appear in $N(u)$. By symmetry, we may assume that $c(v) = 3$ and colors 1 and 2 appear on the remaining vertices in $N(u)$. Recall that the remaining vertices in $N(u)$ are pairwise non-adjacent. Let V_{12} denote the set of vertices $x \in V(G - u)$ with $c(x) \in \{1, 2\}$. Let D denote the union of all connected components of $G[V_{12}]$ that contain vertices $x \in N(u)$ with $c(x) = 1$. If D contains no vertex $y \in N(u)$ with $c(y) = 2$, then we switch colors 1 and 2 on vertices in D and color u by 1 to obtain a 3-coloring of G. Because this is not possible, there exists a path in $G[V_{12}]$ between neighbors of u of different colors. Let Q be a shortest such path. Note that by the minimality of Q and since $N(u) \setminus \{v\}$ is a stable set, u is anticomplete to the internal vertices of Q. Moreover, Q has odd length ≥ 3, since colors on vertices on Q alternate. If Q has length ≥ 7, then it contains an induces P_8.

If Q has length 3, then $u-Q-u$ is an induced 5-cycle. Therefore $Q = v_1-v_2-v_3-v_4-v_5-v_6$ where $c(v_1) = c(v_3) = c(v_5) = 1$, $c(v_2) = c(v_4) = c(v_6) = 2$, and u is complete to $\{v_1, v_6\}$ and anticomplete to $\{v_2, \dots, v_5\}$. Note that v is complete to $\{v_1, v_6, u\}$, since $N(u) \subseteq N(v) \cup \{v\}$. Since $u-v_1-\dots-v_6-u$ is a 7-cycle, it follows from (20a) that v is anticomplete to $\{v_3, v_4\}$ and possibly adjacent to at most one of v_2, v_5. Let $C = G[V(Q) \cup \{u\}]$, and let $v_0 = u$.

First we prove two useful observations:

(33.1) *Let $i \in \{1, \dots, 5\}$. If $w \in V(G) \setminus (V(C) \cup \{v\})$ is complete to $\{v_i, v_{i+1}\}$, then w is adjacent to one of v_{i-1}, v_{i+2}.*

Suppose that w is anticomplete to $\{v_{i-1}, v_{i+2}\}$. Then by (20a), w is a hat for C. Therefore w has a neighbor $z \notin V(C)$. Note that $c(w) = 3$, and so $c(z) \in \{1, 2\}$. In particular, $z \neq v$. Suppose that z is adjacent to v_i. It follows from (29) that z is adjacent to v_{i-1}. Since $c(z) \neq 3$, we deduce that $i = 1$, and therefore z is also adjacent to v. But now $\{u, v, v_1, z\}$ is a K_4, a contradiction. This proves that z is non-adjacent to v_i, and by symmetry z is non-adjacent to v_{i+1}. Now by (29), $N(z) \cap V(C) = \{v_{i-3}\}$. By (32), w has another neighbor z'. Repeating the previous argument for z' we deduce that z' is adjacent to v_{i-3} and nothing else on C. Since $w-z-v_{i-3}-z'-w$ is not a C_4, it follows that z is adjacent to z'. However, if $i \neq 3$, then $c(z) = c(z') \in \{1, 2\} \setminus \{c(v_{i-3})\}$, and if $i = 3$, then both z, z' are adjacent to u and therefore to v, and so $\{z, z', u, v\}$ is a K_4, in both cases a contradiction. Consequently, w has a neighbor in $\{v_{i-1}, v_{i+2}\}$ as claimed. This proves (33.1).

(33.2) *There is no vertex in $V(G) \setminus \{v\}$ complete to $\{v_1, v_2\}$.*

Consider $w \neq v$ adjacent to v_1, v_2. If w is adjacent to u, then w is also adjacent to v because $N(u) \subseteq N(v) \cup \{v\}$, and so $\{v_1, w, u, v\}$ is a K_4, a contradiction. Thus w must be non-adjacent to u, and so by (33.1), we conclude that w is adjacent to v_3.

Suppose first that v is adjacent to v_5. By (27), $N(w) \cap (V(C) \cup \{v\}) = \{v_1, v_2, v_3\}$. By (32), v_5 has a neighbor $a \notin \{v_4, v_6, v\}$. If a is adjacent to v_6, then by the argument of the previous paragraph a is also adjacent to v_4, and $G[V(C) \cup \{v, w, a\}]$ contains F_1 as a subgraph, a contradiction, so a is non-adjacent to v_6. If a is adjacent to v_4, then by (33.1), a is also adjacent to v_3, and again $G[V(C) \cup \{v, w, a\}]$ contains F_1 as a subgraph, a contradiction. Therefore a is non-adjacent to v_4. Now by (20a), a is a leaf for C. By (32), v_3 has a neighbor $b \notin \{w, v_2, v_4\}$. Then b is non-adjacent to v_5 (by the argument applied to a). By (22), b is non-adjacent to v_1. Since $\{b, v_3, v_2, w\}$ is not a K_4, there is $w_2 \in \{v_2, w\}$ such that b is non-adjacent to w_2. Let D be the cycle $G[(V(C) \setminus \{v_2\}) \cup \{w_2\}]$. Now we get a a contradiction to (28) applied to D, a, b. This proves that v is non-adjacent to v_5.

By (32), v_6 has a neighbor $a \notin \{u, v, v_5\}$. If a is adjacent to v_5, then, from the symmetry between a and w we deduce that a is adjacent to v_4, and so $G[V(C) \cup \{v, w, a\}]$ contains F_1 as a subgraph, a contradiction; so a is non-adjacent to v_5. Since there is no K_4, it follows that a is not complete to $\{u, v\}$, and since $N(u) \subseteq N(v) \cup \{v\}$, we deduce that a is non-adjacent to u, and so a is a leaf for C. Now v_5 has a neighbor $b \notin \{v_4, v_6\}$; it follows that b is not adjacent to v_6. If b is adjacent to v_4, then by (33.1), b is also adjacent to v_3, and so $G[V(C) \cup \{b, w, v\}]$ contains F_1 as a subgraph, a contradiction; thus b is non-adjacent to v_4. It follows (using (32) if w is adjacent to v_4) that v_4 has a neighbor $d \notin \{v_3, v_5, w\}$. Then d is non-adjacent to v_5. But (28) applied to C, a and d, it follows that d is adjacent to v_3, and by (33.1), d is also adjacent to v_2. Now (22) implies that w is non-adjacent to v_4, and d is non-adjacent to v_1. Since $\{v_2, v_3, d, w\}$ is not a K_4 in G, it follows that d is non-adjacent to w. Now we get a contradiction to (28) applied to the cycle $G[(V(C) \setminus \{v_2\}) \cup \{w\}]$ and the vertices d and b. This proves (33.2).

By (20a), we may assume that v is non-adjacent to v_2. By (32), there exists $u_1 \notin V(C) \cup \{v\}$, such that u_1 is adjacent to v_1. By (33.2), u_1 is non-adjacent to v_2. Since there is no K_4, we deduce that u_1 is not complete to $\{u, v\}$, and since $N(u) \subseteq N(v) \cup \{v\}$, it follows that u_1 is non-adjacent to u, and so by (20a), u_1 is a leaf for C.

Suppose first that v is non-adjacent to v_5. By (32), there exists u_6 is non-adjacent to v_5. There is symmetry between u_1 and u_6, and so u_6 is anticomplete to to $\{v_5, u\}$. But now both u_1, u_6 are leaves for C, contrary to (28). This proves that v is adjacent to v_5.

By (32), there exists $u_5 \notin V(C) \cup \{v\}$, such that u_5 is adjacent to v_5. By (33.2) and symmetry, u_5 is non-adjacent to v_6. Suppose first that u_5 is adjacent to v_4. By (33.1), it follows that u_5 is adjacent to v_3, and by (20a) and (27), u_5 has no more neighbors in $V(C) \cup \{v\}$. By (32), v_3 has a neighbor $u_3 \notin \{u_5, v_2, v_4\}$. By (22), and by the argument that was applied to u_1, it follows that u_3 is anticomplete to $\{v_5, v_1\}$. Since $\{u_3, v_3, v_4, u_5\}$ is not a K_4 is G, it follows that u_3 has a non-neighbor $w_4 \in \{v_4, u_5\}$. Let D be the cycle $G[(V(C) \setminus \{v_4\}) \cup \{w_4\}]$. Now we get a contradiction to (28) applied to D, u_1, u_3. This proves that u_5 is non-adjacent to v_4, and so $N(u_5) \cap V(C) = \{v_5\}$.

It follows that v_3 has a neighbor $u_3 \notin \{v_2, v_4\}$. By the argument of the previous paragraph applied to u_3, it follows that u_3 is nonadjacent to v_5. By the argument applied to u_1, we deduce that u_3 is non-adjacent to v_1. Therefore, by (20a), we have $N(u_3) \cap V(C) \subseteq \{v_2, v_3, v_4\}$, and by (28) applied to C, u_1, u_3 and to C, u_3, u_5, we deduce that u_3 is complete to $\{v_2, v_3, v_4\}$. Now (27) implies that u_3 has no more neighbors in $V(C) \cup \{v\}$. By (32), there exist $u_2, u_4 \notin V(C) \cup \{u_3\}$ such that u_i is adjacent to v_i. By (33.2) and (22), u_2 is anticomplete to $\{v_1, v_4\}$. By the argument applied to u_5 and (22), u_4 is anticomplete to $\{v_2, v_5\}$. In particular, $u_2 \neq u_4$. By (33.1), v_3 is anticomplete to $\{u_2, u_4\}$. But now we get a contradiction to (28) applied to C, u_2, u_4.

This proves (33).

5.1 Excluding big neighbors of an 8-cycle

In this section, we prove that no vertex has more than 3 neighbors in an induced 8-cycle of G. By (21), such a vertex has exactly 4, or 5 consecutive neighbors, or it is complete to the cycle. We exclude each of the cases as follows.

(34) *Let $C = v_0 - v_1 - \ldots - v_7 - v_0$ be an induced 8-cycle. Then no vertex $v \in V(G) \setminus V(C)$ is complete to $V(C)$.*

Suppose such a vertex v exists. By (33). there exists a vertex y adjacent to v_0 and not to v. By (17) applied to y–v_0–v–v_i where $i \in \{2, \ldots, 6\}$, we conclude that y is anticomplete to $\{v_2, \ldots, v_6\}$. Thus by (21), y is complete to $\{v_7, v_0, v_1\}$, contrary to (22). This proves (34).

(35) *Let $C = v_0 - v_1 - \ldots - v_7 - v_0$ be an induced 8-cycle. If a vertex v is adjacent to v_1, v_2, v_3, v_4 (and possibly other vertices), then there is no vertex u adjacent to v_2, v_3.*

Suppose that such u, v exist. Now by (22) u is anticomplete to $\{v_1, v_4\}$, contrary to (21). This proves (35).

(36) *Let $C = v_0 - v_1 - \ldots - v_7 - v_0$ be an induced 8-cycle. There is no vertex v adjacent to v_1, v_2, v_3, v_4, v_5.*

Suppose such a vertex v exists. By (21) and (34), $N(v) \cap V(C) = \{v_1, v_2, v_3, v_4, v_5\}$. By (33) there exist u_2, u_3, u_4 such that u_i adjacent to v_i and not to v. By (35), we deduce that u_3 is non-adjacent to v_2, v_4, and therefore u_3 is adjacent to v_7 by (21). Also by (35), u_2, u_4 are non-adjacent to v_3. By several applications of (17), it follows that the vertices u_2, u_3, u_4 are pairwise distinct and non-adjacent. If u_2 is non-adjacent to v_1, then by (21), $N(u_2) \cap V(C) = \{v_2, v_6\}$ and $v - v_2 - u_2 - v_6 - v_5 - v$ is an induced 5-cycle, a contradiction. So u_2 is adjacent to v_1, and therefore by (21) to v_0. By (27) applied to $v_3 - u_3 - v_7 - v_0 - v_1 - v_2 - v_3$ and u_2, we deduce that u_2 is non-adjacent to v_7. By symmetry, $N(u_4) \cap V(C) = \{v_4, v_5, v_6\}$.

By (33), there is w_2 adjacent to u_2 and not v_1. Applying (17) to w_2 and the paths $u_2 - v_1 - v - v_5$, $u_2 - v_1 - v - v_4$, and $u_2 - v_1 - v - v_3$ we deduce that w_2 is anticomplete to $\{v, v_3, v_4, v_5\}$. Suppose w_2 is adjacent to v_2. By (21) applied to $G[V(C) \setminus \{v_1\} \cup \{u_2\}]$, we deduce that w_2 is adjacent to v_2, v_0 and not to v_1, which contradicts (21) applied to C and w_2. Thus w_2 is non-adjacent to v_2. Now by (21) applied to $G[V(C) \setminus \{v_1\} \cup \{u_2\}]$ and to C, we deduce that w_2 is adjacent to v_0, v_7, v_6. Similarly, there is w_4 adjacent to u_4, v_6, v_7, v_0 and not to v_4, v_3, v_2, v_1, v. By (22), $w_2 = w_4$. By (21) applied to $G[V(C) \setminus \{v_7\} \cup \{w_2\}]$, u_3 is adjacent to w_2. But now $u_2 - v_2 - v_3 - u_3 - w_2 - u_2$ is an induced 5-cycle, a contradiction. This proves (36).

(37) *Let $C = v_0 - v_1 - \ldots - v_7 - v_0$ be an induced 8-cycle. There is no vertex v adjacent to v_1, v_2, v_3, v_4 (and possibly other vertices).*

Suppose such a vertex v exists. By (36), v has no more neighbors in C. By (33), there exist u_2, u_3 such that $v_i u_i$ is an edge for $i \in \{2, 3\}$, and v is non-adjacent to u_2, u_3. By (35), u_2 is non-adjacent to v_3, and u_3 is non-adjacent to v_2. Since $v_2 - v_3 - u_3 - u_2 - v_2$ is not a C_4, we deduce that u_2 is non-adjacent to u_3.

(37.1) *If a vertex $u \neq v$ is adjacent to both v_1 and v_2, then u_3 is adjacent to v_4 and v_5.*

Consider $u \neq v$ adjacent to v_1, v_2. Clearly, u is non-adjacent to v, or else $\{u, v, v_1, v_2\}$ is K_4. Thus by (22), u is also non-adjacent to v_3, since v is. For contradiction, suppose that u_3 is non-adjacent to v_4. Then by (21), the neighbors of u_3 in $V(C)$ are $\{v_3, v_7\}$. By (27) applied to $v_0 - v_1 - v_2 - v_3 - u_3 - v_7 - v_0$, we find that u is anti-complete to $\{u_3, v_7\}$ and thus by (21), $N(u) \cup V(C) = \{v_0, v_1, v_2\}$. But now $u - v_1 - v - v_4 - v_5 - v_6 - v_7 - u_3$ is an induced P_8. This proves u_3 is adjacent to v_4, and by (21) also to v_5. This proves (37.1).

(37.2) *No vertex $u \neq v$ is adjacent to both v_1 and v_2.*

Consider $u \neq v$ adjacent to v_1, v_2. As before, we deduce that u is non-adjacent to both v and v_3. Thus we may assume that $u = u_2$. By (37.1), u_3 is complete to $\{v_4, v_5\}$. Now there is symmetry between u and u_3, and so we deduce that u is adjacent to v_0.

If v_7 is non-adjacent to u, and v_6 is non-adjacent to u_3, we get a contradiction to (21) applied to v_2–u–v_0–v_7–v_6–v_5–u_3–v_3–v_2 and v, so we may assume by symmetry that u is adjacent to v_7. By (36), we deduce that $N(u) \cap V(C) = \{v_2, v_1, v_0, v_7\}$. By (33), there is u_0 adjacent to v_0 and not to u. By (35), u_0 is non-adjacent to v_1. By (37.1) applied to (u, v, u_0) instead of (v, u, u_3), we deduce that u_0 is adjacent to v_7 and v_6. By (20a), u_0 is non-adjacent to v_4, v, and by (21), $N(u_0) \cap V(C) \subseteq \{v_0, v_7, v_6, v_5\}$. Suppose u_0 is adjacent to u_3. By (21) and (36), it follows that neither of $G[V(C) \setminus \{v_4\} \cup \{u_3\}]$ and $G[V(C) \setminus \{v_7\} \cup \{u_0\}]$ is an induced cycle, and so u_3v_6 and u_0v_5 are both edges. But now $\{v_5, v_6, u_0, u_3\}$ is a K_4, a contradiction. This proves that u_0 is non-adjacent to u_3. By (21) applied to v_2–u–v_0–u_0–v_6–v_5–u_3–v_3–v_2 and v, we deduce that either u_0 is adjacent to v_5, or u_3 is adjacent to v_6. By symmetry (exchanging (u, u_0) and (v, u_3)), we may assume that u_3 is adjacent to v_6. Now there is symmetry between u and u_3, and so there is u_5 adjacent to v_5, v_6, v_7 and possibly v_0 (this is an analogue of u_0). If $u_5 \neq u_0$, then since there is no K_4, it follows that u_5 is non-adjacent to u_0, and we get a contradiction to (21) applied to $G[V(C) \setminus \{v_7\} \cup \{u_0\}]$ and u_5. So $u_5 = u_0$. Since $G[v_0, \dots, v_7, v, u, u_0, u_3]$ is 3-colorable, it follows that there is another vertex $x \in V(G)$. By (20c) applied to v_0–v_1–v–v_4–v_5–v_6–v_7–v_0 and v_0–v_1–v_2–v_3–v_4–v_5–u_0–v_0, and by (27) applied to v_0–v_1–v–v_4–v_5–u_0–v_0, we deduce that x has a neighbor in $\{v_0, \dots, v_7\}$. By symmetry we may assume that x is adjacent to v_0. By (35), x is non-adjacent to v_1. If x is adjacent to v_7, then by (21), x is adjacent to v_6 contrary to (22), so x is non-adjacent to v_7. Now by (21) $N(x) \cap V(C) = \{v_0, v_4\}$, and we get a contradiction to (27) applied to v_0–v_1–v–v_4–v_5–u_0–v_0. This proves (37.2).

It now follows from (21) and (37.2) that $N(u_2) \cap V(C) = \{v_2, v_6\}$. By symmetry, $N(u_3) \cap V(C) = \{v_3, v_7\}$. By (32), v_1 has a neighbor $u_1 \notin \{v_2, v, v_0\}$. By (37.2), u_1 is non-adjacent to v_2. If $N(u_1) \cap V(C) = \{v_1, v_5\}$, we contradict (17) for u_1 and path v_1–v–v_4–v_5. So by (21), u_1 is adjacent to v_0, v_7. By (17) applied to v_6–u_2–v_2–v_1 and u_1, we conclude that $N(u_1) \cap V(C) = \{v_7, v_0, v_1\}$. By (33), v_0 has a neighbor u_0 non-adjacent to u_1. If u_0 is adjacent to v_1, then by (37.2), we deduce that u_0 is non-adjacent to v_2 and thus by (21), u_0 is adjacent to v_7. This however contradicts (22) when applied to u_0, u_1 and C. Therefore u_0 is non-adjacent to v_1. If $N(u_0) \cap V(C) = \{v_0, v_4\}$, we contradict (17) for u_0 and path v_4–v–v_1–v_0. Thus by (21), we conclude that u_0 is adjacent to v_6, v_7. By (37.2), u_0 is non-adjacent to v_5 and so $N(u_0) \cap V(C) = \{v_0, v_7, v_6\}$. Symmetrically, there is u_4 with $N(u_4) \cap V(C) = \{v_4, v_5, v_6\}$ and u_5 with $N(u_5) \cap V(C) = \{v_5, v_6, v_7\}$ where $u_5 \neq u_0$. Since there is no K_4, we deduce that u_0 is non-adjacent to u_5. But now we get a contradiction to (21) applied to $G[V(C) \setminus \{v_7\} \cup \{v_0\}]$ and u_5. This proves (37).

5.2 The structure of the neighbors of a 7-cycle

In this section, we examine the structure of neighbors of induced 7-cycles. We denote $C = v_0$–v_1–$\dots$–v_6–v_0 an induced 7-cycle of G, and H_i and L_i are the hats and leaves of C as defined earlier. Let $Y = \bigcup_{i=0}^{6} (L_i \cup H_i)$. Our goal is to understand the structure of $G[Y]$.

With (37), we first strengthen (28), (29), and (30) as follows.

(38) *There do not exist vertices $a \in L_0 \cup H_4$ and $b \in L_2 \cup H_5$.*

Suppose that such vertices a, b exist. By (28), we deduce that $ab \in E(G)$, $a \in H_4$, and $b \in H_5$. This contradicts (37) for v_1 and 8-cycle $v_0{-}a{-}b{-}v_2{-}v_3{-}v_4{-}v_5{-}v_6{-}v_0$. This proves (38).

(39) *Let $h \in H_5$, and let $y \in V(G) \setminus V(C)$ be adjacent to h. Then one of the following holds:*

- $\{v_2, v_3, v_4\} \subseteq N(y) \cap V(C) \subseteq \{v_2, v_3, v_4, v_5\}$
- $\{v_0, v_1, v_6\} \subseteq N(y) \cap V(C) \subseteq \{v_0, v_1, v_5, v_6\}$
- $y \in L_5$

By (29), it suffices to show that $N(y) \cap V(C) \neq \{v_2, v_3\}$ and $N(y) \cap V(C) \neq \{v_0, v_1\}$. In other words, we need to show that $y \notin H_4$ and $y \notin H_6$, either of which is excluded by (38), since $h \in H_5$. This proves (39).

(40) *Every vertex $l \in L_1$ has a neighbor in $H_1 \cup L_4 \cup L_5$.*

By (33), l has a neighbor y non-adjacent to v_1. Thus (30) applies and we deduce that $y \in H_1 \cup L_4 \cup L_5$. This proves (40).

We shall use the above claims with symmetry of C in mind. In the following series of claims, we examine the connected components of hats and leaves of C.

(41) *For every i, every connected component of $G[L_i]$ has size at most 2.*

We may assume that $i = 1$; let M be a component of L_1. Since there is no 8-wheel W_8 by (34) and no 6-wheel W_6 by (26), we deduce that M is a tree. Let m_1 be a leaf of M, and let $P = m_1{-}\dots{-}m_k$ be a maximal induced path of M starting at m_1. We may assume that m_1 and P are chosen to maximize k. We may assume that $k \geq 3$.

By (33) there is x in $V(G)$ adjacent to m_1 and not to m_2. Then x is not in M by the maximality of P, and therefore x is not in L_1. By (17) applied to $x{-}m_1{-}m_2{-}m_3$, it follows that x is non-adjacent to m_3. If $k \geq 4$, applying (17) to $m_1{-}m_2{-}m_3{-}m_4$ and x implies that x is non-adjacent to m_4. Suppose that x is adjacent to v_1. Then by (20a), there is an induced path Q, starting at x, and otherwise contained in $V(C) \setminus \{v_1\}$, with $|V(Q)| \geq 5$. But now $Q{-}x{-}m_1{-}m_2{-}m_3$ is an induced P_8, a contradiction. So x is non-adjacent to v_1. By (30), $x \in H_1 \cup L_4 \cup L_5$; from the symmetry we may assume that x is in $H_1 \cup L_4$. Since $v_2{-}v_3{-}v_4{-}x{-}m_1{-}m_2{-}m_3{-}m_4$ is not an induced P_8, we deduce that $k = 3$. Since $v_0{-}v_6{-}v_5{-}v_4{-}x{-}m_1{-}m_2{-}m_3$ is not an induced P_8, it follows that $x \in H_1$.

From the choice of m_1 and P (i.e. that the longest path in M has $k = 3$ vertices), we deduce that M is a star (that is, the graph $K_{1,t}$ for some $t \geq 2$), m_2 is complete to $M \setminus \{m_2\}$, and every vertex of $M \setminus m_2$ has degree one in M. By symmetry, there is $y \in H_1$, adjacent to m_3, and not to m_1, m_2.

By (33) there is w in $V(G)$, adjacent to m_2 and not to v_1. By (30), w is in $L_4 \cup L_5 \cup H_1$. We may assume that w is adjacent to v_5. By (17) applied to $m_2{-}m_1{-}x{-}v_5$ and w, we deduce that w is adjacent to x and m_1. Similarly, w is complete to $\{y, m_3\}$. But now $v_1{-}m_1{-}w{-}m_3{-}v_1$ is an induced 4-cycle, a contradiction. This proves (41).

(42) *Let $M = \{m_1, m_2\}$ be a component of $G[L_i]$. Then either*

(42a) (up to symmetry) there exists a component N of L_{i-3} with $N = \{n_1, n_2\}$ such that n_1 is complete to M, and n_2 is adjacent to m_1 and not to m_2, or

(42b) Same as (42a) with L_{i+3} instead of L_{i-3}, or

(42c) There exist $n_1 \in L_{i+3}$ and $n_2 \in L_{i-3}$ such that m_1 is adjacent to n_1 and not to n_2, and m_2 is adjacent to n_2 and not n_1, and n_1 is non-adjacent to n_2.

Moreover, M is anticomplete to $\bigcup_{j=0}^{6} H_j$.

We may assume that $i = 1$. We prove the first assertion first. By (40), there exist x_1, x_2 such that x_i is adjacent to m_i, and x_i is in $H_1 \cup L_4 \cup L_5$.

Suppose first that x_1, x_2 can be chosen so that x_1 is non-adjacent to m_2, and x_2 is non-adjacent to m_1. Then x_1 is non-adjacent to x_2, since there is no C_4, and since there is no C_5, we deduce that x_1, x_2 have no common neighbor in $\{v_4, v_5\}$, and (42c) holds. So we may assume that $x_1 = x_2$.

By (33), there exist a_1, a_2 such that a_1 is adjacent to m_1 and not to m_2, and a_2 is adjacent to m_2 and not to a_1. Then $a_1, a_2 \neq x_1$. Suppose that a_1 is non-adjacent to v_1. By (30), we deduce that $a_1 \in H_1 \cup L_4 \cup L_5$. By symmetry, we may assume that $x_1 \in H_1 \cup L_4$, i.e. x_1 is adjacent to v_4. Applying (17) to x_1 and $m_2 - m_1 - a_1 - v_4$ (if a_1 is adjacent to v_4) or $m_1 - a_1 - v_5 - v_4$ (if a_1 is non-adjacent to v_4), we deduce that x_1 is adjacent to a_1. Now by (28), (39) and symmetry we may assume that $x_1, a_1 \in L_4$ and (42b) holds. This proves that $\{a_1, a_2\}$ is complete to v_1.

Since $a_1 - v_1 - m_2 - x_1 - a_1$ is not a C_4, it follows that a_1 is non-adjacent to x_1, and similarly a_2 is non-adjacent to x_1. Since $G[\{a_1, m_1, x_1, v_4, v_5\}]$ does not contain C_4, C_5, it follows that a_1 (and symmetrically a_2) is anticomplete to $\{v_4, v_5\}$. By (41), $a_1, a_2 \notin L_1$, and by (22), $\{a_1, a_2\}$ is not complete to $\{v_0, v_2\}$, so by (20), we may assume that a_1 is adjacent to v_0 and not to v_2. By (28), $N(a_1) \cap V(C) = \{v_0, v_1, v_6\}$. But now we get a contradiction to (28) applied to $G[V(C) \setminus \{v_0\} \cup \{a_1\}]$, m_1, and m_2. This proves the first assertion of (42).

Next we prove the second assertion. By symmetry, we may assume that there is $a_1 \in L_5$ adjacent to m_1, v_5, and not to m_2, v_4. Suppose that $h \in \bigcup_{j=1}^{6} H_j$ has a neighbor $m \in M$. By (30), $h \in H_1$. By (28), h is non-adjacent to a_1. But now we get a contradiction to (17) applied to $a_1 - v_5 - h - m$ and m_1. This proves (42).

Recall that $Y = \bigcup_{i=0}^{6} (L_i \cup H_i)$.

(43) *If $l \in L_i$ has a neighbor $h \in H_i$, then $\{l, h\}$ is a connected component of $G[Y]$.*

We may assume that $i = 1$. Suppose there is y in $Y \setminus \{l, h\}$ adjacent to l. By (30), y is in $L_1 \cup H_1 \cup L_4 \cup L_5$. By (42), y is not in L_1, and so we may assume that y is adjacent to v_4. Applying (17) to $v_3 - v_4 - h - l$, we deduce that y is adjacent to h, contrary to (28). So l has no neighbors $Y \setminus \{h\}$.

Next suppose that h has a neighbor y in $Y \setminus \{l, h\}$. Then y is non-adjacent to l. By (39), y is in L_1. But now $h - l - v_1 - y - h$ is an induced 4-cycle, a contradiction. This proves (43).

(44) *Let $m_1, m_2 \in L_1$ and $n_1, n_2 \in L_4 \cup L_5$ as in (42a), (42b) or (42c). Then $\{m_1, m_2, n_1, n_2\}$ is a connected component of $G[Y]$.*

Let $K = \{m_1, m_2, n_1, n_2\}$. Assume for a contradiction that $y \in Y \setminus K$ has a neighbor in K. Suppose first that m_1, m_2, n_1, n_2 are as in (42a). By symmetry, we may assume that y has a neighbor $m \in \{m_1, m_2\}$. By (41), (42) and (30), we conclude that $y \in L_4 \cup L_5$. If $y \in L_4$, then y is adjacent to v_4, non-adjacent to v_5, and we contradict (17) for path $v_4{-}v_5{-}n_1{-}m$ and y. If $y \in L_5$, then y is adjacent to v_5 and non-adjacent to n_1 by (41), and thus $y{-}v_5{-}n_1{-}m{-}y$ is an induced C_4, a contradiction.

Thus we may assume that m_1, m_2, n_1, n_2 are as in (42c). We claim that y has a neighbor in $\{n_1, n_2\}$. Suppose not. By symmetry, assume that y is adjacent to m_2. By (30), (41) and (42), we conclude that y is in $L_4 \cup L_5$. Applying (17) to $v_4{-}v_5{-}n_2{-}m_2$ and y, we deduce that y is anti-complete to $\{v_4, v_5\}$, a contradiction.

This proves that y has a neighbor in $\{n_1, n_2\}$, and we may assume that y is adjacent to n_2. By (30) and (43), we deduce that $y \in L_1 \cup L_2 \cup L_5$. Suppose that y is in $L_1 \cup L_2$. By (28) and (41), y is non-adjacent to m_2, contrary to (17) applied to y and the path $v_2{-}v_1{-}m_2{-}n_2$. This proves that $y \in L_5$. By (17) applied $v_5{-}v_4{-}n_1{-}m_1$ and y, we find that y is anticomplete to $\{m_1, n_1\}$. By (33), there is z adjacent to y and not to n_2. Same argument we applied for y yields by (30), (41), and (43), that $z \in L_1 \cup L_2$. Then by (28) and (41), z is anticomplete to $\{m_1, m_2\}$. If z is adjacent to n_1, then we contradict (17) for z and path $v_2{-}v_1{-}m_1{-}n_1$. Thus z is also non-adjacent to n_1. Now if y is non-adjacent to m_2, we find that $z{-}y{-}n_2{-}m_2{-}m_1{-}n_1{-}v_4{-}v_3$ is an induced P_8, Therefore y is adjacent to m_2, but now we contradict (17) for z and path $v_2{-}v_1{-}m_2{-}y$. This proves (44).

(45) *Let D be a connected component of $G[Y]$. Then one of the following holds:*

- *(45a) $D = \{h\}$ where $h \in H_i$ for some i,*
- *(45b) $D = \{h, l\}$ where $h \in H_i$ and $l \in L_i$ for some i,*
- *(45c) $D = \{m_1, m_2, n_1, n_2\}$ as in (42a) or (42b),*
- *(45d) $D = \{m_1, m_2, n_1, n_2\}$ as in (42c),*
- *(45e) $D = \{l, m\}$ where l is in L_i and m is in L_{i+3} or L_{i-3}.*

Suppose that $h \in D \cap H_i$ for some i. If $D = \{h\}$, then (45a) holds. If h has a neighbor $l \in D$, then $l \in L_i$ by (39), and so (45b) holds by (43). We may therefore assume that D does not contain any hats.

Let us say a component M of $G[L_i]$ is *big* if $|M| = 2$. If D meets a big component of some L_i, then (45c) or (45d) holds by (44). Thus we may assume that D meets no big components of any L_i and that there is l in $L_1 \cap D$. By (40), we may assume that there is m in L_4 adjacent to l. We may assume that there is y in $Y \setminus \{l, m\}$ with a neighbor in $\{l, m\}$, for otherwise (45e) holds. By symmetry, we may assume that y is adjacent to l. Since D meets no big components of L_1, we conclude that y is in $L_4 \cup L_5$. Since D meets no big component of L_4, it follows that y is not complete to $\{m, v_4\}$. But now we get a contradiction to (17) applied to $v_5{-}v_4{-}m{-}l$ and y. This proves (45).

5.3 Excluding neighbors of a 7-cycle

Using the results from the previous section, we can now exclude the existence of vertices with exactly 2 or 4 neighbors in an induced 7-cycle. This will provide the final ingredient for our proof of Theorem 3.

As before, we use the same notation established in the earlier sections.

(46) *Let $C = v_0 - v_1 - \cdots - v_6 - v_0$ be an induced 7-cycle. There is no vertex h adjacent to exactly v_1, v_2 in C.*

Suppose such a vertex h exists. Suppose first that there is $l \in L_5$ adjacent to h. By (45), $\{l, h\}$ is a component of Y. By (32), h has a neighbor $x \notin \{l, v_1, v_2\}$. Since $\{l, h\}$ is a component of Y, we conclude that $x \notin Y$. By (39), we may assume that x is adjacent to v_0, v_1, v_6. But now we get a contradiction to (27) applied to $v_5 - v_6 - v_0 - v_1 - h - l - v_5$ and x. This proves that no such l exists.

By (33) for $i \in \{1, 2\}$, there exists u_i adjacent to h and not to v_i. By (39), u_1 is adjacent to v_2, v_3, v_4 and u_2 is adjacent to v_6, v_0, v_1. Since $G[\{v_4, v_5, v_6, u_1, u_2, h\}]$ contains no C_4, C_5, it follows that u_1 is non-adjacent to u_2, and (using (20a)) that $N(u_1) \cap V(C) = \{v_2, v_3, v_4\}$ and $N(u_2) \cap V(C) = \{v_6, v_0, v_1\}$.

By (32), v_4 has a neighbor $u_4 \notin \{u_1, v_3, v_5\}$, and v_6 has a neighbor $u_6 \notin \{v_5, v_0, u_2\}$. Since $G[V(C) \cup \{u_1, u_2, u_4\}]$ does not contains F_1 as a subgraph, (20a) implies that u_4 is non-adjacent to v_6, and in particular $u_4 \neq u_6$. Similarly, u_6 is non-adjacent to v_4.

Suppose that u_6 is non-adjacent to v_0. If u_4 is non-adjacent to v_3, we contradict (38) for u_4, u_6, and C. Similarly, if u_4 is non-adjacent to u_1, we contradict (38) for u_4, u_6 and $C' = G[V(C) \setminus \{v_3\} \cup \{u_1\}]$. But now $\{u_1, u_4, v_3, v_4\}$ is a K_4, a contradiction.

Therefore we conclude that u_6 is adjacent to v_0, and by the same token also to u_2 (using cycles $C'' = G[V(C) \setminus \{v_0\} \cup \{u_2\}]$, and $C''' = G[V(C) \setminus \{v_0, v_3\} \cup \{u_1, u_2\}]$). But now $\{u_2, u_6, v_0, v_6\}$ is a K_4, a contradiction. This proves (46).

(47) *Let $C = v_0 - v_1 - \cdots - v_6 - v_0$ be an induced 7-cycle. There is no vertex v adjacent to v_1, v_2, v_3, v_4.*

Suppose such v exists. By (33), there exist u_2, u_3 such that u_i is adjacent to v_i, and v is anticomplete to $\{u_2, u_3\}$.

(47.1) *If $u_3 \notin Y$, then u_3 is adjacent to v_3, v_4, v_5, possibly to v_6, and otherwise has no other neighbors in $V(C)$.*

Assume that $u_3 \notin Y$. Then by (20), u_3 has exactly 3 or 4 consecutive neighbors in $V(C)$. By (22), u_3 is non-adjacent to v_1, and so u_3 is adjacent to v_4. Thus again by (22), u_3 is non-adjacent to v_2, and so u_3 must be adjacent to v_5 and possibly to v_6. Since u_3 is non-adjacent to v_2, (20) implies that $N(u_3) \cap V(C) \subseteq \{v_3, v_4, v_5, v_6\}$. This proves (47.1).

Suppose first that $u_2 \in Y$. By (46), it follows that $u_2 \in L_2$. By (40) and (46), there exists $w_2 \in L_5 \cup L_6$ adjacent to u_2. If $w_2 \in L_5$, then by (20a), v is a hat for $v_1 - v_2 - u_2 - w_2 - v_5 - v_6 - v_0 - v_1$, contrary to (46). So $w_2 \in L_6$.

If $u_3 \in Y$, then $u_3 \in L_3$ and by symmetry, there exist $w_3 \in L_6$ adjacent to u_3. By (45)

$w_2 \neq w_3$. If w_2 is non-adjacent to w_3, then v is a hat for $v_2-u_2-w_2-v_6-w_3-u_3-v_3-v_2$, and if w_2 is adjacent to w_3, then w_2 is a hat for $v_0-v_1-v_2-v_3-u_3-w_3-v_6-v_0$, in both cases contrary to (46).

So we must conclude that $u_3 \notin Y$. By (47.1), u_3 is adjacent to v_3, v_4, v_5, possibly v_6, and has no other neighbors in $V(C)$. By (20) applied to $v_2-v_3-v_4-v_5-v_6-w_2-u_2-v_2$, we deduce that u_3 is also anti-complete to $\{u_2, w_2\}$. If u_3 is non-adjacent to v_6, then v is a hat for $v_2-v_3-u_3-v_5-v_6-w_2-u_2-v_2$. Therefore u_3 is adjacent to v_6, but now v_1 is a hat for $v_2-v-v_4-u_3-v_6-w_2-u_2-v_2$, contrary to (46).

This proves that $u_2, u_3 \notin Y$. By (47.1), u_3 is adjacent to v_3, v_4, v_5, possibly to v_6, and similarly, u_2 is adjacent to v_0, v_1, v_2, possibly to v_6. If one of u_2, u_3 is adjacent to v_6, then G contains F_1 as a subgraph, a contradiction. So v_6 is anticomplete to $\{u_2, u_3\}$. But now v is a hat for $v_0-u_2-v_2-v_3-u_3-v_5-v_6-v_0$, contrary to (46). This proves (47).

(48) *Let $C = v_0-v_1-\cdots-v_6-v_0$ be an induced 7-cycle. There is no vertex $v \neq v_1$ adjacent to v_0, v_2.*

Suppose such v exists. By (20a) and (47), $N(v) \cap V(C) = \{v_0, v_1, v_2\}$. By (33), there is u adjacent to v_1 and not to v. By (20c), u has a neighbor in $V(C) \setminus \{v_1\}$. By (22), u is not complete to $\{v_0, v_2\}$, and so by (20a), we may assume that u is adjacent to v_0 and non-adjacent to v_2. Since by (46), u is not a hat for C, it follows that u is adjacent to v_6, and since (again by (46)) u is not a hat for $G[V(C) \setminus \{v_1\} \cup \{v\}]$, we deduce that u is adjacent to v_5. But now u has four neighbors in $V(C)$, contrary to (47). This proves (48).

5.4 Proof of Theorem 3

We now have all pieces to complete the proof of Theorem 3.

Assume that G is a minimum counterexample to the theorem. Since G contains no K_4, C_4, and no C_5, and since G is not 3-colorable, it follows from the Strong Perfect Graph Theorem [4] that G contains an induced 7-cycle. Let C be such an induced 7-cycle in G. By (20a), (20c) and (48), it follows that $V(G) = V(C) \cup Y$ and $Y \neq \emptyset$.

Let D be a connected component of $G[Y]$. Since every vertex of G has degree at least 3, (46) implies that only outcome (45c) is possible. By symmetry we may assume that D is as in (42a). But in this case there is an induced 7-cycle $v_i-v_{i+1}-v_{i+2}-v_{i+3}-v_{i-3}-n_2-m_1-v_i$ where n_1 is adjacent to m_1 and v_{i-3}, contradicting (48). This completes the proof of Theorem 3.

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