INDUCED SUBGRAPHS AND TREE DECOMPOSITIONS XIV. NON-ADJACENT NEIGHBOURS IN A HOLE

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Abstract. A *clock* is a graph consisting of an induced cycle C and a vertex not in C with at least two non-adjacent neighbours in C. We show that every clock-free graph of large treewidth contains a "basic obstruction" of large treewidth as an induced subgraph: a complete graph, a subdivision of a wall, or the line graph of a subdivision of a wall.

1. Introduction

All graphs in this paper are finite, undirected and simple. Given a graph G and a set $X \subseteq V(G)$, we write $G \setminus X$ for the graph arising from G by deleting all vertices in X, and we write $G[X]$ for the subgraph of G induced by X, that is, the graph $G \setminus (V(G) \setminus X)$. If H is isomorphic to $G[X]$ for some X, we say that G *contains* H; otherwise, we say G is H*-free*. For a family $\mathcal{H}$ of graphs, we say that G is $\mathcal{H}$*-free* if G is H-free for all $H \in \mathcal{H}$.

A *tree decomposition* of a graph G is a pair (T, τ) where T is a tree and $\tau : V(T) \to 2^{V(G)}$ assigns to each vertex of T a subset of $V(G)$, such that the following hold:

- $\bigcup_{t \in V(T)} \tau(t) = V(G)$;
- for every edge $xy \in E(G)$, there is a vertex $t \in V(T)$ such that $x, y \in \tau(t)$; and
- for every $v \in V(G)$, the induced subgraph $T[\{t \in V(T) : v \in \tau(t)\}]$ is connected.

The *width* of (T, τ) is $\max_{t \in V(T)} |\tau(t)| - 1$. The *treewidth of* G, denoted $\operatorname{tw}(G)$, is the minimum width of a tree decomposition of G.

Treewidth was defined and used by Robertson and Seymour [15] as part of the Graph Minors series. In particular, from the point of view of graph minors, as well as subgraphs, it is well-known that (subdivided) walls "cause" large treewidth [14].

In the realm of induced subgraphs, this causal role is partly played by the four natural families of graphs:

- the complete graph K_{t+1};
- the complete bipartite graph $K_{t,t}$;
- subdivisions of the $(t \times t)$-wall; and
- line graphs of subdivisions of the $(t \times t)$-wall.

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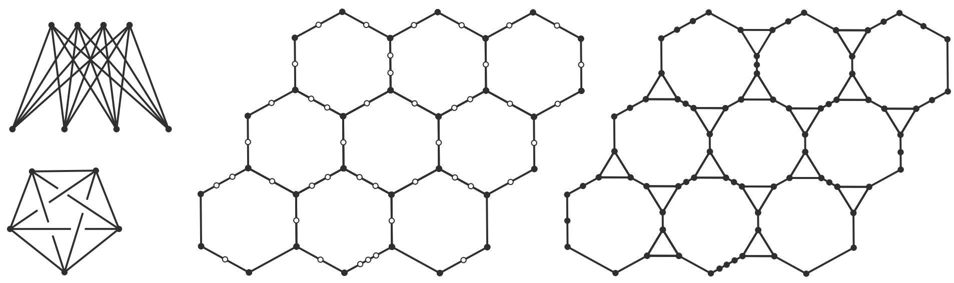

FIGURE 1. The 4-basic obstructions

These graphs, shown in Figure 1 and defined in [3], are called *t-basic obstructions*. For every $t \geq 1$, all t-basic obstructions have treewidth t, and so if a graph G contains a t-basic obstruction, then $\text{tw}(G) \geq t$. The converse, on the other hand, is not true: Let us call a graph *t-clean* if it does not contain a t-basic obstruction. Each of the following constructions are examples of 3-clean graphs of arbitrarily large treewidth:

- Pohoata-Davies graphs [11, 13] (see Figure 2), see also t-sails in [9];
- "Layered wheels" [16];
- "Occultations" [6, 8].

Let us say that a class $\mathcal{C}$ of graphs is *clean* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for all t, every t-clean graph G in $\mathcal{C}$ satisfies $\text{tw}(G) \leq f(t)$. Then, as we saw above, the class of all graphs is not clean.

A *hole* is an induced cycle on four or more vertices. A *wheel* is a graph consisting of a hole C and a vertex v with at least three neighbours in C. Wheels appear to be essential to the construction of two of the above three "non-basic" obstructions: the layered wheels (hence the name) and the occultations. In fact, these two constructions contain a hole with a vertex that has arbitrarily many neighbours in it. The Pohoata-Davies graphs, however, are wheel-free, and so the class of wheel-free graphs is not clean. But the Pohoata-Davies graphs contain a relaxed version of wheels: A *clock* is a graph consisting of a hole C and a vertex v, called the *center* of the clock, such that the neighbours of v in C contain two non-adjacent vertices. Observe that every wheel is a clock, and that clocks are found in abundance in the Pohoata-Davies graphs.

In the present paper, our main result is the following (conjectured in [2]):

Theorem 1.1. *The class of clock-free graphs is clean.*

In [2], with Abrishami, Alecu, and Vušković, we proved a weakening of Theorem 1.1, that graphs in which every vertex v has at most one neighbour in every hole not containing v form a clean class. Explicitly, we proved the following (prisms and pyramids are defined in Section 2):

Theorem 1.2 (Abrishami, Alecu, Chudnovsky, Hajebi, Spirkl, Vušković [2])**.** *The class of (clock, prism, pyramid)-free graphs is clean.*

On the other hand, the following strengthening of Theorem 1.1 might be true (where a *t-clock* is a clock consisting of a hole C and a vertex v with two neighbours $x, y \in V(C)$ where the distance between x and y along C is at least t):

Conjecture 1.3. *For every fixed $t \geq 1$, the family of t-clock-free graphs is clean.*

In the remainder of this section, we give a brief overview of the proof. First, in Section 3, we use results of [2] to reduce our problem to the case of diamond-free graphs with no star cutset. Then, we set up for the central bag method, which works as follows:

- We identify an induced subgraph (sometimes called a *forcer*) that leads to a "nice" cutset. Which forcer we use varies between applications; here we show in Section 4 that paws and certain seagulls (in particular, local configurations that coincide with degree more than 2) give rise to cutsets consisting of a clique plus a vertex. An important feature of this is that the cutset breaks the forcer (that is, we show that certain vertices of the forcer are separated by the cutset we obtain).
- We organize the cutsets. This consists of two steps. First, it is sometimes convenient to be more permissive with cutsets; so even though we obtain cutsets that are a clique plus a vertex in the previous step, here we consider cutsets that are the union of two cliques (with additional restrictions). This change helps with the next goal: We would like to show that these cutsets are "non-crossing."
- To do that, first notice that we may assume that none of these cutsets is balanced; each only cuts off a small part of the graph. Then, we restrict ourselves to cutsets that cut off as much as possibly (the "core"), as described in Section 5.
- In Section 6, we show that the cutsets we chose satisfy the requirements of the central bag method, that is, the cutsets are sufficiently "non-crossing" that we can decompose by all of them simultaneously.
- Section 7 refines what this simultaneous decomposition means; for each cutset, we keep a "marker path" to remember part of what we cut off. Then, we show that the resulting graph, the *central bag*, is simpler than G; intuitively, since the cutsets we used broke all forcers, we expect the central bag to contain no forcers, though some additional care is needed due to marker paths. This helps us prove that the central bag has small treewidth. In this paper, our forcers are related to vertices of big degree, and so we show that, after deleting vertices of degree at most 2, each component has bounded size; then we use a result of [2] to show that treewidth is bounded.
- In the final step, Section 8, we show that G has small treewidth using that the central bag has small treewidth. This is done using balanced separators, which are easier to lift from the central bag to G.

While the general setup of the central bag method has appeared in previous papers of this series, the individual steps outlined above are different depending on which graph class we work with. In particular, in [2], we assumed that pyramids and prisms do not occur; here we instead show that they lead to useful cutsets.

2. Definitions

We begin with some definitions that will be used throughout the paper. For ease of notation, we use graphs and their vertex sets interchangeably. A *k-vertex path* is a graph P with vertex set $\{v_1, \dots, v_k\}$ such that $v_iv_j \in E(P)$ if and only if $|i - j| = 1$. Given a path P, we refer to its vertices of degree at most one as the *ends* of P, and we denote by P^* the *interior* of P, that is, the set obtained from P by deleting the ends of P. If x, y are the ends of P, we also say that P is a *path from x to y*. By a *path in a graph G*, we mean an induced subgraph of G that is a path (and therefore, in particular, "path" always means "induced path"). The *length* of a path is its number of edges. A *prism* is a graph consisting of two triangles with disjoint vertex sets $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, as well as three paths P_1, P_2, P_3 such that:

- P_i has ends a_i and b_i for all $i \in \{1, 2, 3\}$; and
- for distinct $i, j \in \{1, 2, 3\}$, the only edges between P_i and P_j are the edges a_ia_j and b_ib_j.

A *pyramid* is a graph consisting of a vertex a (called the *apex*) and a triangle with vertex set $\{b_1, b_2, b_3\}$ (called the *base*), as well as three paths P_1, P_2, P_3 such that:

- P_i has ends a and b_i for all $i \in \{1, 2, 3\}$;
- for $i \in \{1, 2, 3\}$, each path P_i has length at least one, and there is at most one $i \in \{1, 2, 3\}$ such that the path P_i has length exactly one; and
- for distinct $i, j \in \{1, 2, 3\}$, the only edge between $P_i \setminus \{a\}$ and $P_j \setminus \{a\}$ is the edge a_ia_j.

A *short pyramid* is a pyramid in which one of P_1, P_2, P_3 has length exactly one. We observe:

Lemma 2.1. *If G is clock-free, then G does not contain a short pyramid.*

Proof. Notice that a short pyramid Q is a clock: with P_1, P_2, P_3 and a, b_1, b_2, b_3 as in the definition of a pyramid, let us assume that P_1 has length exactly one. Then $Q \setminus \{b_1\}$ is a hole, and b_1 has three neighbours in it, including a and b_2, which are non-adjacent (since P_2 has length more than one). ■

A *theta* is a graph consisting of two non-adjacent vertices a and b (called its *ends*), as well as three paths P_1, P_2, P_3 such that:

- P_i has ends a and b for all $i \in \{1, 2, 3\}$; and
- for distinct $i, j \in \{1, 2, 3\}$, there are no edges between P_i^* and P_j^*.

A graph is a *three-path configuration* if it is a prism, a pyramid, or a theta. In each case, we refer to P_1, P_2, P_3 as the *paths* of the three-path-configuration.

For a graph G and a vertex $v \in V(G)$, we write $N_G(v)$ for the set of neighbours of v in G, omitting the subscript when there is no danger of confusion. We write $N_G[v]$ for the set $N_G(v) \cup \{v\}$. For $X \subseteq V(G)$, we write $N_G(X) = \bigcup_{x \in X} N_G(x) \setminus X$ and $N_G[X] = X \cup N_G(X)$.

Given a graph G, a set $X \subseteq V(G)$ is a *cutset* of G if $G \setminus X$ is non-empty and not connected. A *clique cutset* is a cutset that is a clique. A *star cutset* is a cutset X such that there is a vertex $x \in X$ with $X \subseteq N[x]$.

Given $X, Y \subseteq V(G)$, we say that X *is anticomplete to* Y if there is no edge in G with one end in X and the other in Y, and we say $x \in V(G)$ *is anticomplete to* Y if $\{x\}$ is anticomplete to Y. A *separation* of G is a triple (A, C, B) of pairwise disjoint subsets of $V(G)$ with union $V(G)$ such that A is anticomplete to B.

3. Diamonds

A *diamond* is a four-vertex graph with exactly five edges. In this section, we use the following results from [2] to reduce Theorem 1.1 to the diamond-free case.

Lemma 3.1 (Abrishami, Alecu, Chudnovsky, Hajebi, Spirkl, Vušković [2])**.** *Let G be a clock-free graph and let (A, C, B) be a separation of G with $A \neq \emptyset$ and $B \neq \emptyset$. Suppose that there exist $v_1, \dots, v_k \in C$ such that $C \subseteq \bigcup_{i=1}^{k} N[v_i]$. Let D_1 be a component of A and let D_2 be a component of B. Then there exist cliques $X_1, \dots, X_k \subseteq C$ of G such that*

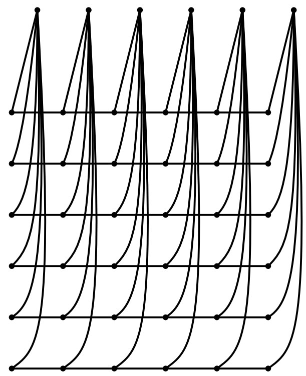

FIGURE 2. A graph from the Pohoata-Davies construction.

every path from a vertex in D_1 to a vertex in D_2 has a vertex in $\bigcup_{i=1}^{k} X_i$. In particular, if G admits a star cutset, then G admits a clique cutset.

Lemma 3.2 (Abrishami, Alecu, Chudnovsky, Hajebi, Spirkl, Vušković [2])**.** *Let G be a clock-free graph and assume that G does not admit a star cutset. Then G is diamond-free.*

Combining Lemmas 3.1 and 3.2, we conclude that every clock-free graph that contains a diamond has a clique cutset; and moreover, every clock-free graph that contains a star cutset has a clique cutset. Since clique cutsets do not affect treewidth (see Lemma 7 in [7]), to prove Theorem 1.1, it now suffices to prove the following:

Theorem 3.3. *The family of (clock, diamond)-free graphs with no star cutset is clean.*

We conclude with a simple observation about diamond-free graphs:

Lemma 3.4. *Let G be diamond-free. Then the following hold:*

- *For every $v \in V(G)$, the graph $G[N(v)]$ is a disjoint union of cliques pairwise anticomplete to each other.*
- *For every edge $xy \in E(G)$, there is exactly one maximal clique of G containing $\{x, y\}$.*

Proof. The first bullet point follows from observing that $G[N(v)]$ does not contain an induced two-edge path; the second follows by observing that $N(x) \cap N(y)$ is a clique. ■

4. PAWS AND SEAGULLS

A *paw* is a graph with vertex set $\{a, a', u, v\}$ and edge set $\{aa', av, a'v, uv\}$. A *seagull* is a graph with vertex set $\{a, u, v\}$ and edge set $\{av, uv\}$.

Our goal in this section is to show that paws and seagulls give rise to particularly nice cutsets in (clock, diamond)-free graphs. To this end, we first show that paws and seagulls are contained in some three-path configuration in a prescribed way; then we show that choosing the right one of these three-path configurations indicates the location of the cutset we are looking for.

We require the following folklore result that appeared, for example, in [4]:

Lemma 4.1. *Let x_1, x_2, x_3 be three distinct vertices of a graph G. Assume that H is a connected induced subgraph of $G \setminus \{x_1, x_2, x_3\}$ such that $V(H)$ contains at least one neighbour of each of x_1, x_2, x_3, and that $V(H)$ is minimal subject to inclusion. Then, one of the following holds:*

(i) For some distinct $i, j, k \in \{1, 2, 3\}$, there exists P that is either a path from x_i to x_j or a hole containing the edge x_ix_j such that
- *$V(H) = V(P) \setminus \{x_i, x_j\}$, and*
- *either x_k has two non-adjacent neighbours in H or x_k has exactly two neighbours in H and its neighbours in H are adjacent.*

(ii) There exists a vertex $a \in V(H)$ and three paths P_1, P_2, P_3, where P_i is from a to x_i, such that
- *$V(H) = (V(P_1) \cup V(P_2) \cup V(P_3)) \setminus \{x_1, x_2, x_3\}$, and*
- *the sets $V(P_1) \setminus \{a\}$, $V(P_2) \setminus \{a\}$ and $V(P_3) \setminus \{a\}$ are pairwise disjoint, and*
- *for distinct $i, j \in \{1, 2, 3\}$, there are no edges between $V(P_i) \setminus \{a\}$ and $V(P_j) \setminus \{a\}$, except possibly x_ix_j.*

(iii) There exists a triangle $a_1a_2a_3$ in H and three paths P_1, P_2, P_3, where P_i is from a_i to x_i, such that
- *$V(H) = (V(P_1) \cup V(P_2) \cup V(P_3)) \setminus \{x_1, x_2, x_3\}$, and*
- *the sets $V(P_1)$, $V(P_2)$ and $V(P_3)$ are pairwise disjoint, and*
- *for distinct $i, j \in \{1, 2, 3\}$, there are no edges between $V(P_i)$ and $V(P_j)$, except a_ia_j and possibly x_ix_j.*

We use Lemma 4.1 to find three-path-configurations, as follows:

Lemma 4.2. *Let G be a (clock, diamond)-free graph with no star cutset. Let $v \in V(G)$, and let $x_1, x_2, x_3 \in N(v)$. If $\{x_1, x_2, x_3\}$ is not a clique of G, then G contains a three-path-configuration Q with $v, x_1, x_2, x_3 \in Q$.*

Proof. Since G is diamond-free and $G[\{v, x_1, x_2, x_3\}]$ is not isomorphic to K_4, we conclude that $G[\{x_1, x_2, x_3\}]$ contains at most one edge.

Since $N[v] \setminus \{x_1, x_2, x_3\}$ is not a star cutset in G, it follows that $V(G) \setminus N[v] \neq \emptyset$. Since $N[v]$ is not a star cutset in G, it follows that $D = G \setminus N[v]$ is connected (and non-empty). Since $\{v\} \cup N(D)$ is not a star cutset in G, we conclude that $N(D) = N(v)$ and so $x_1, x_2, x_3 \in N(D)$.

Let H be a minimal induced subgraph of D such that $\{x_1, x_2, x_3\} \subseteq N(H)$. Then H satisfies one of the outcomes of Lemma 4.1. Note that outcomes (ii) and (iii) of Lemma 4.1 give us the desired three-path configuration. In the case of outcome (i), with i, j, k and P as in Lemma 4.1, either P or $P \cup \{v\}$ is a hole C in G and x_k has at least two neighbours in P^*. Since G is clock-free, it follows that x_k does not have two non-adjacent neighbours in C. Therefore, since x_k is adjacent to v and has a neighbour in P^*, it follows that $v \notin C$. Thus, $x_ix_j \in E(G)$. Moreover, x_k has exactly two neighbours in C (and therefore in P^*), and they are adjacent. Since G is clock-free, and x_k has two neighbours in P^*, it follows that x_k is non-adjacent to x_i, x_j. Therefore, $P \cup \{v, x_k\}$ is a prism in G. This completes the proof. ■

For disjoint sets $X, Y, Z \subseteq V(G)$, we say that X *separates* Y from Z if every path P with one end in Y and the other in Z satisfies $P \cap X \neq \emptyset$.

Theorem 4.3. *Let G be a (clock, diamond)-free graph with no star cutset. Suppose that G contains a paw; and let $a, a', u, v \in V(G)$ be as in the definition of a paw. Then there is a vertex b in $G \setminus N(a)$ and a clique $K \subseteq N[b]$ such that $\{v\} \cup K$ separates $\{u\}$ from $\{a, a'\}$.*

Proof. Let A be the component of $N(v)$ containing a and a'. By Lemma 3.4, it follows that A is a clique. In particular, $u \notin A$, and so for all distinct $a^*, a^{**} \in A$, we obtain a

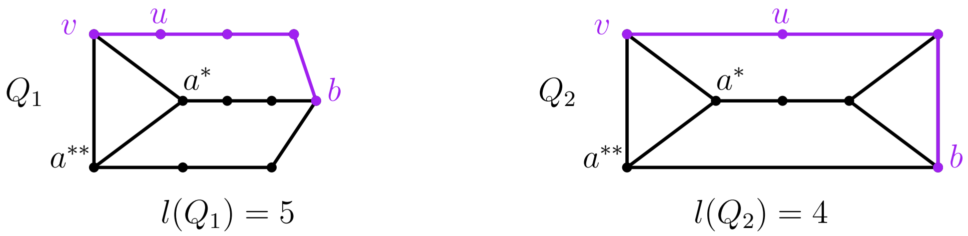

FIGURE 3. In the proof of Theorem 4.3, the choice of $b = b(Q_i)$ and a path with $l(Q_i)$ vertices of the form $P_1(Q_i) \cup \{b\}$ in the case that Q_i is a pyramid ($i = 1$) or Q_i is a prism ($i = 2$).

paw with vertex set $\{a^*, a^{**}, u, v\}$ such that u has degree one and v has degree three in the paw. Since G is diamond-free, no vertex in $V(G) \setminus N[v]$ has more than one neighbour in A.

By Lemma 4.2, for all distinct $a^*, a^{**} \in A$, there is a three-path-configuration in G that contains all of a^*, a^{**}, u and v. Let $\mathcal{Q}_{a^*,a^{**}}$ be the set of all such three-path-configurations. Since $\{a^*, a^{**}, v\}$ is a clique, it follows that every $Q \in \mathcal{Q}_{a^*,a^{**}}$ is a prism or a pyramid. Moreover, for $Q \in \mathcal{Q}_{a^*,a^{**}}$, we define $P_1(Q), P_2(Q), P_3(Q)$ to be the paths of Q such that $v \in P_1(Q), a^* \in P_2(Q)$ and $a^{**} \in P_3(Q)$. Since v is in a triangle of Q, it follows that v is an end of $P_1(Q)$, and $u \in P_1(Q) \setminus \{v\}$. Let us define $b(Q)$ as the end of $P_3(Q)$ which is not equal to a^{**}; that is, $P_3(Q)$ is a path with ends a^{**} and $b(Q)$. Let us define $l(Q) = |P_1(Q) \cup b(Q)|$. See Figure 3.

Let $\mathcal{Q}$ be the union of all sets $\mathcal{Q}_{a^*,a^{**}}$ for distinct $a^*, a^{**} \in A$. Let us pick $Q \in \mathcal{Q}$ with $l(Q)$ minimum; and let $a^*, a^{**} \in A$ such that $Q \in \mathcal{Q}_{a^*,a^{**}}$. By changing the labels of $P_2(Q)$ and $P_3(Q)$ if necessary (noting that this does not change $l(Q)$), we may assume that $a^{**} \neq a$. We claim that $b = b(Q)$ has the desired properties. If Q is a pyramid, then, by Lemma 2.1, it follows that b is non-adjacent to both a^*, a^{**}. If Q is a prism, then b is non-adjacent to a^* since $b \in P_3(Q) \setminus \{a^{**}\}$. Moreover, b has no neighbour in $A \setminus \{a^*, a^{**}\}$; for suppose otherwise, letting $\hat{a}$ be such a neighbour. Then $\hat{a}$ has two non-adjacent neighbours (v and b) in the hole $P_1(Q) \cup P_3(Q)$, a contradiction as G is clock-free. Therefore, the only possible neighbour of b in A is a^{**}, and in particular, since we relabeled $P_2(Q)$ and $P_3(Q)$ if necessary, it follows that b is non-adjacent to a.

Next, we need to show that $b \neq u$. Suppose that $b = u$. Since $b \in P_3(Q)$ and $u \in P_1(Q)$, it follows that $P_3(Q) \cap P_1(Q) \neq \emptyset$, and hence Q is a pyramid with apex $b = u$. But then, $P_1(Q) = \{u, v\}$, and so Q is a short pyramid, contrary to Lemma 2.1.

In what follows, we will show:

(1) *In G, the set $X = \{v\} \cup (N[b] \setminus (A \cup P_1(Q)))$ separates $\{u\}$ from A.*

Let us first show that (1) implies the statement of the theorem: By Lemma 3.1 applied to $G \setminus \{v\}$, it follows that $G \setminus \{v\}$ has a clique cutset K contained in $N[b] \setminus (A \cup P_1(Q))$ separating $\{u\}$ from A in $G \setminus \{v\}$; but then $K \cup \{v\}$ is the desired cutset of G.

It remains to prove (1). Suppose that (1) does not hold. Then $G \setminus X$ contains a path from $Y = P_1(Q) \setminus \{v, b\}$ to $Z = A \cup (P_2(Q) \cup P_3(Q)) \setminus N[b]$ (see Figure 4) with interior disjoint from X; let R be a shortest such path.

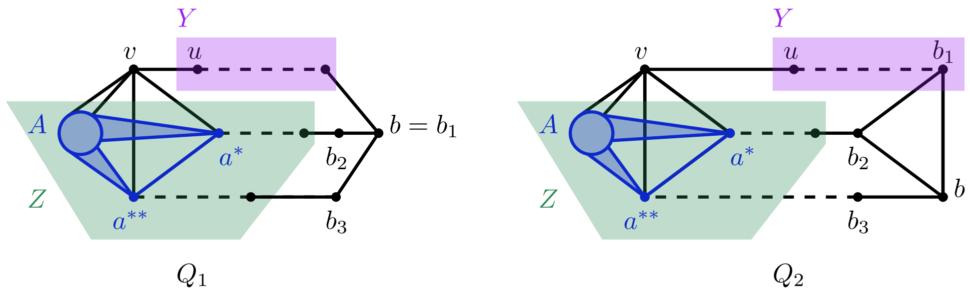

FIGURE 4. The sets A, Y and Z in the proof of Theorem 4.3, in the case that Q_i is a pyramid ($i = 1$) or Q_i is a prism ($i = 2$). Dashed lines represent paths of arbitrary length (possibly zero).

(2) *The path R^* is non-empty.*

Suppose not; then R consists of an edge yz with $y \in Y$ and $z \in Z$. Since $Y = P_1(Q) \setminus \{b, v\}$ is anticomplete to $Q \cap Z$, it follows that $z \in Z \setminus Q = A \setminus \{a^*, a^{**}\}$. But then z is adjacent to two non-adjacent vertices, namely a^* and y, in the hole $P_1(Q) \cup P_2(Q)$, which violates the assumption that G is clock-free. This proves (2).

(3) *The path R^* is disjoint from $N[b]$.*

Suppose that there is a vertex $r \in R^* \cap N[b]$. Since $r \notin X$, it follows that $r \in A \cup P_1(Q)$. This contradicts the fact that R^* is disjoint from $Y \cup Z$, and proves (3).

Let $r_1, \ldots, r_t$ be the vertices of R^* in order, such that r_1 has a neighbour in Y and r_t has a neighbour in Z. There are three vertices in Q that may have neighbours in $R^* \setminus \{r_1, r_t\}$: the vertex v, the neighbour b_2 of b in $P_2(Q)$, and the neighbour b_3 of b in $P_3(Q)$. Let us write b_1 for the end of $P_1(Q)$ which is not equal to v; so $b_1 = b$ if Q is a pyramid, and $b_1 \in N(b)$ if Q is a prism. See Figure 4. By considering the holes $P_i(Q) \cup P_j(Q)$ for distinct $i, j \in \{1, 2, 3\}$, we conclude that $N(x) \cap Q$ is a clique for all $x \in V(G) \setminus Q$.

(4) $N(r_1) \cap Q \subseteq P_1(Q)$.

First note that $N(r_1) \cap Q$ is a clique. If r_1 has a neighbour in $P_1(Q)^*$, then $N(r_1) \cap Q \subseteq P_1(Q)$, as desired. Otherwise, r_1 is adjacent to b_1 as well as at least one of b, b_2, and Q is a prism. Since G is diamond-free, it follows that r_1 is adjacent to both b and b_2. But this contradicts (3). We conclude that $N(r_1) \cap Q \subseteq P_1(Q)$, and (4) holds.

(5) *There exists $i \in \{2, 3\}$ such that r_t has no neighbour in $P_i(Q)$.*

Suppose not. If $N(r_t) \cap Q = \{a^*, a^{**}\}$, then $r_t \in N(v)$ as G is diamond-free, and so $r_t \in A$; however, this contradicts that R^* is disjoint from Z. Since $N(r_t) \cap Q$ is a clique not equal to $\{a^*, a^{**}\}$, the only possibility is that r_t is adjacent to b; but this contradicts (3) and thus proves (5).

Let i be as in (5), and let $j \in \{2, 3\} \setminus \{i\}$. We will fix i and j throughout the remainder of the proof. From the choice of R, it follows that r_t has a neighbour in $(A \setminus P_i(Q)) \cup (P_j(Q) \setminus N[b])$; let R' be a path from r_t to $\hat{a} \in \{a^*, a^{**}\} \cap P_i(Q)$ with interior in $A \cup (P_j(Q) \setminus N[b])$. Note that the neighbour of $\hat{a}$ in R' is a vertex in A.

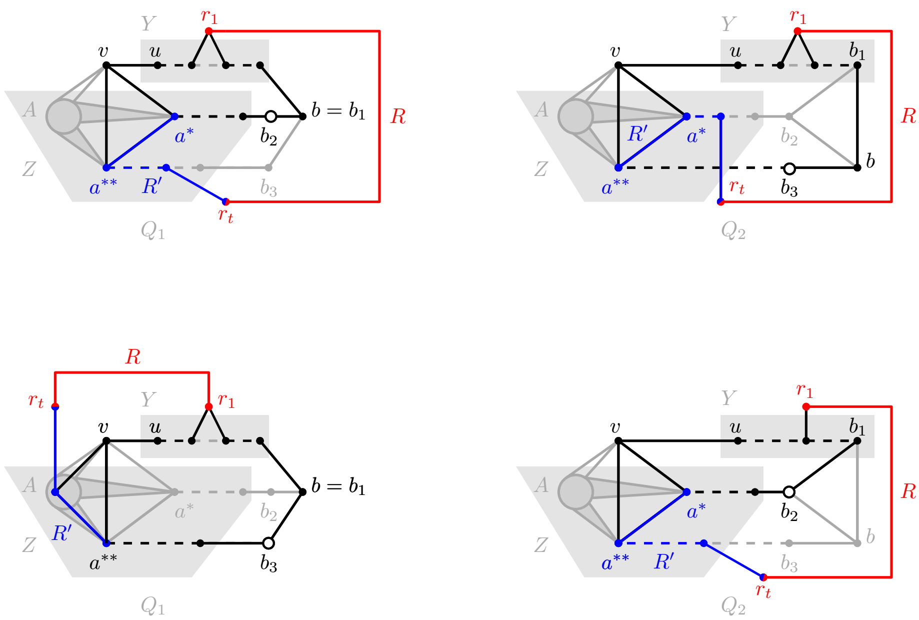

FIGURE 5. Some of the cases for (7), assuming neither b_i nor v has a neighbour in R^*. In each case, b_i is depicted by a hollow node, and a better choice of Q has been highlighted (consisting of R, R' and bits of the previous choice of Q). The vertex r_1 may have one neighbour or two consecutive neighbours in $P_1(Q)$; this only affects whether the better choice for Q is a pyramid or a prism.

(6) *If b_i has no neighbour in R^*, then v does not have a neighbour in R.*

Suppose not, that is, v has a neighbour in R, but b_i has no neighbour in R^*. Traversing $P_1(Q)$ from b_1 to v, let r_0 be the first neighbour of r_1. Then there is a hole $H = r_t$-R'-$\hat{a}$-$P_i(Q)$-b_i-b_1-$P_1(Q)$-r_0-r_1-R-r_t, and v has two non-adjacent neighbours in H, namely $\hat{a} \in P_i(Q)$ and a neighbour in R, contrary to G being clock-free. This proves (6).

(7) *The vertex b_i has a neighbour in R^*. Moreover, if Q is a prism and b_3 has a neighbour in R^*, then, traversing R^* from r_1 to r_t, the first neighbour of b_3 appears at the same time or after the first neighbour of b_2.*

Suppose not. We will show that there is a better choice of Q; see Figure 5. Let us define two paths T, T' as follows:

- If b_i has no neighbour in R^*, then $T = r_1$-R^*-r_t-$(R' \setminus \{\hat{a}\})$ and $T' = P_i(Q)$.
- Otherwise, Q is a prism and b_3 has a neighbour in R^*; let $\hat{R}$ be a path from r_1 to b_3 with interior in R^*. From our assumption, it follows that $\hat{R}$ contains no neighbour of b_2. We set $T = r_1$-$\hat{R}$-b_3-$P_3(Q)$-a^{**} and $T' = b_2$-$P_2(Q)$-a^*.

In either case, the paths T, T' are disjoint, each having an end in A.

We claim that $N(v) \cap T \subseteq A$. If b_i has no neighbour in R^*, then this follows from (6). Otherwise, traversing $P_1(Q)$ from b_1 to v, let r_0 be the first neighbour of r_1. Then $H = r_1$-T-a^{**}-a^*-$P_2(Q)$-b_2-b_1-$P_1(Q)$-r_1 is a hole, and since v is adjacent to a^*, a^{**} in H, it follows that $N(v) \cap T = \{a^{**}\}$, as desired.

We now construct a better choice of Q, which is either a pyramid (if $|N(r_1) \cap Q| = 1$) or a prism (if $|N(r_1) \cap Q| = 2$. By (4), we have $N(r_1) \cap Q \subseteq P_1(Q)$. See Figure 5.

If $N(r_1) \cap Q = \{b'\}$ is a single vertex, then there is a pyramid Q' with paths $P_1(Q') = b'$-$P_1(Q)$-v (which contains u), $P_2(Q') = b'$-$P_1(Q)$-b_1-T', and $P_3(Q') = b'$-r_1-T. It follows that $Q' \in \mathcal{Q}$. Since $P_1(Q') \cup \{b(Q')\} = P_1(Q') \subseteq P_1(Q) \cup \{b\}$ and since $b \notin P_1(Q')$, we conclude that $P_1(Q') \cup \{b(Q')\} \subsetneq P_1(Q) \cup \{b\}$, and so $l(Q') < l(Q)$, contrary to the choice of Q. This implies that $N(r_1) \cap Q = \{b', c\}$, where $b'c \in E(Q)$, and we may assume that $P_1(Q)$ traverses v, c, b' in this order. We note that $b' \neq b$ by (3). Now there is a prism Q' in G with paths $P_1(Q') = c$-$P_1(Q)$-v (which contains u), $P_2(Q') = b'$-$P_1(Q)$-T', and $P_3(Q') = T$. It follows that $Q' \in \mathcal{Q}$. Again, we have that $P_1(Q') \cup \{b(Q')\} = P_1(Q') \cup \{b'\} \subseteq (P_1(Q) \setminus \{b\}) \subsetneq P_1(Q) \cup \{b\}$, contradicting the choice of Q. This proves (7).

Let R'' be a shortest path from r_t to b_j with interior in $(A \setminus P_i(Q)) \cup (P_j(Q) \setminus N[b])$. Since we showed that $N(b) \cap A \subseteq \{a^*, a^{**}\}$, it follows that $N(b) \cap R'' \subseteq (N(b) \cap A \cap P_j(Q)) \cup \{b_j\} \subseteq \{b_j\}$. Let $P = r_1$-R-r_t-$(R'' \setminus \{b_j\})$; so in particular, $N[b] \cap P = \emptyset$. Since $N(x) \cap Q$ is a clique for all $x \in V(G) \setminus Q$, and since b is the only vertex of Q adjacent to both b_2 and b_3, it follows that no vertex of P is adjacent to both b_2 and b_3. Let P' be the shortest subpath of P containing r_1 as well as a neighbour of b_i and a neighbour of b_j. Since R'' contains a neighbour of b_j and R^* contains a neighbour of b_i by (7), the path P' is well-defined. Let p be the end of P' not equal to r_1; and let $k \in \{2, 3\}$ such that b_k is adjacent to p. It follows that b_k has no neighbour in $P' \setminus \{p\}$ Let $k' \in \{2, 3\} \setminus \{k\}$.

Suppose first that either Q is a pyramid, or Q is a prism and $k = 3$. Then, there is a hole H' in G, defined as $H' = r_1$-$P_1(Q)$-b-b_k-p-P'-r_1. The vertex $b_{k'}$ has two non-adjacent neighbours in H', namely b and a neighbour in P'. Since G is clock-free, this is a contradiction.

It follows that Q is a prism and $k = 2$. It follows that b_3 has a neighbour in R^*. But now (7) implies that the first neighbour of b_2 along R^*, traversed from r_1 to r_t, appears at the same time or before the first neighbour of b_3, which implies that $k = 3$, a contradiction. This concludes the proof.

■

Next, we show that certain seagulls lead to similar cutsets as in Theorem 4.3. We start with two lemmas. Given a graph G, a vertex $v \in V(G)$ is a *claw center* in G if $N_G(v)$ contains three pairwise non-adjacent vertices.

Lemma 4.4. *Let G be a clock-free graph and let P be a path in G of length at least 1. Let a be an end of P and let y be the neighbour of a in P. Let $x, v \in N(a)$ such that $\{x, y, v\}$ is a stable set. Then at least one of x and v is anticomplete to $P \setminus \{a\}$.*

Proof. Suppose not. We may assume that P is chosen minimal such that $a \in P$ and $P \setminus \{a\}$ contains both a neighbour of x and a neighbour of v. Then $P \cup \{x, v\}$ is a clock: let $w \in \{x, v\}$ be chosen such that the only neighbour of w in $P \setminus \{a\}$ is the end of P not equal to a. Then w-P-a-w is a hole, and the vertex $q \in \{x, v\} \setminus \{w\}$ has at least two non-adjacent neighbours in it, namely a and a vertex in $P \setminus \{a, y\}$. This is a contradiction, proving Lemma 4.4. ■

Lemma 4.5. *Let G be a (clock, diamond)-free graph with no star cutset. Suppose that G contains a seagull; and let $a, u, v \in V(G)$ be as in the definition of a seagull. Suppose further that a is a claw center in G. Then there is a three-path configuration Q in G such that $a, u, v \in Q$ and a is a claw center in Q.*

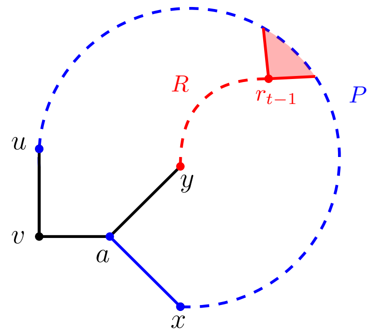

FIGURE 6. Proof of Lemma 4.5. Dashed lines represent paths of arbitrary length (possibly zero).

Proof. Since $N[v] \setminus \{a, u\}$ is not a star cutset in G, it follows that there is a path P in G with ends a and u and with interior disjoint from $N[v]$. Let x be the neighbour of a in P. Then x is non-adjacent to v from the choice of P.

From Lemma 3.4, since a is a claw center, it follows that $N(a)$ has at least three components. Let us pick $y \in N(a)$ such that $\{x, y, v\}$ is an independent set.

Since $N[a] \setminus \{y\}$ is not a star cutset, it follows that there is a path from y to $P \setminus \{x\}$ with interior disjoint from $N[a]$. Let R be a shortest such path. See Figure 6.

Let $r_1, \ldots, r_t$ denote the vertices of R in order such that $y = r_1$, and $r_t \in P$. Traversing P from x to u, let w and w' denote the first and last neighbour of r_{t-1} in P, respectively. Then, $P' = a\text{-}y\text{-}R\text{-}w'\text{-}P\text{-}u$ is an induced path (due to the choice of R and w'). Applying Lemma 4.4 to P' and a, x, y, v implies that not both x and v have a neighbour in P'. But v does have a neighbour in P', namely u; so x is anticomplete to P'.

Now let us consider the hole $H = a\text{-}y\text{-}R\text{-}w\text{-}P\text{-}x\text{-}a$. Then, since $a \in N(v) \cap H$ and $x, y \notin N(v)$, it follows that v has no neighbour in $H \setminus \{a\}$ as G is clock-free.

Since $a\text{-}x\text{-}P\text{-}u\text{-}v\text{-}a$ is a hole, it follows that r_{t-1} either has exactly one neighbour in P, or exactly two neighbours in P and they are adjacent. In the former case, we get a theta Q with ends a and $w = w'$; in the latter case, we get a pyramid Q with apex a and base $\{w, w', r_{t-1}\}$. In both cases, $a, u, v \in Q$ and a is a claw center in Q. This concludes the proof. ■

The proof of the next result follows the same structure as the proof of Theorem 4.3:

Theorem 4.6. *Let G be a (clock, diamond)-free graph with no star cutset. Suppose that G contains a seagull with $a, u, v \in V(G)$ as in the definition of a seagull. Suppose further that a is a claw center in G. Then there is a vertex $b \in V(G)$ and a clique $K \subseteq N[b]$ such that $\{v\} \cup K$ separates $\{a\}$ from $\{u\}$, and a is non-adjacent to b.*

Proof. We first show:

(8) *We may assume that the set $N(a) \cap N(v)$ is empty.*

Suppose that there is a vertex $a' \in N(a) \cap N(v)$. Then, the vertices a, a', u, v form a paw (using that G is diamond-free), and so Theorem 4.6 follows from Theorem 4.3, and we are done. Therefore, we may assume that $N(a) \cap N(v)$ is empty. This proves (8).

Let $\mathcal{Q}$ be the set of all three-path configuration of the form guaranteed by Lemma 4.5; that is, every $Q \in \mathcal{Q}$ is a three-path configuration containing a, u, v, and a is a claw center in Q. By Lemma 4.5, the set $\mathcal{Q}$ is non-empty. Let $Q \in \mathcal{Q}$. Since Q contains

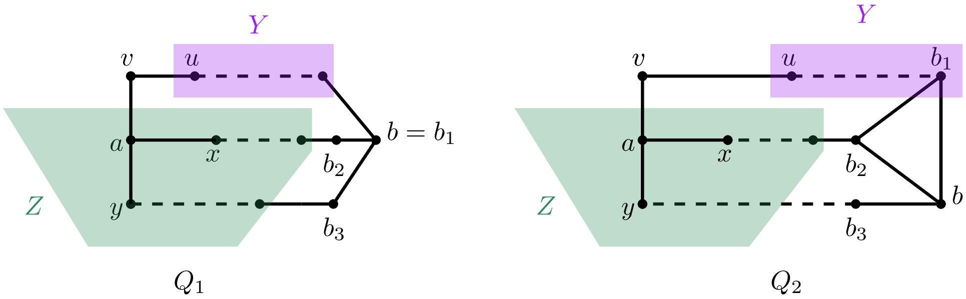

FIGURE 7. Name conventions for vertices in the proof of Theorem 4.6 in the case when $Q = Q_i$ is a theta ($i = 1$) and when $Q = Q_i$ is a pyramid ($i = 2$). Dashed lines represent paths of arbitrary length (possibly zero).

a claw center, it follows that Q is a pyramid with apex a, or a theta with end a. Let $P_1(Q), P_2(Q), P_3(Q)$ be the paths of Q, labelled in such a way that $v \in P_1(Q)$. We define $b(Q)$ to be the end of $P_3(Q)$ which is not equal to a, and we define $l(Q) = |P_1(Q) \cup b(Q)|$. See Figure 7.

Now, let $Q \in \mathcal{Q}$ be chosen with $l(Q)$ minimum. We claim that $b = b(Q)$ is the desired vertex. Note that a and b are not adjacent (if Q is a theta, this is true from the definition of a theta; if Q is a pyramid, it follows from Lemma 2.1). As in Theorem 4.3, it is sufficient to prove:

(9) *The set* $\{v\} \cup (N[b] \setminus P_1(Q))$ *separates* $\{u\}$ *from* $\{a\}$ *in* G.

Assuming (9) to be true, by applying Lemma 3.1 to $G \setminus \{v\}$ and the star cutset $N[b] \setminus P_1(Q)$, we obtain the desired clique K.

It remains to prove (9); so we suppose for a contradiction that (9) does not hold. We first show that $u \neq b$. In the case that Q is a pyramid, this is immediate; when Q is a theta, it follows from the fact that v does not have two non-adjacent neighbours (namely a and $u = b$) in the hole $P_2(Q) \cup P_3(Q)$. Therefore, $u \neq b$.

Next, let us define some notation. We denote the neighbour of a in $P_2(Q)$ as x, and the neighbour of a in $P_3(Q)$ as y. Moreover, let us write b_1 for the end of $P_1(Q)$ not equal to a, and for $i \in \{2, 3\}$, let us write b_i for the unique neighbour of b in $P_i(Q)$; see Figure 7.

Let $X = \{v\} \cup (N[b] \setminus P_1(Q))$. Let $Y = P_1(Q) \setminus \{v, a, b\}$; it follows that $u \in Y$ as $u \neq b$. Let $Z = (P_2(Q) \cup P_3(Q)) \setminus N[b]$; it follows that $a \in Z$. Since we assumed that (9) does not hold, it follows that there is a path from Y to Z with interior disjoint from X; let R be a shortest such path. It follows that one end of R is in Y, and the other is in Z, and R^* is disjoint from $X \cup Y \cup Z$.

(10) *The set* R^* *is non-empty, and* $N[b] \cap R^* = \emptyset$.

Since Y is anticomplete to Z, it follows that R^* is non-empty. Moreover, since $N[b] \subseteq X \cup Y$, it follows that $N[b] \cap R^* = \emptyset$. This proves (10).

Let $r_1, \ldots, r_t$ denote the vertices of R^* in order, such that r_1 has a neighbour in Y and r_t has a neighbour in Z. By (10), the only vertices of Q that may have a neighbour in $R^* \setminus \{r_1, r_t\}$ are v, b_2, b_3.

By considering the holes $P_i(Q) \cup P_j(Q)$ for distinct $i, j \in \{1, 2, 3\}$, we conclude that $N(w) \cap Q$ is a clique for all $w \in V(G) \setminus Q$.

(11) $N(r_1) \cap Q \subseteq P_1(Q)$.

Note that $N(r_1) \cap Q$ is a clique. If r_1 has a neighbour in $P_1(Q)^*$, then $N(r_1) \cap Q \subseteq P_1(Q)$, as desired. Otherwise, r_1 is adjacent to b_1 as well as at least one of b, b_2 and Q is a pyramid. Since G is diamond-free, it follows that r_1 is adjacent to both b and b_2. But this contradicts (10). We conclude that $N(r_1) \cap Q \subseteq P_1(Q)$, and (11) holds.

(12) *There exists* $j \in \{2, 3\}$ *such that* $N(r_t) \cap Q \subseteq P_j(Q)$.

Suppose not. Then, since r_t is non-adjacent to b by (10), and since $N(r_t) \cap Q$ is a clique, it follows that r_t is adjacent to v and a. This contradicts (8) and proves (12).

In the remainder of this proof, let us fix j as in (12) and let $i \in \{2, 3\} \setminus \{j\}$. Since $N(r_t) \cap Q$ is a clique and b is non-adjacent to r_t, it follows that $N(r_t) \cap P_i(Q) \subseteq \{a\}$.

(13) *The vertex* b_i *has a neighbour in* R^**. Furthermore, if* Q *is a pyramid and* b_3 *has a neighbour in* R^**, then, traversing* R^* *from* r_1 *to* r_t*, the first neighbour of* b_3 *appears at the same time or after the first neighbour of* b_2*.*

The proof of (13) is similar to the proof of (7). Suppose that (13) does not hold. Let R' be a path from r_t to a with interior in $P_j(Q) \setminus N[b]$. Traversing $P_1(Q)$ from b_1 to a, let r_0 be the first neighbour of r_1. Let us define a hole H and paths T, T' as follows (the purpose of H will be to show that v does not have neighbours in the part of R that we are interested in, and T, T' will be used to construct a better choice of Q):

- If b_i has no neighbour in R^*, then we let $H = r_t\text{-}R'\text{-}a\text{-}P_i(Q)\text{-}b_i\text{-}b_1\text{-}P_1(Q)\text{-}r_0\text{-}r_1\text{-}R^*\text{-}r_t$ and $T = r_1\text{-}R^*\text{-}r_t\text{-}R'$ and $T' = P_i(Q)$.
- Otherwise, Q is a pyramid and b_3 has a neighbour in R^*; let $\hat{R}$ be a path from r_1 to b_3 with interior in R^*. From our assumption, it follows that $\hat{R}$ contains no neighbour of b_2. We set $H = r_1\text{-}\hat{R}\text{-}b_3\text{-}P_3(Q)\text{-}a\text{-}P_2(Q)\text{-}b_2\text{-}b_1\text{-}P_1(Q)\text{-}r_0\text{-}r_1$ and $T = r_1\text{-}\hat{R}\text{-}b_3\text{-}P_3(Q)\text{-}a$ and $T' = P_2(Q)$.

In either case, we have $T \cap T' = \{a\}$.

Suppose first that v has a neighbour q in $H \setminus \{a\}$. Then, since G does not contain a clock and v is adjacent to $a \in H$, it follows that $qa \in E(G)$. But now $q \in N(a) \cap N(v)$, contrary to (8). Therefore, v has no neighbour in $H \setminus \{a\}$.

By (11), we have $N(r_1) \cap Q \subseteq P_1(Q)$. If $N(r_1) \cap Q = \{b'\}$ is a single vertex, then there is a theta Q' with paths $P_1(Q') = b'\text{-}P_1(Q)\text{-}a$ (which contains u since r_1 has a neighbour in Y), $P_2(Q') = b'\text{-}P_1(Q)\text{-}b_1\text{-}T'\text{-}a$, and $P_3(Q') = b'\text{-}r_1\text{-}T\text{-}a$. It follows that $Q' \in \mathcal{Q}$. Since $P_1(Q') \cup \{b(Q')\} = P_1(Q') \subsetneq P_1(Q) \cup \{b\}$ since $b \notin P_1(Q')$, we conclude that $l(Q') < l(Q)$, contrary to the choice of Q. This implies that $N(r_1) \cap Q = \{b', c\}$, where $b'c \in E(P_1(Q))$, and we may assume that $P_1(Q)$ traverses v, c, b' in this order. We note that $b' \neq b$ by (10). Also, since r_1 has a neighbor in Y, $b' \neq v$. Now there is a pyramid Q' in G with paths $P_1(Q') = c\text{-}P_1(Q)\text{-}a$ (which contains u, since Q' is not a short pyramid by Lemma 2.1), $P_2(Q') = b'\text{-}P_1(Q)\text{-}T'$, and $P_3(Q') = T$. It follows that $Q' \in \mathcal{Q}$. Again, we have that $P_1(Q') \cup \{b(Q')\} = P_1(Q') \cup \{b'\} \subsetneq P_1(Q) \cup \{b\}$, contradicting the choice of Q. This proves (13).

Let R'' be a shortest path from r_t to b_j with interior in $P_j(Q) \setminus N[b]$). Let $P = r_1$-R-r_t-$(R'' \setminus \{b_j\})$; so in particular, $N(b) \cap P = \emptyset$. Let P' be the shortest subpath of P containing r_1 as well as a neighbour of b_i and a neighbour of b_j. Since R'' contains a neighbour of b_j and R^* contains a neighbour of b_i by (13), the path P' is well-defined. Let p be the end of P' not equal to r_1; and let $k \in \{2, 3\}$ be maximum such that b_k is adjacent to p and b_k has no neighbour in $P' \setminus \{p\}$ (such k exists by the choice of P', as otherwise $P' \setminus \{p\}$ would be a better choice that P'). Let $k' \in \{2, 3\} \setminus \{k\}$.

Suppose first that either Q is a theta, or Q is a pyramid and $k = 3$. Then, there is a hole H' in G, defined as $H' = r_1$-$P_1(Q)$-b-b_k-p-P'-r_1. The vertex $b_{k'}$ has two non-adjacent neighbours in H', namely b and a neighbour in P'. Since G is clock-free, this is a contradiction.

It follows that Q is a pyramid and $k = 2$. Since at least one of b_2, b_3 has a neighbour in R^* by (13), it follows from the choice of k that b_3 has a neighbour in R^*. But now (13) implies that the first neighbour of b_2 along R^*, traversed from r_1 to r_t, appears at the same time or before the first neighbour of b_3, which implies that $k = 3$, a contradiction. This concludes the proof. ■

5. Central bags

In the previous section, we showed that paws and certain seagulls lead to cutsets in (clock, diamond)-free graphs. In this section, we will set up the "central bag method," (see [1]) which, under certain circumstances, allows us to decompose a graph along several cutsets simultaneously, obtaining a much simplified graph – the *central bag* – as a result. Then, using the structure of the central bag, we show that it has small treewidth. Finally, we "lift" a certificate of small treewidth for the central bag to a certificate for the original graph by carefully reversing the decompositions.

As a first step, let us describe the "certificate" of small treewidth. Rather than working with a tree decomposition, we will work with balanced separators: Let G be a graph. A *weight function* on G is a function $w : V(G) \to [0, 1]$ such that $\sum_{v \in V(G)} w(v) = 1$. For $X \subseteq V(G)$, we write $w(X)$ for $\sum_{x \in X} w(x)$.

Let $c \in [0, 1]$ and let G be a graph with weight function w. A set $X \subseteq V(G)$ is a (w, c)-*balanced separator* for G if for every component D of $G \setminus X$, we have $w(D) \leq c$. The following two lemmas show that treewidth is closely related to the existence of balanced separators:

Lemma 5.1 (Harvey and Wood [12]; stated in this form in [3])**.** *Let G be a graph, let $c \in [\frac{1}{2}, 1)$, and let k be a positive integer. If G has a (w, c)-balanced separator of size at most k for every weight function w on G, then* $\mathrm{tw}(G) \leq \frac{1}{1-c}k$.

Lemma 5.2 (Cygan, Fomin, Kowalik, Lokshtanov, Marx, Pilipczuk, Pilipczuk, and Saurabh [10])**.** *Let G be a graph and let k be a positive integer. If* $\mathrm{tw}(G) \leq k$, *then G has a (w, c)-balanced separator of size at most $k + 1$ for every $c \in [\frac{1}{2}, 1)$ and for every weight function w on G.*

Therefore, to prove Theorem 3.3, it suffices to show that for every $t \in \mathbb{N}$, there exists a $k = k(t)$ such that for every t-clean (clock, diamond)-free graph G and every weight function w on G, there is a $\left(w, \frac{1}{2}\right)$-balanced separator of size at most k in G. In particular, we can fix a weight function w and assume (for a contradiction) that for every set $X \subseteq V(G)$ of size at most k, at least one (and therefore exactly one) component of $G \setminus X$ has weight more than $\frac{1}{2}$.

Let us now turn to the cutsets we use to create the central bag. Let G be a graph, and let w be a weight function on G. For a set X that is not a $\left(w, \frac{1}{2}\right)$-balanced separator of G, we define its *canonical separation* $S_{w,G}(X) = (A, C, B)$ where B is the unique component D of $G \setminus X$ with $w(D) > \frac{1}{2}$, $C = X$, and $A = V(G) \setminus (B \cup C)$. We note that A is anticomplete to B.

Let G be a graph, and let w be a weight function on G. Let $\mathcal{X}$ be a set of subsets of $V(G)$, none of which is a $\left(w, \frac{1}{2}\right)$-balanced separator of G. Then, we define the *central bag* of G and w with respect to $\mathcal{X}$ as

$$\beta(G, w, \mathcal{X}) = \bigcap_{X \in \mathcal{X} \text{ with } S_{w,G}(X)=(A,C,B)} (B \cup C).$$

In other words, we delete all the "A-sides" of canonical separations corresponding to sets $X \in \mathcal{X}$.

We would like to say that each component of $G \setminus \beta(G, w, \mathcal{X})$ is contained in A for some $S_{w,G}(X) = (A, C, B)$ with $X \in \mathcal{X}$. To arrange this, we define the following. Let us say that two separations (A, C, B) and (A', C', B') of a graph G are *loosely non-crossing* if there is no path in G with one end in $A \cap C'$, the other end in $A' \cap C$, and with interior in $A \cap A'$. We observe:

Lemma 5.3. *Let G be a graph, and let w be a weight function for G. Let $\mathcal{X}$ be a set of subsets of $V(G)$, none of which is a $\left(w, \frac{1}{2}\right)$-balanced separator of G. Suppose that for all $X, X' \in \mathcal{X}$, the canonical separations $S_{w,G}(X)$ and $S_{w,G}(X')$ are loosely non-crossing. Then, for every component D of $G \setminus \beta(G, w, \mathcal{X})$, there is an $X \in \mathcal{X}$ such that $D \subseteq A$, where $(A, C, B) = S_{w,G}(X)$.*

Proof. From the definition of $\beta(G, w, \mathcal{X})$, it follows that there is an $X \in \mathcal{X}$ such that $D \cap A \neq \emptyset$, where $(A, C, B) = S_{w,G}(X)$. Among all such X, let us choose X and a component D_X of $D \cap A$ with $|D_X|$ as large as possible; and fix $(A, C, B) = S_{w,G}(X)$.

If $D_X = D$, then the lemma holds; so we may assume that there is a vertex $v' \in D \cap N(D_X)$ (as D is connected). Again from the definition of $\beta(G, w, \mathcal{X})$, it follows that there is an $X' \in \mathcal{X}$ such that $v' \in A'$, where $(A', C', B') = S_{w,G}(X')$.

Let $D_{X'}$ be the component of $(D_X \cup \{v'\}) \cap A'$ containing v' (which exists, as $v' \in A'$). From the choice of X and D_X, it follows that $D_{X'}$ does not contain all vertices of D_X. Since $D_X \cup \{v'\}$ is connected and $D_{X'}$ is a proper subset of $D_X \cup \{v'\}$, it follows that there is a vertex $v \in D_X \setminus D_{X'}$ with a neighbour in $D_{X'}$. From the choice of $D_{X'}$, it follows that $v \notin A'$.

Now, let P be a path from v to v' with interior in $D_{X'}$. Then:

- $P \setminus \{v\} \subseteq D_{X'} \subseteq A'$ and $P \setminus \{v'\} \subseteq D_{X'} \setminus \{v'\} \subseteq D_X \subseteq A$;
- v is not in A', but v has a neighbour in A' (along P), and so $v \in C'$;
- v' is not in A, but v' has a neighbour in A (along P), and so $v' \in C$.

This is shows that (A, C, B) and (A', C', B') are not loosely non-crossing, contrary to our assumption. ■

Given two sets X, X' that are not $\left(w, \frac{1}{2}\right)$-balanced separators of G, we say that X is a (w, G)-*shield* for X' (a notion also used in, for example, [3, 4, 5]) if, writing $S_{w,G}(X) = (A, C, B)$ and $S_{w,G}(X') = (A', C', B')$, we have that one of the following holds:

- $B \cup C \subsetneq B' \cup C'$; or
- $B \cup C = B' \cup C'$ and $B' \subsetneq B$.

From this definition, it is immediate that for every set $\mathcal{X}$ of subsets of $V(G)$ none of which is a $\left(w, \frac{1}{2}\right)$-balanced separator of G, being a (w, G)-shield is a partial order on $\mathcal{X}$. We now consider the set of all "minimal" elements of $\mathcal{X}$ with respect to this order. Explicitly, we define

$$\text{core}_{w,G}(\mathcal{X}) = \{X' \in \mathcal{X} : \text{there is no } X \in \mathcal{X} \text{ which is a } (w, G)\text{-shield for } X'\}.$$

It follows that for every $X' \in \mathcal{X} \setminus \text{core}_{w,G}(\mathcal{X})$, there is an $X \in \text{core}_{w,G}(\mathcal{X})$ which is a shield for X'.

The idea is that if X is a shield for X', then X is "strictly more useful" than X' (given that the sets B and B' are the large components, and we would like the large components to be as small as possible); so it suffices to consider cutsets $X \in \text{core}_{w,G}(\mathcal{X})$.

So far, this description is common to applications of the central bag method; the main difference is in the types of sets X we use for canonical separations, which we describe in the next section.

6. 2-clique cutsets

Let G be a diamond-free graph. For a clique $K \subseteq V(G)$ and a set $A \subseteq V(G)$, let us define $c_A(K)$ as follows:

- If $|K| \leq 1$, then $c_A(K) = K$.
- Otherwise, let $x, y \in K$ be distinct. Then $c_A(K) = K \cup (N(x) \cap N(y) \cap A)$.

Note that, by Lemma 3.4, the set $c_A(K)$ is a (well-defined) clique.

Let G be a diamond-free graph and let w be a weight function on G. Let K_1, K_2 be cliques in G, and suppose that $X = K_1 \cup K_2$ is not a $\left(w, \frac{1}{2}\right)$-balanced separator of G. Let $(A, C, B) = S_{w,G}(X)$. Then, we define $\text{closure}(K_1, K_2) = c_{A\cup C}(K_1 \cap N(B)) \cup c_{A\cup C}(K_2 \cap N(B))$. We observe that:

- The set $\text{closure}(K_1, K_2)$ is the union of at most two cliques K_1', K_2' where $\text{closure}(K_1', K_2') = \text{closure}(K_1, K_2)$.
- B is a component of $G \setminus \text{closure}(K_1, K_2)$, and therefore, $\text{closure}(K_1, K_2)$ is not a $\left(w, \frac{1}{2}\right)$-balanced separator of G.
- Writing $(A', C', B') = S_{w,G}(\text{closure}(K_1, K_2))$, we have $B' = B$.

For a graph G, we denote by $\omega(G)$ the size of the largest clique in G. For a diamond-free graph G and a weight function w such that G has no $\left(w, \frac{1}{2}\right)$-balanced separator of size at most $4\omega(G)$, let us define

$$\mathcal{X}(G) = \{\text{closure}(K_1, K_2) : K_1, K_2 \text{ are cliques in } G\}.$$

Theorem 6.1. *Let G be a (clock, diamond)-free graph, and let w be a weight function on G. Suppose that G contains no $\left(w, \frac{1}{2}\right)$-balanced separator of size at most $4\omega(G)$ and that G has no star cutset. Let $X, X' \in \text{core}_{w,G}(\mathcal{X}(G))$. Then $S_{w,G}(X)$ and $S_{w,G}(X')$ are loosely non-crossing.*

Proof. Suppose not. Let $(A, C, B) = S_{w,G}(X)$ and $(A', C', B') = S_{w,G}(X')$. Let K_1, K_2 be cliques such that $C = K_1 \cup K_2 = \text{closure}(K_1, K_2)$, and let K_1', K_2' be cliques such that $C' = K_1' \cup K_2' = \text{closure}(K_1', K_2')$.

Since (A, C, B) and (A', C', B') are not loosely non-crossing, and by symmetry, we may assume that there is a path P from $r' \in A \cap K_1'$ to $r \in A' \cap K_1$ with interior in $A \cap A'$.

Since G has no $\left(w, \frac{1}{2}\right)$-balanced separator of size at most $4\omega(G)$, it follows that there is a component B^* of $G \setminus (C \cup C')$ with $w(B^*) > \frac{1}{2}$.

Since $w(B) > \frac{1}{2}$ and $w(B') > \frac{1}{2}$ from the definition of $S_{w,G}(\cdot)$, it follows that $B^* \cap B' \neq \emptyset$ and $B^* \cap B \neq \emptyset$, and therefore, since B^* is disjoint from $C \cup C'$, it follows that $B^* \subseteq B \cap B'$.

(14) *We have that* $N(B^*) \subseteq K_2 \cup K_2' \cup (K_1 \cap K_1')$.

Suppose not. Since $B^* \subseteq B \cap B'$, it follows that $N(B^*) \subseteq (B \cap C') \cup (B' \cap C) \cup (C \cap C')$. Since $r \in A' \cap K_1$, it follows that $B' \cap C \subseteq K_2$; similarly, $B \cap C' \subseteq K_2'$. Finally, $(C \cap C') \setminus (K_2 \cup K_2') \subseteq K_1 \cap K_1'$. This implies (14).

(15) *The set* $K_1 \cap K_1'$ *is non-empty.*

Suppose not. By (14), it follows that $N(B^*) \subseteq K_2 \cup K_2'$. Let K_3, K_4 be cliques such that $C^* = K_3 \cup K_4 = \text{closure}(K_3, K_4) = \text{closure}(K_2 \cap N(B^*), K_2' \cap N(B^*))$. Letting $A^* = V(G) \setminus (C^* \cup B^*)$, we have that $S_{w,G}(C^*) = (A^*, C^*, B^*)$. We also have $C^* \in \mathcal{X}(G)$. We notice that:

- $C^* \cup B^* \subseteq B \cup C$: Suppose not. Clearly, $B^* \cup N(B^*) \subseteq B \cup C$. Therefore, it follows that there is a vertex x in $K_3 \cup K_4$ which is not in $B \cup C$. From the definition of K_3 and K_4, it follows that either x has at least two neighbours in $K_2 \cap N(B^*)$ (but then $x \in K_2 \subseteq C$) or x has at least two neighbours in $K_2' \cap N(B^*)$ (but then $x \in K_2'$). We conclude that $x \in K_2' \cap A$. Since $r' \in A \cap K_1'$, and since K_1', K_2' are cliques, it follows that $C' \subseteq A \cup C$. Therefore, $C' \cap B = \emptyset$. Since B is connected and $B \cap B' \neq \emptyset$, we conclude that $B \subseteq B'$. Now, since $r \in A' \cap K_1$ has no neighbour in B, it follows that $r \in c_{A \cup C}(N(B) \cap K_1)$, and so r has at least two neighbours in $N(B) \cap K_1 \subseteq C \cap C'$. Since $K_1 \cap K_1' = \emptyset$, it follows that $N(B) \cap K_1 \subseteq K_2'$. But then $r \in c_{A' \cup C'}(K_2')$, and so $r \in C'$, a contradiction.
- The vertex r is in A^*: Suppose not. Since $r \notin B^* \cup N(B^*)$ (as $B^* \subseteq B$ and $N(B^*) \subseteq B \cup C$), it follows that $r \in C^*$ and r has at least two neighbours in one of $K_2 \cap N(B^*)$ and $K_2' \cap N(B^*)$. Since $r \in A'$, it follows that $r \notin c_{A' \cup C'}(K_2')$, and so r does not have two or more neighbours in K_2'. It follows that r has two or more neighbours in $K_2 \cap N(B^*)$. Then, as G is diamond-free, it follows that r is adjacent to every vertex in K_2. Since r is also adjacent to every vertex in K_1 (as $r \in K_1$ and K_1 is a clique), it follows that C is a star cutset (separating r' from B). This contradicts the assumption that G has no star cutset.

Putting the above two items together, it follows that $C^* \cup B^* \subsetneq B \cup C$, and so C^* is a (w, G)-shield for $C = X$, a contradiction to the assumption that $X \in \text{core}_{w,G}(\mathcal{X}(G))$. This proves (15).

Since $r \in A'$, it follows that r has at most one neighbour in K_1'. But r is adjacent to every vertex in $K_1 \cap K_1'$ as $r \in K_1 \setminus K_1'$; and so by (15), it follows that $|K_1 \cap K_1'| = 1$. Let $v \in K_1 \cap K_1'$.

Next, let us define $M = B \cup B'$. Since B and B' are connected, and have non-empty intersection B^*, it follows that M is connected. Moreover, since $K_1 \cup K_1' \subseteq (C \cup A) \cap (C' \cup A')$, it follows that $K_1 \cup K_1'$ is disjoint from M.

We define a vertex t as follows: Since $r \in c_{A \cup C}(K_1 \cap N(B))$, it follows that either r has a neighbour in B, and we let $t = r$; or the set $K_1 \cap N(B)$ contains at least two vertices, and we choose $t \in (K_1 \cap N(B)) \setminus \{v\}$. Analogously, we define a vertex t' as

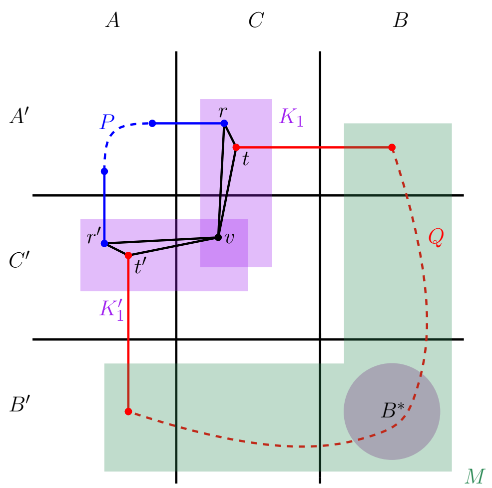

FIGURE 8. The proof of Theorem 6.1. Dashed lines represent paths of arbitrary length (possibly zero).

follows: Since $r' \in c_{A' \cup C'}(K_1' \cap N(B'))$, it follows that either r' has a neighbour in B', and we let $t' = r'$; or the set $K_1' \cap N(B')$ contains at least two vertices, and we choose $t' \in (K_1' \cap N(B')) \setminus \{v\}$.

We note that if $r \neq t$, then r has no neighbour in M (as r has no neighbour in B from the choice of t, and r has no neighbour in B' since $r \in A'$); an analogous statement holds for r'. Let Q be a path from t to t' with interior in M; see Figure 8.

Since both t and t' have a neighbour in P, we choose P' to be a path from t to t' with interior contained in P.

We claim that $H = t\text{-}P'\text{-}t'\text{-}Q\text{-}t$ is a hole: No vertex in P^* has a neighbour in M, and if r is in the interior of P', then $r \neq t$ and so r has no neighbour in M; and similarly for r'. Therefore, there are no edges from P'^* to Q^*; and both P' and Q are induced paths.

Moreover, we claim that $v \notin H$: From the choices of t and t', we have that $v \neq t, t'$; and since both P and Q are disjoint from $C \cap C'$, it follows that P' and Q are disjoint from $\{v\}$.

Since $t \in K_1$ and $t' \in K_1'$, it follows that t, t' are two distinct neighbours of v in H. Moreover, since $t \notin c_{A' \cup C'}(K_1')$, it follows that t has at most one neighbour in K_1', namely v; so t is non-adjacent to t'. This implies that $\{v\} \cup H$ is a clock, which is a contradiction and completes the proof. ∎

7. Inside the central bag

Throughout this section, we make the following assumption:

Assumption 7.1. *Let G be a (clock, diamond)-free graph with no star cutset and let w be a weight function on G. Suppose that G contains no $\left(w, \frac{1}{2}\right)$-balanced separator of size at most $4\omega(G)$. Let $\mathcal{X}(G)$ be as in the previous section, and let $\beta = \beta(G, w, \text{core}_{w,G}(\mathcal{X}(G)))$.*

Following the central bag method, we now have two goals:

- Show that β has a small balanced separator; and
- "Lift" this separator to G.

In pursuit of the second goal, it is helpful to extend the central bag β to incorporate certain bits from each component of $G \setminus \beta$. This will help with lifting, as it prevents these components from creating arbitrary connections that we do not "see" in β. Instead, we keep certain "marker paths" to record those connections. To make our lives easier, we would like to choose marker paths that are as simple as possible; the following lemma helps with this.

Lemma 7.2. *We assume that Assumption 7.1 holds. Let $X \in \mathcal{X}(G)$. Let $(A, C, B) = S_{w,G}(X)$ and let D be a component of A. Let K_1, K_2 be cliques such that $X = K_1 \cup K_2 = \text{closure}(K_1, K_2)$. Then there is a path P with ends $x, y \in X$ and interior in D such that the following hold:*

- *The path P has at least three vertices (in other words, x and y are not adjacent); and*
- *Every vertex in P^* has degree 2 in the graph $G[P \cup X]$ (and is therefore not contained in a triangle in $G[P \cup X]$).*

Proof. Since G has no star cutset, we find that $K_1 \cap K_2 = \emptyset$. We start with the following:

(16) *No vertex in K_1 has two or more neighbours in K_2, and vice versa.*

Suppose for a contradiction that $v \in K_1$ has at least two neighbours in K_2. Then, since G is diamond-free, it follows that v is adjacent to every vertex in K_2. But then $X \subseteq N[v]$, and so G has a star cutset, a contradiction. This proves (16).

Since G has no star cutset, and hence no clique cutset, it follows that $N(D)$ is not a clique. Therefore, there is a path with interior in D and non-adjacent ends in $N(D) \subseteq X$. Let P be a shortest such path, and let x and y be its ends. We may assume, by symmetry, that $x \in K_1$ and $y \in K_2$.

(17) *Let $v \in X \setminus \{x, y\}$ such that v has a neighbour in P^*. Then v is adjacent to both x and y.*

Suppose not. We may assume that $v \in K_1$, and so v is adjacent to x and non-adjacent to y. From the choice of P, since the path from v to y with interior in P^* is not a better choice for P, it follows that v is adjacent to the neighbour x^* of x in P. But now $x^* \in \text{closure}(K_1, K_2)$, a contradiction. This proves (17).

Let us pick x' as follows: If x has a neighbour in B, then $x' = x$. Otherwise, at least two vertices in $K_1 \setminus \{x\}$ have a neighbour in B. Note that y has at most one neighbour in K_1 by (16). We choose $x' \in K_1 \setminus N(y)$ such that x' has a neighbour in B.

Let us now pick y'. If y has a neighbour in B, then we let $y' = y$. Otherwise, at least two vertices in $K_2 \setminus \{y\}$ have a neighbour in B. Since x has at most one neighbour in K_2, we pick y' to be a vertex in $K_2 \setminus N(x)$ with a neighbour in B.

From our choices described above, it follows that:

- the vertices x, x' are in K_1 (and possibly equal);
- the vertices y, y' are in K_2 (and possibly equal);
- y is anticomplete to $\{x, x'\}$ and x is anticomplete to $\{y, y'\}$;
- if $x \neq x'$, then x' has no neighbour in P^* (by (17), since x' is non-adjacent to y); and
- If $y \neq y'$, then y' has no neighbour in P^* (by (17), since y' is non-adjacent to x).

Now, let Q be a path from x' to y' with interior in B; this exists by the choice of x' and y'. Then, $H = x'\text{-}Q\text{-}y'\text{-}y\text{-}P\text{-}x\text{-}x'$ is a hole in G. If some vertex $v \in (K_1 \cup K_2) \setminus \{x, y\}$ has a neighbour in P^*, then by (17), v has two non-adjacent neighbours in H, namely x and

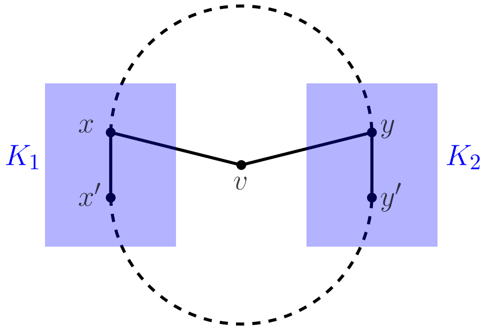

FIGURE 9. Proof of Lemma 7.2. A clock in the case that some vertex in X is adjacent to both x and y. Dashed lines represent paths of arbitrary length (possibly zero).

y, a contradiction as G is clock-free. It follows that no vertex in $(K_1 \cup K_2) \setminus \{x, y\}$ has a neighbour in P^*. Since P is induced, each of x and y has exactly one neighbour in P^*. This completes the proof. ■

By Theorem 6.1, the separations $S_{w,G}(X)$ and $S_{w,G}(X')$ are loosely non-crossing for all $X, X' \in \text{core}_{w,G}(\mathcal{X}(G))$. By Lemma 5.3, we may define, for each component D of $G \setminus \beta$, a set $X(D) \in \text{core}_{w,G}(\mathcal{X}(G))$ such that, writing $(A, C, B) = S_{w,G}(X(D))$, we have that $D \subseteq A$ (if more than one valid choice for $X(D)$ exists, we pick one arbitrarily). It follows that D is a component of A.

Given $X \in \text{core}_{w,G}(\mathcal{X}(G))$, we write $D(X)$ for the union of all components D of $G \setminus \beta$ with $X(D) = X$. It follows that

$$\bigcup_{X \in \text{core}_{w,G}(\mathcal{X}(G))} D(X) = G \setminus \beta,$$

where the union is a disjoint union.

Now, for every $X \in \text{core}_{w,G}(\mathcal{X}(G))$, let us define $P(X)$ as follows. If $D(X) = \emptyset$, then $P(X) = \emptyset$. Otherwise, let us pick a component D of $D(X)$, and let $P(X)$ be a path with interior in D and ends in X as guaranteed by Lemma 7.2. We call $P(X)$ the *marker path* for X.

We define

$$\beta^* = \beta \cup \bigcup_{X \in \text{core}_{w,G}(\mathcal{X}(G))} P(X).$$

From the choice of paths $P(X)$ as in Lemma 7.2, and since D is a component of $G \setminus \beta$, it follows that the ends of $P(X)$ are in β, and $(P(X))^*$ is disjoint from β. Therefore, every vertex in $\beta^* \setminus \beta$ has degree two in β^* and is not contained in a triangle in β^*.

In preparation for the next section, let us also define a weight function w^* on β^*, as follows:

- For every $v \in \beta$, we let $w^*(v) = w(v)$.
- For every $X \in \text{core}_{w,G}(\mathcal{X}(G))$ with $D(X) \neq \emptyset$, we pick a vertex $a_X \in (P(X))^*$ arbitrarily; and we set $w^*(a_X) = w(D(X))$ and $w^*(v) = 0$ for all $v \in (P(X))^* \setminus \{a_X\}$.

Then, for every vertex $v \in \beta$, its weight remains the same; for every component D of $G \setminus \beta$, we move its total weight to a_X where $X = X(D)$. From this, it is easy to see that w^* is a weight function on β^*.

The following results help us describe ways in which β^* is structurally simpler than G.

Lemma 7.3. *Assuming Assumption 7.1, and with the definition of β^* as above, the following holds. Suppose that β^* contains a seagull with a, u, v as in the definition of a seagull. Then one of the following holds:*

- *At least one of a or u is in $G \setminus \beta$.*
- *$N_G(u) \setminus \{v\}$ is a clique anticomplete to $\{v\}$.*
- *$N_G(a) \setminus \{v\}$ is a clique anticomplete to $\{v\}$.*

Proof. First we show:

(18) *If $N_G(u) \setminus \{v\}$ is a clique, then the lemma holds.*

Suppose $N_G(u) \setminus \{v\}$ is a clique K and v has a neighbour in K. Then $N_G(u)$ is connected, and therefore a clique by Lemma 3.4. Since G contains a seagull, G is not a complete graph. But now $N_G(u)$ is a clique cutset in G, contradicting Assumption 7.1. This proves (18).

By (18), we may assume that $N_G(u) \setminus \{v\}$ is not a clique, and similarly $N_G(a) \setminus \{v\}$ is not a clique. We may assume that the first outcome does not hold, and it follows that a and u are in β.

(19) *There is a cutset in G of the form $\{v\} \cup K$ where K is a clique and which separates $\{a\}$ from $\{u\}$.*

We consider two cases. Suppose first that a is a claw center in G. Then, by Theorem 4.6, there is a vertex b non-adjacent to a and a clique $K \subseteq N[b]$ in G such that $\{v\} \cup K$ separates $\{a\}$ from $\{u\}$, and (19) holds. Therefore, we may assume that a is not a claw center, and therefore (using Lemma 3.4), we have that $N_G(a) = K_1 \cup K_2$, where K_1 and K_2 are cliques. We may assume that $v \in K_1$. Since $N_G(a) \setminus \{v\}$ is not a clique, we may assume that K_1 contains a vertex $a' \neq v$. Since G is diamond-free, we have $a'u \notin E(G)$. Now, by Theorem 4.3 applied to the paw induced by $\{u, v, a, a'\}$, we once again obtain the desired cutset. This proves (19).

It follows that $X = \text{closure}(K, \{v\}) \in \mathcal{X}(G)$, and so $\text{core}_{w,G}(\mathcal{X}(G))$ contains either $X' = X$ or a (w, G)-shield X' for X. Then, writing $(A, C, B) = S_{w,G}(X)$ and $(A', C', B') = S_{w,G}(X')$, it follows that $A \subseteq A'$ and $\beta \cap A = \emptyset$.

It follows that $a, u \notin A$, and therefore $a \in X$ or $u \in X$. There is symmetry, and so we may assume that $u \in X$. Since $u \notin K$, it follows that u has at least two neighbours in K; consequently u is adjacent to all vertices of K. Let Q be the component of $G \setminus (\{v\} \cup K)$ containing u. Since $K \cup \{v\} \subseteq N[u]$, and since G has no star cutset by Assumption 7.1, it follows that $Q = \{u\}$, and so $N(u) = K \cup \{v\}$, contrary to our assumption that $N(u) \setminus \{v\}$ is not a clique. ■

A *simplicial* vertex in a graph is a vertex whose neighbourhood is a clique. Let us say that a vertex u in a graph G is *near-simplicial* in G if there is a vertex $v \in V(G)$ such that $N_G(u) \setminus \{v\}$ is a clique. Then, the second and third outcomes of Lemma 7.3 imply that u and a, respectively, are near-simplicial.

Lemma 7.4. *Assuming Assumption 7.1, and with the definition of β^* as above, the following hold:*

(i) For every $x \in \beta^*$, there is a set $Z_1 \subseteq N(x)$ such that $Z_1 = K \cup \{x'\}$ where K is a clique, with the property that for every $v \in N_{\beta^*}(x) \setminus Z_1$, we have $\deg_{\beta^*}(v) \leq 2$.
(ii) For every x in β^* which is a claw center in β^*, there is a set $Z_2 \subseteq N(x)$ such that $Z_2 = K \cup \{x', x''\}$ where K is a clique, with the property that for every neighbour y of x in $\beta^* \setminus Z_2$, there is no $X \in \mathrm{core}_{w,G}(\mathcal{X}(G))$ with $X = K_1 \cup K_2 = \mathrm{closure}(K_1, K_2)$ such that y has a neighbour in a component of $D(X)$ and $x, y \in K_1$.

Proof. By our choice of $P(X)$ and by Lemma 7.2, it follows that all vertices in $\beta^* \setminus \beta$ have degree two in β^* and are not contained in triangles in β^*.

Let $x \in \beta^*$. If $x \in \beta^* \setminus \beta$, then $\deg_{\beta^*}(x) = 2$, and both statements hold. So we may assume that $x \in \beta$. Moreover, if x is near-simplicial in β^*, then again both (i) and (ii) hold; so we may assume that this is not the case.

(20) *The set $N_{\beta^*}(x)$, which is a disjoint union of cliques by Lemma 3.4, contains at most one component of size more than one.*

Suppose not; let $a, u \in N_{\beta^*}(x)$ be non-adjacent such that both a and u are in a clique of size at least two in $N_{\beta^*}(x)$; say a' is a common neighbour of x and a in β^*. We apply Lemma 7.3 to the seagull with vertex set $\{a, x, u\}$. Then, since a, u are each in a triangle in β^* (with x and a neighbour of x), it follows that $a, u \in \beta$, so the second or third outcome of Lemma 7.3 holds. By symmetry, we may assume that $N_G(a) \setminus \{x\}$ is a clique K containing a' and anticomplete to $\{x\}$. But a' is adjacent to x, a contradiction; this proves (20).

Let K be defined as follows. If $N_{\beta^*}(x)$ has no component of size more than one, then $K = \emptyset$. Otherwise, by (20), we define K to be the unique component of $N_{\beta^*}(x)$ of size more than one; by Lemma 3.4, we have that K is a clique.

(21) *There is at most one vertex in $N_{\beta^*}(x) \setminus K$ with degree more than two in β^*.*

Suppose that $a, u \in N_{\beta^*}(x) \setminus K$ both have degree more than two in β^*. Every neighbour of x which is in $\beta^* \setminus \beta$ has degree two in β^*, and hence $a, u \in \beta$. Applying Lemma 7.3 to the seagull with vertex set $\{a, x, u\}$, we conclude that the second or third outcome holds. By symmetry, we may assume that there is a clique X such that $N_G(a) = X \cup \{v\}$ with X anticomplete to $\{v\}$. Since a has degree at least three in β^*, it follows that there are two distinct vertices $y, y' \in X \cap \beta^*$. Since both are in the triangle $\{a, y, y'\}$, it follows that $y, y' \in \beta$. But now, by applying Lemma 7.3 to the seagull with vertex set $\{x, a, y\}$, we conclude that one of the following holds:

- The vertex x is near-simplicial in G (contrary to our assumption that x is not near-simplicial in β^*); or
- $N_G(y) \setminus \{a\}$ is a clique anticomplete to $\{a\}$ (contrary to the fact that y' is in $N_G(y) \setminus \{a\}$ and $y'a \in E(G)$).

This is a contradiction, and proves (21).

By (21), there is at most one vertex in $N_{\beta^*}(x) \setminus K$ of degree more than two in β^*; let us choose x' to be this vertex if it exists (letting x' be an arbitrary vertex in $N_{\beta^*}(x)$ otherwise). Then $Z_1 = K \cup \{x'\}$ satisfies (i).

It remains to prove (ii). Let K, Z_1, x' be as above. Suppose that $a, u \in N_{\beta^*}(x) \setminus Z_1$ are distinct and non-adjacent vertices in β. Then, by Lemma 7.3, at least one of a, u is near-simplicial in G and simplicial in $G \setminus \{x\}$. We define a vertex x'' as follows: If every vertex in $N_\beta(x) \setminus Z_1$ is simplicial in $G \setminus \{x\}$, then $x'' = x'$. Otherwise, x'' is the unique vertex in $N_\beta(x)$ which is not simplicial in $G \setminus \{x\}$. Now let $Z_2 = Z_1 \cup \{x''\} = K \cup \{x', x''\}$.

We claim that Z_2 satisfies (ii). Suppose for a contradiction that (ii) does not hold, that is, there is a vertex $y \in N_{\beta^*}(x) \setminus Z_2$ and a set $X \in \text{core}_{w,G}(\mathcal{X}(G))$ with $X = K_1 \cup K_2 = \text{closure}(K_1, K_2)$ such that y has a neighbour in a component of $D(X)$ and $x, y \in K_1$.

Suppose first that $y \in \beta$. From the choice of Z_2, it follows that $N_G(y) = \{x\} \cup T$, where T is a clique anticomplete to $\{x\}$. It follows that $K_1 = \{x, y\}$. We note that $|T \cap K_2| \leq 1$, as otherwise y is adjacent to all of K_2 (since G is diamond-free), but then $X \subseteq N[y]$ is a star cutset, contradicting Assumption 7.1.

Letting $(A, C, B) = S_{w,G}(X)$, and using that T is a clique, it follows that either $T \cap A = \emptyset$ or $T \cap B = \emptyset$. Since $\emptyset \neq N(y) \cap D(X) \subseteq N(y) \cap A$, and since $x \in K_1 \subseteq X$, it follows that $T \cap A \neq \emptyset$, and so $T \subseteq C \cup A$. Then, y is not in $N(B)$, and y does not have two neighbours in either K_1 or K_2, contrary to the fact that $X = K_1 \cup K_2 = \text{closure}(K_1, K_2)$, a contradiction.

It follows that $y \in \beta^* \setminus \beta$. Then, $y \in P(X')$ for some $X' \in \text{core}_{w,G}(\mathcal{X}(G))$; and in particular, there is a component D' in $D(X')$ such that $y \in D'$ and D' is a component of $G \setminus \beta$. Since y has a neighbour in a component D of $D(X)$, and D is a component of $G \setminus \beta$, it follows that $D = D'$. But then, once again letting $(A, C, B) = S_{w,G}(X)$, we conclude that $y \in A$, a contradiction as we had assumed that $y \in K_1 \subseteq C$. This proves (ii). ■

8. Putting everything together

We require the following result and definition of [2]. For a graph G and positive integer d, we denote by $\gamma_d(G)$ the maximum degree of the subgraph of G induced by the set of vertices with degree at least d in G.

Theorem 8.1 (Abrishami, Alecu, Chudnovsky, Hajebi, Spirkl, Vušković [2])**.** *For all $t, \gamma > 0$, there exists $q = q(t, \gamma)$ such that every graph G with $\gamma_3(G) \leq \gamma$ and treewidth more than q contains a subdivision of $W_{t\times t}$ or the line graph of a subdivision of $W_{t\times t}$ as an induced subgraph.*

Now we can prove:

Lemma 8.2. *For every $t \in \mathbb{N}$, there exists a constant $n = n(t)$ such that the following holds.*

Let G be a t-clean graph and assume that Assumption 7.1 holds. With the definition of β^* and w^* as in the previous section, we have that β^* has a $\left(w^*, \frac{1}{2}\right)$-balanced separator of size at most n.

Proof. Let q be as in Theorem 8.1 applied with t and with $\gamma = t$; let $n = q + 1$. Then, by Lemma 7.4(i), every vertex in β^* has at most $\omega(G) \leq t$ (as G is t-clean) neighbours of degree more than two, it follows that $\gamma_3(\beta^*) \leq t$. Therefore, by Theorem 8.1, and since G is t-clean, it follows that $\text{tw}(\beta^*) \leq q$. Now, by Lemma 5.2, it follows that β^* has a $\left(w^*, \frac{1}{2}\right)$-balanced separator of size at most $q + 1 = n$. ■

Our final step is to lift the balanced separator from β^* to G. Given two sets $X, Y \subseteq V(G)$, let us say that X *has a neighbour in* Y if there is an edge $xy \in E(G)$ with $x \in X$ and $y \in Y$.

Theorem 8.3. *Let $t \in \mathbb{N}$. There exists a $c = c(t)$ such that the following holds. Let G be a t-clean (clock, diamond)-free graph with no star cutset. Let w be a weight function on G. Then G has a $\left(w, \frac{1}{2}\right)$-balanced separator of size at most c.*

Recall that the above, together with Lemma 5.1, implies Theorem 3.3, which in turn implies Theorem 1.1, as discussed in Section 3. It remains to prove Theorem 8.3.

Proof of Theorem 8.3. Let $n = n(t)$ be as in Lemma 8.2. We define $c = \max\{4t, n(2t+1)\}$.

Suppose that G contains no $\left(w, \frac{1}{2}\right)$-balanced separator of size at most c. Since G is t-clean, it follows that $\omega(G) \leq t$. Since $c \geq 4t$, it follows that G contains no $\left(w, \frac{1}{2}\right)$-balanced separator of size at most $4\omega(G)$. Therefore, Assumption 7.1 holds, and in particular, we can define β^* and w^* as in Section 7.

By Lemma 8.2, it follows that β^* has a $\left(w^*, \frac{1}{2}\right)$-balanced separator S of size at most n.

Let us define a set Y as follows. For every $v \in S$ such that $v \in (P(X))^*$ for some $X \in \text{core}_{w,S}(\mathcal{X}(G))$, we set $Y(v) = X$. For every $v \in S$ which is not a claw center in β^*, we let $Y(v) = N_{\beta^*}(v)$. For every $v \in S$ which is a claw center in β^*, we let $Y(v) = Z_2$ where Z_2 is as is Lemma 7.4(ii). Finally, we let $Y = S \cup \bigcup_{v \in S} Y(v)$. From the definition of $Y(v)$ above, it follows that $Y(v)$ is either the union of two cliques or the union of a clique and two vertices for every $v \in S$; therefore, $|Y| \leq |S|(2t+1) \leq n(2t+1) \leq c$, as desired.

It remains to show that Y is a $\left(w, \frac{1}{2}\right)$-balanced separator of G. In order to accomplish this, we would like to show, roughly speaking, that adding back components of $G \setminus \beta$ does not "merge" two components of $\beta^* \setminus S$, and that for each component of $G \setminus \beta$, its weight is accounted for, in the corresponding component of $\beta^* \setminus S$, by a vertex of the form a_X (as defined in Section 7). Both of these statements are formalized in (22) below.

Let D be a component of $G \setminus \beta$. Writing $X = X(D)$ and $S_{w,G}(X) = (A, C, B)$, we have that $D \subseteq A$ and so $w(D) \leq \frac{1}{2}$. Let D' be a component of $D \setminus Y$. If D' has no neighbour in $\beta \setminus Y$, then D' is a component of $G \setminus Y$ and of weight at most $\frac{1}{2}$, as desired. Thus, from now on, we will focus on the case in which D' has a neighbour in $\beta \setminus Y$.

Given a component R of $\beta \setminus Y$, let us define $\hat{R}$ as the component of $\beta^* \setminus S$ containing R.

(22) *Let D be a component of $G \setminus \beta$, and let D' be a component of $D \setminus Y$. Let $X = X(D)$. If D' has a neighbour y in a component R of $\beta \setminus Y$, then $a_X \in \hat{R}$. In particular, $\hat{R}$ is the same for all components R of $\beta \setminus Y$ in which D' has a neighbour.*

Suppose not, that is, D' has a neighbour y in a component R of $\beta \setminus Y$, but $a_X \notin \hat{R}$. If $(P(X))^* \cap S \neq \emptyset$, say $q \in (P(X))^* \cap S$, then $X = Y(q) \subseteq Y$, and so D' has no neighbour in $\beta \setminus Y \subseteq \beta \setminus X$. Therefore, we may assume that $a_X \in (P(X))^*$ is in a component R' of $\beta^* \setminus S$ with $R' \neq \hat{R}$.

Writing $X = K_1 \cup K_2 = \text{closure}(K_1, K_2)$, and using that $y \in \beta \cap N_G(D') \subseteq \beta \cap N_G[D] = X$, we may assume that $y \in K_1$. Let x, x' be the ends of $P(X)$. Since $(P(X))^*$ is disjoint from S and contains $a_X \in R'$, it follows that $(P(X))^* \subseteq R'$. But one of x, x', say x, is in K_1 and therefore adjacent to $y \in \hat{R}$ (which is anticomplete to R'); so we conclude that $x \in S$ (as x has a neighbour in both $\hat{R}$ and R', two different components of $\beta^* \setminus S$); see Figure 10. But then, as $y \in N_{\beta^*}(x) \setminus Y(x)$, it follows from the choice of

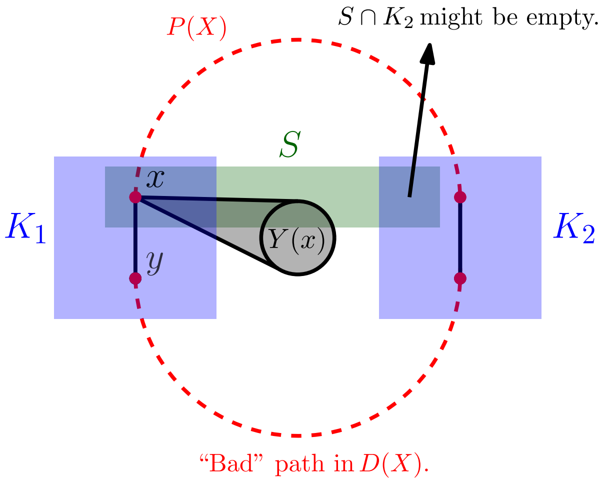

FIGURE 10. Proof of Theorem 8.3. We have shown that $P(X)$ is disjoint from S, and that x has neighbours in both R' and $\hat{R}$, and so $x \in S$. Dashed lines represent paths of arbitrary length (possibly zero).

$Y(x)$ that x is a claw center in β^*. By Lemma 7.4 and the choice of $Y(x)$, there is no $X \in \text{core}_{w,G}(\mathcal{X}(G))$ such $X = K_1 \cup K_2 = \text{closure}(K_1, K_2)$ such that $x, y \in K_1$ and y has a neighbour in $D' \subseteq D(X)$. This is a contradiction (because X, x, y, K_1, K_2 have precisely those properties), and proves (22).

Let M be a component of $G \setminus Y$ with $M \cap \beta \neq \emptyset$. Suppose first that there are two distinct components of $\beta^* \setminus S$ such that $M \cap \beta$ has a non-empty intersection with each of them. Then, M contains a path Q with interior in $G \setminus \beta$ such that the ends of Q are in β, and in two different components of $\beta^* \setminus S$. It follows that Q^* is contained in a component D' of $D \setminus Y$ for some component D of $G \setminus \beta$. However, this contradicts (22), which states that both ends of Q are in the same component of $\beta^* \setminus S$.

It follows that for every component M of $G \setminus Y$ with $M \cap \beta \neq \emptyset$, there is a component $\hat{R}$ of $\beta^* \setminus S$ such that:

- $M \cap \beta \subseteq \hat{R}$; and
- For every $x \in G \setminus \beta$ such that x is in a component D of $G \setminus \beta$ and $x \in M$, we have that $a_{X(D)} \in \hat{R}$ (by (22)).

Now it follows that $w(M) \leq w^*(\hat{R}) \leq \frac{1}{2}$; but then Y is a $\left(w, \frac{1}{2}\right)$-balanced separator, as desired. ■

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