Induced subgraphs of graphs with large chromatic number.
XI. Orientations

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Abstract

Fix an oriented graph $H$, and let $G$ be a graph with bounded clique number and very large chromatic number. If we somehow orient its edges, must there be an induced subdigraph isomorphic to $H$? Kierstead and Rödl [12] raised this question for two specific kinds of digraph $H$: the three-edge path, with the first and last edges both directed towards the interior; and stars (with many edges directed out and many directed in). Aboulker et al. [1] subsequently conjectured that the answer is affirmative in both cases. We give affirmative answers to both questions.
1 Introduction

All graphs in this paper are finite and simple. If $G$ is a graph, $\chi(G)$ denotes its chromatic number, and $\omega(G)$ denotes its clique number, that is, the cardinality of the largest clique of $G$. This paper is concerned with the digraphs that can be obtained by orienting the edges of a graph, and in particular, digraphs in this paper have no “antiparallel” pairs of edges, that is, no directed cycles of length two, as well as no loops or parallel edges. If $G$ is a digraph, $G^*$ means the underlying graph. For a digraph $G$, we say $u$ is $G$-adjacent to $v$ or from $v$ to indicate the direction of the edge between $u$ and $v$, and $u$ is $G^*$-adjacent with $v$ to mean adjacency in $G^*$. The chromatic number $\chi(G)$ and clique number $\omega(G)$ of a digraph $G$ mean the corresponding quantities for $G^*$.

Let $H$ be a graph. We say that $H$ is $\chi$-bounding if there is a function $f$ such that $\chi(G) \leq f(\omega(G))$ for every graph $G$ not containing $H$ as an induced subdigraph. If $H$ is $\chi$-bounding then it cannot contain a cycle, since (as shown by Erdős [5]) there are graphs with large girth and large chromatic number. Thus the only possible $\chi$-bounding graphs are forests. The Gyárfás-Sumner conjecture [6, 17] asserts that every forest is $\chi$-bounding. Despite considerable work, the conjecture is only known for comparatively few families (see [7, 9, 10, 11, 15, 16, 3]).

Now let $H$ be an oriented graph. When is there a function $f$ such that $\chi(G) \leq f(\omega(G))$ for every oriented graph $G$ not containing $H$ as an induced subdigraph? As in the graph case, we call a digraph $H$ with this property $\chi$-bounding. Then every $\chi$-bounding digraph $H$ is an oriented forest, because we can take $G$ to be any orientation of a graph with large girth and large chromatic number. However, for digraphs it is not the case that $H$ is $\chi$-bounding whenever $H^*$ is a forest. Indeed, this is false even for digraphs $H$ such that $H^*$ is a three-edge path.

There are four ways to orient the edges of a three-edge path, up to reversing the path, and we denote the corresponding digraphs by $\rightarrow\rightarrow\rightarrow$, $\rightarrow\leftarrow\rightarrow$, $\rightarrow\leftarrow\leftarrow$, $\leftarrow\rightarrow\rightarrow$ with the natural meaning. Then

- Kierstead and Trotter [13] showed that $\rightarrow\rightarrow\rightarrow$ is not $\chi$-bounding, by constructing triangle-free graphs with arbitrarily large chromatic number together with a suitable orientation;
- Gyárfás [8] noted that $\rightarrow\leftarrow\rightarrow$ is not $\chi$-bounding: let $D$ be the natural orientation of the shift graph on pairs, so $D$ has vertex set $[n][2]$, with edges from $\{i,j\}$ to $\{j,k\}$ whenever $i < j < k$.
  Then $D$ is triangle-free, has large chromatic number, and does not contain an induced copy of $\rightarrow\leftarrow\rightarrow$.

The two remaining orientations are equivalent under reversing all edges, so both or neither are $\chi$-bounding. Thus it is enough to consider $\rightarrow\leftarrow\leftarrow$.

Oriented graphs with no induced $\rightarrow\leftarrow\leftarrow$ have been considered by several authors. In the special case of acyclic orientations, Chvátal [4] showed that if $G$ is an acyclic oriented graph with no induced copy of $\rightarrow\leftarrow\leftarrow$ then $G$ is perfect, and so $\chi(G) = \omega(G)$. Kierstead and Rödl [12] asked whether $\rightarrow\leftarrow\leftarrow$ is $\chi$-bounding, and showed that the class of oriented graphs with no induced copy of $\rightarrow\leftarrow\leftarrow$ and no cyclic triangle is $\chi$-bounded. Aboulker et al. [1] conjectured that $\rightarrow\leftarrow\leftarrow$ is in fact $\chi$-bounding, and proved some further special cases. Our first main result resolves the question.

1.1 The digraphs $\rightarrow\leftarrow\leftarrow$ and $\leftarrow\rightarrow\rightarrow$ are $\chi$-bounding.
This is proved in the next section.

Which forests have the property that every orientation is $\chi$-bounding? The results of Gyárfás and of Kierstead and Trotter mentioned above show that such a forest cannot contain a three-edge path, and so every component must be a star.

A digraph $H$ is an oriented star if $H^{*}$ is a star, that is, isomorphic to the complete bipartite graph $K_{1,t}$ for some $t \geq 0$. As with paths, there have been several previous results on the chromatic number of graphs with a forbidden oriented star. Gyárfás [8] asked whether, for every oriented star $H$, the class of acyclic oriented graphs with no induced $H$ is $\chi$-bounded. Kierstead and Rödl [12] proved the stronger result that the class of oriented graphs with no induced $H$ and no cyclic triangle is $\chi$-bounded; they further asked whether every oriented star is $\chi$-bounding. Aboulker et al. [1] conjectured that oriented stars are indeed $\chi$-bounding, and showed that for every oriented star $H$ the class of oriented graphs with no induced $H$ and no transitive triangle has bounded chromatic number (note that every orientation of $K_4$ has a transitive triangle, so if $G$ has no transitive triangle then $\omega(G)$ is at most 3). Our second main result answers this question.

1.2 Every oriented star is $\chi$-bounding.

It is easy to prove this for stars in which every edge is directed away from the centre, or every edge is directed towards the centre, but the case when there are edges of both types is more difficult. It follows from Theorem 1.2 that if $F$ is a forest such that every component is a star then every orientation of $F$ is $\chi$-bounding.

2 An oriented three-edge path

If $X \subseteq V(G)$, $G[X]$ denotes the subgraph or subdigraph induced on $X$, and we write $\chi(X)$ for $\chi(G[X])$ when there is no danger of ambiguity. If $G$ is a digraph and $v \in V(G)$, we denote the set of vertices with distance at most $r$ (in $G^*$) from $v$ by $N^{r}[v]$ or $N^r_{G}[v]$, and the set with distance exactly $r$ by $N^r_{G}(v)$. We denote by $\chi^r(G)$ the maximum of $\chi(N^{r}[v])$ over all $v \in V(G)$ (or zero for the null digraph.)

In this section we prove our first main result, that $\rightarrow\leftarrow\leftarrow$ is $\chi$-bounding. In fact, with a very little extra work we can prove a stronger statement, which we now explain. A hole in a graph is an induced cycle of length at least four, and when $G$ is a digraph, by a “hole” of $G$ we mean an induced subgraph $C$ such that $C^*$ is a hole of $G^*$. By a long hole we mean (just in this paper) a hole of length at least five. A hole of a digraph $C$ is

- **directed** if each of its vertices has outdegree one in $C$;
- **alternating** if each of its vertices has outdegree two or zero in $C$ (and therefore $C$ has even length); and
- **disoriented** if it is neither directed nor alternating.

It is easy to see that if some long hole of $G$ is disoriented, then $G$ contains $\rightarrow\leftarrow\leftarrow$ as an induced subdigraph. (Some two consecutive edges of $G$ make a two-edge directed path, but $C$ is not a directed cycle; grow the path to a maximal directed path of $C$ and look at its ends.) Thus the following theorem implies that $\rightarrow\leftarrow\leftarrow$ is $\chi$-bounding. (A useful feature of this strengthening is that now we are proving something invariant under reversing all edges of $G$, which reduces the case analysis.)
2.1 For all \( \kappa \) there exists \( c \) such that if \( G \) is a digraph with \( \omega(G) \leq \kappa \) and \( \chi(G) > c \) then some long hole of \( G \) is disoriented.

\textbf{Proof.} We proceed by induction on \( \kappa \); thus we may assume that \( \chi(J) \leq \tau \) for every digraph \( J \) with with \( \omega(J) < \kappa \) and no disoriented long hole. Let \( c = 2(3\tau)^5 \); we claim that \( c \) satisfies the theorem. Let \( G \) be a digraph with \( \omega(G) \leq \kappa \) and with no disoriented long hole.

(1) For each vertex \( z \) and integer \( r \geq 0 \), \( \chi(N^r(z)) \leq 3\tau \chi(N^{r-1}(z)) \). Consequently \( \chi(N^r(z)) \leq \tau(3\tau)^{r-1} \) and \( \chi(N^s(z)) \leq (3\tau)^{s-r} \chi(N^r(z)) \) for all \( s \geq r \).

Let us write \( L_i \) for \( N^i(z) \) \((i \geq 0)\). Since \( \omega(G[L_1]) \leq \kappa \) the result holds if \( r = 1 \), so we assume \( r \geq 2 \). Let \( I \subseteq L_{r-1} \) be stable. Let \( I_1 \) be the set of vertices in \( I \) with no in-neighbours in \( L_{r-2} \); \( I_2 \) the set with no out-neighbours in \( L_{r-2} \); and \( I_3 = I \setminus (I_1 \cup I_2) \). Let \( J_i \) be the set of vertices in \( L_r \) with a neighbour in \( I_i \) for \( i = 1, 2, 3 \). Let \( i \in \{1, 2, 3\} \), and suppose that \( \omega(G[J_i]) = \kappa \). Choose a clique \( K \) of \( G[J_i] \) with cardinality \( \kappa \), and take a minimal subset \( I_0 \) of \( I_i \) such that every vertex in \( K \) has a neighbour in \( I_0 \). Since \( \omega(G) = |K| \), it follows that \( |I_0| \geq 2 \); choose distinct \( v_1, v_2 \in I_0 \). From the minimality of \( I_0 \), there exists \( u_1 \in K \) \( G^r \)-adjacent with \( v_1 \) and not with \( v_2 \), and \( u_2 \) \( G^r \)-adjacent with \( v_2 \) and not with \( v_1 \). Thus \( v_1-u_1-u_2-v_2 \) is an induced path of \( G^r \). By reversing all edges of \( G \) if necessary (this is legitimate since what we are proving is invariant under this reversal) we may assume that \( i \in \{1, 3\} \).

Since \( G^r[L_0 \cup \cdots \cup L_{r-2}] \) is connected, there is an induced path of \( G^r \) joining \( v_1, v_2 \) with interior in this set, and its union with \( v_1-u_1-u_2-v_2 \) is a hole. We may assume this hole is either directed or alternating in \( G \), and in either case exactly one of the edges \( u_1v_1, u_2v_2 \) of \( G^r \) is oriented in \( G \) from \( L_r \) to \( L_{r-1} \). Consequently we may assume that \( v_1v_1 \) and \( u_2v_2 \) are edges of \( G \). Since \( i \in \{1, 3\} \), both \( v_1, v_2 \) have out-neighbours in \( L_{r-2} \), say \( w_1, w_2 \) respectively. If \( w_1 \) is \( G^r \)-adjacent with \( v_2 \), then adding \( w_1 \) to \( v_1, v_1, v_2, u_2 \) gives a hole of length five that is not directed, a contradiction; so \( v_1, v_2 \) are \( G^r \)-nonadjacent. In particular \( w_1 \neq w_2 \), so \( r \geq 3 \). If \( v_1, v_2 \) are \( G^r \)-nonadjacent, there is an induced path of \( G^r \) between \( w_1 \) and \( w_2 \) with interior in \( L_0 \cup \cdots \cup L_{r-3} \), and its union with \( w_1-v_1-u_1-u_2-v_2 \) yields a disoriented hole of \( G \), a contradiction; so \( v_1, v_2 \) are \( G^r \)-adjacent. This provides a hole of length five, which is therefore directed; so \( u_1w_2 \) and \( w_1v_2 \) are edges of \( G \). In particular \( v_1 \) has both an in-neighbour and an out-neighbour in \( L_{r-2} \), and so \( i = 3 \), and therefore \( v_2 \) has an in-neighbour \( x_2 \) say in \( L_{r-2} \). Since the path \( u_2v_2-w_2-x_2 \) is not directed, it follows that \( x_2, v_1 \) are \( G^r \)-nonadjacent, and in particular \( x_2 \neq w_1 \). Join \( w_1, x_2 \) by an induced path with interior in \( L_0 \cup \cdots \cup L_{r-3} \); then the union of this with the path \( w_1-v_1-u_1-u_2-v_2 \) yields a disoriented hole, a contradiction. This proves that \( \omega(G[J_i]) < \kappa \), and so \( \chi(J_i) \leq \tau \). Consequently \( \chi(J_1 \cup J_2 \cup J_3) \leq 3\tau \). Applying this to each colour class of a \( \chi(L_{r-1}) \)-colouring of \( G[L_{r-1}] \), we deduce the first assertion of (1). The second follows from the first by induction on \( r \), since \( \omega(G[L_1]) < \kappa \) and so \( \chi(L_1) \leq \tau \); and the third follows from the first by induction on \( s - r \). This proves (1).

Suppose that \( \chi(G) > c = 2(3\tau)^5 \). We may assume that \( G^r \) is connected; choose a vertex \( z \), and let \( L_i = N^i(z) \) for all \( i \geq 0 \). Choose \( s \) such that \( \chi(L_s) \geq \chi(G)/2 \). Since \( \chi(G) > 2(3\tau)^5 \), it follows that \( \chi(L_s) > (3\tau)^5 \) and so \( s \geq 6 \) by (1). Let \( S \) be the vertex set of a component of \( G[L_s] \) with maximum chromatic number. Let \( r = s - 4 \), and choose \( R \subseteq L_r \) minimal such that every vertex in \( S \) is joined to a vertex in \( R \) by a path in \( G^r \) of length 4. Let \( G' = G \setminus (L_r \setminus R) \). Thus \( N^r_{G'}(z) = R \), and \( S \subseteq N^r_{G'} \). By the third assertion of (1) applied to \( G' \), \( \chi(N^r_{G'}) \leq (3\tau)^4 \chi(N^r_{G'}) \), and since \( \chi(N^r_{G'}) > 3(3\tau)^4 \) it
follows that $\chi(R) > 2$ (indeed, $\chi(R) > 3$).

If $a \in R$ and $v \in S$, we say that $a$ is an ancestor of $v$ if there is a path of $G^*$ between $v$ and $a$ of length 4. From the minimality of $R$, for each $a \in R$ there is a vertex $v$ in $S$ such that $a$ is its unique ancestor; let $P_a$ be a path between $a$ and some such $v$ of length 4.

Let $R_1$ be the set of vertices $a \in R$ such that the edge of $P_a$ incident with $a$ has head $a$, and $R_2$ the set for which this edge has tail $a$. Since $\chi(R) > 2$, not both $R_1, R_2$ are stable, and by reversing all edges if necessary, we may assume that $R_1$ is not stable. Let $a_r, b_r \in R_1$ be $G^*$-adjacent. Let the vertices of $P_a_r$ be $a_r-a_{r+1}-\cdots-a_s$ in order, and let those of $P_{b_r}$ be $b_r-b_{r+1}-\cdots-b_s$ in order. Since $b_r$ is the unique ancestor of $b_s$, $b_i$ is $G^*$-nonadjacent with $a_{i-1}$ for $r+1 \leq i \leq s$, and similarly $a_i$ is $G^*$-nonadjacent with $b_{i-1}$ for $r+1 \leq i \leq s$. In particular, $a_i \neq b_i$ for $r \leq i \leq s$. (However, $a_i, b_i$ may be adjacent in $G^*$.)

(2) There is an induced path of $G^*$ between $a_{r+2}$ and $b_r$ containing no neighbours of $a_{r+1}, b_{r+1}, b_{r+2}$, and an induced path between $b_{r+2}$ and $a_r$ containing no neighbours of $a_{r+1}, b_{r+1}, a_{r+2}$.

Since $\chi(S) > 3\chi^5(G)$, there is a vertex $v \in S$ with distance at least 6 from each of $a_{r+1}, b_{r+1}, b_{r+2}$. Let $u \in R$ be an ancestor of $v$, and let $P$ be a path of length 4 between $u$ and $v$. Thus none of $a_{r+1}, b_{r+1}, b_{r+2}$ have neighbours in $V(P)$. Also there is an induced path between $u$ and $b_r$ with interior in $L_0 \cup \cdots \cup L_{r-1}$, an induced path between $a_s$ and $v$ with interior in $S$, and the path $a_{r+2} \cdots a_s$. The union of these paths gives a path of $G^*$ (not necessarily induced) between $a_{r+2}$ and $b_r$, and so there is an induced path of $G^*$ using a subset of the same vertices between $a_{r+2}$ and $b_r$. None of $a_{r+1}, b_{r+1}, b_{r+2}$ has a neighbour in any of these paths, and this proves the first statement. The second follows by symmetry. This proves (2).

Since $G^*[S]$ is connected, there is an induced path of $G^*$ between $a_{r+1}$ and $b_{r+1}$ with interior in $L_{r+2} \cup L_{r+3} \cup S$, and since the union of this path with the path $a_{r+1}a_r b_r b_{r+1}$ does not give a disoriented hole, it follows that $a_{r+1}, b_{r+1}$ are $G^*$-adjacent. By exchanging $a_r, b_r$ if necessary, we may assume that the edge $a_{r+1}b_{r+1}$ has head $b_{r+1}$. If the edge $a_{r+2}a_{r+1}$ has head $a_{r+2}$, the path $b_r b_{r+1} a_{r+1} a_{r+2}$ together with the first path of (2) gives a disoriented hole, a contradiction. So $a_{r+2} a_{r+1}$ has head $a_{r+1}$. If the edge $b_{r+2}b_{r+1}$ has head $b_{r+2}$, then the path $a_{r+1} a_{r+1} b_{r+1} b_{r+2}$ together with the second path of (2) gives a disoriented hole; so $b_{r+2} b_{r+1}$ has head $b_{r+1}$. There is an induced path joining $a_{r+2}, b_{r+2}$ with interior in $L_{r+3} \cup S$, and its union with $a_{r+2} a_{r+1} b_{r+1} b_{r+2}$ does not give a disoriented hole; so $a_{r+2}, b_{r+2}$ are $G^*$-adjacent. If the edge $a_{r+2} b_{r+2}$ has head $a_{r+2}$, the union of the path $a_{r+2} b_{r+2} b_{r+1} b_r$ with the first path of (2) gives a disoriented hole; while if $a_{r+2} b_{r+2}$ has head $b_{r+2}$, the union of $b_{r+2} a_{r+2} a_{r+1} a_r$ with the second path of (2) gives a disoriented hole. This proves 2.1.

3 Oriented stars

Now we turn to the proof of 1.2. If $v$ is a vertex of a digraph $G$, $N^+(v)$ denotes the set of out-neighbours of $V$, and $N^-(v)$ denotes the set of in-neighbours. Let us say a digraph $G$ is $\lambda$-spread if for every vertex $v$ of $G$, and for all $A \subseteq N^+(v)$ and $B \subseteq N^-(v)$ with $|A| = |B| = \lambda$, some vertex of $A$ is $G^*$-adjacent with some vertex of $B$. If $S$ is an oriented star and $\kappa \geq 0$, choose $\lambda$ such that every graph with at least $\lambda$ vertices has either a clique of cardinality more than $\kappa$ or a stable set.
of cardinality $|V(S)|$. It follows then that every digraph $G$ with $\omega(G) \leq \kappa$ not containing $S$ as an induced subdigraph is $\lambda$-spread. (For otherwise, with $v, A, B$ as above, there are stable subsets of $A, B$ each of cardinality $|V(S)|$, and an appropriate subset of $A \cup B$ together with $\{v\}$ induces $S$, a contradiction.) It is more convenient to replace the hypothesis that $G$ does not contain $S$ as an induced subdigraph with the hypothesis that $G$ is $\lambda$-spread. Thus, now we need to prove: for all $\kappa, \lambda \geq 0$, if $G$ is a $\lambda$-spread digraph with $\omega(G) \leq \kappa$ then the chromatic number of $G$ is bounded by some function of $\kappa, \lambda$. We will prove this by induction on $\kappa$, for fixed $\lambda$; so we will assume (throughout this section) that $\kappa, \lambda$ and $\tau$ are fixed integers satisfying

- $\kappa \geq 2, \lambda \geq 0$, and $\chi(J) \leq \tau$ for every $\lambda$-spread digraph $J$ with $\omega(J) < \kappa$.

If $G$ is a graph and $A, B \subseteq V(G)$, we say $A$ is $G$-complete with $B$ if $A \cap B = \emptyset$ and every vertex in $A$ is $G$-adjacent with every vertex in $B$. If $G$ is a digraph, we say $A$ is $G$-complete to $B$ and $B$ is complete from $A$ if $A \cap B = \emptyset$ and every vertex in $A$ is $G$-adjacent to every vertex of $B$. We need the following, which is an easy application of Ramsey’s theorem [14] and its bipartite version [2], and we omit its proof:

**3.1** For all $k, m$ there exists $n \geq 0$ with the following property. Let $A_1, \ldots, A_n, B_1, \ldots, B_n$ be pairwise disjoint subsets of the vertex set of a graph $G$, each of cardinality $m$. Then either

- there exist $A \subseteq A_1 \cup \cdots \cup A_n$ and $B \subseteq B_1 \cup \cdots \cup B_n$ with $|A| = |B| = \lambda$ such that no vertex in $A$ has a neighbour in $B$, or

- there exist $I, J \subseteq \{1, \ldots, n\}$ with $|I| = |J| = k$ such that $\bigcup_{i \in I} A_i$ is $G$-complete with $\bigcup_{j \in J} B_j$.

A $k$-clique means a clique of cardinality $k$. If $X$ is a clique of a digraph $G$, a vertex in $X$ is a source of $X$ if it is $G$-adjacent to every other vertex in $X$, and a sink if it is $G$-adjacent from every other vertex in $X$. If $k, m \geq 1$ are integers, a vertex $v$ of a digraph $G$ is $(k, m)$-rich if there exist $k$ pairwise disjoint $m$-cliques $A_1, \ldots, A_k \subseteq N^+(v)$, and $k$ pairwise disjoint $m$-cliques $B_1, \ldots, B_k \subseteq N^-(v)$, such that $A_1 \cup \cdots \cup A_k$ is $G^*$-complete with $B_1 \cup \cdots \cup B_k$.

**3.2** For all integers $k, m \geq 1$ there exists $t$ with the following property. Let $G$ be a $\lambda$-spread digraph such that no vertex of $G$ is $(k, m)$-rich. Then $V(G)$ can be partitioned into $t$ sets $X_1, \ldots, X_t$ such that for $1 \leq i \leq t$, either no $(m+1)$-clique of $G[X_i]$ has a source or no $(m+1)$-clique of $G[X_i]$ has a sink.

**Proof.** Choose $n$ such that 3.1 holds, and let $t = 4nm$. We claim that $t$ satisfies the theorem. For let $G$ be as in the theorem. Let $P$ be the set of vertices of $G$ such that there do not exist $n$ pairwise disjoint $m$-cliques in $N^+(v)$, and let $Q$ be the set such that there do not exist $n$ pairwise disjoint $m$-cliques in $N^-(v)$. Suppose first that some vertex $v$ belongs to neither of $P, Q$. Then there exist $n$ pairwise disjoint $m$-cliques $A_1, \ldots, A_n \subseteq N^+(v)$, and there exist $n$ pairwise disjoint $m$-cliques $B_1, \ldots, B_n \subseteq N^-(v)$. Since $G$ is $\lambda$-spread, 3.1 implies that there exist $I, J \subseteq \{1, \ldots, n\}$ with $|I| = |J| = k$ such that $\bigcup_{i \in I} A_i$ is $G^*$-complete with $\bigcup_{j \in J} B_j$, that is, $v$ is $(k, m)$-rich, a contradiction. This proves that $P \cup Q = V(G)$.

For each vertex $v \in P$, choose a maximal set of pairwise disjoint $m$-cliques included in $N^+(v)$, and let the union of the members of this set be $P_v$. Then each $|P_v| < nm$, and has nonempty intersection with every $m$-clique included in $N^+(v)$. Let $H$ be the digraph with vertex set $P$ and
Choose a function \( \phi \) such that for all \( k, m \geq 0 \), setting \( t = \phi(k, m) \) satisfies 3.2. Let \( \phi \) be fixed for the remainder of this section.

Next we prove 1.2 for acyclic digraphs (a digraph is acyclic if it has no directed cycle).

3.3 There exists \( c_0 \) such that \( \chi(G) \leq c_0 \) for every acyclic \( \lambda \)-spread digraph \( G \) with \( \omega(G) \leq \kappa \).

Proof. Let \( t = \phi(1, \kappa - 1) \), and let \( c_0 = t\tau \). We claim that \( c_0 \) satisfies the theorem.

Let \( G \) be a \( \lambda \)-spread acyclic digraph with \( \omega(G) \leq \kappa \). Now no vertex of \( G \) is \((1, \kappa - 1)\)-rich, because then \( G \) would have a clique of cardinality \( 2\kappa - 1 \geq \kappa \). By 3.2, \( V(G) \) can be partitioned into \( t \) sets \( X_1, \ldots, X_t \), such that for \( 1 \leq i \leq t \), either no \( \kappa \)-clique of \( G[X_i] \) has a source or no \( \kappa \)-clique of \( G[X_i] \) has a sink. But every \( \kappa \)-clique has both a source and a sink, since \( G \) is acyclic, and so \( \omega(G[X_i]) < \kappa \), and consequently \( \chi(X_i) \leq \tau \). Hence \( \chi(G) \leq t\tau = c_0 \). This proves 3.3.

A digraph \( G \) is \((h, k)\)-out-orderable if there is a partition \( X_1, \ldots, X_n \) of its vertex set, such that for \( 1 \leq i \leq n \), \( \chi(X_i) \leq h \), and each vertex of \( X_i \) has at most \( k - 1 \) out-neighbours in \( X_{i+1} \cup \cdots \cup X_n \).

We define \((h, k)\)-in-orderable similarly. If \( G \) is a digraph, we say \( X \subseteq V(G) \) is acyclic if \( G[X] \) is acyclic.

3.4 If the digraph \( G \) is \((h, k)\)-out-orderable, then there is a partition of \( V(G) \) into \( hk \) acyclic sets.

Proof. Let \( X_1, \ldots, X_n \) be as in the definition of \((h, k)\)-out-orderable. Let \( J \) be the graph with vertex set \( V(G) \) in which \( u, v \) are \( J \)-adjacent if \( u \) is \( G \)-adjacent to \( v \) and \( i \leq j \) where \( u \in X_i \) and \( v \in X_j \).

Since \( J[X_i] \) is \( h \)-colourable for each \( i \), there is a partition \( Y_1, \ldots, Y_h \) of \( V(J) \) such that \( X_i \cap Y_j \) is stable for \( 1 \leq i \leq n \) and \( 1 \leq j \leq h \). But every nonempty induced subgraph of \( J[Y_j] \) has a vertex with degree in \( J \) less than \( k \) (choose a vertex of \( Y_1 \) in \( X_i \) for the smallest \( i \) with \( X_i \cap Y_j \) nonempty); and so \( J[Y_j] \) is \((k - 1)\)-degenerate and hence \( k \)-colourable. Consequently \( \chi(J) \leq hk \). But for each stable set \( Y \) of \( J \), \( G[I] \) is acyclic. This proves 3.4.

A digraph \( G \) is \((h, k)\)-robust if for every nonempty subset \( Z \subseteq V(G) \) with \( \chi(Z) \leq h \), some vertex of \( Z \) has at least \( k \) out-neighbours in \( V(G) \setminus Z \) and at least \( k \) in-neighbours in \( V(G) \setminus Z \).

3.5 Let \( h, k \geq 0 \); then for every digraph \( G \) there is a partition of \( V(G) \) into three sets \( P, Q, R \) such that \( G[P] \) is \((h, k)\)-out-orderable, \( G[Q] \) is \((h, k)\)-in-orderable and \( G[R] \) is \((h, k)\)-robust.

Proof. We proceed by induction on \( |V(G)| \). If \( G \) is \((h, k)\)-robust we are done, so we may assume that there is a nonempty subset \( Z \subseteq V(G) \) with \( \chi(Z) \leq h \), such that for each \( v \in Z \), either \( |N^+(v) \setminus Z| < k \) or \( |N^-(v) \setminus Z| < k \).

Let \( X_1 \) be the set of vertices \( v \in Z \) such that \( |N^+(v) \setminus X_1| < k \), and \( Y_1 = Z \setminus X_1 \).

From the inductive hypothesis there is a partition \( P, Q, R \) of \( V(G) \setminus Z \) such that \( G[P] \) is \((h, k)\)-out-orderable, \( G[Q] \) is \((h, k)\)-in-orderable and \( G[R] \) is \((h, k)\)-robust. Let \( X_2, \ldots, X_n \) be a partition of
\[ P \text{ such that for } 2 \leq i \leq n, \chi(X_i) \leq h, \text{ and each vertex of } X_i \text{ has at most } k - 1 \text{ out-neighbours in } \bigcup_{i=1}^{n} X_i. \text{ Then the sequence } X_1, \ldots, X_n \text{ shows that } G[P \cup X_1] \text{ is } (h, k)\text{-out-orderable. Similarly } G[Q \cup Y_1] \text{ is } (h, k)\text{-in-orderable, and so the partition } P \cup X_1, Q \cup Y_1, R \text{ satisfies the theorem. This proves 3.5.} \]

We recall that \( \kappa, \lambda \) and \( \tau \) are fixed integers satisfying \( \kappa \geq 2, \lambda \geq 0, \) and \( \chi(J) \leq \tau \) for every \( \lambda \)-spread digraph \( J \) with \( \omega(J) < \kappa \). Let us define \( \Lambda = 2\lambda^2 + \lambda \) (throughout the remainder of this section).

3.6 Let \( G \) be a \( \lambda \)-spread digraph and let \( X \subseteq V(G) \) be nonempty. If every vertex in \( X \) has at least \( \Lambda \) out-neighbours in \( X \) and at least \( \Lambda \) in-neighbours in \( X \), then \( G \) is not \( (|X|\tau, |X| + \Lambda) \)-robust.

Proof. We claim first:

1. For each vertex \( v \) of \( G \), if \( A \subseteq N^+(v) \) and \( B \subseteq N^-(v) \) with \( |A| = \lambda \), then some vertex of \( A \) is \( G^* \)-adjacent with at least \( |B|/\lambda - 1 \) members of \( B \).

For there are fewer than \( \lambda \) members of \( B \) that have no \( G^* \)-neighbour in \( A \), since \( G \) is \( \lambda \)-spread. So all the others have at least one \( G^* \)-neighbour in \( A \); and so some vertex in \( A \) is \( G^* \)-adjacent with at least \( (|B| - \lambda)/\lambda \) of them. This proves (1).

Now let \( X \subseteq V(G) \) be nonempty, such that every vertex in \( X \) has at least \( \Lambda \) out-neighbours in \( X \) and at least \( \Lambda \) in-neighbours in \( X \). Let \( P \) be the set of vertices not in \( X \) with at least \( 2\lambda \) \( G^* \)-neighbours in \( X \).

2. For each \( u \in X \), \( u \) is \( G^* \)-adjacent with fewer than \( 2\lambda \) vertices in \( V(G) \setminus (P \cup X) \).

For suppose not; then from the symmetry we may assume that there is a set \( A \) of in-neighbours of \( u \) in \( V(G) \setminus (P \cup X) \), with \( |A| = \lambda \). But \( u \) has at least \( \Lambda \) out-neighbours in \( X \); and so by (1), some vertex in \( V(G) \setminus (P \cup X) \) has at least \( 2\lambda \) neighbours in \( X \), and therefore belongs to \( P \), a contradiction. This proves (2).

Suppose that there exists \( v \in P \) with at least \(|X| + \Lambda \) out-neighbours in \( V(G) \setminus P \) and at least \(|X| + \Lambda \) in-neighbours in \( V(G) \setminus P \). Since \( v \) has at least \( 2\lambda \) \( G^* \)-neighbours in \( X \), from the symmetry we may assume that \( v \) has at least \( \lambda \) out-neighbours in \( X \). Let \( Y \) be the set of vertices in \( V(G) \setminus (P \cup X) \) that are in-neighbours of \( v \). Then \(|Y| \geq \Lambda \). Since \( v \) has at least \( \lambda \) out-neighbours in \( X \), (1) implies that one of these out-neighbours, say \( u \), is \( G^* \)-adjacent with at least \(|Y|/\lambda - 1 \geq 2\lambda \) vertices in \( Y \), contrary to (2). Thus there is no such \( v \). But \( \chi(P) \leq |X|\tau \) since every vertex in \( P \) has a neighbour in \( X \); and so \( G \) is not \( (|X|\tau, |X| + \Lambda) \)-robust. This proves 3.6.

If \( G \) is a digraph and \( u, v, w \) are vertices, pairwise \( G^* \)-adjacent, such that one of them is \( G \)-adjacent from the other two, we call \( \{u, v, w\} \) a transitive triangle. Next we need:

3.7 There exists \( k_0 \) with the following property. Every non-null \( (3\Lambda\tau, k_0) \)-robust \( \lambda \)-spread digraph has a transitive triangle.
Proof. By the bipartite version of Ramsey’s theorem [2], for all \( n \geq 0 \) there exists \( f(n) \geq n \) such that for every partition of the edges of the complete bipartite graph \( K_{f(n), f(n)} \) into two sets, either the first set includes the edge set of a \( K_{n,n} \) subgraph, or the second set includes the edges of a \( K_{\lambda,\lambda} \)-subgraph. Let \( k_0 = f(f(\Lambda)) \). Suppose that \( G \) is a \((3\lambda, k_0)\)-robust \( \lambda \)-spread digraph with no transitive triangle. Since \( G \) is \((3\lambda, k_0)\)-robust, every vertex has at least \( k_0 \) out-neighbours and \( k_0 \) in-neighbours. Let \( v \in V(G) \); then since \( G \) is \( \lambda \)-spread, there exist \( A_1 \subseteq N^+(v) \) and \( B_1 \subseteq N^-(v) \) with \( |A_1| = |B_1| = f(f(\Lambda)) \) such that \( A_1 \) is \( G^* \)-complete with \( B_1 \); and since there is no transitive triangle it follows that every vertex in \( A_1 \) is \( G \)-adjacent to every vertex in \( B_1 \). Choose \( a \in A_1 \). Since \( a \) has at least \( k_0 \) in-neighbours, and none of them belong to \( A_1 \cup B_1 \) (because \( A_1 \) is stable since there is no transitive triangle) there is a set \( C \) of vertices in \( V(G) \setminus (A_1 \cup B_1) \) all \( G \)-adjacent to \( a \), with \( |C| = k_0 \). Since \( a \) is \( G \)-adjacent to every vertex in \( B_1 \), and \( |B_1|, |C| \geq f(f(\Lambda)) \), and there is no transitive triangle, there exist \( B_2 \subseteq B_1 \) and \( C_1 \subseteq C \) with \( |B_2| = \lambda \) and \( |C_1| = f(\Lambda) \) such that every vertex in \( B_2 \) is \( G \)-adjacent to every vertex in \( C_1 \). Choose \( b \in B_2 \). Since \( b \) is \( G \)-adjacent to every vertex in \( C_1 \) and from every vertex in \( A_1 \), and \( |A_1|, |C_1| \geq f(\Lambda) \), there exist \( A_2 \subseteq A_1 \) and \( C_2 \subseteq C_1 \) with \( |A_2|, |C_2| = \lambda \) such that \( A_2 \) is \( G \)-complete from \( C_2 \). Since \( |A_2|, |B_2|, |C_2| = \lambda \), every vertex in \( A_2 \cup B_2 \cup C_2 \) has at least \( \lambda \) out-neighbours and \( \Lambda \) in-neighbours in \( A_2 \cup B_2 \cup C_2 \), contrary to 3.6. This proves 3.7. \( \square \)

A tournament \( H \) is regular if all its vertices have the same outdegree, and they all have the same indegree; and it follows that \( |V(H)| \) is odd, \( |V(H)| = 2m + 1 \) say, and all vertices have indegree and outdegree \( m \). A tournament \( H \) is cyclic if it has an odd number of vertices, say \( 2m + 1 \), and its vertex set can be ordered as \( \{v_1, \ldots, v_{2m+1}\} \) such that for \( 1 \leq i < j \leq 2m + 1 \), \( v_i \) is \( H \)-adjacent to \( v_j \) if and only if \( j - i \leq m \).

3.8 Let \( H \) be a regular tournament with \( 2m + 1 \) vertices, and let \( v \in V(H) \); and suppose there is no directed cycle with vertices \( p-q-r-s-p \) in order such that \( p, r \) are out-neighbours of \( v \) and \( q, s \) are in-neighbours of \( v \). Then \( H \) is cyclic.

Proof. Let \( J \) be the subdigraph of \( H \) with vertex set \( V(H) \) and edge set all edges between \( N^+(v) \) and \( N^-(v) \). If \( J \) has a directed cycle, take the shortest such directed cycle \( C \); then \( C \) is induced and so has length four, a contradiction. Thus \( J \) has no directed cycle, and so \( V(H) \setminus \{v\} \) can be ordered as \( \{v_1, \ldots, v_{2m}\} \) such that for every edge of \( J \), its head is earlier than its tail. We claim that

- for \( i \) odd, \( v_i \in N^+(v) \) and \( v_i \) is \( H \)-adjacent from all vertices of \( N^+(v) \) except \( v_1, v_3, \ldots, v_{i-2} \); and
- for \( i \) even, \( v_i \in N^-(v) \) and \( v_i \) is \( H \)-adjacent from all vertices of \( N^-(v) \) except \( v_2, v_4, \ldots, v_{i-2} \).

We prove this claim by induction on \( i \). Suppose then that it holds for all smaller values of \( i \), and first suppose that \( v_i \in N^+(v) \). Then \( v_i \) is \( H \)-adjacent to \( v_h \) for all odd \( h < i \) (from the inductive hypothesis applied to \( v_h \)), and there are \( \lceil i/2 \rceil \) such values of \( h \). Moreover, \( v_i \) is \( H \)-adjacent to all vertices \( v_j \in N^-(v) \) with \( j > i \), from the property of the ordering. But \( |N^-(v)| = m \), and there are exactly \( \lceil i/2 \rceil - 1 \) values of \( j \) with \( j < i \) such that \( v_j \in N^-(v) \), from the inductive hypothesis, and so \( v_i \) has at least \( m + 1 - \lceil i/2 \rceil \) outneighbours in \( N^-(v) \). Since \( v_i \) has outdegree exactly \( m \) in \( H \), it must be the case that \( \lceil i/2 \rceil + m + 1 - \lceil i/2 \rceil \leq m \). So \( i \) is odd, and moreover \( v_i \) has no further outneighbours; and so \( v_i \) is \( H \)-adjacent from all vertices of \( N^+(v) \) except \( v_1, v_3, \ldots, v_{i-2} \) as claimed.
Now suppose that \( v_i \in N^-(v) \). Thus \( v_i \) is \( H \)-adjacent to \( v_h \) for all even \( h < i \) by the inductive hypothesis, and to every vertex of \( N^+(v) \) except for the vertices \( v_j \) with \( j \) odd and \( j < i \), because of the ordering. There are \( \lceil i/2 \rceil - 1 \) outneighbours of the first kind, and \( m - \lfloor i/2 \rfloor \) of the second kind; and in addition \( v_i \) is \( H \)-adjacent to \( v \). Consequently \( \lceil i/2 \rceil - 1 + m - \lfloor i/2 \rfloor + 1 \leq m \), and so \( i \) is even, and \( v_i \) is \( H \)-adjacent from every vertex of \( N^-(v) \) except \( v_2, v_4, \ldots, v_{i-2} \). This proves the inductive statement. But then the result follows. This proves 3.8.

\[ \]
from the definition of $t$ there exist $A \subseteq \{a_1^1, \ldots, a_n^1\}$ and $C \subseteq \{a_1^m, \ldots, a_n^m\}$ with $|A| = |C| = \Lambda$, such that $A$ is $G$-complete to $C$. Let $B = \{b_1^1, \ldots, b_1^\Lambda\}$; then $B$ is $G$-complete to $A$, and $C$ is $G$-complete to $B$. Consequently, every vertex in $A \cup B \cup C$ has at least $\Lambda$ out-neighbours and $\Lambda$ in-neighbours in this set. But this contradicts 3.6. This proves (1).

By 3.2, $V(G)$ can be partitioned into $\phi(k, m)$ subsets such that for each such subset $Y$ say, either no $(m+1)$-clique of $G[Y]$ has a source, or none has a sink. In either case it follows that $\omega(G[Y]) < \kappa$ and so $\chi(Y) \leq \tau$ and hence $\chi(G) \leq \phi(k, m)\tau = c_1$. This proves 3.9.

**Proof of 1.2.** As discussed at the beginning of this section, it suffices to show that for some $c \geq 0$, $\chi(G) \leq c$ for every $\lambda$-spread digraph $G$ with $\omega(G) \leq \kappa$. Let $c_0$ satisfy 3.3, let $c_1, k_1$ satisfy 3.9, and let $c = 4\Lambda\tau k_1 c_0 + \phi(1, \lceil \kappa/2 \rceil)c_1$.

Now let $G$ be a $\lambda$-spread digraph $G$ with $\omega(G) \leq \kappa$. By 3.5 and 3.4 there is a subset $R \subseteq V(G)$ such that $G[R]$ is $(4\Lambda\tau, k_1)$-robust and $V(G) \setminus R$ can be partitioned into $4\Lambda\tau k_1$ acyclic sets; and each of the latter induces a $c_0$-colourable digraph by 3.3. Thus $\chi(G) \leq 4\Lambda\tau k_1 c_0 + \chi(R)$, so it remains to bound $\chi(R)$.

No vertex is $(1, \lceil \kappa/2 \rceil)$-rich, since that would imply that $G$ contains a $(\kappa + 1)$-clique. By 3.2, $V(G)$ can be partitioned into $\phi(1, \lceil \kappa/2 \rceil)$ subsets such that for each such subset $Y$ say, either no $(\lceil \kappa/2 \rceil + 1)$-clique of $G[Y]$ has a source or none has a sink. From 3.9 it follows that $\chi(Y \cap R) \leq c_1$ for each $Y$, and so $\chi(R) \leq \phi(1, \lceil \kappa/2 \rceil)c_1$. Consequently $\chi(G) \leq 4\Lambda\tau k_1 c_0 + \phi(1, \lceil \kappa/2 \rceil)c_1 = c$. This proves 1.2.

**References**


