INDUCTED SUBGRAPHS AND TREE DECOMPOSITIONS
VI. ONE NEIGHBOR IN A HOLE
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Abstract. What are the unavoidable induced subgraphs of graphs with large treewidth? It
is well-known that the answer must include a complete graph, a complete bipartite graph, all
subdivisions of a wall and line graphs of all subdivisions of a wall (we refer to these graphs as
the “basic treewidth obstructions.”) So it is natural to ask whether graphs excluding the basic
treewidth obstructions as induced subgraphs have bounded treewidth. Sintiari and Trotignon
answered this question in negative. Their counterexamples, the so-called “layered wheels,”
contain wheels, where a wheel consists of a hole (i.e. an induced cycle of length at least four)
along with a vertex with at least three neighbors in the hole. This leads one to ask whether
wheels are essential to the construction of layered wheels, that is, whether graphs excluding
wheels and the basic treewidth obstructions as induced subgraphs have bounded treewidth.
This also turns out to be false due to Davies’ recent example of graphs with large treewidth and
no induced subgraph isomorphic to (fairly small) basic treewidth obstructions, in which every
vertex has at most two neighbors in every hole.
Here we prove that a hole with a vertex with at least two neighbors in it is inevitable in
a counterexample: graphs in which every vertex has at most one neighbor in every hole (that
does not contain it) and with the basic treewidth obstructions excluded as induced subgraphs
have bounded treewidth.

1. Introduction
All graphs in this paper are finite and simple. Let $H$ and $G$ be graphs. We say $G$ contains $H$ if $G$ has an induced subgraph isomorphic to $H$. We say $G$ is $H$-free if $G$ does not contain $H$. For a family of graphs $\mathcal{H}$, we say that $G$ is $\mathcal{H}$-free if $G$ is $H$-free for every $H \in \mathcal{H}$. A tree decomposition $(T, \chi)$ of $G$ consists of a tree $T$ and a map $\chi : V(T) \to 2^{V(G)}$ such that the following hold:

(i) For every vertex $v \in V(G)$, there exists $t \in V(T)$ such that $v \in \chi(t)$.
(ii) For every edge $v_1v_2 \in E(G)$, there exists $t \in V(T)$ such that $v_1, v_2 \in \chi(t)$.
(iii) For every $v \in V(G)$, the subgraph of $T$ induced by $\{t \in V(T) \mid v \in \chi(t)\}$ is connected.

If $(T, \chi)$ is a tree decomposition of $G$ and $V(T) = \{t_1, \ldots, t_n\}$, the sets $\chi(t_1), \ldots, \chi(t_n)$ are called the bags of $(T, \chi)$. The width of a tree decomposition $(T, \chi)$ is $\max_{t \in V(T)} |\chi(t)| - 1$. The treewidth of $G$, denoted $\text{tw}(G)$, is the minimum width of a tree decomposition of $G$.

Treewidth is an extensively-studied graph parameter, mostly due to the fact that graphs of bounded treewidth exhibit interesting structural [13] and algorithmic [7] properties. It is thus

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of interest the understand the unavoidable substructures emerging in graphs of large treewidth (these are often referred to as “obstructions to bounded treewidth”). For instance, for each \( k \), the \( (k \times k) \)-wall, denoted by \( W_{k \times k} \), is a planar graph with maximum degree three and with treewidth \( k \) (see Figure 1; a precise definition can be found in [2]). Every subdivision of \( W_{k \times k} \) is also a graph of treewidth \( k \). The unavoidable subgraphs of graphs with large treewidth are fully characterized by the Grid Theorem of Robertson and Seymour, the following.

**Theorem 1.1 ([12])**. There is a function \( f : \mathbb{N} \rightarrow \mathbb{N} \) such that every graph of treewidth at least \( f(k) \) contains a subdivision of \( W_{k \times k} \) as a subgraph.

![Figure 1. \( W_{5 \times 5} \)](image)

Following the same line of thought, our motivation in this series is to study induced subgraph obstructions to bounded treewidth. Together with subdivided walls mentioned above, complete graphs and complete bipartite graphs are easily observed to have arbitrarily large treewidth: the complete graph \( K_{t+1} \) and the complete bipartite graph \( K_{t,t} \) both have treewidth \( t \). Line graphs of subdivided walls form another family of graphs with unbounded treewidth, where the line graph \( L(F) \) of a graph \( F \) is the graph with vertex set \( E(F) \), such that two vertices of \( L(F) \) are adjacent if the corresponding edges of \( G \) share an end.

We call a family \( \mathcal{H} \) of graphs **useful** if there exists \( c(\mathcal{H}) \) such that every \( \mathcal{H} \)-free graph has treewidth at most \( c(\mathcal{H}) \). The discussion above can be summarized as follows:

**Theorem 1.2.** If \( \mathcal{H} \) is a useful family of graphs, then there exists an integer \( t \) such that \( \mathcal{H} \) contains \( K_t, K_{t,t} \), an induced subgraph of every subdivision of \( W_{t \times t} \) and an induced subgraph of the line graph of every subdivision of \( W_{t \times t} \).

The following was conjectured in [1] and proved in [11]:

**Theorem 1.3.** For all \( k, \Delta > 0 \), there exists \( c = c(k, \Delta) \) such that every graph with maximum degree at most \( \Delta \) and treewidth more than \( c \) contains a subdivision of \( W_{k \times k} \) or the line graph of a subdivision of \( W_{k \times k} \) as an induced subgraph.

The bounded-degree condition of Theorem 1.3 implies that \( K_{\Delta+2} \) and \( K_{\Delta+1, \Delta+1} \) are excluded. However, Theorem 1.3 does not hold if “bounded degree” is replaced by excluding \( K_{\Delta+2} \) and \( K_{\Delta+1, \Delta+1} \), as is evidenced by the constructions of [9] and [14]. Thus a natural question arises: what can replace this condition? Let us call a family \( \mathcal{F} \) of graphs **helpful** if the following holds: for all \( t > 0 \), there exists \( c = c(t) \) such that every \( \mathcal{F} \)-free graph with treewidth more than \( c \) contains \( K_t, K_{t,t} \), a subdivision of \( W_{t \times t} \) or the line graph of a subdivision of \( W_{t \times t} \).

A **hole** in a graph is an induced cycle of length at least four. The **length** of a hole is the number of vertices in it. A **wheel** is a graph consisting of a hole \( H \) and a vertex \( v \) with at least three neighbors in \( H \) (in the literature, sometimes further restrictions are placed on the location of the neighbors of \( v \) in \( H \)). In view of the prevalence of wheels in the construction of [14], one might ask if the family of all wheels is helpful. The answer to this question is negative, because of the construction of [9] (see Figure 2 for an example; we omit the precise definition), but the following weaker statement is true, and it is the main result of this paper. Let \( \mathcal{T}_t \) be the family of all graphs consisting of an (induced) cycle \( C \) of length at least four, and a vertex outside of \( C \) with at least two neighbors in \( C \).
Our main result is the following:

**Theorem 1.4.** The family $\mathcal{T}_1$ is helpful.

In fact, we prove something stronger. In the following, the length of a path is its number of edges. A **pyramid** is a graph consisting of a vertex $a$, a triangle $\{b_1, b_2, b_3\}$, and three paths $P_i$ from $a$ to $b_i$ for $1 \leq i \leq 3$ of length at least one, such that for $i \neq j$ the only edge between $P_i \setminus \{a\}$ and $P_j \setminus \{a\}$ is $b_i b_j$, and at most one of $P_1, P_2, P_3$ has length exactly one.

A **prism** is a graph consisting of two triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, and three paths $P_i$ from $a_i$ to $b_i$ for $1 \leq i \leq 3$, all of length at least one, and such that for $i \neq j$ the only edges between $P_i$ and $P_j$ are $a_i a_j$ and $b_i b_j$.

Let $\mathcal{T}_2$ be the family of all graphs consisting of an (induced) cycle $C$ of length at least four, and a vertex outside of $C$ with at least two non-adjacent neighbors in $C$, together with all prisms and all pyramids. Note that each graph in $\mathcal{T}_2$ contains a graph in $\mathcal{T}_1$. We prove:

**Theorem 1.5.** The family $\mathcal{T}_2$ is helpful.

Let us next restate Theorem 1.5 more explicitly. A graph is **sparse** if for every hole $H$ of $G$ and vertex $v \notin H$, there is an edge $ab$ of $H$ such that $N(v) \cap H \subseteq \{a, b\}$. A graphs is **very sparse** if it is sparse and also (pyramid, prism)-free. It follows that if $G$ is very sparse, then $G$ does not contain $K_{3,3}$ or the line graph of a subdivision of $W_{3 \times 3}$. Let $\mathcal{F}$ be the family of all very sparse graphs, and let $\mathcal{F}_t$ be the family of all very sparse graphs with no clique of size at least $t+1$.

We prove:

**Theorem 1.6.** For all $t > 0$, there exists $c = c(t)$ such that every graph in $\mathcal{F}_t$ with treewidth more than $c$ contains a subdivision of $W_{t \times t}$ as an induced subgraph.

The rough outline of the proof of Theorem 1.6 is as follows. Our first step is to show that if a graph in $\mathcal{F}_t$ contains a triangle, then it admits a clique cutset. Thus it is enough to prove the result for graphs in $\mathcal{F}_2$. Now let $G \in \mathcal{F}_2$. A **heavy seagull** in $G$ is an induced three-vertex path both of whose ends have degree at least three in $G$. First we prove that every heavy seagull of $G$ is “borken” by a two-clique-separation (this means that for every heavy seagull $H$ of $G$, there exist two cliques $K_1, K_2 \in G$ such that no component of $G \setminus (K_1 \cup K_2)$ contains $H$). Now the idea is to use the central bag method, developed in earlier papers in this series, to identify an induced subgraph $\beta$ of $G$ that contains no heavy seagull, and such that the treewidth of $G$ is not much larger than the treewidth of $\beta$. The key difference of our situation here from the earlier papers is that the cutsets we use to break the heavy seagulls are not connected, a property that was crucial in the earlier proofs. To deal with this difficulty, we change the definition of a central bag, including in it a path between the two cliques of the cutset whose interior is in $G \setminus \beta$ (this is in the spirit of, but different from, “marker paths” for 2-joins). We then modify the previously known central bag tools to work in this new setting. By “breaking” heavy seagulls, we arrange
that in $\beta$, vertices of degree at least three occur in components of bounded size. In this setting, we apply a theorem from [5] to get a bound on the treewidth of $\beta$, and the theorem follows.

1.1. Definitions and notation. Let $G$ be a graph. For $X \subseteq V(G)$, we denote by $G[X]$ the induced subgraph of $G$ with vertex set $X$, and $G \setminus X$ denotes $G[V(G) \setminus X]$. In this paper we use the set $X$ and the subgraph $G[X]$ of $G$ interchangeably. If $F$ is a graph and $G[X]$ is isomorphic to $F$, we say that $X$ is an $F$ in $G$. Let $v \in V(G)$. The open neighborhood of $v$, denoted $N(v)$, is the set of all vertices in $V(G)$ adjacent to $v$. We denote the degree of $v$ in $G$ by $\deg_G(v) = |N(v)|$. The closed neighborhood of $v$, denoted $N[v]$, is $N(v) \cup \{v\}$. Let $X \subseteq V(G)$. The open neighborhood of $X$, denoted $N(X)$, is the set of all vertices in $V(G) \setminus X$ with a neighbor in $X$. The closed neighborhood of $X$, denoted $N[X]$, is $N(X) \cup X$. If $H$ is an induced subgraph of $G$ and $X \subseteq V(G)$, then $N_H(X) = N(X) \cap H$. Let $Y \subseteq V(G)$ be disjoint from $X$. Then, $X$ is complete to $Y$ if every vertex of $X$ is adjacent to every vertex of $Y$, and $X$ is anticomplete to $Y$ if there are no edges between $X$ and $Y$. We use $X \cup v$ to mean $X \cup \{v\}$, and $X \setminus v$ to mean $X \setminus \{v\}$.

Given a graph $G$, a path in $G$ is an induced subgraph of $G$ that is a path. If $P$ is a path in $G$, we write $P = p_1 \cdots p_k$ to mean that $p_i$ is adjacent to $p_j$ if and only if $|i - j| = 1$. We call the vertices $p_1$ and $p_k$ the ends of $P$, and say that $P$ is from $p_1$ to $p_k$. The interior of $P$, denoted by $P^*$, is the set $P \setminus \{p_1, p_k\}$. The length of a path $P$ is the number of edges in $P$.

A theta is a graph $T$ containing two vertices $a, b$ and three paths $P_1, P_2, P_3$ from $a$ to $b$ of length at least two, such that $P_1 \setminus \{a, b\}, P_2 \setminus \{a, b\}, P_3 \setminus \{a, b\}$ are disjoint and anticomplete to each other. We call $a, b$ the ends of $T$.

1.2. Organization of the paper. This paper is organized as follows. In Section 2, we give general background and definitions related to separations in graphs; we also discuss connections between different kinds of separations in the special case of sparse graphs. In Section 3, we reduce Theorem 1.6 to the case of triangle-free sparse graphs. In Section 4, we discuss balanced separators in graphs, and develop our main tool, Theorem 4.5, which allows us to use the central bag method. In Section 5, we prove results about two-clique-separations, which are the cutsets that will used to form the central bag. In Section 6, we prove structural results that allow us to break every heavy seagull in a triangle-free sparse graph and produce a central bag that contains no heavy seagulls. In Section 7, we use the tools of Section 4 to prove our main result for graphs in $\mathcal{F}_2$. Finally, in Section 8, we prove Theorem 1.6.

2. Separations

A separation of a graph $G$ is a triple $(A, C, B)$, where $A, B, C \subseteq V(G)$, $A \cup C \cup B = V(G)$, $A$, $B$, and $C$ are pairwise disjoint, and $A$ is anticomplete to $B$. If $S = (A, C, B)$ is a separation, we let $A(S) = A$, $B(S) = B$, and $C(S) = C$. We say that $C \subseteq V(G)$ is a cutset of $G$ if there exists a separation $(A, C, B)$ of $G$ with $A \neq \emptyset$ and $B \neq \emptyset$. We say that $G$ admits a clique cutset of $G$ that is a clique. A separation $(A, C, B)$ is a star separation if there exists $v \in C$ such that $C \subseteq N[v]$ (we say that $v$ is a center of $C$). A star separation $(A, C, B)$ is proper if $A \neq \emptyset$ and $B \neq \emptyset$. We say that $G$ admits a star cutset if there is a proper star separation in $G$.

First we observe:

Lemma 2.1. Let $G$ be a sparse graph and $(A, C, B)$ be a separation of $G$ with $A \neq \emptyset$ and $B \neq \emptyset$. Suppose that there exist $v_1, \ldots, v_k \in C$ such that $C \subseteq \bigcup_{i=1}^k N[v_i]$. Let $D_1$ be a component of $A$ and let $D_2$ be a component of $B$. Then there exist cliques $X_1, \ldots, X_k \subseteq C$ of $G$ such that every path from a vertex of $D_1$ to a vertex of $D_2$ meets $\bigcup_{i=1}^k X_i$. In particular, if $G$ admits a star cutset, then $G$ admits a clique cutset.
Lemma 3.3. Let $N_1 = N(D_1) \subseteq C$, and let $D'_2$ be the component of $G \setminus (N_1 \cup \{v_1, \ldots, v_k\})$ such that $D_2 \subseteq D'_2$. Let $X = N(D'_2) \cup \{v_1, \ldots, v_k\}$. Then $X \subseteq N_1 \cup \{v_1, \ldots, v_k\} \subseteq C$, and every path from a vertex of $D_1$ to a vertex of $D'_2$ in $G$ meets $X$. We claim that for every $i \in \{1, \ldots, k\}$ the set $X \cap N[v_i]$ is a clique. Suppose not, and let $x, y \in X \cap N[v_i]$ (say) be non-adjacent (and so in particular, $x, y \not\in v_1$). It follows that $x, y \in N(D_1) \cap N(D'_2)$. Let $P_1$ be a path from $x$ to $y$ with $P_1 \subseteq D_1$ and let $P_2$ be a path from $x$ to $y$ with $P_2 \subseteq D'_2$. Then $H = x-P_1-y-P_2-x$ is a hole and $v_1 \not\in H$ since $v_1 \in X$. But now $v_1$ has two non-adjacent neighbors in $H$, contrary to the fact that $G$ is sparse. This proves Lemma 2.1.

A special case of Lemma 3.1 from [6] shows that clique cutsets do not affect treewidth. Now, by Lemma 2.1, it follows that in order to prove Theorem 1.6 it is enough to prove the following:

Theorem 2.2. For all $t > 0$, there exists $c = c(t)$ such that every graph in $F_t$ with treewidth more than $c$ and with no star cutset contains a subdivision of $W_{1\times 1}$ as an induced subgraph.

3. REDUCING TO THE TRIANGLE-FREE CASE

In this section we show how to deduce Theorem 1.6 from the special case of triangle-free graphs. A diamond is the graph obtained from $K_4$ by removing an edge.

Lemma 3.1. Let $G$ be a sparse graph and assume that $G$ does not admit a star cutset. Then $G$ is diamond-free.

Proof. Suppose first $\{a, b, c, d\}$ is a diamond in $G$. We may assume that the pairs $ab, bc, cd, ad, ac$ are adjacent and the pair $bd$ is non-adjacent. Since $b$ is not the center of a star cutset in $G$, it follows that there exists a path from $a$ to $c$ with no neighbor of $b$ in its interior. Let $P$ be such a path. Then $d$ is not a vertex of $P$, since $d$ is adjacent to $b$. Moreover, $a-P-c-b-a$ is a hole, and $d$ has two non-adjacent neighbors in it, namely $a$ and $c$. This contradicts that $G$ is diamond-free.

We also need the following folklore result that appeared in [3]:

Lemma 3.2. Let $x_1, x_2, x_3$ be three distinct vertices of a graph $G$. Assume that $H$ is a connected induced subgraph of $G \setminus \{x_1, x_2, x_3\}$ such that $V(H)$ contains at least one neighbor of each of $x_1, x_2, x_3$, and that $V(H)$ is minimal subject to inclusion. Then, one of the following holds:

(i) For some distinct $i, j, k \in \{1, 2, 3\}$, there exists $P$ that is either a path from $x_i$ to $x_j$ or a hole containing the edge $x_ix_j$ such that

- $V(H) = V(P) \setminus \{x_i, x_j\}$, and
- either $x_k$ has two non-adjacent neighbors in $H$ or $x_k$ has exactly two neighbors in $H$ and its neighbors in $H$ are adjacent.

(ii) There exists a vertex $a \in V(H)$ and three paths $P_1, P_2, P_3$, where $P_i$ is from $a$ to $x_i$, such that

- $V(H) = (V(P_1) \cup V(P_2) \cup V(P_3)) \setminus \{x_1, x_2, x_3\}$, and
- the sets $V(P_1) \setminus \{a\}, V(P_2) \setminus \{a\}$ and $V(P_3) \setminus \{a\}$ are pairwise disjoint, and
- for distinct $i, j \in \{1, 2, 3\}$, there are no edges between $V(P_i) \setminus \{a\}$ and $V(P_j) \setminus \{a\}$, except possibly $x_ix_j$.

(iii) There exists a triangle $a_1a_2a_3$ in $H$ and three paths $P_1, P_2, P_3$, where $P_i$ is from $a_i$ to $x_i$, such that

- $V(H) = (V(P_1) \cup V(P_2) \cup V(P_3)) \setminus \{x_1, x_2, x_3\}$, and
- the sets $V(P_1), V(P_2)$ and $V(P_3)$ are pairwise disjoint, and
- for distinct $i, j \in \{1, 2, 3\}$, there are no edges between $V(P_i)$ and $V(P_j)$, except $a_ia_j$ and possibly $x_ix_j$.

Lemma 3.3. Let $G$ be a complete graph. Then either $G \in F_2$, $G$ is a complete graph, or $G$ admits a star cutset.
Lemma 4.1

of (how balanced separator not admit a clique cutset, it follows that \( D \) is connected, non-empty, and every vertex of \( K \) has a neighbor in \( D \). By Lemma 3.1, it follows that \( G \) does not contain a diamond.

(1) Let \( v \in D \). Then \( v \) has at most one neighbor in \( K \).

Suppose not. Let \( v \in D \) and assume that \( v \) has at least two neighbors in \( K \), say \( k_1 \) and \( k_2 \). Since \( K \) is a maximal clique, there exists \( k_3 \in K \) non-adjacent to \( v \). But now \( \{v, k_1, k_2, k_3\} \) is a diamond, a contradiction. This proves (1).

Now let \( x_1, x_2, x_3 \in K \). Apply Lemma 3.2 to \( \{x_1, x_2, x_3\} \) and a minimal connected subgraph \( F \) of \( D \) containing at least one neighbor of each of \( x_1, x_2, x_3 \). By (1), we have that \( |V(F)| \geq 3 \).

Now the first outcome of Lemma 3.2 gives a hole and a vertex with two non-adjacent neighbors in it, the second outcome gives a pyramid, and the third gives a prism. In all cases we get a contradiction to the fact that \( G \in F \). This proves Lemma 3.3.

Now, by Lemma 3.3, in order to prove Theorem 2.2 it is enough to prove:

**Theorem 3.4.** For all \( k \), there exists \( c = c(k) \) such that every graph in \( F_2 \) with no star cutset and with treewidth more than \( c \) contains a subdivision of \( W_{k \times k} \) as an induced subgraph.

4. Balanced separators and central bags

Let \( G \) be a graph, and let \( w : V(G) \to [0, 1] \). For \( X \subseteq V(G) \), we write \( w(X) \) for \( \sum_{x \in X} w(x) \).

We call \( w \) a weight function on \( G \) if \( w|_V = 1 \). Now let \( c \in [\frac{1}{2}, 1) \). A set \( X \subseteq V(G) \) is a \((w, c)\)-balanced separator if \( w(D) \leq c \) for every component \( D \) of \( G \setminus X \). The next two lemmas show how \((w, c)\)-balanced separators relate to treewidth. The first result was originally proved by Harvey and Wood in [10] using different language, and was restated and proved in the language of \((w, c)\)-balanced separators in [2].

**Lemma 4.1 ([2, 10]).** Let \( G \) be a graph, let \( c \in [\frac{1}{2}, 1) \), and let \( k \) be a positive integer. If \( G \) has a \((w, c)\)-balanced separator of size at most \( k \) for every weight function \( w \) on \( G \), then \( tw(G) \leq \frac{1}{1-c} k \).

**Lemma 4.2 ([8]).** Let \( G \) be a graph and let \( k \) be a positive integer. If \( tw(G) \leq k \), then \( G \) has a \((w, c)\)-balanced separator of size at most \( k + 1 \) for every \( c \in [\frac{1}{2}, 1) \) and for every weight function \( w \) on \( G \).

A pair \((G, w)\) is \( d\)-unbalanced if \( w \) is a weight function on \( G \), and \( G \) has no \((w, \frac{1}{2})\)-balanced separator of size at most \( d \). Let \( K \) be an integer, let \( G \) be a graph and let \( K_1, K_2 \) be two cliques of \( G \), each of size at most \( K \). Let \((G, w)\) be a \( 2K \)-unbalanced pair. Following [4], we define the canonical two-clique-separation for \( \{K_1, K_2\} \), as follows. Let \( B(K_1, K_2) \) be a component of \( G \setminus (K_1 \cup K_2) \) with \( w(B(K_1, K_2)) \) maximum. Since \( (G, w) \) is \( 2K \)-unbalanced, it follows that \( w(B(K_1, K_2)) > \frac{1}{2} \), and so the choice of \( B(K_1, K_2) \) is unique. Let \( A(K_1, K_2) = G \setminus (B(K_1, K_2) \cup K_1 \cup K_2) \) and \( C(K_1, K_2) = K_1 \cup K_2 \). Now \( S(K_1, K_2) = (A(K_1, K_2), K_1 \cup K_2, B(K_1, K_2)) \) is the canonical two-clique-separation corresponding to \( \{K_1, K_2\} \).

Let \( K_1^1, K_1^2, K_2^1, K_2^2 \) be cliques in \( G \). For \( i \in \{1, 2\} \), let \( S_i = (A_i, C_i, B_i) \) be the canonical two-clique-separation for \( \{K_1^i, K_2^i\} \). We say that \( (A_1, C_1, B_1) \) and \( (A_2, C_2, B_2) \) are non-crossing if \( A_1 \cup C_1 \subseteq B_2 \cup C_2 \) and \( A_2 \cup C_2 \subseteq B_1 \cup C_1 \), and that \( (A_1, C_1, B_1) \) and \( (A_2, C_2, B_2) \) are loosely non-crossing if \( A_1 \cap C_2 = A_2 \cap C_1 = \emptyset \). Clearly, if \( S_1 \) and \( S_2 \) are non-crossing, then they are loosely non-crossing. (Note that here we break the symmetry between \( A_i \) and \( B_i \), and so our definition is slightly different from the classical definition of [12].)

For the remainder of this section, let \( K \) be an integer, and let \((G, w)\) be a \( 2K \)-unbalanced pair. We observe:
Lemma 4.3. Assume that $G$ does not admit a star cutset. Let $K_1, K_2$ be cliques of size at most $K$ in $G$ such that $A(K_1, K_2) \neq \emptyset$. The following hold.

(1) $K_1 \cap K_2 = \emptyset$.

(2) Let $D$ be a component of $G \setminus (K_1 \cup K_2)$. Then $N(D) \cap K_i \neq \emptyset$ for all $i \in \{1, 2\}$, and so there is a path from a vertex of $K_1$ to a vertex of $K_2$ with interior in $D$.

Throughout this section, let $S$ be a set of sets $\{K_1, K_2\}$ where each of $K_1, K_2$ is a clique of size at most $K$ of $G$, and let $\mathcal{T}$ be the set of canonical two-clique-separations corresponding to members of $S$. Assume for the remainder of this section that every pair of separations in $\mathcal{T}$ is loosely non-crossing. Let $\pi$ be a total order on $S$. Since every pair of separations in $\mathcal{T}$ is loosely non-crossing, every component of $\bigcup_{S \in \mathcal{T}} A(S)$ is contained in some $A(K_1, K_2)$ with $\{K_1, K_2\} \in S$. For $\{K_1, K_2\} \in S$, let $A^+(K_1, K_2)$ be the union of all components $D$ of $A(K_1, K_2)$ such that $D \subseteq A(K_1, K_2)$ and $\{K_1, K_2\}$ is $\pi$-minimal with this property. By Lemma 4.3, for every $\{K_1, K_2\}$, $A^+(K_1, K_2) \neq \emptyset$, there exists a path $P_{K_1, K_2}$ from a vertex of $K_1$ to a vertex of $K_2$ with interior in $A^+(K_1, K_2)$. Let $\mathcal{S}' = \{\{K_1, K_2\} \in S \mid A^+(K_1, K_2) \neq \emptyset\}$.

Write $\beta = \bigcap_{\{K_1, K_2\} \in \mathcal{S}'} (B(K_1, K_2) \cup K_1 \cup K_2) \cup \bigcup_{\{K_1, K_2\} \in \mathcal{S}'} P_{K_1, K_2}$.

We call $\beta$ a central bag for $S$. Note that the choice of $\beta$ is not unique since the choice of the paths $P_{K_1, K_2}$ is not unique.

Let $w_\beta$ be the function on defined as follows. For $v \in \bigcap_{\{K_1, K_2\} \in \mathcal{S}'} (B(K_1, K_2) \cup K_1 \cup K_2)$, we set $w_\beta(v) = w(v)$. Next let $\{K_1, K_2\} \in \mathcal{S}'$, and let $a_{K_1, K_2}$ be a neighbor of a vertex of $K_1$ in $P_{K_1, K_2}$; set $w_\beta(a_{K_1, K_2}) = w(A^+(K_1, K_2))$. Let $w_\beta(v) = 0$ for every $v \in \beta$ where $w_\beta$ has not been defined yet. We call $w_\beta$ the weight function inherited from $w$.

Lemma 4.4. The function $w_\beta$ is a weight function, that is, $w_\beta(\beta) = 1$.

Proof. Since every pair of separations in $\mathcal{T}$ is loosely non-crossing, it follows that the sets $\bigcap_{\{K_1, K_2\} \in \mathcal{S}'} (B(K_1, K_2) \cup K_1 \cup K_2)$ and $A^+(K_1, K_2)$ for $\{K_1, K_2\} \in \mathcal{S}'$ are pairwise disjoint and have union $V(G)$. It follows that the sets $P_{K_1, K_2}$ are pairwise disjoint, and $\Sigma_{\{K_1, K_2\} \in \mathcal{S}'} w_\beta(a_{K_1, K_2}) = \Sigma_{\{K_1, K_2\} \in \mathcal{S}'} w(A^+(K_1, K_2))$. Consequently $w(\beta) = w(G) = 1$, as required. \hfill \blacksquare

For $v \in V(G)$, let $\delta_S(v) = \bigcup_{K: v \in K \text{ and there exists } L \text{ such that } \{K, L\} \in S} K$.

Theorem 4.5. Let $d, \Delta$ be integers. Assume that $|\delta_S(v)| \leq \Delta$ for every $v \in G$. Assume also that $(\beta, w_\beta)$ is $d$-balanced. Then $(G, w)$ is max$(2Kd, \Delta d)$-balanced.

Proof. Suppose that $X$ is a $(w_\beta, \frac{1}{2})$-balanced separator in $\beta$ with $|X| \leq d$. Let

$$Y_1 = X \cap \left( \bigcap_{\{K_1, K_2\} \in \mathcal{S}'} (B(K_1, K_2) \cup K_1 \cup K_2) \right).$$

For $x \in Y_1$, let $Y(x) = \delta_S(x)$. Now let $x \in X \setminus Y_1$. Since every two separations in $\mathcal{T}$ are loosely non-crossing, it follows that $x \in P_{K_1, K_2}$ for exactly one $\{K_1, K_2\} \in \mathcal{S}'$; let $Y(x) = K_1 \cup K_2$. Let $Y = \bigcup_{x \in X} Y(x)$. Then $|Y| \leq \Delta |Y_1| + 2K(d - |Y_1|) \leq \max(\Delta d, 2Kd)$, as required. Next we prove that $Y$ is a $(w, \frac{1}{2})$-balanced separator of $G$. 


(2) Let $F$ be a component of $G \setminus \beta$. Then, there exists $\{K_1, K_2\} \in S$ such that $F \subseteq A^*(K_1, K_2)$.

By construction of $\beta$, it holds that $G \setminus \beta \subseteq \bigcup_{(K_1, K_2) \in S} A(K_1, K_2)$. Since every pair of separations in $T$ is loosely non-crossing, it follows that there exists $\{K_1, K_2\} \in S$ such that $F \subseteq A^*(K_1, K_2)$. This proves (2).

From now on, let $D$ be a component of $G \setminus Y$. We will show that $w(D) \leq \frac{1}{2}$. Since $(G, w)$ is $2K$-unbalanced, it follows that $w(A(K_1, K_2)) < \frac{1}{2}$ for all $\{K_1, K_2\} \in S$, and so if $D$ is a component of $G \setminus \beta$, then by (2), it follows that $w(D) \leq \frac{1}{2}$. Thus we may assume that $D \cap \beta = \emptyset$.

Suppose first that $D \cap A(K_1, K_2) \neq \emptyset$ for some $\{K_1, K_2\} \in S$ such that $K_1 \cup K_2 \subseteq Y$. Since $N(A(K_1, K_2)) \subseteq K_1 \cup K_2$ and $K_1 \cup K_2 \subseteq Y$, it follows that $D \subseteq A(K_1, K_2)$, and so $w(D) < \frac{1}{2}$. Therefore, we may assume that $D \cap A(K_1, K_2) = \emptyset$ for all $\{K_1, K_2\} \in S$ such that $K_1 \cup K_2 \subseteq Y$. Next, suppose $D \cap A(K_1, K_2) = \emptyset$ for $\{K_1, K_2\} \in S'$ such that $P^*_{K_1, K_2} \cap X \neq \emptyset$.

Let $x \in P^*_{K_1, K_2} \cap X$. Now, $x \in X \setminus Y_1$, and so $Y(x) = K_1 \cup K_2 \subseteq Y$, a contradiction. Therefore, we may assume that for all $\{K_1, K_2\} \in S'$ such that $D \cap A(K_1, K_2) = \emptyset$, it holds that $P^*_{K_1, K_2}$ is disjoint from $X$, and thus $P^*_{K_1, K_2}$ is contained in a component of $\beta \setminus X$. Let $Q_1, \ldots, Q_m$ be the components of $\beta \setminus X$.

(3) Let $\{K_1, K_2\} \in S'$, and suppose that $P^*_{K_1, K_2} \subseteq Q_k$. Then $K_1 \cup K_2 \subseteq Q_k \cup Y$.

Since $N(P^*_{K_1, K_2}) \cap K_i \neq \emptyset$ for each $i \in \{1, 2\}$, it follows that each of $K_1, K_2$ either is contained in $Q_k$ or has a vertex in $X$. Since every two separations in $S$ are loosely non-crossing, it follows that each of $K_1, K_2$ is either contained in $Q_k$ or has a vertex in $Y_1$. Since $\delta_S(x) \subseteq Y$ for every $x \in Y_1$, it follows that for $i \in \{1, 2\}$, if $K_i \cap Y_1 \neq \emptyset$, then $K_i \subseteq Y$. This proves (3).

(4) Let $\{K_1, K_2\} \in S'$, and suppose that $N(A(K_1, K_2)) \cap Q_k \neq \emptyset$. Then either $K_1 \cup K_2 \subseteq Y$, or $P^*_{K_1, K_2} \subseteq Q_k$. In particular, if $K_1 \cup K_2 \subseteq Y$, then there is at most one $k' \in \{1, \ldots, m\}$ with $N(A(K_1, K_2)) \subseteq Q_{k'} \cup Y$, and thus $N(A(K_1, K_2)) \subseteq Q_{k'} \cup Y$, a contradiction. This proves (4).

Since $D \cap \beta = \emptyset$, it follows that for each $\{K_1, K_2\} \in S'$ with $D \cap A(K_1, K_2) = \emptyset$, we have $D \cap N(A(K_1, K_2)) = \emptyset$, and in particular $(K_1 \cup K_2) \cap D = \emptyset$, so $K_1 \cup K_2 \subseteq Y$. Moreover, from (4), it follows that $P^*_{K_1, K_2} \subseteq Q_k$ for some $k \in \{1, \ldots, m\}$, and $N(A(K_1, K_2)) \cap Q_{k'} = \emptyset$ for all $k' \neq k$. Since $D$ is connected, it follows that there is a $k \in \{1, \ldots, m\}$ such that for every $\{K_1, K_2\} \in S'$ with $D \cap A(K_1, K_2) = \emptyset$, we have $N(A(K_1, K_2)) \subseteq Q_k \cup Y$, and $P^*_{K_1, K_2} \subseteq Q_k$. It follows that $D \cap \beta \subseteq Q_k$, and $d_{K_1, K_2} \subseteq Q_k$ for all such $\{K_1, K_2\} \in S'$, and therefore $w(D) \leq w_\beta(Q_k) \leq \frac{1}{2}$. This concludes the proof.

Let $K_1, K_2$ be cliques of size at most $K$ in $G$. We say that $S(K_1, K_2)$ is proper (or that the pair $\{K_1, K_2\}$ is proper) if some component $D$ of $A(K_1, K_2)$ satisfies $K_1 \cup K_2 \subseteq N(D)$ and moreover, if $|K_1| = |K_2| = 1$, then $A(K_1, K_2) \cup K_1 \cup K_2$ is not a path from the vertex of $K_1$ to the vertex of $K_2$. We observe:

**Lemma 4.6.** Let $K_1, K_2$ be cliques of size at most $K$ in $G$ and assume that $S(K_1, K_2)$ is a proper two-clique-separation in $G$. Then either some vertex of $A(K_1, K_2)$ has at least three neighbors in $A(K_1, K_2) \cup K_1 \cup K_2$, or some vertex of $K_1 \cup K_2$ has at least two neighbors in $A(K_1, K_2)$. 
theorem holds. Thus we may assume that $K$ for every pair of cliques $A$ and $B$ such that $A \cap B = \emptyset$.

Proof. Let $D$ be a component of $A(K_1, K_2)$ such that $K_1 \cup K_2 \subseteq \mathcal{N}(D)$. Then $\mathcal{N}[D]$ has a spanning tree $T$ such that every vertex of $K_1 \cup K_2$ is a leaf of $T$. If $|K_1| > 1$, then $T$ has at least three leaves, and therefore some vertex of $D$ has degree at least three in $\mathcal{N}[D]$ as required. Thus we may assume that $|K_1| = |K_2| = 1$. If $\mathcal{N}[D]$ is not a path from the vertex of $K_1$ to the vertex of $K_2$, then some vertex of $D$ has at least three neighbors in $\mathcal{N}[D]$, and again the theorem holds. Thus we may assume that $\mathcal{N}[D]$ is a path from the vertex of $K_1$ to the vertex of $K_2$. Since $S(K_1, K_2)$ is proper, $A(K_1, K_2) \neq D$. Let $D'$ be a component of $A(K_1, K_2) \setminus D$. By Lemma 4.3, we have that $K_1 \subseteq \mathcal{N}(D')$. But then the vertex of $K_1$ has at least two neighbors in $A_1 \cup A_2$ as required. This proves Lemma 4.6.

We say that $S(K_1, K_2)$ is active (or that the pair $\{K_1, K_2\}$ is active) if it is proper and for every pair of cliques $K_1', K_2'$ of size at most $K$ in $G$ such that $S(K_1', K_2')$ is proper and $K_1 \cup K_2 \neq K_1' \cup K_2'$, it holds that

- $B(K_1', K_2') \cup K_1' \cup K_2' \neq \emptyset$ is not a proper subtree of $B(K_1, K_2) \cup K_1 \cup K_2$; and
- if $B(K_1', K_2') \cup K_1' \cup K_2' = B(K_1, K_2) \cup K_1 \cup K_2$, then $B(K_1', K_2') \subset B(K_1, K_2)$.

Lemma 4.7. Let $K_1, K_2$ be cliques of $G$ of size at most $K$. If $S(K_1, K_2)$ is active, then $K_1 \cup K_2 \subseteq \mathcal{N}(B(K_1, K_2))$.

Proof. Suppose not. We may assume that there exists $x \in K_1$ such that has no neighbor in $B(K_1, K_2)$. Then $(A(K_1, K_2) \cup \{x\}, (K_1 \cup K_2) \setminus \{x\}, B(K_1, K_2))$ is a proper two-clique-separation of $G$ contrary to the fact that $S$ is active. This proves Lemma 4.7.

5. Two-clique-separations

The main result of this section will allow us to apply Theorem 4.5 with $K = 2$:

Theorem 5.1. Let $G \in \mathcal{F}_2$ and let $(G, w)$ be an 8-unbalanced pair. Let $K_1, K_2, K_1', K_2'$ be cliques of $G$ such that the separations $S = S(K_1, K_2)$ and $S' = S(K_1', K_2')$ are active in $G$. Assume also that $G$ admits no star cutset. Then $S$ and $S'$ are loosely non-crossing.

Proof. Suppose that $S$ and $S'$ are not loosely non-crossing. Then $(C(K_1, K_2) \cup C(K_1', K_2')) \cap (A(K_1, K_2) \cup A(K_1', K_2')) = \emptyset$. Since $w(B(K_1, K_2)) > \frac{1}{2}$ and $w(B(K_1', K_2')) > \frac{1}{2}$, it follows that $B(K_1, K_2) \cap B(K_1', K_2') = \emptyset$.

(5) $C(K_1, K_2) \cap B(K_1', K_2') = \emptyset$.

Suppose $C(K_1, K_2) \cap B(K_1', K_2') = \emptyset$. Since $B(K_1', K_2')$ is connected, it follows that $A(K_1, K_2) \cap B(K_1', K_2') = \emptyset$. Since by Lemma 4.7 every vertex of $K_1' \cup K_2'$ has a neighbor in $B(K_1, K_2)$ it follows that $C(K_1, K_2) \cap B(K_1', K_2') = \emptyset$. But now $B(K_1', K_2') \cup K_1' \cup K_2' \subseteq B(K_1, K_2) \cup K_1 \cup K_2$. Since $S$ is active, it follows that $B(K_1', K_2') \cup K_1' \cup K_2' = B(K_1, K_2) \cup K_1 \cup K_2$. But now one of $S, S'$ is not active by the second bullet of the definition of being active, a contradiction. This proves (5).

(6) $C(K_1', K_2') \cap A(K_1, K_2) = \emptyset$.

Suppose $C(K_1', K_2') \cap A(K_1, K_2) = \emptyset$. Then $C(K_1, K_2) \cap A(K_1', K_2') = \emptyset$. By (5), $C(K_1, K_2) \cap B(K_1', K_2') = \emptyset$. Let $D(K_1, K_2)$ be a component of $A(K_1, K_2)$ such that $K_1 \cup K_2 \subseteq \mathcal{N}(D(K_1, K_2))$. Since $C(K_1', K_2') \cap A(K_1, K_2) = \emptyset$ it holds that either $D(K_1, K_2) \subseteq B(K_1', K_2')$ or $D(K_1', K_2') \subseteq A(K_1', K_2')$. In the former case $D(K_1, K_2)$ is anticomplete to $C(K_1, K_2) \cap A(K_1', K_2')$, and in the latter case $D(K_1, K_2)$ is anticomplete to $C(K_1, K_2) \cap B(K_1', K_2')$; in both cases a contradiction. This proves (6).

By (5), (6), and symmetry each of the four sets $C(K_1, K_2) \cap A(K_1', K_2'), C(K_1, K_2) \cap B(K_1', K_2'),$
Suppose it is. Then every vertex of $a$ theta through it in $Y$. Let $D$ be a separation of $D'$ such that $1 \subseteq B(K_1', K_2') \cup K_1' \cup K_2', K_1' \subseteq B(K_1, K_2) \cup K_1 \cup K_2$, and $K_2' \subseteq A(K_1, K_2) \cup K_1 \cup K_2$. 

(7) There is a component $D$ of $A(K_1, K_2) \cup A(K_1', K_2')$ such that $1 \subseteq N[D]$. 

Let $D(K_1, K_2)$ be a component of $A(K_1, K_2)$ such that $1 \subseteq N(D(K_1, K_2))$ and let $D(K_1', K_2')$ be a component of $A(K_1', K_2')$ such that $1 \subseteq N(D(K_1', K_2'))$. Since $C(K_1, K_2) \cap B(K_1', K_2') \neq \emptyset$ and $C(K_1', K_2) \cap A(K_1', K_2') \neq \emptyset$ it follows that $D(K_1, K_2) \subseteq A(K_1', K_2')$ and $D(K_1, K_2) \subseteq B(K_1', K_2')$, and therefore $D(K_1, K_2) \cap C(K_1', K_2') = \emptyset$. Similarly $D(K_1', K_2') \cap C(K_1, K_2) = \emptyset$. Consequently $D(K_1, K_2) \cap D(K_1', K_2')$ is connected. Now set $D$ to be the component of $A(K_1, K_2) \cup A(K_1', K_2')$ that contains $D(K_1, K_2) \cap D(K_1', K_2')$, and (7) holds.

Since $(G, w)$ is 8-unbalanced, there is a component $B$ of $G \setminus (K_1 \cup K_1' \cup K_2 \cup K_2')$ with $w(B) > \frac{1}{2}$. Then $B \subseteq B(K_1, K_2) \cup B(K_1', K_2')$. Let $C = N(B)$ and let $A = G \setminus (B \cup C)$. Then $(A, C, B)$ is a separation of $G$.

(8) $K_2 \cap K_2' = \emptyset$ and $C \cap (K_1 \cup K_1')$ is not a clique.

Suppose $K_2 \cap K_2' = \emptyset$ or $K_1 \cup K_1'$ is a clique. Then $C$ is the union of two cliques, say $X$ and $Y$, and so $(A, C, B)$ is a two-clique-separation of $G$. We claim that $(A, C, B)$ is proper. By (7) there is a component $D$ of $A$ such that $1 \subseteq N(D)$, and therefore $C \subseteq N(D)$. If $|C| > 2$, the claim follows. Since $G$ does not admit a clique cutset, we may assume that $X = \{x\}$ and $Y = \{y\}$ and $x$ is non-adjacent to $y$. We need to show that $A$ is not a path from $x$ to $y$. Suppose it is. Then every vertex of $A$ has exactly two neighbors in $A \cup X \cup Y$, and each of $x, y$ has exactly one neighbor in $A$. Since $A(K_1, K_2) \cup A(K_1', K_2') \subseteq A$, this contradicts Lemma 4.6. This proves the claim that $(A, C, B)$ is proper.

Observe that $B \cup C \subseteq B(K_1, K_2) \cup K_1 \cup K_2$. Since $C(K_1, K_2) \cap A(K_1', K_2') = \emptyset$, we get a contradiction to the fact that $S$ is active. This proves (8).

In view of (8), we write $K_2 \cap K_2' = \{s\}$, $K_2 = \{s, t\}$ and $K_2' = \{s, r\}$. Also by (8), there exist non-adjacent $k_1 \in K_1 \cap C$ and $k_1' \in K_1' \cap C$. Let $P$ be a path from $k_1$ to $k_1'$ with $P^* \subseteq B$. Let $Q$ be a path from $k_1$ to $k_1'$ with $Q^* \subseteq D$ where $D$ is as in (7). Then $H = k_1-P-k_1'-Q-k_1$ is a hole.

(9) $A(K_1, K_2) \cap A(K_1', K_2') = \emptyset$.

Suppose that $A(K_1, K_2) \cap A(K_1', K_2') = \emptyset$. Since $S'$ is proper, it follows that $N_{A(K_1', K_2')}(r) = \{t\}$. But now $\{s, t, r\}$ is a triangle, contrary to the fact that $G \in F_2$. This proves (9).

Since $N(A(K_1, K_2) \cap A(K_1', K_2')) \subseteq K_2 \cup K_2' \cup (K_1 \cap K_1')$, and since $K_2 \cup K_2'$ is not a star cutset in $G$, it follows that $K_1 \cap K_1' = \emptyset$. Let $x \in K_1 \cap K_1'$. Now $x$ has two non-adjacent neighbors in $H$, namely $k_1$ and $k_1'$, contrary to the fact that $G$ is sparse. This proves Theorem 5.1.

6. Heavy seagulls

A seagull is a graph that is a three-vertex path. Given a seagull $F = a-v-u$ in $G$, an induced subgraph $T$ of $G$ is a theta through $F$ if $T$ is a theta, one of $a, u$ is an end of $T$, and $F \subseteq T$. A seagull $a-v-u$ is heavy if $\deg_G(a) > 2$ and $\deg_G(u) > 2$. Heavy seagull is extendable if there is a theta through it in $G$. The goal of this section is to show that every heavy seagull is “broken” by some two-clique-separation. We start with a lemma.
Lemma 6.1. Let $G \in \mathcal{F}_2$, let $F = a-v_1-v_2$ be a seagull in $G$ and let $T$ be a theta through $F$ in $G$. Let the ends of $T$ be $a, b$ and let the paths of $T$ be $P_1, P_2, P_3$ where $F \subseteq P_1$. Assume that $T$ is chosen with $|P_1|$ minimum among all thetas through $F$ with end $a$ in $G$. Let $P$ be a path from $u_1$ to $\{ P_2 \cup P_3 \} \setminus N(b)$. Then $P^*$ contains a vertex of $N(b) \cup N[v_1]$.

Proof. Suppose for a contradiction that Lemma 6.1 is false, and let $P$ be a path from $u_1$ to $(P_2 \cup P_3) \setminus N(b)$ such that $P^* \cap (N[b] \cup N[v_1]) = \emptyset$. Let $N_T(b) = \{ w_1, w_2, w_3 \}$ where $w_i \in P_i$. Then $P$ contains a path $Q = q_1, \ldots, q_k$ such that $q_1$ has a neighbor in $P_1 \setminus \{ a, v_1, b \}$, $q_k$ has a neighbor in $(P_2 \cup P_3) \setminus \{ b, w_2, w_3 \}$ and $Q \cap T = \emptyset$. We may assume that $Q$ is chosen in such a way that $k$ is minimum. We may also assume that $q_k$ has a neighbor $s$ in $P_2 \setminus \{ b, w_3 \}$. Since $G \in \mathcal{F}_2$, it follows that $N_T(q_k) = \{ s \}$. Let $t$ be a neighbor of $q_1$ in $P_1^* \setminus \{ v_1 \}$; similarly $N_T(q_1) = \{ t \}$. In particular $k > 1$. It follows from the minimality of $k$ that $Q^*$ is anticomplete to $T \setminus \{ w_2, w_3 \}$. Moreover, since $s-Q-t-P_1-a-P_2-s$ is a hole, it follows that each of $w_2, w_3$ has at most one neighbor in $Q$.

(10) Not both $w_2$ and $w_3$ have a neighbor in $Q$.

Suppose not. Let $i,j \in \{ 1, \ldots, k \}$ such that $q_i$ is adjacent to $w_2$ and $q_j$ is adjacent to $w_3$. Since $N_T(q_k) = \{ s \}$, it follows that $i,j \neq k$. Now, $w_3-P_3-a-P_2-w_2-q_j-Q-q_i-w_3$ is a hole, and $b$ has two neighbors in it, a contradiction. This proves (10).

(11) $w_3$ is anticomplete to $Q$.

Suppose not. Let $i \in \{ 1, \ldots, k \}$ such that $q_i$ is adjacent to $w_3$. Then, by (10), it follows that $w_2$ has no neighbor in $Q$, and so $s-P_2-b-P_1-t-Q-s$ is a hole and $w_3$ has two neighbors $b$ and $q_i$ in it. This proves (11).

(12) $w_2$ is anticomplete to $Q$.

Suppose $w_2$ or has a neighbor in $Q$: let $i \in \{ 1, \ldots, k \}$ be such that $w_2$ is adjacent to $q_i$. Let $S$ be the path $w_1-P_1-t-q_1-Q-q_k$. Since $t \neq v_1$, we have that $v_1 \notin S$. Now $H = b-w_1-S-q_k-s-P_2-a-P_3-b$ is a hole and $q_1 \in N_H(w_2)$, contrary to the fact that $G$ is sparse. This proves (12).

Since $s \neq w_2$ and $t \neq v_1$ the paths $t-P_1-a, t-q_1-Q-q_k-s-P_2-a$ and $t-P_1-b-P_3-a$ form a theta through $\{ a, v_1, u_1 \}$ that contradicts the choice of $T$ with $|P_1|$ minimum. This proves Lemma 6.1. ■

The next result allows us to use Lemma 6.1 to handle heavy seagulls.

Lemma 6.2. Let $G \in \mathcal{F}_2$ and let $F$ be a heavy seagull in $G$. Assume that $G$ does not admit a star cutset. Then $F$ is extendable.

Proof. Let $F = a-v-u$. Since $F$ is heavy, there exist $x_1, x_2 \in N(a) \setminus \{ v \}$. Since $G \in \mathcal{F}_2$ the set $\{ x_1, v, x_2 \}$ is stable. Since $G$ does not admit a star cutset, it follows that for $i \in \{ 1, 2 \}$, there exists a path $P_i$ from $x_i$ to $u$ with $P_i^* \cap N[a] = \emptyset$. By choosing $P_1, P_2$ with $P_1 \cup P_2$ minimal, and permuting the indices if necessary, we may assume that one of the following two cases holds.

1. $P_1 \subseteq P_2$ and $x_1$ has a neighbor in $P_2^*$.

2. There exists a vertex $q \in V(G) \setminus \{ v, a, x_1, x_2 \}$ and a path $Q$ from $u$ to $q$ such that $P_i = u-Q-q-P_i^*-x_i$ and $P_i^* \setminus q$ is disjoint from and anticomplete to $P_2^* \setminus q$.

We handle the former case first. Let $P_1 = p_1, \ldots, p_k$ where $p_1 = u$ and $p_k = x_2$. Let $i$ be maximum such that both $x_i$ and $v$ have neighbors in $P_i$. Then there exists $x \in \{ x_1, v \}$ such that $x$ is anticomplete to $\{ p_{i+1}, \ldots, p_k \}$, and consequently $H = x-p_{i+1}-P_{i+1}-a-x$ is a hole. Let $y \in \{ x_1, v \} \setminus \{ x \}$. Since $y$ is adjacent to $a$ and has a neighbor in $\{ p_1, \ldots, p_k \}$, it follows that $y$ has at least two neighbors in $H$, contrary to the fact that $G$ is sparse. This proves that the first case is impossible, and so the second case holds. Now let $H'$ be the hole $q-P_2^*-x_2-a-x_1-P_1^*-q$. Since
v is adjacent to a and G ∈ F₂, it follows that v is anticomplete to P₁' ∪ P₂', and in particular, u /∈ V(H').

Now let R be a shortest path from u to a vertex u' with a neighbor in H' such that R is contained in G \ (N[v] \ \{a\}). Such a path exists, since v is not a star cutset center. Since G ∈ F₂, it follows that u' has a unique neighbor h in H'. If h /∈ \{x₁, x₂, a\}, then H' ∪ R ∪ \{v\} is a theta in G with ends h and a, and paths a-v-u-R-u'-h and the the two paths from h to a in H'. So (by symmetry) we may assume that h ∈ \{x₁, a\}.

Let R' be the path from h to q with interior in R ∪ Q. Write R' = r₁ - - - - - rₜ, where r₁ = h, rₜ = q, and there exists i ∈ \{2, ..., t - 1\} such that r₁, ..., rᵢ ∈ R and rᵢ₊₁, ..., rₜ ∈ Q. Suppose first that v has a neighbor w in \{rᵢ₊₁, ..., rₜ\}. Then h-R' - q-P₂'-x₂-a-h is a hole, and v has two neighbors in it (namely a and w), contrary to the fact that G ∈ F₂. So v is anticomplete to \{rᵢ₊₁, ..., rₜ\}.

If v is anticomplete to Q \ u, then H' ∪ Q ∪ \{v\} is a theta with ends a, q and paths a-v-u-Q-q and the the two paths from a to q in H', and so F is extendable. Thus we may assume that v has a neighbor in Q \ u, and therefore u is distinct from and non-adjacent to rᵢ₊₁.

Next suppose that rᵢ is adjacent to a. Then i = 2 and h = a. Let Q' be the path from a to q with interior in Q (thus Q' is obtained from a-v-u-Q-q by shortcutting through an edge incident with v). Then a, rᵢ₊₁ ∈ Q'. Now a-Q' - q-P₂'-x₂-a is a hole, and rᵢ has two neighbors in it (namely a and rᵢ₊₁), contrary to the fact that G ∈ F₂. This proves that rᵢ is non-adjacent to a.

Now there is a path S from u to q with S' ⊆ u-R-rᵢ ∪ rᵢ₊₁-Q-q. It follows that \{a, v\} is anticomplete to S' \ u. Consequently, a-v-u-S is a path from a to q. If x₁ has a neighbor s ∈ S, then x₁ has two neighbors in the hole a-S-q-P₂'-x₂-a (namely a and s), contrary to the fact that G ∈ F₂. This proves that x₁ is anticomplete to S. But now H' ∪ S is a theta with ends a, q and paths S and the the two paths from a to q in H', and so F is extendable. This proves 6.2.

Now we deal with extendable seagulls.

**Theorem 6.3.** Let G ∈ F₂ and let (G, w) be a 4-unbalanced pair. Assume that G does not admit a star cutset. Let F = a-v₁-u₁ be a heavy seagull in G. Then there are two cliques K₁, K₂ of G such that S(K₁, K₂) is active and A(K₁, K₂) ∩ \{a, u₁\} ≠ ∅.

**Proof.** Let T' be a theta through F. We may assume that a is an end of T'; let the other end be b'. Let the paths of T' be P₁', P₂', P₃' with v₁ ∈ P₁', and T' is chosen with |P₁'| minimum among all thetas through F in G with end a.

(13) Let D be a component of G \ ((N[b'] \ \{v₁\}) \ \{a, u₁\}). Then |D ∩ \{a, u₁\}| ≤ 1.

Since G ∈ F₂, we have that |V(P')| ≥ 4 and so v₁, u₁ ∈ P₁' \ \{b'\}. Suppose for a contradiction that u₁, a ∈ D. Then there is a path P from u₁ to a with P⁺ ⊆ D. Consequently P⁺ contains no vertex of N[b'] ∪ N[v₁]. Since a ∈ (P₂' ∪ P₃') \ N[b'] we get a contradiction to Lemma 6.1 applied to F, T' and P. This proves (13).

(14) There are cliques X, Y of G and a separation (A, X ∪ Y, B) such that a ∈ A and u₁ ∈ B.

Let Dₐ, Dᵤ be the components of G \ ((N[b'] \ ∪ N[v₁]) \ \{a, u₁\}) with a ∈ Dₐ and u₁ ∈ Dᵤ. By (13), we have that Dₐ ≠ Dᵤ. It follows that there is a separation S = (A, (N[b'] \ ∪ N[v₁]) \ \{a, u₁\}, B) of G with now Dₐ ⊆ A and Dᵤ ⊆ B. Now (14) follows from Lemma 2.1 applied to S.

Let X, Y be as in (14). Since G ∈ F₂ and since (G, w) is a 4-unbalanced pair, the canonical two-clique-separation corresponding to \{X, Y\} is defined, and |B(X, Y) ∩ \{a, u₁\}| ≤ 1. Since |B(X, Y) ∩ \{a, u₁\}| ≤ 1, we deduce that A(X, Y) ∩ \{a, u₁\} = ∅; let p ∈ A(X, Y) ∩ \{a, u₁\}. Let
\[ D \text{ be the component of } A(X, Y) \text{ containing } p, \text{ and let } N = N(D). \text{ Then } N \text{ is the union of two cliques } K_1, K_2. \]

(15) The pair \( \{K_1, K_2\} \) is proper.

Observe that \( B(X, Y) \subseteq B(K_1, K_2) \) and \( D \subseteq A(K_1, K_2). \) Since \( G \) does not admit a clique cutset, both \( K_1 \) and \( K_2 \) are non-empty. If \( |K_1 \cup K_2| \geq 3 \), then \( D \) is a component of \( A(K_1, K_2) \) with \( K_1 \cup K_2 \subseteq N(D) \), and the claim holds. Thus we may assume that \( |K_1| = |K_2| = 1. \) Since \( a \) is an end of \( T' \) and since \( \deg_G(u_1) > 2 \), it follows that \( \deg_G(p) > 2 \), and therefore \( D \cup K_1 \cup K_2 \) is not a path from \( K_1 \) to \( K_2 \), and again the claim holds. This proves (15).

Now among all proper pairs \((K'_1, K'_2)\) with \( B(K'_1, K'_2) \subseteq B(K_1, K_2) \cup K_1 \cup K_2 \) choose \( K'_1, K'_2 \) with \( B(K'_1, K'_2) \cup K'_1 \cup K'_2 \) inclusion-wise minimal, and subject to that with \( B(K'_1, K'_2) \) inclusion-wise maximal. Then \((K'_1, K'_2)\) is active and \( A(K'_1, K'_2) \cap \{a, u_1\} \neq \emptyset \). This proves Theorem 6.3.

7. Proof of Theorem 3.4

First we need a theorem from [5]. For a graph \( G \) we denote by \( \gamma(G) \) the maximum size of a connected component of the graph induced by \( G \) on the set of vertices whose degree in \( G \) is at least three.

Theorem 7.1 ([5]). For all \( k, \gamma > 0 \), there exists \( c = c(k, \gamma) \) such that every graph \( G \) with \( \gamma(G) \leq \gamma \) and treewidth more than \( c \) contains a subdivision of \( W_{k \times k} \) or the line graph of a subdivision of \( W_{k \times k} \) as an induced subgraph.

We deduce:

Theorem 7.2. For all \( t \), there exists \( M = M(t) \) such that every graph in \( F_t \) with no heavy seagull and with treewidth more than \( c \) contains a subdivision of \( W_{t \times t} \) as an induced subgraph.

Proof. We may assume, by increasing \( t \), that \( t \geq 3 \). Since \( G \) contains no heavy seagull, it follows that no two vertices of degree at least three in \( G \) are at distance two in \( G \). This implies that every connected component of the graph induced by \( G \) on the vertices of degree at least three in \( G \) is a clique, and therefore has size at most \( t \). Since \( G \in F_t \), no induced subgraph of \( G \) is the line graph of a subdivision of \( W_{3 \times 3} \). Now Theorem 7.2 follows from Theorem 7.1.

We are now ready to prove Theorem 3.4, the main result of this section, which we restate.

Theorem 7.3. For all \( k \), there exists \( c = c(k) \) such that every graph in \( F_2 \) with no star cutset and with treewidth more than \( c \) contains a subdivision of \( W_{k \times k} \) as an induced subgraph.

Proof. Let \( M = M(k) \geq 1 \) be as in Theorem 7.2. Let \( G \in F_2 \). We show that \( \text{tw}(G) \leq 8(M + 1) \). Suppose not. By Lemma 4.1, there is a function \( w : V(G) \to [0, 1] \) such that \((G, w)\) is \( 4(M + 1) \)-unbalanced, and in particular \( 8 \)-unbalanced. Let \( \mathcal{H} \) be the set of all heavy seagulls of \( G \). By Lemma 6.2, every seagull in \( \mathcal{H} \) is extendable. Let \( \mathcal{S} \) be the set of all pairs of cliques \( \{K_1, K_2\} \) obtained by applying Theorem 6.3 to each member of \( \mathcal{H} \). Let \( \mathcal{T} \) be the set of the canonical two-clique-separations corresponding to the members of \( \mathcal{S} \). By Theorem 5.1 every pair of members of \( \mathcal{T} \) is loosely non-crossing. Let \( \beta \) be a central bag for \( \mathcal{T} \).

(16) There is no heavy seagull in \( \beta \).

Suppose \( X = a-b-c \) is a heavy seagull in \( \beta \). Then \( X \in \mathcal{H} \), and so there is a separation \((A, B, C) \in \mathcal{T} \) such that \( \{a, c\} \cap A \neq \emptyset \). We may assume that \( a \in A \). It follows from the definition
of \( \beta \) that there exists a pair \( \{K_1, K_2\} \in S' \) such that \( a \in P^*_{K_1K_2} \). But then \( \text{deg}_\beta(a) = 2 \), contrary to the fact that \( X \) is a heavy seagull of \( \beta \). This proves (16).

(17) \(|\delta_S(v)| \leq 2 \) for every \( v \in \beta \).

Suppose \(|\delta(v)| > 2 \) for some \( v \in \beta \). Then there exist pairs \( \{K_1, K_2\}, \{K'_1, K'_2\} \in S \) such that \( v \in K_1 \cap K'_1 \). Let \( K_1 = \{k_1, v\} \) and \( K'_1 = \{k'_1, v\} \). Since \( G \in F_2 \), it follows that \( k_1-v-K_2 \) is a seagull in \( G \). Since \( k_1 \in K_1 \), it follows from Lemma 4.7 that \( k_1 \) has a neighbor in \( A(K_1, K_2) \) and a neighbor in \( B(K_1, K_2) \). Since \( v \in C(K_1, K_2) \), we deduce that \( \text{deg}_G(k_1) > 2 \). Similarly, \( \text{deg}_G(k'_1) > 2 \). Consequently, \( k_1-v-K'_2 \) is a heavy seagull of \( G \). It follows that there exists a pair \( \{L_1, L_2\} \in T \) such that \( A(L_1, L_2) \cap \{k_1, k'_1\} \neq \emptyset \), say \( k_1 \in A(L_1, L_2) \). But then \( k_1 \in A(L_1, L_2) \cap C(K_1, K_2) \), contrary to Theorem 5.1. This proves (17).

It follows from (16) that there is no heavy seagull in \( \beta \). By Theorem 7.2, we have that \( \text{tw}(\beta) \leq M \). Let \( w_\beta \) be the inherited weight function on \( \beta \). Since \( \text{tw}(\beta) \leq M \), Lemma 4.2 implies that \((\beta, w_\beta)\) is \((M+1)\)-balanced. Now, by Theorem, 4.5 \((G, w)\) is \(\max(4(M+1), 2(M+1))\)-balanced, and therefore \((G, w)\) is \(4(M+1)\)-balanced, a contradiction. This proves Theorem 7.3.

8. Putting everything together

In this section, we prove Theorem 1.6, which we restate.

**Theorem 8.1.** For all \( t > 0 \), there exists \( c = c(t) \) such that every graph in \( F_t \) with treewidth more than \( c \) contains a subdivision of \( W_{1 \times t} \) as an induced subgraph.

**Proof.** Let \( c = c(t) \) be as in Theorem 7.3. By increasing \( t \), we may assume that \( c(t) \geq t \). Let \( G \in F_1 \), and suppose that \( \text{tw}(G) > c \). A special case of Lemma 3.1 from [6] shows that clique cutsets do not affect treewidth, and so we may assume that \( G \) does not admit a clique cutset. Now we deduce from Lemma 2.1 that \( G \) does not admit a star cutset. By Lemma, 3.3 it follows that either \( G \in F_2 \), or \( G \) is a complete graph (and so \( \text{tw}(G) \leq t \)). So we may assume that \( G \in F_2 \). But now the result follows from Theorem 7.3.

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**References**


