The Structure of Metrizable Graphs

Maria Chudnovsky* Daniel Cizma† Nati Linial‡

November 19, 2023

Abstract

A *consistent path system* in a graph G is an intersection-closed collection of paths, with exactly one path between any two vertices in G. We call G *metrizable* if every consistent path system in it is the system of geodesic paths defined by assigning some positive lengths to its edges. We show that metrizable graphs are, in essence, subdivisions of a small family of basic graphs with additional compliant edges. In particular, we show that every metrizable graph with 11 vertices or more is outerplanar plus one vertex.

1 Introduction

Let $G = (V, E)$ be a connected graph, and let $w : E \to \mathbb{R}_{>0}$ be a positive weight function on its edges. This induces a metric on V, where the distance between any two vertices is the least w-length of a path between them. What can be said about such a system of geodesics? E.g., what does the collection of w-geodesics tell us about w? Is it possibly true that *every* collection of paths in a graph constitute the system of geodesics corresponding to some graph metric? To simplify matters, suppose that w is such that the shortest path between any two vertices is unique. Clearly, any subpath of a geodesic in G is itself a geodesic. With this in mind, we define the notion of a *consistent path system* $\mathcal{P}$ in G. This is a collection of paths that is closed under taking subpaths, with a unique uv path in $\mathcal{P}$ for each pair $u, v \in V$. So, we refine the question and ask if every consistent path system coincides with the set of geodesics that corresponds to some positive weight function on the edges. We already know [1] that this is not necessarily so, and here we seek to understand *metrizable* graphs. Namely,

---

*Princeton University, Princeton, NJ 08544, USA. e-mail: mchudnov@math.princeton.edu. Supported by NSF-EPSRC Grant DMS-2120644 and by AFOSR grant FA9550-22-1-0083

†Einstein Institute of Mathematics, Hebrew University, Jerusalem 91904, Israel. e-mail: daniel.cizma@mail.huji.ac.il.

‡School of Computer Science and Engineering, Hebrew University, Jerusalem 91904, Israel. e-mail: nati@cs.huji.ac.il . Supported by NSF-BSF US-Israel Grant 2021690 "The Global Geometry of Graphs"

graphs in which *every* consistent path system is induced by some graph metric. For questions of a similar flavor that come from physics see, e.g., [2].

The study of metrizable graphs was initiated in [1] where it was shown that metrizable graphs are in fact quite rare. For example, all large metrizable graphs are planar and not 3-connected. On the other hand, that paper exhibits an infinite family of metrizable graphs, viz., all outerplanar graphs. In this paper we hone in on the structure of metrizable graphs. In particular, we show that every large 2-connected metrizable graph can be obtained by taking some basic graph, subdividing its edges and iteratively adding edges between vertices connected by flat paths. (Recall a path in G is said to be *flat* if every internal vertex in it has degree 2 in G.) For instance, every 2-connected outerplanar graph can be constructed using this procedure starting with a cycle. Explicitly, if G is outerplanar, then there exist graphs $G_0, G_1, \ldots, G_t = G$ where G_0 is an cycle and $G_{i+1} = G_i \cup e_i$ is obtained from G_i by adding an edge e_i between two vertices which are connected by a flat path in G_i. We show here that every metrizable graph on 11 vertices or more can be constructed by this procedure starting with $G_0 = K_{2,n}$ or is else a subdivision of $K_{2,3}, C_3, K_4, W_4, W_4'$, see fig. 1. This, in particular, implies that every large metrizable graph can be made outerplanar by removing at most one vertex.

2 Preliminaries and Overview

All graphs in this paper are finite and simple. We recall some definitions and concepts from [1]. A *consistent path system* in a graph $G = (V, E)$ is a collection of paths $\mathcal{P}$ in G satisfying two properties:

1) For every $u, v \in V$ there is exactly one uv-path in $\mathcal{P}$

2) Every two paths in $\mathcal{P}$ are either: (i) vertex disjoint, or (ii) have exactly one vertex in common, or (iii) their intersection is a path in $\mathcal{P}$.

A path system $\mathcal{P}$ of $G = (V, E)$ is *metric* if there is a positive weight function $w : E \to \mathbb{R}_{>0}$ such that each path in $\mathcal{P}$ is a w-shortest path. We call G *metrizable* if every consistent path system in G is metric. It is known

Proposition 2.1 ([1]). *The family of metrizable graphs is closed under topological minors.*

(Recall that H is a topological minor of G if G contains a subdivision of H as a subgraph.) Therefore, every graph that contains a subdivision of a non-metrizable graph is itself non-metrizable.

The length of a path P, denoted $|P|$, is the number of its edges. A vertex is called a *branch vertex* if it has degree at least 3. A *flat path*, aka a suspended path in G is a path of length at least 2 all of whose internal vertices are of degree 2 in G. We call an edge xy in G *compliant* if x and y are also connected by a flat path. The role of compliant edges in the present context is captured by the following result:

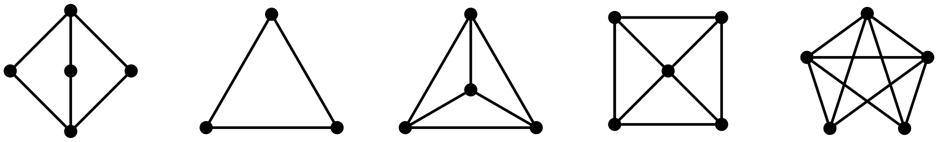


Figure 1: Up to adding compliant edges, metrizable graphs are either $K_{2,n}$ or subdivisions of one of the above graphs.

Proposition 2.2 ([1]). *If e is a compliant edge in G, then G is metrizable if and only if $G - e$ is metrizable.*

Our main result is that every metrizable graph can be constructed by subdividing some basic graph, fig. 1, and adding compliant edges along its subdivided edges. As usual, W_4 is the 4-wheel. We denote the 4-wheel plus one edge by W_4'.

Theorem 2.3. *If a 2-connected metrizable graph G with at least 11 vertices has no compliant edges, then it is either $K_{2,n}$ for some $n \geq 4$ or a subdivision of one of the following: $K_{2,3}$, K_4, W_4 or W_4'.*

The proof of this theorem builds on the very restricted structure of metrizable graphs.

Theorem 2.4. *If a graph G with at least 11 vertices is (i) 2-connected, (ii) has no compliant edges, (iii) has at least two disjoint cycles, then G is non-metrizable.*

The basic method for showing that a graph $G = (V, E)$ is non-metrizable, was developed in [1]: Namely, we consider a consistent path system $\mathcal{P}$ in G. Associated with $\mathcal{P}$ is a system of linear inequalities, and $\mathcal{P}$ is metric iff this system is feasible. So if the chosen $\mathcal{P}$ is non-metric, we can use LP-duality to create a *hand-checkable certificates* of this. Thus, using a computer, we created a "zoo" of 16 non-metrizable graphs along with such path systems and the corresponding certificates, fig. 7 and appendix A. We will refer throughout to graphs from the zoo as 7a-7p, as they appear in fig. 7. Our proof of theorem 2.4 shows that any graph satisfying these conditions contains a subdivision of some graph from the zoo. Our argument splits according to whether G contains two disjoint cycles or not. If it does, then we use these cycles and paths between them to find subdivisions of zoo graphs. On the other hand, as shown by Lovász (lemma 4.1), graphs with no two disjoint cycles have a very restricted structure. This allows us to derive a proof of theorem 2.3 from theorem 2.4. Finally, we show that every metrizable graph is nearly outerplanar.

Corollary 2.5. *Every 2-connected metrizable graph with at least 11 vertices can be made outerplanar by removing at most one vertex.*

This corollary follows from theorem 2.3 and the fact that outerplanarity is preserved under the addition of compliant edges (proposition 2.2). Note that the statement of corollary 2.5 need not hold for smaller graphs. For example, K_6 is metrizable but can be made outerplanar only by removing 3 vertices.

This paper is organized as follows. In section 3 we prove several lemmas needed for our main proofs. In section 4 we prove our main results: theorem 2.4, theorem 2.3 and corollary 2.5. Section 5 is devoted to open questions. Following section 5 is a figure with our zoo of non-metrizable graphs, and appendix A contains certificates of their non-metrizability.

3 Lemmas

We begin with some relevant preliminary facts:

Proposition 3.1. *Let $G = (V, E)$ be a graph which is not a cycle.*

1. *If G is 2-connected and outerplanar then it has a compliant edge. In particular, G has at least two edge disjoint flat paths whose endpoints span a compliant edge.*
2. *If $G - e$ is outerplanar for some compliant edge $e \in E(G)$, then G is outerplanar.*
3. *If G is 2-connected and $e \in E(G)$ is a compliant edge, then $G - e$ is 2-connected.*

Proof.

1. The edge set of every 2-connected outerplanar graph has the form $C \sqcup M$, with C a Hamiltonian cycle and M cyclically non-crossing edges. We argue by induction on $|M|$. If $|M| = 1$ then G is a cycle with one additional edge xy which splits C into two edge disjoint flat xy paths. For the induction step, pick an edge $xy \in M$. By the induction hypothesis $G - xy$ has two edge disjoint flat paths P_1 and and P_2 whose endpoints span a compliant edge. If neither x nor y is an internal vertex of P_1 or P_2 then the same two paths work for G as well.

   Otherwise, say x is an internal vertex of P_1, a flat path in $G - e$ between u and v, where, by assumption $uv \in M$. We conclude that also y belongs to P_1, for otherwise $C \cup uv \cup xy$ forms a subdivided K_4, contrary to G being outerplanar. Now consider the xy subpath P_1' of P_1. Clearly, P_1' is a flat path in G whose ends span the compliant edge xy. The paths P_1' and P_2 satisfy the claim.

2. Suppose toward a contradiction that G is not outerplanar, while $G - e$ is, where $e = xy$ is a compliant edge that connects between the two ends of the flat path P. Since G is not outerplanar, it must have a subgraph H that is either a subdivided K_4 or a subdivided $K_{2,3}$, and since $G - e$ is outerplanar, e must belong to $E(H)$. If x, y are the only common vertices of P and H, then by removing the edge e from H and adding instead the path P, we obtain a subdivision of H, which is impossible, since $G - e$ is outerplanar.

   But P is flat and has no branch vertices other that possibly x, y, so that if P and H have a non-trivial intersection, i.e. H contains an internal vertex of P, then necessarily $P \subseteq H$. The cycle $C = P \cup xy$ is contained in H. If H is a subdivided K_4, then each of its cycles has 3 branch vertices, while C has no branch vertices other than (possibly) x and y. Similarly, if H is a subdivision of $K_{2,3}$ then C must have two non-adjacent branch vertices, again a contradiction.

3. Again let P be a flat xy path, and $e = xy$ the corresponding compliant edge. Since G is not a cycle there must be another vertex $z \in G - C$, where $C = P \cup e$. Since G is 2-connected, there exist two paths Q_1, Q_2 from z to C which are disjoint except at z. Since P is flat, Q_1 and Q_2 must go from z to $\{x, y\}$, respectively. To see that $G - e$ is 2-connected we exhibit two internally disjoint xy-paths in $G - e$, namely P and the concatenation of Q_1 and Q_2.

□

Thus item 3 of proposition 3.1 implies that, given any 2-connected G, we can iteratively remove compliant edges, which process ends with a cycle or a 2-connected subgraph with no compliant edges.

To prove our main results we need some groundwork, starting with the following technical lemma regarding compliant edges in 2-connected graphs:

Lemma 3.2. *Let $G = (V, E)$ be a 2-connected graph which is not a cycle, and which has no compliant edges. Let $e = xy \in E$, let S be the vertex set of a connected component of $G - \{x, y\}$, and let H be the subgraph spanned by $S \cup \{x, y\}$. Then H is 2-connected, and not outerplanar.*

Proof. Since G is 2-connected and has no compliant edges, it is non-outerplanar by proposition 3.1. This yields our claim when $H = G$, so we can and will assume henceforth that $H \neq G$. That H is 2-connected follows from a standard argument. For a vertex $z \in H$ we need to show that $H - z$ is connected. For any two vertices $u, v \in H - z$ we find a (uv)-path in $H - z$. Since G is 2-connected, $G - z$ is connected and there exists a path $P \subseteq G - z$ between u and v. Since x and y separates H from the rest of G, either $P \subseteq H - z$ or $x, y \in P$ with, say, the (ux)-subpath and the (vy)-subpath of P both contained in H. In the latter case, we can concatenate these two subpaths with edge xy to obtain a (uv)-path in $H - z$.
We now show H is not outerplanar. If H is a cycle, then $P = H - xy$ is a flat path in H with xy the corresponding compliant edge. But P is flat in G as well, because other than x and y no vertex in P has neighbors outside of H. However, by assumption G has no compliant edges, so we can and will assume that H is not a cycle. If H is outerplanar, then by proposition 3.1, H has two edge disjoint flat paths P_1, P_2 whose endpoints form compliant edges. By our assumption these paths are not flat in G. Since x and y are the only vertices in H that have neighbors outside of H, necessarily x and y are internal vertices of P_1 and P_2. But then P_1 cannot be flat in H, because x has a neighbor outside of P_1, namely y. □

Next, we make a simple observation on the structure of non-outerplanar graphs.

Lemma 3.3. *If G is a 2-connected graph which is not outerplanar, then every edge of G is in some subgraph which is a subdivision of K_4 or of $K_{2,3}$.*

Proof. Let H be a subgraph of G which is a subdivision of K_4 or $K_{2,3}$. Such H exists, since G is not outerplanar. Let $xy \in E(G)$, and wlog $x \notin V(H)$. We proceed according to whether $y \in V(H)$ or not.

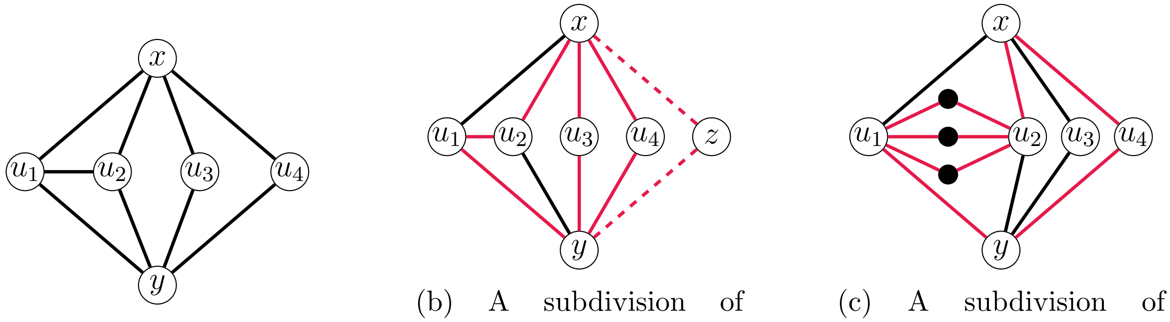
Consider first the case $y \in H$. Since G is 2-connected, there is a path P from x to a vertex $u \in H$, $u \neq y$. But then yxP is a yu path whose internal vertices are disjoint from H. If u and y are neighbors, we may replace the edge $yu \in E(H)$ with the yu-path yxP to obtain a graph isomorphic to a subdivision of H containing both vertices x and y. If $uy \notin E(H)$ then there exists two internally disjoint yu paths in H, of length at least 2. These two yu paths along with the path yxP constitute a subdivided $K_{2,3}$ containing x and y.

Next suppose that neither x nor y is in H. Since G is 2-connected, there exist two disjoint paths between $\{x, y\}$ and H. Namely, P_1 is a xu path, P_2 is a yv path. Two paths that are disjoint from H except at their endpoints u and v. The concatenated path $Q = P_1^{-1}xyP_2$, is a uv path whose internal vertices are disjoint from H. As before, if $uv \in E(H)$ then we may replace the edge uv by the path Q to obtain a subgraph of G that is isomorphic to a subdivision of H. Similarly, if $uv \notin E(H)$ then there are two internally disjoint uv paths in H, with length at least 2. These two paths together with Q form a subdivided $K_{2,3}$ that contains both x and y. □

Let $K'_{2,4}$ denote the graph obtained by adding to $K_{2,4}$ an edge between two vertices on the side of 4. We prove:

Lemma 3.4. *If a 2-connected graph G with at least 9 vertices has a subgraph H that is a subdivision of $K'_{2,4}$, then G is not metrizable.*

Proof. Since H contains a subdivision of $K_{2,4}$, there exist $x, y \in H$ with 4 internally disjoint xy paths P_1, P_2, P_3, P_4 between them of length at least 2. Observe that if the length of any P_i strictly exceeds 2, then we get a subdivision of Graph 7a. We can therefore assume that H looks as follows: The above paths are $P_i = xu_iy$, $i = 1, 2, 3, 4$ and in addition there is a u_1u_2 path Q, disjoint from $\{x, y, u_3, u_4\}$. Note that if $|Q| > 2$ then we obtain a subdivision of Graph 7k. Therefore, we may assume that Q is either an edge or has length 2. We argue the case where Q is an edge, fig. 2a, and note the same argument applies as well when Q has length 2. Since H has 6 vertices there is a vertex $z \in G - H$, and since G is 2-connected, there are two paths from z to H which are disjoint except at z. Concatenating these paths at z, we obtain a path of length at least 2 which intersects H only at its endpoints. If these endpoints are x and y, then we obtain a subdivision of Graph 7a, see fig. 2b. Similarly, if the endpoints of this path is x and u_i (or y and u_i) for some $i = 1, 2, 3, 4$ we again obtain a subdivision of Graph 7a. Therefore, the only possibility left is that the path connects two vertices u_i, u_j for $i \neq j$. (For the following arguments we may ignore the edge u_1u_2 so that the argument is symmetric w.r.t. vertices u_1, u_2, u_3, u_4.) If the length of this u_iu_j path is strictly larger than 2 we obtain a subdivision of Graph 7k. Therefore, there is a path u_izu_j in G. Since $|H| = 6$ and $|G| \geq 9$ there must be at least two other vertices z' and z'' not in H, and the same argument yields paths $u_{i'}z'u_{j'}$ and $u_{i''}z''u_{j''}$ in G. We claim that if, say, $|\{i, j\} \cap \{i', j'\}| = 1$ then G is not metrizable. Indeed, if say $i = 1$, $j = j' = 2$ and $i' = 3$ then Graph 7l is a subgraph of $H \cup u_1zu_2 \cup u_2z'u_3$. So we may assume that the sets $\{i, j\}$,$\{i', j'\}$, $\{i'', j''\} \subset \{1, 2, 3, 4\}$ are pairwise equal or disjoint.



(a) A $K'_{2,4}$ subgraph

(b) A subdivision of Graph 7a.

(c) A subdivision of Graph 7a.

Figure 2

There are two subcases to consider: i) all three sets coincide, ii) two sets are equal while the third is disjoint. For i) suppose the three sets are equal to $\{1, 2\}$. In this case G contains a subdivision of Graph 7a, see fig. 2c. For ii) suppose two of sets are equal to $\{1, 2\}$ while the third is $\{3, 4\}$. Then G contains a subdivision of Graph 7j, see fig. 3a. In either case, G is not metrizable. □

4 Proof of the Main Results

We now prove the main results starting with theorem 2.4.

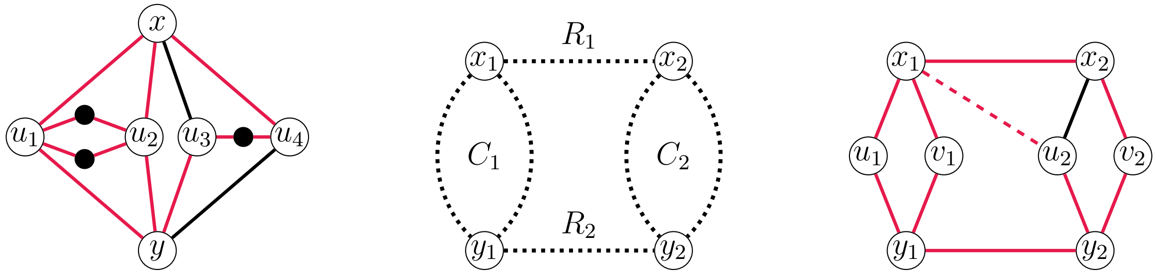
Proof. [Theorem 2.4] Let C_1 and C_2 be two disjoint cycles in G. Since G is 2-connected, there exists two disjoint paths, R_1 and R_2, (possibly edges) between them. Say R_1 connects between $x_1 \in C_1$ to $x_2 \in C_2$, and R_2 connects $y_1 \in C_1$ to $y_2 \in C_2$.
Set $H = C_1 \cup C_2 \cup R_1 \cup R_2$, fig. 3b. There are three cases to consider:

Case 1. Neither x_1y_1 nor x_2y_2 are edges of C_1 and C_2, respectively.

Case 2. x_1, y_1 are not adjacent in C_1 but x_2y_2 is an edge of C_2

Case 3. Both x_1y_1 and x_2y_2 are adjacent in C_1 and C_2, respectively

Case 1: The vertices x_i, y_i split C_i into two arcs ($i = 1, 2$). If any of these four arcs is strictly longer than 2, then we we obtain a subdivision of Graph 7n. So we can assume that C_1, C_2 are 4-cycles, with x_i, y_i antipodal in C_i. Also, if either $|R_1| > 1$ or $|R_2| > 1$ we get a subdivision of Graph 7m. So we can assume that R_1, R_2 are in fact edges. Let u_i, v_i be the two vertices in $C_i - \{x_i, y_i\}$. Since G is 2-connected and has at least 11 vertices and H contains 8, we can, as before, find an H-*ear* in G. Namely, a path Q of at least length 2 between two vertices in H. We can assume that these two vertices are non-adjacent since, as discussed above, subdividing any edge in H yields a non-metrizable graph. If Q runs between a vertex in C_1 and a non-neighbor of it in C_2, then up to symmetry three subcases may occur: Q connects i) x_1 and y_2, ii) x_1 and u_2, and iii) u_1 and u_2. Subcases i) and ii) yield a subdivision of Graph 7b, see fig. 4a.



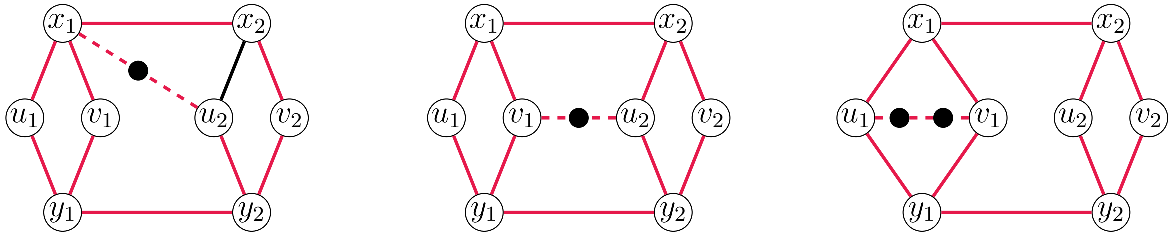
(a) A subdivision of Graph 7j.

(b) The subgraph H of disjoint cycles.

(c) A subdivision of Graph 7b.

Figure 3

Lastly, case iii) gives a subdivision of the prism, Graph 7d, which is not metrizable, see fig. 4b. So we may assume that Q connects two non-adjacent vertices in the same C_i. Notice a path between x_i and y_i gives a subdivision of Graph 7a. So the only option left is that Q connects u_i and v_i. In this case, if $|Q| > 2$ we then obtain a subdivision of Graph 7p, see fig. 4c, which leaves only the case $Q = u_i z v_i$. However, since $|H| = 8$ and $|G| \geq 11$, there exists at least three distinct vertices $z, z', z'' \in G - H$ such that $u_i z v_i$, $u_{i'} z' v_{i'}$ and $u_{i''} z'' v_{i''}$ are paths in G. Since $i, i', i'' \in \{1, 2\}$ at least two of these indices must coincide, say $i = i' = 1$, which yields a subdivision of Graph 7j, see fig. 5a, and G is not metrizable.



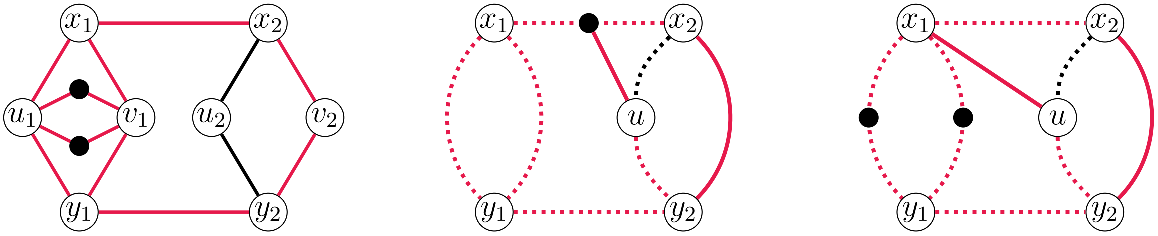
(a) A path between x_1 and u_2 yields a subdivision of Graph 7b.

(b) A path between x_1 and y_2 yields a subdivision of Graph 7d.

(c) A path between u_1 and v_1 gives a Graph 7p subdivision.

Figure 4

Case 2: Here we view C_2 as the union of an (x_2y_2)-path P, $|P| \geq 2$, and the edge x_2y_2, which, by assumption is non-compliant. Let us consider first the case where x_2, y_2 separate C_2 from $H - C_2$. By lemma 3.2, C_2 is contained in a 2-connected non-outerplanar subgraph which is disjoint from $H - C_2$. By lemma 3.3 the vertices x_2 and y_2 are contained a subdivision F, of either K_4 or $K_{2,3}$, disjoint from H. If x_2 and y_2 are not neighbors in F then they are contained in some cycle of F, and we are back to Case 1 which we have already settled. Similarly, if x_2, y_2 are adjacent in F, then it is easily seen that we can find a cycle $C' \subset F - x_2y_2$ and two disjoint paths from x_2, y_2 to C', so we can again reduce the situations to Case 1.



(a) A subdivision of Graph 7j.

(b) A path from Q to R_1 recovers the previous case.

(c) An path from P to x_1 gives a subdivision of Graph 7b.

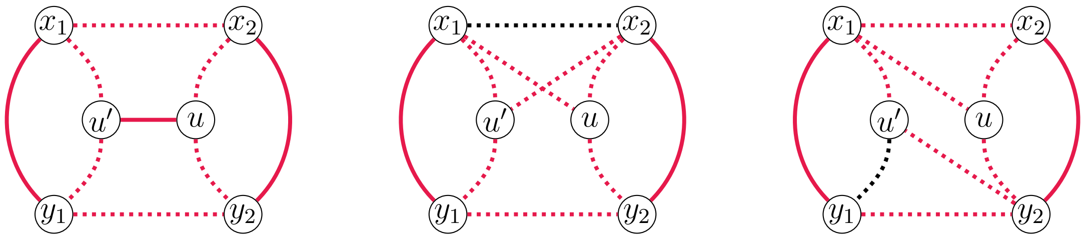
Figure 5

So we may assume that x_2, y_2 do not disconnect C_2 from the rest of H. In particular, there exists a path Q (possibly an edge) from an internal vertex, u, of P to a vertex v in $H - P$ which is otherwise disjoint from H. Up to symmetries there are three possibilities to consider: Q connects between u and i) an internal vertex of R_1. ii) x_1, iii) $C_1 - \{x_1, y_1\}$. In i), the situation reduces to Case 1, fig. 5b , which is already settled. In ii), G contains a subdivision of Graph 7b, see fig. 5c. In iii), G contains a subdivision of Graph 7e.

Case 3: Now each C_i is the union of the edge x_iy_i and an (x_iy_i)-path P_i of length at least 2. First assume that x_2, y_w separate C_2 from C_1. As argued above, by lemma 3.2 and lemma 3.3 the vertices x_2 and y_2 are contained in a subdivision F of $K_{2,3}$ or K_4 which is disjoint from C_1. If x_2 and y_2 are not neighbors in F then we can take C_1 and any cycle containing x_2 and y_2 and apply case 2. Similarly, if x_2y_2 is an edge in F then we can take C_1 and a cycle in $F - x_2y_2$ and again apply case 2.

So we may assume that x_2, y_2 do not separate C_2 from C_1. This means that there is a path Q (possibly an edge) connecting a vertex u in $C_2 - \{x_2, y_2\}$, i.e. an internal vertex of P_2, to a vertex in $H - C_2$. First assume that Q is a path from u to an internal vertex of R_1 (or R_2). (Recall that R_1 and R_2 are disjoint paths between C_1 and C_2, fig. 3b.) This creates the same situation as in Case 2, see e.g. fig. 5b.

So we may assume that Q is a path from u to C_1. Next suppose that Q is a path connecting u to an internal vertex of P_1. This gives us a subdivision of the prism, fig. 6a, and as shown in [1], a 2-connected graph on 7 vertices or more which contains a proper subdivision of the prism is not metrizable. We can therefore assume that the path Q connects u to the set $\{x_1, y_1\}$. The above discussion allows us to assume as well that x_1, y_1 do not separate C_1 from C_2, and that there is a path Q' from a vertex $u' \in C_1 - \{x_1, y_1\}$ to the set $\{x_2, y_2\}$, which is internally disjoint from H. There are now up to symmetries 2 subcases to consider: i) Q is a (ux_1)-path and Q' is a $(u'x_2)$-path, ii) Q is a (ux_1)-path and Q' is $(u'y_2)$ path. In subcase i), G contains a subdivision of the prism and has at least 7 verices, fig. 6b, and as noted above, it is consequently non-metrizable. In subcase ii), notice that this graph contains a $K'_{2,4}$ subdivision, fig. 6c, which is not metrizable by lemma 3.4.



(a) A path between u and u' gives a subdivision of the prism.

(b) Paths from u' to x_1 and u to x_2 yield a subdivision of the prism.

(c) A subdivision of $K'_{2,4}$, with x_1, y_2 on the 2-part.

Figure 6

□

Our proof of theorem 2.3 uses the following result of Lovász, [3]. Consider the graphs obtained by adding 1, 2 and 3 edges to the side of 3 vertices in $K_{3,n}$. Call them $K'_{3,n}$, $K''_{3,n}$ and $K'''_{3,n}$, respectively.

Lemma 4.1. *If a graph G contains no disjoint cycles and satisfies $\delta(G) \geq 3$, then it is either K_5, a wheel, $K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$ or $K'''_{3,n}$.*

Proof. [Theorem 2.3] Let G be a 2-connected graph with no compliant edges and at least 11 vertices. We may assume that G does not contain disjoint cycles, for otherwise it is not metrizable by theorem 2.4.
In order to use lemma 4.1, we need to eliminate all vertices of degree 2 in G, since the lemma assumes $\delta(G) \geq 3$. If we suppress all vertices of degree 2 in G, the resulting graph $\tilde{G}$ indeed satisfies $\delta(\tilde{G}) \geq 3$, though it need not be a *simple* graph. It is not difficult to see that $\tilde{G}$ is simple if and only if G does not contain any parallel flat paths. In this view let us consider what happens if G contains parallel flat paths. Namely, there are at least two flat (uv)-paths in G between a pair of vertices $u, v \in G$. We show that there cannot be an additional vertex pair $\{x, y\} \neq \{u, v\}$ that is connected by parallel flat paths. Now $\{x, y\}$, $\{u, v\}$ cannot be disjoint, or else these four flat paths form two disjoint cycles. So suppose that there are parallel flat paths between x, y and x, v. Since G is 2-connected, there must be another path from v to y disjoint from x. But this forms a subdivision of Graph 7b.
So say that x, y is the (unique) pair of vertices between which there are parallel flat paths. If x and y are the only branch vertices of G then G is a subdivision of $K_{2,n}$, $n \geq 3$. But as shown in [1], for $n \geq 4$ the graph $K_{2,n}$ is metrizable, whereas every proper subdivision of $K_{2,n}$ is not metrizable, since it contains a subdivision of Graph 7a. In the remaining case $n = 3$ and G is a subdivided $K_{2,3}$.
So we may assume that G has another branch vertex $z \neq x, y$. Since G is 2-connected there is an (xy)-path in G containing z. Let P be such a path of largest length. Note that every cycle Z in G must contain either x or y. Otherwise, Z along with the parallel xy paths, form two disjoint cycles. It follows that if and Q is a (uv)-path internally

disjoint from P with $u, v \in P$, then $\{u, v\} \cap \{x, y\} \neq \emptyset$. Otherwise, Q along with the (uv)-subpath of P forms a a cycle containing neither x nor y.
We first suppose that there exists a vertex $u \in P - \{x, y\}$ with a neighbor $v \notin P$. Since G is 2-connected there are two paths from v to P disjoint except at v. At least one of these paths does not contain u, and gluing this path with the edge uv we obtain a path Q from u to another vertex in P. As was observed above, this other endpoint must be either x or y. Say Q is a (ux)-path, and note that u and x are not neighbors in P for otherwise we can use Q to replace P by a longer (xy)-path, contradicting the maximality of P's length. Then the graph $H \cup P \cup Q$ contains a subdivision of Graph 7b, implying G is not metrizable.
In the remaining case every internal vertex of P has all its neighbors in P. Given an internal vertex of u in P, let us call w an additional neighbor of u if uw is an edge in G and but not an edge in P. Note by assumption, all additional neighbors are all also vertices in P. Moreover, by our above observations if w is an additional neighbor of u then $w \in \{x, y\}$. (Recall that otherwise we would obtain disjoint cycles in G.) Recall that P has at least one vertex z, of degree at least 3. So z has at least one additional neighbor, say x. Let u be the vertex in P such that x is an additional neighbor of u and the distance between u and x along P is minimized. Since ux is not a compliant edge, the (ux)-subpath of P is not flat. In particular, there must be a vertex, v, of degree 3 along this subpath. This vertex v has x or y as an additional neighbor. Also, since u was chosen to minimizes the distance to x, v must have y as an additional neighbor. Along with the flat (xy)-paths we obtained a subdivision of $K'_{2,4}$. It follows that G is not metrizable by lemma 3.4.
In the eventual case G has no parallel flat paths, so we may suppress all its degree-2 vertices to obtain a simple graph $\tilde{G}$ such that $\delta(\tilde{G}) \geq 3$. By lemma 4.1 $\tilde{G}$ is either K_5, W_n, $K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$ or $K'''_{3,n}$. As shown in [1], every 2-connected non-planar graph with at least 8 vertices is non-metrizable. Since G has at least 11 vertices this excludes K_5, $K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$ or $K'''_{3,n}$ for $n \geq 3$. To exclude W_n for $n \geq 5$, recall from [1] that a 2-connected graph containing W_5 with at least 7 vertices is non-metrizable. The possibilities for $\tilde{G}$ that remain are precisely K_4, W_4 and W'_4.

□

Next we prove corollary 2.5.

Proof. [Corollary 2.5] To prove the claim we use the following procedure: we first iteratively remove all compliant edges in G to obtain a graph G_t with no compliant edges and conclude that there exists a vertex x such that $G_t - x$ is outerplanar. The last step is to recover the graph $G - x$ by iteratively adding back the compliant edges we removed while maintining outerplanarity. In this last step, we use the fact that an outerplanar graph remains outerplanar after adding a compliant edge, proposition 3.1 (2).
We first observe that every compliant edge e in G, is also compliant in $G - x$, for every vertex $x \in G$ of degree at least 3 that is not incident with e. Start with $G_0 = G$

and define the graphs $G_0, G_1, \dots, G_t$, where G_{i+1} is obtained from G_i by deleting some compliant edge e_i of G_i. This sequence ends with G_t that has no compliant edges. Suppose that there exists a vertex $x \in G_t$ of degree at least 3 such that $G_t - x$ is outerplanar. By our initial observation, if e_i is not incident with x then e_i is a compliant edge in $G_i - x$. It follows that $G - x$ can be constructed from $G_t - x$ by iteratively adding compliant edge. By proposition 3.1, G is also outerplanar.
Why is there such a vertex x in G_t? By proposition 3.1, G_t is 2-connected and so by theorem 2.3 it is either $K_{2,n}$ or a subdivision of C_3, $K_{2,3}$, K_4, W_4 or W_4'. Unless G_t is a subdivision of W_4' it is clear such a vertex exists and so it suffices to consider this case. But W_4' is the wheel W_4 plus an additional edge e, so $G_t = H \cup P$ where H is the subdivision of W_4 and P is the subdivision of e. Notice that if $|P| \geq 2$ then G_t contains a subdivision of $K_{2,4}'$, and by lemma 3.4, G_t is not metrizable. It follows that P is just an edge and G_n can be made outerplanar by removing the single vertex of degree 4 in H. ☐

5 Open Questions

While the structure of metrizable graphs is now much better understood, many questions remain open. For instance, theorem 2.3 shows that, up to compliant edges, large metrizable graphs are either $K_{2,n}$ or subdivisions of $C_3, K_4, K_{2,3}, W_4$ or W_4', but the converse is far from true. Indeed, many of the non-metrizable graphs in fig. 7 are subdivisions of these graphs. Our ignorance is perhaps best illustrated by Θ-graphs. Recall that $\Theta_{a,b,c}$ consists of two vertices u and v and three internally disjoint (uv)-paths of length a, b and c. As mentioned already in [1], we still do not know in full which Θ-graphs are metrizable.

Open Problem 5.1. *Which subdivisions of $K_4, K_{2,3}, W_4$ and W_4' are metrizable and which are not?*

In [1] it was shown that graph metrizability can be decided in polynomial time. The proof of this relied heavily on the graph minor theory of Robertson and Seymour, [4], and therefore a practical metrizability algorithm still remains elusive. On the other hand, our results here show that metrizable graphs have a very simple structure. In this view we could hope to use these insights and obtain a practically efficient metrizability algorithm. Unfortunately, even such simple graphs have exponentially many consistent path systems and so an exhuastive checking of all consistent path systems is still inefficient.

Open Problem 5.2. *Find a practically efficient algorithm for graph metrizability.*

It is suggestive to go beyond decision problems into the realm of *promise problems*, and ask for example:

Open Problem 5.3. *Given a non-metrizable graph, can we efficiently find a consistent non-metrizable path system?*

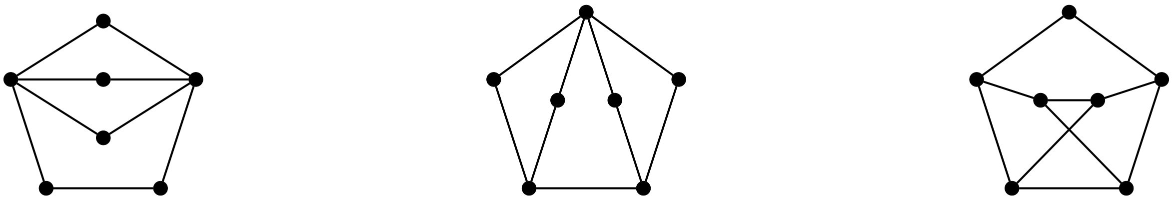
As mentioned in section 2, given non-metric path system we can find a "hand-checkable" certificate of non-metrizability using LP-duality. On the other hand, it is not clear that such a certificate exists for metrizable graphs.

Open Problem 5.4. *Do there exist "humanly verifiable" certificates of metrizability?*

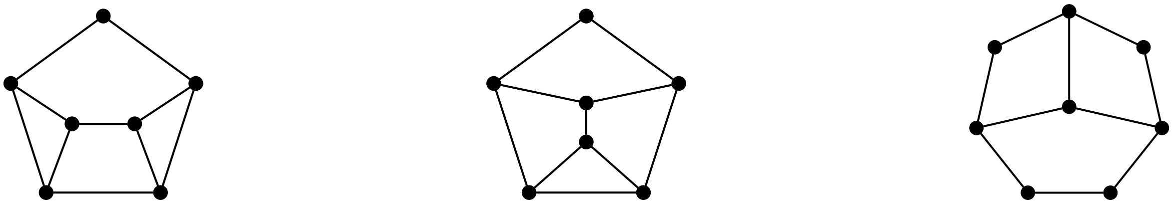
Our definitions do not posit that path systems use all the edges in the graph under consideration. In this view we find the concept of neighborly metrizability particularly appealing. A *neighborly consistent path system* is a consistent path system where every edge is the chosen path between its two vertices. We say a graph G is *neighborly metrizable* if every consistent neighborly path system in G is induced by a metric. (For example, K_n is always neighborly metrizable but never metrizable for $n \geq 7$.)

Open Problem 5.5. *Characterize the family of graph which are neighborly metrizable.*

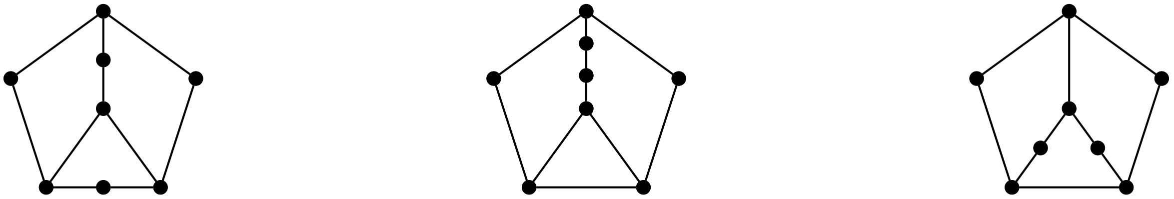
Unlike metrizable graphs, neighborly metrizable graphs are not closed under topological minors. Therefore, different tools are necessary to tackle this problem.



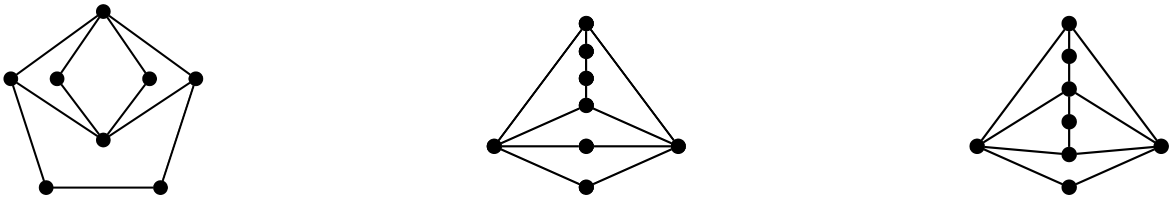
(a) Graph 1 (b) Graph 2 (c) Graph 3



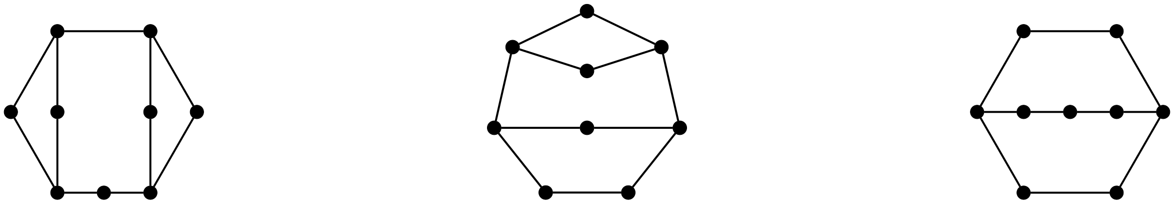
(d) Graph 4 (e) Graph 5 (f) Graph 6



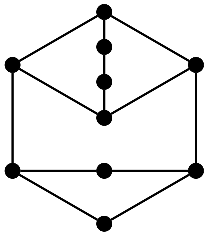
(g) Graph 7 (h) Graph 8 (i) Graph 9



(j) Graph 10 (k) Graph 11 (l) Graph 12



(m) Graph 13 (n) Graph 14 (o) Graph 15

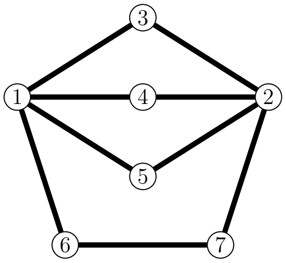


(p) Graph 16

Figure 7: Currently known topologically minimal non-metrizable graphs

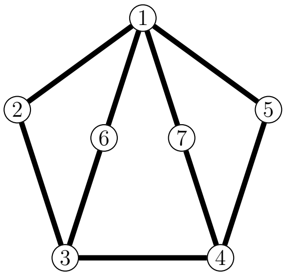
A Certificates of non-metrizability

For each graph G in Figure 7 we give a path system in G along with a system of inequalities a weight function inducing this path system must satisfy. In each case, these inequalities imply at least one edge in the graph must have a non-positive weight, showing the graph in not metrizable.



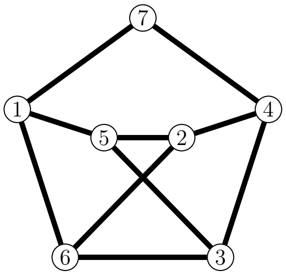
$(1,3,2),\ (1,6,7),\ (2,7,6),\ (3,1,4)\ (3,2,5),\ (3,1,6),$
$(3,1,6,7),\ (4,1,5),\ (4,2,7,6),\ (4,2,7),\ (5,1,6),\ (5,2,7)$

$$\begin{aligned} w_{2,3} + w_{2,5} &\leq w_{1,3} + w_{1,5} \\ w_{1,4} + w_{1,5} &\leq w_{2,4} + w_{2,5} \\ w_{2,4} + w_{2,7} + w_{6,7} &\leq w_{1,4} + w_{1,6} \\ w_{1,3} + w_{1,6} + w_{6,7} &\leq w_{2,3} + w_{2,7} \end{aligned} \quad \Longrightarrow \quad w_{6,7} \leq 0$$



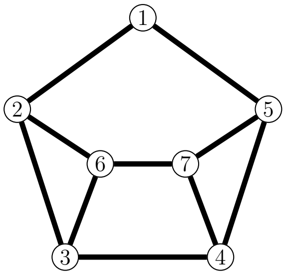
$(1,6,3),\ (1,7,4),\ (2,3,4),\ (2,3,4,5),$
$(2,1,6),\ (2,1,7),\ (3,4,5),\ (3,4,7),$
$(4,3,6),\ (5,4,3,6),\ (5,1,7),\ (6,3,4,7)$

$$\begin{aligned} w_{2,3} + w_{3,4} + w_{4,5} &\leq w_{1,2} + w_{1,5} \\ w_{1,2} + w_{1,6} &\leq w_{2,3} + w_{3,6} \\ w_{1,5} + w_{1,7} &\leq w_{4,5} + w_{4,7} \\ w_{3,6} + w_{3,4} + w_{4,7} &\leq w_{1,6} + w_{1,7} \end{aligned} \quad \Longrightarrow \quad w_{3,4} \leq 0$$



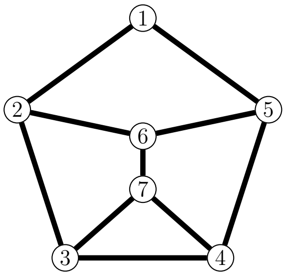
$(1,7,4,2),\ (1,6,3),\ (1,7,4),$
$(2,6,3), (2,4,7),\ (3,4,7),\ (4,3,5),$
$(4,3,6),\ (5,1,6),\ (5,3,4,7),\ (6,1,7)$

$$\begin{aligned} w_{2,6}+w_{3,6} &\le w_{2,4}+w_{3,4}\\ w_{1,5}+w_{1,6} &\le w_{3,5}+w_{3,6}\\ w_{1,7}+w_{4,7}+w_{2,4} &\le w_{1,6}+w_{2,6}\\ w_{3,5}+w_{3,4}+w_{4,7} &\le w_{1,5}+w_{1,7} \end{aligned} \qquad \Longrightarrow \qquad w_{4,7}\le 0$$



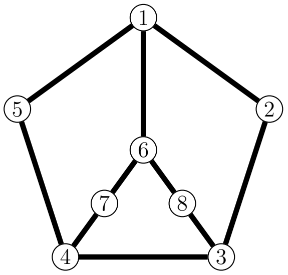
$(1,2,3)$, $(1,2,3,4)$, $(1,2,6)$, $(1,5,7)$,
$(2,3,4)$, $(2,1,5)$, $(2,1,5,7)$, $(3,4,5)$,
$(3,6,7)$, $(4,7,6)$, $(5,1,2,6)$

$$\begin{aligned} w_{1,2}+w_{2,3}+w_{3,4} &\le w_{1,5}+w_{4,5}\\ w_{1,2}+w_{1,5}+w_{5,7} &\le w_{2,6}+w_{6,7}\\ w_{1,5}+w_{1,2}+w_{2,6} &\le w_{5,7}+w_{6,7}\\ w_{3,6}+w_{6,7} &\le w_{3,4}+w_{4,7}\\ w_{4,7}+w_{6,7} &\le w_{3,4}+w_{3,6}\\ w_{3,4}+w_{4,5} &\le w_{1,5}+w_{1,2}+w_{2,3} \end{aligned} \qquad \Longrightarrow \qquad w_{1,2}\le 0$$



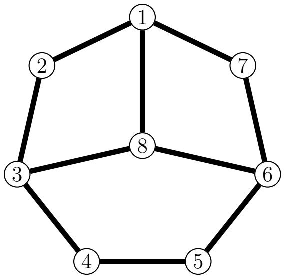
$(1,5,4,3)$, $(1,5,4)$, $(1,5,6)$, $(1,5,6,7)$,
$(2,1,5,4)$, $(2,1,5)$, $(2,3,7)$, $(3,4,5)$
$(3,2,6)$, $(4,7,6)$, $(5,6,7)$

$$\begin{aligned} w_{1,5}+w_{4,5}+w_{3,4} &\le w_{1,2}+w_{2,3}\\ w_{1,2}+w_{1,5}+w_{4,5} &\le w_{2,3}+w_{3,4}\\ w_{2,3}+w_{3,7} &\le w_{2,6}+w_{6,7}\\ w_{2,3}+w_{2,6} &\le w_{3,7}+w_{6,7}\\ w_{5,6}+w_{6,7} &\le w_{4,5}+w_{4,7}\\ w_{4,7}+w_{6,7} &\le w_{4,5}+w_{5,6} \end{aligned} \qquad \Longrightarrow \qquad w_{1,5}\le 0$$



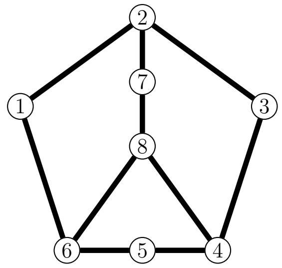
$(1,5,4,3),\ (1,5,4),\ (1,6,7),\ (1,6,8),$
$(2,3,4),\ (2,3,4,5),\ (2,1,6),\ (2,1,6,7),\ (2,1,6,8),\ (3,4,5),$
$(3,8,6),\ (3,4,7),\ (4,5,1,6),\ (4,3,8),$
$(5,1,6),\ (5,4,7),\ (5,1,6,8),\ (7,4,3,8)$

$$\begin{aligned} w_{4,7}+w_{3,4}+w_{3,8} &\le w_{6,7}+w_{6,8} \\ w_{2,3}+w_{3,4}+w_{4,5} &\le w_{1,2}+w_{1,5} \\ w_{1,5}+w_{1,6}+w_{6,8} &\le w_{4,5}+w_{3,4}+w_{3,8} \\ w_{1,2}+w_{1,6}+w_{6,7} &\le w_{2,3}+w_{3,4}+w_{4,7} \end{aligned} \quad \Longrightarrow \quad w_{1,6}\le 0$$



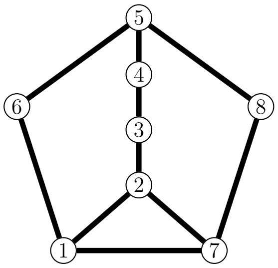
$(1,2,3),\ (1,7,6,5,4),\ (1,7,6,5),\ (1,7,6),$
$(2,3,4),\ (2,3,4,5),\ (2,3,4,5,6),\ (2,1,7),\ (2,1,8),$
$(3,4,5),\ (3,4,5,6),\ (3,2,1,7),\ (4,5,6),\ (4,5,6,7),$
$(4,3,8),\ (5,6,7),\ (5,4,3,8),\ (7,6,8)$

$$\begin{aligned} w_{4,5}+w_{3,4}+w_{3,8} &\le w_{5,6}+w_{6,8} \\ w_{1,2}+w_{1,8} &\le w_{2,3}+w_{3,8} \\ w_{6,7}+w_{6,8} &\le w_{1,7}+w_{1,8} \\ w_{1,7}+w_{6,7}+w_{5,6}+w_{4,5} &\le w_{1,2}+w_{2,3}+w_{3,4} \\ w_{2,3}+w_{3,4}+w_{4,5}+w_{5,6} &\le w_{1,2}+w_{1,7}+w_{6,7} \\ w_{2,3}+w_{1,2}+w_{1,7} &\le w_{3,4}+w_{4,5}+w_{5,6}+w_{6,7} \end{aligned} \quad \Longrightarrow \quad w_{4,5}\le 0$$



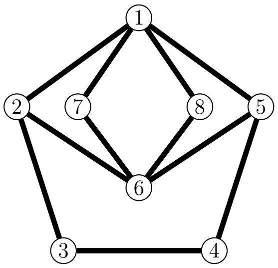
$(1,2,3),\ (1,2,3,4),\ (1,6,5),\ (1,6,8,7),\ (1,6,8),$
$(2,3,4),\ (2,1,6,5),\ (2,1,6),\ (2,7,8),$
$(3,4,5),\ (3,4,5,6),\ (3,2,7),\ (3,2,7,8),\ (4,5,6),\ (4,8,7),$
$(5,4,8,7),\ (5,4,8)$

$$\begin{aligned} w_{1,6}+w_{6,8}+w_{7,8} &\le w_{1,2}+w_{2,7} \\ w_{2,3}+w_{2,7}+w_{7,8} &\le w_{3,4}+w_{4,8} \\ w_{4,5}+w_{4,8} &\le w_{5,6}+w_{6,8} \\ w_{1,2}+w_{2,3}+w_{3,4} &\le w_{1,6}+w_{5,6}+w_{4,5} \\ w_{1,2}+w_{1,6}+w_{5,6} &\le w_{2,3}+w_{3,4}+w_{4,5} \\ w_{3,4}+w_{4,5}+w_{5,6} &\le w_{2,3}+w_{1,2}+w_{1,6} \end{aligned} \qquad \Longrightarrow \qquad w_{7,8} \le 0$$



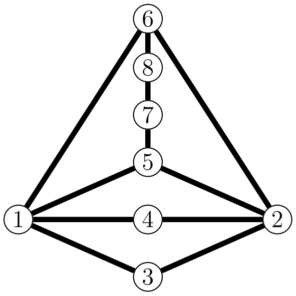
$(1,6,5,4,3),\ (1,6,5,4),\ (1,6,5),\ (1,7,8),$
$(2,3,4),\ (2,3,4,5),\ (2,1,6),\ (2,3,4,5,8),$
$(3,4,5),\ (3,4,5,6),\ (3,2,7),\ (3,4,5,8),\ (4,5,6),\ (4,3,2,7),$
$(4,5,8),\ (5,8,7),\ (6,5,8,7),\ (6,5,8)$

$$\begin{aligned} w_{1,6}+w_{5,6}+w_{4,5}+w_{3,4} &\le w_{1,2}+w_{2,3} \\ w_{2,3}+w_{3,4}+w_{4,5}+w_{5,8} &\le w_{2,7}+w_{7,8} \\ w_{5,6}+w_{5,8}+w_{7,8} &\le w_{1,6}+w_{1,7} \\ w_{3,4}+w_{2,3}+w_{2,7} &\le w_{4,5}+w_{5,8}+w_{7,8} \\ w_{1,7}+w_{7,8} &\le w_{1,6}+w_{5,6}+w_{5,8} \\ w_{1,2}+w_{1,6} &\le w_{2,3}+w_{3,4}+w_{4,5}+w_{5,6} \end{aligned} \qquad \Longrightarrow \qquad w_{3,4} \le 0$$



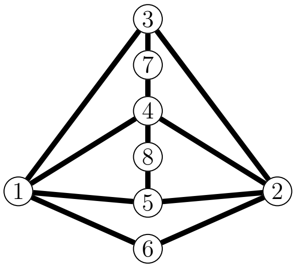
$(1,5,4,3),\ (1,5,4),\ (1,5,6),\ (2,3,4),\ (2,6,5), (2,1,7),$
$(2,1,8),\ (3,4,5),\ (3,2,6),\ (3,4,5,1,7),\ (3,4,5,1,8),$
$(4,3,2,6),\ (4,5,1,7),\ (4,5,1,8),\ (5,1,7),\ (5,1,8),\ (7,6,8)$

$$\begin{aligned} w_{1,5}+w_{1,8}+w_{3,4}+w_{4,5} &\le w_{2,3}+w_{2,6}+w_{6,8} \\ w_{3,4}+w_{2,3}+w_{2,6} &\le w_{4,5}+w_{5,6} \\ w_{2,6}+w_{5,6} &\le w_{1,2}+w_{1,5} \\ w_{1,2}+w_{1,7} &\le w_{2,6}+w_{6,7} \\ w_{6,7}+w_{6,8} &\le w_{1,7}+w_{1,8} \end{aligned} \qquad \Longrightarrow \qquad w_{3,4} \le 0$$



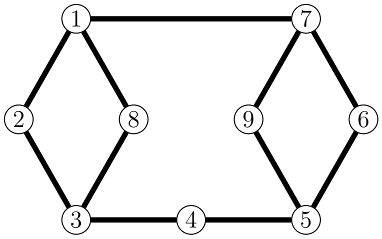
$$(1,6,2),\ (1,6,8,7),\ (1,6,8),\ (2,5,7),\ (2,5,7,8),$$
$$(3,1,4),\ (3,2,5),\ (3,2,6),\ (3,2,5,7),\ (3,2,5,7,8),$$
$$(4,2,5),\ (4,2,6),\ (4,2,5,7),\ (4,2,5,7,8),$$
$$(5,1,6),\ (5,7,8),\ (6,8,7)$$

$$\left.\begin{aligned} w_{1,6}+w_{6,8}+w_{7,8} &\le w_{1,5}+w_{5,7}\\ w_{2,3}+w_{2,5}+w_{5,7}+w_{7,8} &\le w_{1,3}+w_{1,6}+w_{6,8}\\ w_{1,3}+w_{1,4} &\le w_{2,3}+w_{2,4}\\ w_{2,4}+w_{2,6} &\le w_{1,4}+w_{1,6}\\ w_{1,5}+w_{1,6} &\le w_{2,5}+w_{2,6} \end{aligned}\right. \quad\Longrightarrow\quad w_{7,8}\le 0$$



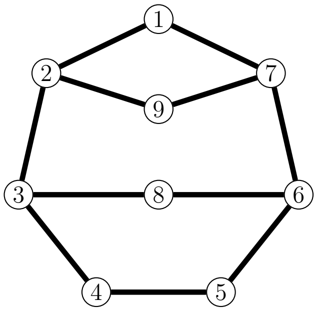
$$(1,5,2),\ (1,3,7),\ (1,5,8),\ (2,4,7),\ (2,5,8),$$
$$(3,2,4),\ (3,1,5),\ (3,1,6),\ (3,1,5,8),$$
$$(4,8,5),\ (4,1,6),\ (5,2,6),\ (5,1,3,7),$$
$$(6,1,3,7),\ (6,2,5,8),\ (7,3,1,5,8)$$

$$\left.\begin{aligned} w_{2,4}+w_{4,7} &\le w_{2,3}+w_{3,7}\\ w_{2,3}+w_{2,4} &\le w_{1,3}+w_{1,4}\\ w_{4,8}+w_{5,8} &\le w_{2,4}+w_{2,5}\\ w_{1,4}+w_{1,6} &\le w_{2,4}+w_{2,6}\\ w_{2,5}+w_{2,6} &\le w_{1,5}+w_{1,6}\\ w_{1,3}+w_{1,5}+w_{3,7}+w_{5,8} &\le w_{4,7}+w_{4,8} \end{aligned}\right. \quad\Longrightarrow\quad w_{5,8}\le 0$$



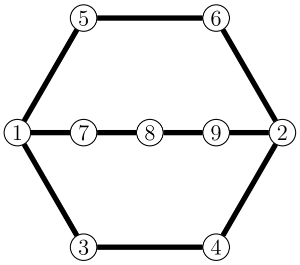
$$(1,8,3),\ (1,8,3,4),\ (1,7,6,5),\ (1,7,6),\ (1,7,9),$$
$$(2,3,4),\ (2,3,4,5),\ (2,3,4,5,6),\ (2,1,7),\ (2,1,8),\ (2,1,7,9),$$
$$(3,4,5),\ (3,4,5,6),\ (3,8,1,7),\ (3,4,5,9),$$
$$(4,5,6),\ (4,3,8,1,7),\ (4,3,8),\ (4,5,9),\ (5,6,7),\ (5,4,3,8),$$
$$(6,7,1,8),\ (6,7,9),\ (7,1,8),\ (8,3,4,5,9)$$

$$\left.\begin{aligned}
w_{2,3} + w_{3,4} + w_{4,5} + w_{5,6} &\le w_{1,2} + w_{1,7} + w_{6,7}\\
w_{3,4} + w_{3,8} + w_{1,8} + w_{1,7} &\le w_{4,5} + w_{5,6} + w_{6,7}\\
w_{3,8} + w_{3,4} + w_{4,5} + w_{5,9} &\le w_{1,8} + w_{1,7} + w_{7,9}\\
w_{1,7} + w_{6,7} + w_{5,6} &\le w_{1,8} + w_{3,8} + w_{3,4} + w_{4,5}\\
w_{1,2} + w_{1,8} &\le w_{2,3} + w_{3,8}\\
w_{6,7} + w_{7,9} &\le w_{5,6} + w_{5,9}
\end{aligned}\right. \quad\Longrightarrow\quad w_{3,4} \le 0$$



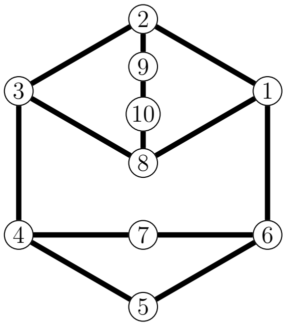
$$(1,2,3),\ (1,2,3,4),\ (1,2,3,4,5),\ (1,7,6),\ (1,7,6,8),$$
$$(1,2,9),\ (2,3,4),\ (2,3,4,5),\ (2,9,7,6),\ (2,9,7),\ (2,3,8),$$
$$(3,4,5),\ (3,8,6),\ (3,2,9,7),\ (3,2,9),\ (4,5,6),$$
$$(4,5,6,7),\ (4,5,6,8),\ (4,5,6,7,9),\ (5,6,7),$$
$$(5,6,8),\ (5,6,7,9),\ (6,7,9),\ (7,6,8),\ (8,3,2,9)$$

$$\left.\begin{aligned}
w_{1,2} + w_{2,3} + w_{3,4} + w_{4,5} &\le w_{1,7} + w_{6,7} + w_{5,6}\\
w_{4,5} + w_{5,6} + w_{6,7} + w_{7,9} &\le w_{2,3} + w_{3,4} + w_{2,9}\\
w_{1,7} + w_{6,7} + w_{6,8} &\le w_{1,2} + w_{2,3} + w_{3,8}\\
w_{2,3} + w_{2,9} + w_{3,8} &\le w_{6,8} + w_{6,7} + w_{7,9}
\end{aligned}\right. \quad\Longrightarrow\quad w_{4,5} \le 0$$



$$(1,7,8,9,2),\ (1,3,4),\ (1,5,6),\ (1,7,8),\ (1,7,8,9),$$
$$(2,4,3),\ (2,6,5),\ (2,9,8,7),\ (2,9,8),$$
$$(3,1,5),\ (3,1,5,6),\ (3,1,7),\ (3,1,7,8),\ (3,4,2,9),\ (4,2,6,5),\ (4,2,6),$$
$$(4,3,1,7),\ (4,3,1,7,8),\ (4,2,9),\ (5,1,7),\ (5,1,7,8),$$
$$(5,1,7,8,9),\ (6,2,9,8,7),\ (6,2,9,8),\ (6,2,9),\ (7,8,9)$$

$$\left.\begin{aligned}
w_{2,6} + w_{2,9} + w_{8,9} + w_{7,8} &\le w_{5,6} + w_{1,5} + w_{1,7}\\
w_{3,4} + w_{1,3} + w_{1,7} + w_{7,8} &\le w_{2,4} + w_{2,9} + w_{8,9}\\
w_{1,5} + w_{1,7} + w_{7,8} + w_{8,9} &\le w_{5,6} + w_{2,6} + w_{2,9}\\
w_{2,4} + w_{2,6} + w_{5,6} &\le w_{3,4} + w_{1,3} + w_{1,5}\\
w_{1,3} + w_{1,5} + w_{5,6} &\le w_{3,4} + w_{2,4} + w_{2,6}\\
w_{3,4} + w_{2,4} + w_{2,9} &\le w_{1,3} + w_{1,7} + w_{7,8} + w_{8,9}
\end{aligned}\right. \quad\Longrightarrow\quad w_{7,8} \le 0$$



$$\begin{gathered}(1,2,3),\ (1,2,3,4),\ (1,6,5),\ (1,6,7),\ (1,2,9),\\ (1,2,9,10),\ (2,3,4),\ (2,1,6,5),\ (2,1,6),\ (2,1,6,7),\\ (2,3,8),\ (2,9,10),\ (3,4,5),\ (3,4,5,6),\ (3,4,7),\\ (3,8,10,9),\ (3,8,10),\ (4,5,6),\ (4,3,8),\ (4,3,8,10,9),\\ (4,3,8,10),\ (5,4,7),\ (5,6,1,8),\ (5,6,1,2,9),\\ (5,6,1,2,9,10),\ (6,1,8),\ (6,1,2,9),\ (6,1,2,9,10),\\ (7,4,3,8),\ (7,6,1,2,9),\ (7,6,1,2,9,10),\ (8,10,9)\end{gathered}$$

$$\begin{aligned} w_{6,7}+w_{1,6}+w_{1,2}+w_{2,9}+w_{9,10} &\le w_{4,7}+w_{3,4}+w_{3,8}+w_{8,10}\\ w_{3,8}+w_{8,10}+w_{9,10} &\le w_{2,3}+w_{2,9}\\ w_{3,4}+w_{4,5}+w_{5,6} &\le w_{2,3}+w_{1,2}+w_{1,6}\\ w_{5,6}+w_{1,6}+w_{1,8} &\le w_{4,5}+w_{3,4}+w_{3,8}\\ w_{1,2}+w_{2,3}+w_{3,4} &\le w_{1,6}+w_{5,6}+w_{4,5}\\ w_{2,3}+w_{3,8} &\le w_{1,2}+w_{1,8}\\ w_{4,5}+w_{4,7} &\le w_{5,6}+w_{6,7}\end{aligned} \quad\Longrightarrow\quad w_{9,10}\le 0$$

References

[1] D. Cizma, N. Linial, Geodesic geometry on graphs, Discrete & Computational Geometry 68 (2022) 298–347.

[2] V. S. Matveev, Geodesically equivalent metrics in general relativity, Journal of Geometry and Physics 62 (3) (2012) 675–691.

[3] L. Lovász, On graphs not containing independent circuit, Mat. Lapok. 16 (1965) 289–299.

[4] N. Robertson, P. D. Seymour, Graph minors. XIII. The Disjoint Path Problems, Journal of Combinatorial Theory. Series B 65 (1990) 65–110.