Detecting an induced net subdivision

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Abstract

A *net* is a graph consisting of a triangle C and three more vertices, each of degree one and with its neighbour in C, and all adjacent to different vertices of C. We give a polynomial-time algorithm to test whether an input graph has an induced subgraph which is a subdivision of a net. Unlike many similar questions, this does not seem to be solvable by an application of the "three-in-a-tree" subroutine.

1 Introduction

In this paper, all graphs are simple and finite. Let H be the graph with six vertices $a_1, a_2, a_3, b_1, b_2, b_3$ and the following edges: $a_1a_2, a_2a_3, a_3a_1, b_1a_1, b_2a_2, b_3a_3$. This is called a *net*. A *doily* is a graph consisting of a cycle H and three more vertices b_1, b_2, b_3, pairwise non-adjacent, such that each b_i has a unique neighbour a_i in $V(H)$, and a_1, a_2, a_3 are all different. Thus some induced subgraph of a graph G is a doily if and only if some induced subgraph of G is a subdivision of a net. A *doily of a graph G* is an induced subgraph of G that is a doily. We say a graph *contains* a doily if some induced subgraph is a doily.

In this paper, we give a polynomial-time algorithm to test whether an input graph G contains a doily. Before we go on, let us motivate this a little. For any fixed graph H, we can test if an input graph G contains H as a subgraph, or as an induced subgraph, in time $O(n^{|V(H)|})$, just by checking all sets of vertices of G of cardinality $|V(H)|$. (When we give the running time of an algorithm whose input is a graph G, n stands for the number of vertices of G.) And these count as polynomial-time algorithms, if H is a fixed graph. One can also check in polynomial time (again, with H fixed) whether some subgraph of G is a subdivision of H, as a consequence of the results of the Graph Minors series [4].

On the other hand, checking whether G contains an *induced* subgraph isomorphic to a subdivision of H is much more complicated. Let us call this the "induced H subdivision problem". For some graphs H this can be solved in polynomial time, and for some it is NP-complete, and we are far from identifying the border between the two. For instance, the following seem to be open:

- Can it be solved in polynomial time for every graph H with maximum degree at most three?
- Can it be solved in polynomial time when H is K_4?
- Can it be solved in polynomial time when H consists of two disjoint triangles?

Here are some results:

1.1. *The induced H subdivision problem is NP-complete when H is either*

- *the graph obtained from the complete bipartite graph $K_{2,3}$ by adding an edge joining the two vertices of degree three, or*
- *the graph with seven vertices $1, \ldots, 7$ and edges $12, 13, 23, 14, 15, 26, 27$, or*
- *the tree obtained by adding nine leaves to a three-vertex path P, three adjacent to each vertex of P.*

It is polynomial-time solvable when H is either

- *the complete bipartite graph $K_{2,3}$, or*
- *the graph with six vertices $1, \ldots, 6$ and edges $12, 13, 23, 14, 15, 26$, or*
- *a tree H that can be obtained as follows: let T_1 be a tree with at most four vertices, let T_2 be obtained from T_1 by adding arbitrarily many leaves each adjacent to some leaf of T_1, and let H be a subdivision of T_2.*

The first three results are consequences of a result of [3]. The last three are all by application of an algorithm of [2], that we call the "three-in-a-tree" algorithm; given a graph G and three vertices of G, it tests if there is a subset of $V(G)$ inducing a tree that contains the three vertices. The three-in-a-tree algorithm was given several more applications in [3]. Indeed, to date all the non-trivial polynomial-time instances of the induced H subdivision problem were solved by application of the three-in-a-tree algorithm. The result of this paper is *not* such an application. Our main result is that when H is a net, the induced H subdivision problem is polynomial-time solvable. More explicitly:

1.2. *There is an $O(n^{16})$-time algorithm whose input is a graph G and whose output is a doily of G if such a doily exists.*

In fact, one can easily modify the algorithm we present so that it outputs a doily of G with the minimum number of vertices, when one exists, but we omit those details.

2 Outline of the algorithm

A *frame* in a graph G is a twelve-tuple

$$(b_1, b_2, b_3, a_1, a_2, a_3, a'_1, a'_2, a'_3, a''_1, a''_2, a''_3)$$

of vertices from $V(G)$ such that

- a_1, a_2, a_3, b_1, b_2, and b_3 are distinct,
- for $i = 1, 2, 3$, b_i has degree one and $a_ib_i \in E(G)$, and
- for $i = 1, 2, 3$, a_i has degree three and its neighbours are b_i, a'_i, a''_i.

Let K be a doily of some graph G. (In what follows, all computations with indices are modulo three). A frame

$$(b_1, b_2, b_3, a_1, a_2, a_3, a'_1, a'_2, a'_3, a''_1, a''_2, a''_3)$$

in G is a *frame for K* if:

- $b_1, b_2, b_3 \in V(K)$, and therefore $a_1, a_2, a_3, a'_1, a'_2, a'_3, a''_1, a''_2, a''_3 \in V(K)$
- for $i = 1, 2, 3$, a''_{i+1} and a'_{i-1} belong to the path of K between a_{i-1} and a_{i+1} not containing a_i.

If v is a vertex of a graph G, $N(v)$ denotes the set of neighbours of v in G. Let G be a graph and $X, Y \subseteq V(G)$ be disjoint non-empty sets. A vertex $v \in V(G) \setminus (X \cup Y)$ is *the centre of a star cutset that separates X from Y* if for some $S \subseteq N(v) \setminus (X \cup Y)$, the graph $G \setminus (\{v\} \cup S)$ contains no path from X to Y (in particular, $G \setminus (\{v\} \cup S)$ is disconnected).

A doily of a graph G is *minimum* if its number of vertices is minimum over all doilies of G. A pair (G, F), where G is a graph and $F = (b_1, b_2, b_3, a_1, a_2, a_3 \ldots)$ is a frame in G, is *trackable* if the following hold:

- every doily in G has at least nine vertices;
- G contains a doily if and only if F is a frame for some minimum doily of G;

- no vertex of G is the centre of a star cutset that, for some $i \in \{1,2,3\}$, separates $\{a_i\}$ from $\{a_1, a_2, a_3\} \setminus \{a_i\}$.

The following is easy to prove with brute force enumeration.

2.1. *There is an $O(n^{16})$-time algorithm whose input is a graph G, and whose output is a doily of G with at most eight vertices if there is one, and otherwise $k \leq n^{12}$ pairs $(G_1, F_1), \dots, (G_k, F_k)$ such that:*

- *for all $i = 1, \dots, k$, G_i is an induced subgraph of G and F_i is a frame in G_i;*
- *G contains a doily if and only if for some $i \in \{1, \dots, k\}$, (G_i, F_i) is trackable and G_i contains a doily.*

Proof. First, check all subsets of cardinality at most eight from $V(G)$, and stop if one of them induces a doily. This takes time $O(n^8)$. Now, generate all twelve-tuples $F_1, \dots, F_k$ from $V(G)^{12}$. For each F_i in turn, let

$$F_i = (a_1, a_2, a_3, b_1, b_2, b_3, a'_1, a''_1, a'_2, a''_2, a'_3, a''_3),$$

and let X be the set

$$\{a_1, a_2, a_3, b_1, b_2, b_3, a'_1, a''_1, a'_2, a''_2, a'_3, a''_3\}.$$

First we check whether

- b_1, b_2, b_3 are distinct, and different from all of $a_1, a_2, a_3, a'_1, a''_1, a'_2, a''_2, a'_3, a''_3$
- for $i = 1, 2, 3$, b_i is adjacent to a_i, and has no other neighbour in X
- a_1, a_2, a_3 are distinct,
- for $i = 1, 2, 3$, a'_i, a''_i are distinct neighbours of a_i, and a_i has no neighbours in X except b_i, a'_i, a''_i
- for $i = 1, 2, 3$, if a_{i-1}, a_{i+1} are adjacent then $a'_{i-1} = a_{i+1}$ and $a''_{i+1} = a_{i-1}$, and if a_{i-1}, a_{i+1} are non-adjacent then a'_{i-1}, a''_{i+1} are different from all of

$$a_{i-1}, a''_{i-1}, a'_i, a_i, a''_i, a'_{i+1}, a_{i+1}.$$

If one of these is false, go to the next twelve-tuple. Otherwise build G_i as follows. Initally set $G_i = G$. For $i = 1, 2, 3$, delete from G_i all neighbours of b_i except a_i, and delete all neighbours of a_i except b_i, a'_i, a''_i.

Now, go through the following loop. While there exists a vertex v that is the centre of a star cutset of G_i that, for some $j \in \{1, 2, 3\}$, separates $\{a_j\}$ from $\{a_1, a_2, a_3\} \setminus \{a_j\}$, put $G_i \leftarrow G_i \setminus v$. Note that this loop can be performed in time $O(n^2|E(G)|)$, because for each v, computing the connected components of $G_i \setminus (N(v) \setminus \{a_1, a_2, a_3\})$ decides whether v is the centre of a star cutset that separates $\{a_j\}$ from $\{a_1, a_2, a_3\} \setminus \{a_j\}$. If some vertex of F_i is erased during the loop, go to the next 12-tuple. Clearly, after the loop, F_i is a frame in G_i and no vertex of G_i is the centre of a star cutset that separates some $\{a_j\}$ from $\{a_1, a_2, a_3\} \setminus \{a_j\}$.

If G contains no doily, then clearly the same is true for G_i for all $i \in \{1, \dots, k\}$. Suppose conversely that G contains a doily. Let K be a minimum doily of G. Since no vertex of K is the centre of a star cutset that separates vertices of the cycle of K, it follows that for some twelve-tuple F_i made of vertices of K, a pair (G_i, F_i) is generated, such that G_i contains K and F_i is a frame for K. So, (G_i, F_i) is a trackable pair and G_i contains a doily. □

The following is less obvious.

2.2. *There is an $O(n^2)$-time algorithm, whose input is a pair (G, F), where G is a graph and F a frame in G, and whose output is an induced subgraph K of G, such that if (G, F) is a trackable pair and G contains a doily, then K is a doily.*

The proof of 2.2 is postponed to the next two sections: in Section 3, we show that when (G, F) is a trackable pair, every vertex attaches "locally" to every minimum doily of G with frame F (this will be defined formally); and in Section 4, we take advantage of this to prove 2.2 with the shortest path detector method. Assuming all this, we can now prove our main result.

Proof of 1.2

Here is an algorithm. Step 1: run the algorithm from 2.1 for G. If a doily on at most eight vertices is found, then stop. Otherwise, go to Step 2: run the algorithm from 2.2 for all pairs (G_i, F_i) generated in Step 1. If some doily is found, then stop. Otherwise output "G contains no doily" and stop.

Let us prove the correctness of this algorithm. If the algorithm outputs a doily, then G obviously contains a doily (so the answer is correct). Suppose conversely that G contains a doily. If some doily of G has fewer than nine vertices, it is detected in Step 1. Otherwise, Step 1 provides a trackable pair (G_i, F_i) that contains a doily. So, in Step 2, when (G_i, F_i) is considered (or possibly before), a doily is found. This proves 1.2. □

3 Cleaning major vertices

When P is a path or cycle, its *length* is the number of edges in P. When H is an induced cycle and $b \notin V(H)$ is a vertex with a unique neighbour $a \in V(H)$, we say that the vertex b is a *tuft for H at a*. We say that vertices b_1, b_2, b_3 form a *tufting* for H if b_1, b_2, b_3 are pairwise non-adjacent, each is a tuft for H, and no two of them have the same neighbour in $V(H)$. So, a doily is an induced cycle with a tufting. A *hole* is an induced cycle with at least four vertices. When K is a subgraph of some graph G and v a vertex of G, we define $N_K(v) = N(v) \cap V(K)$.

When K is a doily of G and F is a frame for K, we use the following notation and definitions. Let $F = (b_1, b_2, b_3, a_1, a_2, a_3, \ldots)$ (we do not need to name the other vertices of the frame in this section). We denote by H_K the unique cycle of K. So, H_K is the union of three disjoint chordless paths P_1, P_2, P_3, where for $i = 1, 2, 3$, P_i is the path of $K \setminus a_i$ from a_{i+1} to a_{i-1}

In what follows, all computations with indices are modulo three. Let us assign an orientation "clockwise" to the cycle H_K, such that a_1, a_2, a_3 are in clockwise order. For any vertex v of H_K, let v^+ be the vertex of H_K that follows v in clockwise order, and let v^- be the vertex that precedes v. A vertex $v \in V(G) \setminus V(K)$ is *minor (with respect to K)* if for some $i = 1, 2, 3$, $N_K(v)$ is a subset of some subpath of P_i of length at most two. A vertex $v \in V(G) \setminus V(K)$ is *major (with respect to K)* if it has neighbours in P_i for all $i = 1, 2, 3$. For $i = 1, 2, 3$, when v has at least one neighbour in P_i, we define $y_{i+1}(v)$ as the neighbour of v in P_i that is closest to a_{i+1} (along P_i). Note that v is non-adjacent to a_{i-1}, a_{i+1} because the latter both have degree three, and so $y_{i+1}(v)$ is an internal vertex of P_i. Similarly, for $i = 1, 2, 3$, when v has at least one neighbour in P_i, we define $x_{i-1}(v)$ as the neighbour of v in P_i that is closest to a_{i-1} (along P_i). For $i = 1, 2, 3$, when v has neighbours in P_{i-1} and P_{i+1}, we define $W_i(v)$ to be the path $x_i(v)$-P_{i+1}-a_i-P_{i-1}-$y_i(v)$.

Our goal in this section is the following statement.

3.1. *If G is a graph and (G, F) is a trackable pair, and K is a minimum doily of G, with frame F, then every vertex in $V(G) \setminus V(K)$ is minor with respect to K in G.*

Throughout this section, G is a graph that contains a doily and (G, F) is a trackable pair, and K is some minimum doily of G, with frame F. The proof goes through several lemmas. The idea is that if there are major vertices for K, then one of them is the centre of a star cutset that separates a_1 from $\{a_2, a_3\}$, which contradicts that (G, F) is trackable. Note that from the definition of a trackable pair, K has at least nine vertices (so H_K is of length at least six, but possibly some P_i is of length one).

3.2. *If $v \in V(G) \setminus V(K)$, then either v is minor or v is major. Suppose that v is major. Then for all $i \in \{1, 2, 3\}$, $x_i(v)$, $x_i(v)^-$, $x_i(v)^{--}$ are internal vertices of P_{i+1} and are all adjacent to v; and $y_i(v)$, $y_i(v)^+$, $y_i(v)^{++}$ are internal vertices of P_{i-1}, and all adjacent to v.*

Proof. First we show that if $N_K(v) \subseteq V(P_i)$ for some $i = 1, 2, 3$, then v is minor. For if v has neighbours in P_i that are not in a three-vertex path of P_i, then $P_i \cup \{v\}$ contains a chordless path P_i' from a_{i-1} to a_{i+1}, shorter than P_i. Replacing P_i by P_i' in K, we obtain a doily smaller than K, a contradiction.

Hence, from the symmetry, we may assume that v has neighbours in P_1 and P_2. Suppose that v has no neighbours in P_3. Then

$$H = v\text{-}x_1(v)\text{-}P_2\text{-}a_1\text{-}P_3\text{-}a_2\text{-}P_1\text{-}y_2(v)\text{-}v$$

is a hole and b_1, b_2 are tufts for H. We claim that H is smaller than H_K. Suppose not; then $x_1(v)$ and $y_2(v)$ are neighbours of a_3, and so $H' = a_3$-$x_1(v)$-v-$y_2(v)$-a_3 is a hole of length four. Since H_K has length at least six (because every doily has at least nine vertices), it follows that $y_2(v)^-$, $x_1(v)^+$ are non-adjacent. Hence b_3, $y_2(v)^-$ and $x_1(v)^+$ form a tufting for H', giving a seven-vertex doily, a contradiction. This proves our claim that H is smaller than H_K. So, since b_1, b_2 are tufts for H and H is smaller than H_K, it cannot be that H has a third tuft at some vertex different from a_1, a_2. In particular, v is adjacent to $y_2(v)^+$ (for otherwise $y_2(v)^+$ is a tuft at $y_2(v)$). Symmetrically, v is adjacent to $x_1(v)^-$, and, in particular, $x_1(v)^-$ is different from a_3, and so $x_1(v)$ is non-adjacent to $y_2(v)^{++}$. Now, v is non-adjacent to $y_2(v)^{++}$ (for otherwise $y_2(v)^{++}$ is a tuft for H at v). Hence, v-$y_2(v)$-$y_2(v)^+$-v is a triangle for which $x_1(v)$, $y_2(v)^-$ and $y_2(v)^{++}$ form a tufting, giving a six-vertex doily, a contradiction. Thus we have proved that v has neighbours in P_3, so v is major.

Let us now prove the second statement of the theorem, and we may assume that $i = 1$. Assume that v is major; then the hole H formed by $W_1(v)$ and v is smaller than H_K, and $b_1, y_2(v)$ are tufts for H. So, H cannot have a third tuft at some vertex different from a_1, v. Now $x_1(v)^-$ is non-adjacent to $y_2(v)$ because otherwise v-$x_1(v)$-a_3-$y_2(v)$-v is a hole for which $x_1(v)^+$, $y_2(v)^-$ and b_3 form a tufting, giving a seven-vertex doily, a contradiction. Hence, v is adjacent to $x_1(v)^-$ for otherwise $x_1(v)^-$ would be a third tuft for H at $x_1(v)$. In particular, $x_1(v)^- \neq a_3$. Symmetrically, v is adjacent to $y_2(v)^+$, and $y_2(v)^+ \neq a_3$. Consequently $x_1(v)^{--}$ is non-adjacent to $y_2(v)$. Now, v is adjacent to $x_1(v)^{--}$, for otherwise, v-$x_1(v)$-$x_1(v)^-$-v is a triangle for which $y_2(v)$, $x_1(v)^+$ and $x_1(v)^{--}$ form a tufting. It follows that $x_1(v)^{--}$ is an internal vertex of P_2. Thus we have proved that v is adjacent to $x_1(v)$, $x_1(v)^-$ and $x_1(v)^{--}$. Symmetrically, $y_1(v)$, $y_1(v)^+$, $y_1(v)^{++}$ are all internal vertices of P_{i-1}, and v is adjacent to them all. □

Let u and v be two major vertices and $i \in \{1, 2, 3\}$. We say that u and v *disagree at i* when $x_i(u)$, $x_i(v)$, $y_i(u)$, $y_i(v)$ are pairwise distinct and appear along the path a_{i-1}-P_{i+1}-a_i-P_{i-1}-a_{i+1} in one of the following orders:

- $x_i(u)$, $x_i(v)$, $y_i(u)$, $y_i(v)$ or
- $x_i(v)$, $x_i(u)$, $y_i(v)$, $y_i(u)$.

Note that, if u, v are non-adjacent, then u and v disagree at i if and only if there exists an induced path from u to v that goes through a_i and whose interior is in a_{i-1}-P_{i+1}-a_i-P_{i-1}-a_{i+1}. Moreover, this induced path is unique and we denote it by $W_i(u, v)$.

A vertex $z \in V(H_K)$ is a *tie at i for u and v* if $i \in \{1, 2, 3\}$ and either $z = x_i(u) = x_i(v)$ or $z = y_i(u) = y_i(v)$.

We say that u *beats v at i* when $x_i(u)$, $x_i(v)$, $y_i(u)$, $y_i(v)$ are pairwise distinct and appear along a_{i-1}-P_{i+1}-a_i-P_{i-1}-a_{i+1} in the following order: $x_i(v)$, $x_i(u)$, $y_i(u)$, $y_i(v)$.

It is clear that when u and v are major vertices, then for each $i = 1, 2, 3$ exactly one of the following holds: u and v disagree at i; or there is a tie for u and v at i; or u beats v at i; or v beats u at i.

3.3. *If u and v are two non-adjacent major vertices, then they disagree at at most one $i \in \{1, 2, 3\}$.*

Proof. Suppose that u and v disagree at 1 and 2 say. Then $W_2(u, v) \cup W_1(u, v)$ is a hole smaller than H_K for which b_1, b_2 and one of $x_3(u), x_3(v)$ are non-adjacent tufts unless $x_3(u) = x_3(v)$. In this last case, $W_1(u, v) \cup \{x_3(u)\}$ is a hole for which b_1 and the second and penultimate vertices of $W_2(u, v)$ are non-adjacent tufts, a contradiction to the minimality of K. □

3.4. *Let u and v be two non-adjacent major vertices. If there is a tie for u, v at distinct $i, j \in \{1, 2, 3\}$, then $N_K(u) = N_K(v)$.*

Proof. If z_i is a tie for u, v at some $i \in \{1, 2, 3\}$, let $z'_i = z_i^+$ if $z_i \in V(P_{i+1})$, and $z'_i = z_i^-$ if $z_i \in V(P_{i-1})$. Suppose that for some i, z_i and z_{i+1} are ties for u, v at i and $i+1$, but that there is no tie for u, v at $i - 1$. So $H = u$-z-v-z'-u is a hole. Then z'_i and z'_{i+1} are tufts for H. Since there is no tie for u, v at $i - 1$, from the symmetry we may assume that $x_{i-1}(u)$ is closer to a_{i-1} than $x_{i-1}(v)$. Hence, $x_{i-1}(u)$ is a third tuft for H at u, forming a seven-vertex doily, a contradiction.

Thus there exist z_1, z_2, z_3 such that for $i = 1, 2, 3$, z_i is a tie for u, v at i. Suppose that $N_K(u) \neq N_K(v)$, and let

$$w \in (N_K(u) \cup N_K(v)) \setminus (N_K(u) \cap N_K(v)).$$

By 3.2, w is adjacent to at most one of a_1, a_2, a_3, and so we may assume that w is non-adjacent to a_1, a_2. So the subgraph H_S induced on $\{z_1, z_2, u, v\}$ is a hole, and z'_1, z'_2, w form a tufting for H_S, giving a seven-vertex doily, a contradiction. □

3.5. *Let u and v be two non-adjacent major vertices. If there is a tie for u, v at $i \in \{1, 2, 3\}$, then either $N_K(u) \setminus N_K(a_i) \subseteq N_K(v)$ or $N_K(v) \setminus N_K(a_i) \subseteq N_K(u)$. In particular u and v do not disagree at any $j \in \{1, 2, 3\}$.*

Proof. By 3.4, we may assume that there is a tie for u, v at 1, but not at 2 or 3. From the symmetry, we may assume that $y_1(u) = y_1(v)$. By 3.2, u and v are adjacent to $y_1(u)^+$ and $y_1(u)^{++}$. So, $H = u$-$y_1(u)$-v-$y_1(u)^{++}$-u is a hole and $y_1(u)^-$ is a tuft for H at $y_1(u)$.

Since there is no tie for u, v at 2, we may assume that $y_2(u)$ is strictly between a_2 and $y_2(v)$ on P_1. We claim that $N_K(v) \setminus N_K(a_1) \subseteq N_K(u)$. For suppose that there exists $w \in N_K(v) \setminus N_K(a_1)$ such that $w \notin N_K(u)$. By 3.2, w is non-adjacent to $y_2(u)$, and so $w, y_1(u)^-, y_2(u)$ are pairwise non-adjacent (in particular, $w, y_1(u)^-$ are non-adjacent since $w \notin N_K(a_1)$). But $w, y_1(u)^-, y_2(u)$ are not a tufting for H, since they would give a seven-vertex doily; and so w is adjacent to $y_1(u)^{++}$. In particular, $w \neq x_3(v)$, and so $x_3(v)$ is adjacent to u. Hence u-$y_1(u)$-v-$x_3(v)$-u is a hole and $w, y_1(u)^-, y_2(u)$ form a tufting, giving a seven-vertex doily, a contradiction. □

3.6. *Let u and v be two non-adjacent major vertices that disagree at some $i \in \{1, 2, 3\}$. Then either $N_K(u) \setminus N_K(v)$ is a clique (and hence has one or two members) or $N_K(v) \setminus N_K(u)$ is a clique.*

Proof. Let $i = 1$ say. By 3.5, there is no tie for u and v at 1, 2 or 3, and by 3.3, u and v do not disagree at 2 or at 3. So, we may assume up to symmetry that v beats u at 2. Let u', v' be the neighbours of u, v respectively in $W_1(u, v)$. Now at most one vertex in $N_K(u) \setminus (N_K(v) \cup \{u'\})$ is adjacent to u'; so, since we may assume that $N_K(u) \setminus N_K(v)$ is not a clique, it follows that there exists $w \in N_K(u) \setminus (N_K(v) \cup \{u'\})$ nonadjacent to u'. Hence w is not in $W_1(u, v)$. By 3.2, v is adjacent to the neighbour of v' in K not in $W_1(u, v)$; so w is nonadjacent to v', and hence w has no neighbour in $W_1(u, v)$ except u.

Suppose that $w \notin V(P_1)$. If $y_2(u)$ is adjacent to $y_2(v)$ and hence v is adjacent to $y_2(u)^+$ by 3.2, let Q be the path u-$y_2(u)^+$-v, and otherwise let Q be the induced path from u to v with interior in $y_2(u)$-P_1-a_2. In either case, since v beats u at 2, it follows that $y_2(v)$ has no neighbour in $V(Q)$ except v. But then $Q \cup W_1(u, v)$ is a hole and $b_1, y_2(v), w$ are three tufts for it, contrary to the minimality of K.

This proves that $w \in V(P_1)$. But $w, y_2(v)$ are nonadjacent since v is adjacent to $y_2(v)^+$. Let Q' be the induced path between u, v with interior in $x_2(u)$-P_3-a_2. Then $Q' \cup W_1(u, v)$ is a hole and $b_1, y_2(v), w$ are three tufts for it, contrary to the minimality of K. □

We denote the length of a path P by $|P|$.

3.7. *Let v be a major vertex such that $|W_1(v)|$ is minimum and, subject to that, such that $|W_2(v)| + |W_3(v)|$ is minimum. If u is a major vertex non-adjacent to v that has neighbours in the interior of $W_1(v)$, then $N_K(u) \setminus N_K(v)$ is a clique, and v beats u at 2 and 3.*

Proof. From the minimality of $|W_1(v)|$ and the fact that u has neighbours in the interior of $W_1(v)$, we know that u and v disagree at 1. Hence, by 3.5, there is no tie between u, v at 1, 2 or 3. So, by 3.6, we may assume that $N_K(v) \setminus N_K(u)$ is a clique. It follows that u beats v at 2 and 3, so $y_2(u)$ is not adjacent to v. We may assume that $x_1(u)$, $x_1(v)$, $y_1(u)$, $y_1(v)$ appear in this order along a_3-P_2-a_1-P_3-a_2.

If $y_1(u)^+ = y_1(v)$ then u-$y_1(u)$-$y_1(v)$-u is a triangle for which $y_1(u)^-, v, y_2(u)$ form a tufting, giving a six-vertex doily, a contradiction. So, $y_1(u)^+ \neq y_1(v)$. From the minimality of $|W_1(v)|$, it follows that $x_1(u)$-P_2-$x_1(v)$ has length at least two. If it has length exactly two, then $|W_1(u)| = |W_1(v)|$ and, since u beats v at 2 and 3, there is a contradiction to the optimality of v. So, the length of $x_1(u)$-P_2-$x_1(v)$ is at least three. Now let Q be an induced path from u to v whose interior is in $y_2(v)$-P_1-a_2. Then $Q \cup W_1(u,v)$ is a hole for which $x_3(u)$, $x_1(v)^{--}$ and b_1 form a tufting, a contradiction to the minimality of K. □

Proof of 3.1. Let (G, F) be a trackable pair. By 3.2, we only need to prove that no minimum doily of G has frame F and has a major vertex. To do so, we prove that if some minimum doily has frame F and has a major vertex, then there is a star cutset that separates $\{a_1\}$ from $\{a_2, a_3\}$, which is a contradiction to the trackability of (G, F).

We assume therefore that there is a minimum doily K of G that has frame F and has a major vertex v; and let us choose K, v as follows.

- **(i)** Among all such choices of K, v, let us choose K, v such that $|W_1(v)|$ is minimum.
- **(ii)** Among all such choices of K, v satisfying condition (i), let us choose K, v such that $|W_2(v)| + |W_3(v)|$ is minimum.
- **(iii)** Among all choices of K, v satisfying (i) and (ii) above, since v is not the centre of a star cutset that separates $\{a_1\}$ from $\{a_2, a_3\}$, there is a path $P = p_1\text{-}\cdots\text{-}p_k$ disjoint from K such that $k \geq 1$, p_1 has neighbours in the interior of $W_1(v)$, p_k has neighbours in $V(K) \setminus (W_1(v) \cup N_K(v))$, and no vertex of P is a neighbour of v. Let us choose K, v such that such a path $P = p_1\text{-}\cdots\text{-}p_k$ exists with k minimum.

The first two conditions will later be referred to as the *optimality* of v, and the third the *minimality* of P. We now look for a contradiction. (This will prove 3.1.)

(1) *P is induced, and no vertex of $P \setminus p_1$ has a neighbour in the interior of $W_1(v)$, and no vertex of $P \setminus p_k$ has a neighbour in $V(K) \setminus (W_1(v) \cup N_K(v))$.*

This is immediate from the minimality of P.

(2) *$k \geq 2$; and if p_1 is major then $N_K(p_1) \setminus N_K(v)$ is a clique and v beats p_1 at 2 and 3.*

The second assertion follows from 3.7, so it remains to prove that $k \geq 2$. Suppose that $k = 1$. Then p_1 has a neighbour in $V(K) \setminus (W_1(v) \cup N(v))$; let Q be a minimal subpath of $H_K \setminus a_1$ containing a neighbour of p_1 in the interior of $W_1(v)$ and a neighbour of p_1 in $K \setminus (W_1(v) \cup N(v))$. Then Q has at least three internal vertices by 3.2. Consequently p_1 is major, contrary to the second assertion. This proves (2).

(3) *p_2 is adjacent to both or neither of $x_1(v), y_1(v)$.*

For suppose it is adjacent to exactly one, say $y_1(v)$. Let $t \in \{y_2(v), x_3(v)\}$. Now, $W_1(v)$ and v form a hole for which b_1, p_2 and t are tufts; so p_2 is adjacent to t, and so p_2 is adjacent to both $y_2(v), x_3(v)$. In particular, p_2 is major. Since $x_1(v)$ is non-adjacent to p_2, and there is a tie for v, p_2 at 1, 3.4 implies that there is no tie for v, p_2 at 2, and so v is non-adjacent to $y_2(p_2)$.

Then v-$y_1(v)$-p_2-$x_3(v)$-v is a hole for which $y_1(v)^-$, $x_1(v)$ and $y_2(p_2)$ form a tufting (note that $y_1(v)^-, x_1(v)$ are non-adjacent because p_1 has a neighbour in the interior of $W_1(v)$ different from a_1) contrary to the minimality of K. This proves (3).

From the optimality of v, not both the paths a_1-P_3-$y_1(v)$ and a_1-P_2-$x_1(v)$ contain neighbours of p_1. Thus we may assume from the symmetry that p_1 has no neighbours in a_1-P_2-$x_1(v)$. It follows that p_1 has a neighbour in the interior of a_1-P_3-$y_1(v)$, and so this path has length at least two.

(4) *If p_2 is adjacent to both $x_1(v), y_1(v)$, then p_2 is non-adjacent to both of $y_2(v), x_3(v)$.*

For suppose that p_2 is adjacent to one of $y_2(v), x_3(v)$, say t. Let $t' = y_2(v)^-$ if $t = y_2(v)$, and $t' = x_3(v)^+$ if $t = x_3(v)$. Then v-$x_1(v)$-p_2-t-v is a hole, with a tufting $x_1(v)^+, t', p_1$, a contradiction. This proves (4).

(5) *If p_2 is adjacent to both $x_1(v), y_1(v)$, then $W_1(v)$ has length three.*

For the hole v-$y_1(v)$-p_2-$x_1(v)$-v has three tufts $x_1(v)^+$, $y_1(v)^-$, and $y_2(v)$. Since these do not form a tufting, we deduce that $x_1(v)^+$, $y_1(v)^-$ are adjacent, and so $W_1(v)$ has length three. This proves (5).

Thus, in the case that p_2 is adjacent to both $x_1(v), y_1(v)$, since p_1 has a neighbour in the interior of a_1-P_3-$y_1(v)$, it follows that $x_1(v) = a_1^-$, and a_1^+ is adjacent to $y_1(v)$. Note also that in this case p_2 is major.

(6) *If p_2 is adjacent to both $x_1(v), y_1(v)$, then p_1 is adjacent to a_1^+ and to $y_1(v)$, and p_1 is major.*

For from the definition of p_1 it follows that p_1 is adjacent to a_1^+. Since $a_1, p_1, y_2(v)$ is not a tufting for the hole v-$x_1(v)$-p_2-$y_1(v)$-v, it follows that p_1 is adjacent to $y_1(v)$. Let $u = x_1(v)^{--}$. Since $a_1, x_3(v), p_1$ is not a tufting for the hole v-$x_1(v)$-p_2-u-v, it follows that p_1 is adjacent to u and hence p_1 is major. This proves (6).

(7) *If p_2 is adjacent to both $x_1(v), y_1(v)$, then p_2 is non-adjacent to $y_2(p_1), x_3(p_1)$.*

For suppose that p_2 is adjacent to one of $y_2(p_1), x_3(p_1)$, say t. Let $t' = y_2(p_1)^-$ if $t = y_2(p_1)$, and $t' = x_3(p_1)^+$ if $t = x_3(p_1)$. Then the subgraph induced on $\{p_1, p_2, t, a_1^+, x_1(v), t'\}$ is a six-vertex doily, a contradiction. This proves (7).

(8) *If p_2 is adjacent to both $x_1(v), y_1(v)$, then $k \geq 3$, and p_3 is adjacent to $x_1(v)$ and to $y_1(v)^+$.*

For since there is a tie for v, p_2 at 1, and $y_2(v) \in N_K(v) \setminus N_K(p_2)$, 3.5 implies that $N_K(p_2) \subseteq N_K(v)$. Consequently $k \geq 3$. Now a_1-$x_1(v)$-p_2-p_1-a_1^+-a_1 is a hole, and $x_1(v)$ is the only neighbour of v in this hole, and a_1 is the only neighbour of b_1 in this hole. Since v, b_1, p_3 are pairwise non-adjacent, and every doily has at least nine vertices, it follows that p_3 is adjacent to $x_1(v)$. Since

$\{p_1, p_2, y_1(v)^+, a_1^+, v, p_3\}$ does not induce a six-vertex doily, it follows that p_3 is adjacent to $y_1(v)^+$. This proves (8).

(9) *p_2 is non-adjacent to $x_1(v), y_1(v)$.*

For otherwise by (3), p_2 is adjacent to both $x_1(v), y_1(v)$. Since $\{p_2, p_3, x_1(v), p_1, x_3(v), a_1\}$ does not induce a six-vertex doily, and p_1, p_2 are non-adjacent to $x_3(v)$ by (2) and (4), it follows that p_3 is non-adjacent to $x_3(v)$. But then $p_1, a_1, x_3(v)$ form a tufting for the hole v-$y_1(v)^+$-p_3-$x_1(v)$-v, a contradiction. This proves (9).

(10) *p_1 is major with respect to K.*

For suppose that p_1 is minor. If p_1 has a unique neighbour r in $W_1(v)$, then r is in the interior of $W_1(v)$, so $W_1(v)$ and v form a hole for which b_1, p_1 and $x_3(v)$ are non-adjacent tufts, a contradiction to the minimality of K.

Suppose that p_1 has exactly two neighbours in $W_1(v)$, say q and r, and they are adjacent. We may assume that a_1, q, r and $y_1(v)$ appear in this order along P_3 (possibly $r = y_1(v)$). Let $r' = r^+$ if $r \neq y_1(v)$, and $r' = v$ if $r = y_1(v)$. By (9), q^-, r', p_2 form a tufting for the cycle q-r-p_1-q, a contradiction.

It follows that p_1 has two non-adjacent neighbours q, s in $W_1(G)$, and $N(v) \cap V(K) \subseteq \{q, r, s\}$ where r is the common neighbour of q, s in K. Then, we obtain another minimum doily K' of G, still with frame F, by replacing r by p_1 in K. The doily K' together with v and p_2-P-p_k contradicts the minimality of P. From 3.2, this proves (10).

(11) *p_1 has two non-adjacent neighbours in $W_1(v)$.*

For suppose not. Since p_1 is major, it follows from 3.2 that p_1 has exactly two neighbours in $W_1(v)$, namely $y_1(v)$ and $y_1(v)^- = y_1(p_1)$. But now by (9) the triangle p_1-$y_1(p_1)$-$y_1(v)$-p_1 has a tufting $y_1(p_1)^-$, p_2 and v. This proves (11).

(12) *p_2 is adjacent to all of $x_2(v)$, $y_2(v)$, $x_3(v)$, $y_3(v)$, and in particular p_2 is major.*

For by (9), p_2 is non-adjacent to both $x_1(v)$ and $y_1(v)$. Let s be the neighbour of p_1 closest to $y_1(v)$ along $W_1(v)$. So,

$$p_1\text{-}s\text{-}W_1(v)\text{-}y_1(v)\text{-}v\text{-}x_1(v)\text{-}W_1(v)\text{-}y_1(p_1)\text{-}p_1$$

is a hole for which b_1 and p_2 are non-adjacent tufts. Since $x_2(v)$, $y_2(v)$, $x_3(v)$, $y_3(v)$ are tufts at v, p_2 is adjacent to all of them, for otherwise one of them would be a third tuft at p_1. In particular, p_2 is major. This proves (12).

(13) *p_2 beats both v and p_1 at both 2 and 3.*

For suppose there is a tie z for p_2 and v at 2. Let $z' = z^+$ if $z \in V(P_3)$, and $z' = z^-$ if $z \in V(P_1)$. Then

$$p_1\text{-}p_2\text{-}z\text{-}v\text{-}x_1(v)\text{-}W_1(v)\text{-}y_1(p_1)\text{-}p_1$$

is a hole for which z' and b_1 are non-adjacent tufts. By 3.4, there is no tie for v and p_2 at 3, so $x_3(p_2)$ is non-adjacent to v, and therefore non-adjacent to p_1 by the minimality of P, it follows that $x_3(p_2)$ is a third tuft, a contradiction. Hence, there is no tie for v and p_2 at 2, and similarly none at 3. Thus p_2 beats v at both 2 and 3. Since v beats p_1 by (2), it follows that p_2 beats p_1 at 2 and 3. This proves (13).

Now, to finally obtain a contradiction:

- If $x_1(p_1)$ and $x_1(p_2)$ are distinct and appear in this order along a_3-P_2-a_1, then

$$p_1\text{-}p_2\text{-}x_1(p_2)\text{-}W_1(p_2)\text{-}y_1(p_1)\text{-}p_1$$

  is a hole for which b_1, $y_1(p_1)^{++}$ and $x_3(p_2)$ are non-adjacent tufts, a contradiction.

- If $x_1(p_1) = x_1(p_2)$ then p_1-p_2-$x_1(p_1)$-p_1 is a triangle for which $x_1(p_1)^+$, $y_1(p_1)$ and $x_3(p_2)$ is a tufting, a contradiction.

- Finally, suppose $x_1(p_2)$ and $x_1(p_1)$ are distinct and appear in this order along a_3-P_2-a_1. From (11) and the optimality of v, it follows that $x_1(p_1), x_1(v)$ are non-adjacent. Let u be the neighbour of v in $W_1(p_1)$ closest to $x_1(p_1)$. By 3.2, $u, x_1(v)$ are non-adjacent. Then by (2), p_1 is non-adjacent to $y_2(v)$; by (12), p_2 is adjacent to $y_2(v)$; and by (13), p_1 is non-adjacent to $x_3(p_2)$. Consequently $x_3(p_2), y_1(p_1), x_1(v)$ is a tufting for the hole

$$p_1\text{-}x_1(p_1)\text{-}W_1(p_1)\text{-}u\text{-}v\text{-}y_2(v)\text{-}p_2\text{-}p_1,$$

  a contradiction.

This proves 3.1.

4 Shortest path detector

Our goal in this section is to prove 2.2. We need the following lemma.

4.1. *Let K be a minimum doily in a graph G, and let F be a frame for K. Suppose that G contains no major vertex with respect to K. With our usual notation, let $1 \le i \le 3$, and let $s, t \in V(H_K) \setminus a_i$. Let Q be a path in $G \setminus a_i$ between s, t, and let P be the (unique) path of $K \setminus a_i$ between s, t. Then $|Q| \ge |P|$, and if equality holds then no vertex of the interior of Q belongs to or has a neighbour in $V(H_K) \setminus V(P)$.*

Proof. We proceed by induction on $|Q|$ (for all minimum doilies with frame F), and for $|Q|$ fixed, by induction on $|V(H_K) \setminus V(P)|$. We may assume that $i = 1$ from the symmetry, and that Q is an induced path from the first inductive hypothesis.

(1) *We may assume that no internal vertex of Q belongs to K.*

For suppose that some internal vertex r of Q belongs to K, and hence to $V(H_K \setminus a_1)$. Let

P_1 be the path in $K \setminus a_1$ between s, r, and let Q_1 be the subpath of Q between s, r. Define P_2, Q_2 between r, t similarly. From the first inductive hypothesis, $|Q_j| \geq |P_j|$ for $j = 1, 2$; and since

$$|Q| = |Q_1| + |Q_2| \geq |P_1| + |P_2| \geq |P|,$$

it follows that $|Q| \geq |P|$, and we may assume that equality holds. Hence $|P_j| = |Q_j|$ for $j = 1, 2$, and $|P_1| + |P_2| = |P|$. The latter implies that $r \in V(P)$. From the first inductive hypothesis, for $j = 1, 2$, no internal vertex of Q_j belongs to or has a neighbour in $V(H_K) \setminus V(P_j)$, and in particular, no internal vertex of Q_j belongs to or has a neighbour in $V(H_K) \setminus V(P)$. Moreover, r does not belong to $V(H_K) \setminus V(P)$ (since $r \in V(P)$), and r has no neighbour in $V(H_K) \setminus V(P)$, because it has precisely two neighbours in $V(H_K)$ and they both belongs to $V(P)$ (one is in $V(P_1)$ and the other in $V(P_2)$). Thus in this case the result holds. This proves (1).

If Q has length at most one the result is clear. If it has length two, then since its internal vertex is minor by hypothesis, again the result holds. We may therefore assume that Q has length at least three. Let u, v be the neighbours of s, t in Q, respectively. Then $u, v \notin V(K)$. We may assume that a_1, s, t are in clockwise order in H_K. Now $H_K \setminus a_1$ is a path R say, between a_1^- and a_1^+. For all $p, q \in V(R)$, $R[p, q]$ denotes the subpath of R between p, q. For each $w \in V(G) \setminus V(K)$ with a neighbour in $V(R)$, let $x(w)$ be the neighbour of w in $V(R)$ that is closest (in R) to a_1^-, and let $y(w)$ be the neighbour closest to a_1^+. Since w is not major, it follows that none of a_1, a_2, a_3 belong to the path $R[x(w), y(w)]$.

(2) *We may assume that no vertex of the interior of Q has a neighbour in $V(H_K) \setminus V(P)$; and in particular $s = y(u)$ and $t = x(v)$.*

For suppose that some internal vertex q of Q is adjacent to some $r \in V(H_K) \setminus V(P)$. We may assume that $r \in V(R[a_1^+, s]) \setminus \{s\}$ from the symmetry. Let Q' be the path t-Q-q-r; then $|Q'| \leq |Q|$. But from the second inductive hypothesis, $|Q'| \geq |R[r, t]| > |P|$, and so $|Q| > |P|$ as required. This proves (2).

If $|Q| > |P|$ there is nothing to prove, and if $|P| = |Q|$ then the result holds by (2). Thus we may assume that $|Q| < |P|$, and we need to prove that this is impossible. In particular, P has at least five vertices, and so K has at least nine; and from the minimality of K, it follows that there is no doily in G with at most eight vertices. Let T be the path of H_K between s, t that passes through a_1, and let H be the hole $Q \cup T$. Thus $|V(H)| < |V(H_K)|$.

(3) *The subpaths $R[x(u), y(u)]$ and $R[x(v), y(v)]$ are disjoint, and one of a_2, a_3 (say a_h) belongs to the interior of $R[x(u), y(v)]$.*

For if a_2, a_3 both belong to $V(H)$ then b_1, b_2, b_3 are three tufts for H, and since $|V(H)| < |V(H_K)|$, this contradicts the minimality of K. Thus we may assume that one of a_2, a_3, say a_h, belongs to the interior of P. Since a_h does not belong to $R[x(u), y(u)]$ or to $R[x(v), y(v)]$ (because u, v are minor), it follows that these subpaths are disjoint, and a_h belongs to the interior of $R[x(u), y(v)]$. This proves (3).

Let u', v' be the neighbours of u, v in the interior of Q, respectively. We recall that, since a_h has degree three, it has no neighbours in the interior of Q.

(4) *$x(u), y(u)$ are either equal or adjacent, and so are $x(v), y(v)$.*

For suppose that $x(u), y(u)$ are distinct and non-adjacent. Then they have a common neighbour r in H_K. Replacing r by u in K gives another minimum doily K' of G, also with frame F; and $Q \setminus s$ is a path between u, t. Now every major vertex for K' is also major for K, and so there are no major vertices with respect to K'. From the first inductive hypothesis, it follows that the length of $Q \setminus s$ is at least one more than the length of $R[x(u), t]$. But then it follows that $|Q| \geq |P|$, a contradiction. This proves (4).

(5) *If $x(u) \neq s$ then s^{++} is adjacent to u' and to no other vertex of H, and u, v are non-adjacent. Similarly if $y(v) \neq t$ then t^{--} is adjacent to v' and to no other vertex of H, and u, v are non-adjacent.*

For suppose that $x(u) \neq s$. By (4), $x(u) = s^+$. From the first inductive hypothesis, s^+ has no neighbours in Q except s, u; and s^{++} has no neighbours in Q except possibly u'. Now u' is non-adjacent to s^- by (2), and non-adjacent to s^+ as we have seen, and non-adjacent to s since Q is induced. Since the subgraph induced on $\{u, s, s^+, u', s^-, s^{++}\}$ is not a six-vertex doily, it follows that u' is adjacent to s^{++}. Since a_h does not belong to the path $R[y(v), x(v)]$, it follows that v is non-adjacent to s^{++}, and so $v \neq u'$, and therefore u, v are non-adjacent. This proves the first statement of (5), and the second follows from the symmetry.

Now if $x(u) = s$ and $y(v) = t$ then s^+, t^-, b_1 form a tufting for H, a contradiction. Thus from the symmetry, we may assume that $x(u) \neq s$. By (4) and (5), s^{++} is adjacent to u' and to no other vertex of H, and u, v are non-adjacent. In particular $a_h \neq s^{++}$ (since a_h has no neighbours in the interior of Q), and so a_h belongs to the interior of $R[s^{++}, y(v)]$. If $y(v) = t$ then s^{++}, t^-, b_1 form a tufting for H, a contradiction; so $y(v) \neq t$. By (4) and (5), $y(v) = t^-$ and t^{--} is adjacent to v' and to no other vertex of H. Consequently a_h belongs to the interior of $R[s^{++}, t^{--}]$, and in particular s^{++}, t^{--} are non-adjacent. Since a_h does not belong to the path $R[x(u'), y(u')]$, it follows that $u' \neq v'$. We deduce that s^{++}, t^{--}, b_1 form a tufting for H, a contradiction. This proves 4.1. □

4.2. *Let G be a graph, and let F be a frame for a minimum doily K of G, with the usual notation. Suppose that G contains no major vertex with respect to K. Let $i \in \{1, 2, 3\}$, and let Q_i be a shortest path from a_{i-1} to a_{i+1} in $G \setminus a_i$. The graph obtained from K by replacing P_i by Q_i is a minimum doily of G, and has frame F, and no vertex is major with respect to it.*

Proof. From the choice of Q_i it follows that $|Q_i| \leq |P_i|$; and so from 4.1, equality holds, and no vertex of the interior of Q_i belongs to or has a neighbour in $V(H_K) \setminus V(P_i)$. Consequently the graph obtained from K by replacing P_i by Q_i is a minimum doily of G, say K', and has frame F. Since no vertex has neighbours in the interiors of both P_{i-1}, P_{i+1} except a_i, it follows that no vertex is major with respect to K'. □

Proof of 2.2. Suppose that we are given a pair (G, F) where

$$F = (b_1, b_2, b_3, a_1, a_2, a_3, a_1', a_2', a_3', a_1'', a_2'', a_3'')$$

is a frame in G. Here is an algorithm:

- For $i = 1, 2, 3$, compute a shortest path Q_i from a'_{i-1} to a''_{i+1} (if such a path does not exist, let Q_i be the null graph).
- Output the subgraph of G induced on a_1, a_2, a_3, b_1, b_2, b_3 and the vertices of Q_1, Q_2 and Q_3.

This algorithm obviously outputs an induced subgraph K of G. It remains to prove that if (G, F) is a trackable pair and G contains a doily, then K is a doily. Assume therefore that (G, F) is a trackable pair and G contains a doily. Consequently, there is a minimum doily K' of G such that F is a frame for K'. By 3.1, G contains no major vertex with respect to K'. By applying Lemma 4.2 three times, we see that K is a doily.

References

[1] M. Chudnovsky, G. Cornuéjols, X. Liu, P. Seymour, and K. Vušković, "Recognizing Berge graphs", *Combinatorica*, 25 (2005), 143–186.

[2] M. Chudnovsky and P.D. Seymour, "The three-in-a-tree problem", *Combinatorica*, 30 (2010), 387–417.

[3] B. Lévêque, D. Lin, F. Maffray, and N. Trotignon, "Detecting induced subgraphs", *Discrete Applied Mathematics*, 157 (2009), 3540–3551.

[4] N. Robertson and P.D. Seymour, " "Graph Minors. XIII. The disjoint paths problem", *J. Combinatorial Theory, Ser. B*, 63 (1995), 65–110.