Obstructions for three-coloring and list three-coloring H-free graphs

Maria Chudnovsky*[1], Jan Goedgebeur†[2,3], Oliver Schaudt[4], and Mingxian Zhong[5]

[1] *Princeton University, Princeton, NJ 08544, USA. E-mail: mchudnov@math.princeton.edu*

[2] *Ghent University, Ghent, Belgium. E-mail: jan.goedgebeur@ugent.be*,

[3] *Computer Science Department, University of Mons, 7000 Mons, Belgium*

[4] *RWTH Aachen University, Aachen, Germany. E-mail: schaudt@mathc.rwth-aachen.de*

[5] *Lehman College and the Graduate Center, City University of New York, New York, USA. E-mail: mingxian.zhong@lehman.cuny.edu*

In loving memory of Ella.

Abstract

A graph is H-free if it has no induced subgraph isomorphic to H. We characterize all graphs H for which there are only finitely many minimal non-three-colorable H-free graphs. Such a characterization was previously known only in the case when H is connected. This solves a problem posed by Golovach *et al.* As a second result, we characterize all graphs H for which there are only finitely many H-free minimal obstructions for list 3-colorability.

Keywords: graph coloring, critical graph, induced subgraph.

1 Introduction

A *k-coloring* of a graph $G = (V, E)$ is a mapping $c : V \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for all edges $uv \in E$. If a k-coloring exists, we say that G is

*Partially supported by NSF grant DMS-1550991 and U.S. Army Research Office grant W911NF-16-1-0404.

†Supported by a Postdoctoral Fellowship of the Research Foundation Flanders (FWO).

k*-colorable*. The related decision problem – does a given input graph admit a k-coloring? – is called the k*-colorability problem*; it is one of the most famous NP-complete problems. Let L be a mapping that maps each vertex of G to a subset of $\{1, \dots, k\}$. We say that the pair (G, L) is *colorable* if there is a k-coloring c of G with $c(v) \in L(v)$ for each $v \in V(G)$. The *list* k*-colorability problem* is the following: given a pair (G, L) with $L(v) \subseteq \{1, \dots, k\}$ for each $v \in V(G)$, decide whether (G, L) is colorable. Note that the list k-colorability problem generalizes both the k-colorability problem and the precoloring extension problem. In the k-colorability problem we have $|L(v)| = k$ for all $v \in V(G)$, while in the precoloring extension problem we have $|L(v)| \in \{1, k\}$ for all $v \in V(G)$. In this paper we study the minimal obstructions for k-colorability and list k-colorability: minimal subgraphs that prevent a graph from being k-colorable or list k-colorable.

Let H and G be graphs. We say that H is an *induced subgraph* of G if $V(H) \subseteq V(G)$, and $u, v \in V(H)$ are adjacent in H if and only if u, v are adjacent in G. For $X \subseteq V(G)$, we denote by $G|X$ the induced subgraph of G with vertex set X, and we say that X *induces* $G|X$. If $G|X$ is isomorphic to H, we say that X *is an* H *in* G. If $X \neq V(G)$, we say that $G|X$ is a *proper* induced subgraph of G. We say that G is k*-chromatic* if it is k-colorable but not $(k-1)$-colorable. A graph is called $(k+1)$*-vertex-critical* if it is $(k+1)$-chromatic, but every induced proper subgraph is k-colorable. For example, the class of 3-vertex-critical graphs is the family of all odd cycles. In view of the NP-hardness of the k-colorability problem for $k \geq 3$, there is little hope of giving a characterization of the $(k+1)$-vertex-critical graphs that is of use in algorithmic applications. The picture changes if one restricts the structure of the graphs under consideration, and the aim of this paper is to describe this phenomenon.

We use the following notation. Given two graphs G and H, we say that G *contains* H if some induced subgraph of G is isomorphic to H. If G does not contain H, we say that G is H*-free*. For a family $\mathcal{H}$ of graphs, G is $\mathcal{H}$*-free* if G is H-free for every $H \in \mathcal{H}$. Moreover, we write $G_1 + \ldots + G_k$ for the disjoint union of graphs $G_1, \dots, G_k$. For all t, let P_t denote the *path* on t vertices, which is the graph with vertex set $\{p_1, \dots, p_t\}$ such that p_i is adjacent to p_j if and only if $|i - j| = 1$.

In an earlier paper, we proved the following theorem, solving a problem posed by Golovach *et al.* [9] and answering a question of Seymour [24].

Theorem 1 (Chudnovsky *et al.* [4])**.** *Let H be a connected graph. There are only finitely many 4-vertex-critical H-free graphs if and only if H is an induced subgraph of P_6.*

In view of our result, Golovach *et al.* [9] posed the problem of extending the above theorem to a complete dichotomy for arbitrary graphs H. While this seems to be an incremental question at first sight, it requires entirely different machinery to be settled.

A second natural generalization of Theorem 1 is to go from 4-vertex critical graphs to minimal obstructions to list 3-colorability. This more technical generalization is motivated, among other things, by a theorem of Jansen and Kratsch [14], that says that if there is a finite list of H-free minimal obstructions to list-3-colorability, then a polynomial kernelization of the 3-coloring problem exists when parameterized by the number of vertices needed to hit all induced copies of H. This application is outside of the scope of this paper, and so we skip the precise definitions.

1.1 Our contribution

We answer both questions mentioned above. We obtain the following dichotomy theorem which now fully settles the problem of characterizing all graphs H for which there are only finitely many 4-vertex-critical H-free graphs.

Theorem 2. *Let H be a graph. There are only finitely many H-free 4-vertex-critical graphs if and only if H is an induced subgraph of P_6, $2P_3$, or $P_4 + kP_1$ for some $k \in \mathbb{N}$.*

The tools used in [4] to prove Theorem 1 were tailored specifically for the P_6-free case and do not generalize well, while our new approach is significantly more powerful. The idea is to transfer the problem to the more general list setting and solve it there, showing that there is a constant C such that minimal obstructions have bounded size at most C. This generality comes at a certain cost: the upper bound we get is far from sharp.

Our second main result is the analogue of Theorem 2 in the list setting. To state it, we need the following concept.

We call a pair (G, L) with $L(v) \subseteq \{1, 2, 3, \ldots, k\}$, $v \in V(G)$, a *k-minimal list-obstruction* if (G, L) is not colorable but for all proper induced subgraphs A of G the pair (A, L) is colorable. In this paper, we focus on the case when $k = 3$, so by a *minimal list-obstruction* we mean a 3-minimal list-obstruction. Here and throughout the article, if H is an induced subgraph of G, then by (H, L) we mean H with the list system L restricted to $V(H)$. We prove the following theorem.

Theorem 3. *Let H be a graph. There are only finitely many H-free minimal list-obstructions if and only if H is an induced subgraph of P_6, or of $P_4 + kP_1$ for some $k \in \mathbb{N}$.*

Note that there are infinitely many $2P_3$-free minimal list-obstructions while there are only finitely many 4-vertex-critical $2P_3$-free graphs. Thus, Theorem 2 is not a special case of Theorem 3. Moreover, the fact that there are only finitely many P_6-free minimal list-obstructions does not follow from Theorem 1.

1.2 Previous work

It is known that the k-colorability problem is NP-hard for $k \geq 3$ on H-free graphs unless if H is a disjoint union of paths [13, 15, 16, 18]. This motivates the study of graph classes in which a path is forbidden as an induced subgraph in the context of the complexity of the k-colorability problem. Regarding the computational complexity of the 3-colorability problem, the state of the art is the polynomial time algorithm to decide whether a P_7-free graph admits a 3-coloring [1]. The algorithm actually solves the harder list 3-colorability problem. For any $k \geq 8$, it is not known whether deciding 3-colorability for a P_k-free graph is polynomial time solvable or not.

There are quite a few results regarding the number of critical H-free graphs. To describe these results, consider the following definition. If H is a graph, a $(k+1)$-*critical* H*-free graph* is a graph G that is H-free, $(k+1)$-chromatic, and every H-free proper (not necessarily induced) subgraph of G is k-colorable. Note that there are finitely many $(k+1)$-critical H-free graphs if and only if there are finitely many $(k+1)$-vertex-critical H-free graphs [12]. These critical graphs are of special interest since they form a canonical *no-certificate* for k-colorability. Given a decision problem, a solution algorithm is called *certifying* if it provides, together with the yes/no decision, a polynomial time verifiable certificate for this decision. (A canonical *yes-certificate* would be a k-coloring of the graph; since most existing graph coloring algorithms are constructive they can hence easily provide a yes-certificate. A canonical no-certificate would be a $(k+1)$-critical subgraph of bounded size). For all t, we let C_t denote the *cycle* on t vertices, that is a graph with vertex set $c_1, \dots, c_t$ and such that c_i is adjacent to c_j if and only if $|i-j| \in \{1, t-1\}$.

Bruce *et al.* [2] proved that there are exactly six 4-critical P_5-free graphs. Later, Maffray and Morel [19], by characterizing the 4-vertex-critical P_5-free graphs, designed a linear time algorithm to decide 3-colorability of P_5-free graphs. Randerath et al. [22] have shown that the only 4-critical (P_6, C_3)-free graph is the Grötzsch graph. More recently, Hell and Huang [10, 11] proved that there are exactly four 4-critical (P_6, C_4)-free graphs. They also proved that there are only finitely many k-critical (P_6, C_4)-free graphs, for all k. As mentioned earlier, we proved Theorem 1 which says that there are only finitely many 4-vertex-critical P_6-free graphs, namely 80.

Recently [8], two of the authors of this paper developed an enumeration algorithm to automate the case analysis performed in the proofs of the results mentioned above. Using this algorithm, it was shown that there are only finitely many 4-critical (P_7, C_k)-free graphs, for both $k = 4$ and $k = 5$. Since there is an infinite family of 4-critical (P_7, C_6, C_7)-free graphs, only the case of (P_7, C_3)-free graphs remains open. It was also shown that there are only finitely many 4-critical (P_8, C_4)-free graphs. For more details on this line of research we recommend the two excellent survey papers by Hell and Huang [11] and Golovach *et al.* [9].

1.3 Structure of the paper

In Section 2 we state the relevant definitions and the notation used in later sections.

In Section 3 we develop the concept of the so-called *propagation path*, which is the main tool in showing that there are only finitely many H-free minimal list-obstructions (for the right choices of H) with lists of size at most 2. In particular, we show that for every minimal list-obstruction with lists of size at most 2, we can delete at most four vertices so that what remains is the union of four propagation paths.

In Section 4 we prove that there are only finitely many P_6-free minimal list-obstructions. We split the proof into two parts. In the first part we show that there are only finitely many P_6-free minimal list-obstructions where every list is of size at most 2, which amounts to studying P_6-free propagation paths. This step has a computer-aided proof. We also have a computer-free proof of this fact, but it is tedious, and we decided to only include a sketch of it. In the second part of the proof we reduce the general problem to the case solved in the first part. Here we rely on a structural analysis, making use of a structure theorem for P_t-free graphs.

Using a similar approach, in Section 5 we prove that there are only finitely many $2P_3$-free 4-vertex-critical graphs (of course, certain modifications are needed, because the list version is false in this case). In Section 6 we show that there are only finitely many $P_4 + kP_1$-free minimal list-obstructions.

In Section 7 we prove the necessity in the statement of Theorem 2 and Theorem 3, providing infinite families of H-free 4-vertex-critical graphs and minimal list-obstructions.

Our main results are proven in Section 8 where we put together the results mentioned above.

2 Preliminaries

All graphs in this paper are finite and simple. Let G and H be graphs and let X be a subset of $V(G)$. We denote by $G \setminus X$ the graph $G|(V(G) \setminus X)$. If $X = \{v\}$ for some $v \in V(G)$, we write $G \setminus v$ instead of $G \setminus \{v\}$. If $G|X$ is isomorphic to H, then we say that X *is an H in G*. We write $G_1 + \ldots + G_k$ for the disjoint union of graphs $G_1, \ldots, G_k$. The *neighborhood* of a vertex $v \in V(G)$ is denoted by $N_G(v)$ (when there is no danger of confusion, we sometimes write $N(v)$). For a vertex set S, we use $N(S)$ to denote $(\bigcup_{v \in S} N(v)) \setminus S$.

For $n \geq 0$, we denote by P_n *the chordless path on n vertices.* For $n \geq 3$, we denote by C_n *the chordless cycle on n vertices.* By convention, when explicitly describing a path or a cycle, we always list the vertices in order. Let G be a graph. When $G|\{p_1, \ldots, p_n\}$ is the path P_n, we say that p_1-$\ldots$-p_n *is a P_n in G*. Similarly, when $G|\{c_1, c_2, \ldots, c_n\}$ is the cycle C_n, we say that c_1-c_2-$\ldots$-c_n-c_1 *is a C_n in G*. A *Hamiltonian path* is a path that contains all vertices of G.

Let A and B be disjoint subsets of $V(G)$. For a vertex $b \in V(G) \setminus A$, we say that b *is complete to A* if b is adjacent to every vertex of A, and that b *is anticomplete to A* if b is non-adjacent to every vertex of A. If every vertex of A is complete to B, we say A *is complete to B*, and if every vertex of A is anticomplete to B, we say that A *is anticomplete to B*. If $b \in V(G) \setminus A$ is neither complete nor anticomplete to A, we say that b *is mixed on A*. The *complement* $\overline{G}$ of G is the graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if they are non-adjacent in G. If $\overline{G}$ is connected we say that G is *anticonnected*. For $X \subseteq V(G)$, we say that X is *connected* if $G|X$ is connected, and that X is anticonnected if $G|X$ is anticonnected. A *component* of $X \subseteq V(G)$ is a maximal connected subset of X, and an *anticomponent* of X is a maximal anticonnected subset of X. We write *component of* G to mean a component of $V(G)$. A subset D of $V(G)$ is called a *dominating set* if every vertex in $V(G) \setminus D$ is adjacent to at least one vertex in D; in this case we also say that $G|D$ is a *dominating subgraph* of G.

A *list system* L of a graph G is a mapping which assigns each vertex $v \in V(G)$ a finite subset of $\mathbb{N}$, denoted by $L(v)$. A *subsystem* of a list system L of G is a list system L' of G such that $L'(v) \subseteq L(v)$ for all $v \in V(G)$. We say a list system L of the graph G has *order* k if $L(v) \subseteq \{1, \ldots, k\}$ for all $v \in V(G)$. In this article, we will only consider list systems of order 3. Notationally, we write (G, L) to represent a graph G and a list system L of G. We say that c, a coloring of G, is an L-*coloring* of G, or a *coloring of* (G, L) provided $c(v) \in L(v)$ for all $v \in V(G)$. We say that (G, L) is *colorable*, if there exists a coloring of (G, L). A *partial coloring* of (G, L) is a mapping $c : U \to \mathbb{N}$ such that $c(u) \in L(u)$ for all $u \in U$, where U is a subset of $V(G)$. Note that here we allow for edges uv of $G|U$ with $c(u) = c(v)$. If there is no such edge, we call c *proper*.

Let G be a graph and let L be a list system of order 3 for G. We say that (G, L) is a *list-obstruction* if (G, L) is not colorable. As stated earlier, we call (G, L) a *minimal list-obstruction* if, in addition, $(G \setminus x, L)$ is colorable for every vertex $x \in G$.

Let (G, L) be a list-obstruction. We say a vertex $v \in V(G)$ is *critical* if $G \setminus v$ is L-colorable and *non-critical* otherwise. If we repeatedly delete non-critical vertices of G to obtain a new graph, G' say, such that (G', L) is a minimal list obstruction, we say that (G', L) is a minimal list obstruction *induced* by (G, L).

Let u, v be two vertices of a list-obstruction (G, L). We say that u *dominates* v if $L(u) \subseteq L(v)$ and $N(v) \subseteq N(u)$. It is easy to see that if there are such vertices u and v in G, then (G, L) is not a minimal list-obstruction. We frequently use this observation without further reference.

2.1 Updating lists

Let G be a graph and let L be a list system for G. Let $v, w \in V(G)$ be adjacent, and assume that $|L(w)| = 1$. To *update the list of* v *from* w means to delete from $L(v)$ the unique element of $L(w)$. If the size of the list of v is reduced to one, we sometimes say that v is *colored*, and refer to the unique element in the list of v as the *color* of v. Throughout the paper, we make use of distinct updating procedures to reduce the sizes of the lists, and we define them below.

If $P = v_1\text{-}\ldots\text{-}v_k$ is a path and $|L(v_1)| = 1$, then to *update from* v_1 *along* P means to update v_2 from v_1 if possible, then to update v_3 from v_2 if possible, and so on. When v_k is updated from v_{k-1}, we stop the updating.

Let $X \subseteq V(G)$ such that $|L(x)| \leq 1$ for all $x \in X$. For a subset $A \subseteq V(G) \setminus X$, we say that we *update the lists of the vertices in* A *with respect to* X if we update each $a \in A$ from each $x \in X$. We say that we *update the lists with respect to* X if $A = V(G) \setminus X$. Let $X_0 = X$ and $L_0 = L$. For $i \geq 1$ define X_i and L_i as follows.

L_i is the list system obtained from L_{i-1} by updating with respect to X_{i-1}. Moreover, $X_i = X_{i-1} \cup \{v \in V(G) \setminus X_{i-1} : |L_i(v)| \leq 1 \text{ and } |L_{i-1}(v)| > 1\}$. We say that L_i is obtained from L by *updating with respect to X i times.* If $X = \{w\}$ we say that L_i is obtained by *updating with respect to w i times.* If for some i, $W_i = W_{i-1}$ and $L_i = L_{i-1}$, we say that L_i was obtained from L by *updating exhaustively with respect to X (or w).* For simplicity, if X is an induced subgraph of G, by updating with respect to X we mean updating with respect to $V(X)$. We also adopt the following convention. If for some i, if two vertices of X_{i-1} with the same list are adjacent, or $L_i(v) = \emptyset$ for some v, we set $L_i(v) = \emptyset$ for every $v \in V(G) \setminus X_i$. Observe that in this case $(G|X_i, L_i)$ is not colorable, and so we have preserved at least one minimal list obstruction induced by (G, L).

3 Obstructions with lists of size at most two

The aim of this section is to provide an upper bound on the order of the H-free minimal list-obstructions in which every list has at most two entries. Let us stress the fact that we restrict our attention to lists which are (proper) subsets of $\{1, 2, 3\}$. Before we state our lemma, we need to introduce the following technical definition.

Let (G, L) be a minimal list-obstruction such that $|L(v)| \leq 2$ for all $v \in V(G)$. Let $P = v_1\text{-}v_2\text{-}\ldots\text{-}v_k$ be a path in G, not necessarily induced. Assume that $|L(v_1)| \geq 1$ and $|L(v_i)| = 2$ for all $i \in \{2, \ldots, k\}$. Moreover, assume that there is a color $\alpha \in L(v_1)$ such that if we give color α to v_1 and update along P, we obtain a coloring c of P. Please note that c may not be a coloring of the graph $G|V(P)$. For $i \in \{2, \ldots, k\}$ with $L(v_i) = \{\beta, \gamma\}$ and $c(v_i) = \beta$ we define the *shape* of v_i to be $\beta\gamma$, and denote it by $S(v_i)$. If every edge v_iv_j (of G) with $3 \leq i < j \leq k$ and $i \leq j - 2$ is such that

$$S(v_i) = \alpha\beta \text{ and } S(v_j) = \beta\gamma, \tag{1}$$

where $\{1, 2, 3\} = \{\alpha, \beta, \gamma\}$, then we call P a *propagation path* of G and say that P *starts with color* α. As we prove later, (1) implies that the updating process from v_1 along P to v_k cannot be shortcut via any edge v_iv_j with $3 \leq i < j \leq k$ and $i \leq j - 2$.

The next lemma shows that, when bounding the order of our list-obstructions, we may concentrate on upper bounds on the size of propagation paths.

Lemma 4. *Let (G, L) be a minimal list-obstruction, where $|L(v)| \leq 2$ and $L(v) \subseteq \{1, 2, 3\}$ for every $v \in V(G)$. Assume that all propagation paths in G have at most λ vertices for some $\lambda \geq 20$. Then $|V(G)| \leq 4\lambda + 4$.*

In the next section we prove the above lemma. First we show that if G is a minimal list-obstruction in which every list contains at most two colors, then $V(G) = V_1 \cup V_2$, where $|V_1 \cap V_2| \geq 1$, and for some $v \in V_1 \cap V_2$, each $G|V_i$ has a Hamiltonian path P_i starting at v. Moreover, if $L(v) = \{c_1, c_2\}$, then for every $i \in \{1,2\}$ giving v the color c_i and updating along P_i, results in a pair of adjacent vertices of G receiving the same list of size one. Then we prove that the edges of G that are not the edges of P_1, P_2 are significantly restricted, and consequently each P_i is (almost) the union of two propagation paths, thus proving the lemma.

3.1 Proof of Lemma 4

Let $(G = (V, E), L)$ be a minimal list-obstruction such that $|L(v)| \leq 2$ and $L(v) \subseteq \{1, 2, 3\}$ for all $v \in V$. If there is a vertex with an empty list, then this is the only vertex of G and we are done. So, we may assume that every vertex of G has a non-empty list. Let $V_1 = \{v \in V : |L(v)| = 1\}$ and $V_2 = \{v \in V : |L(v)| = 2\}$.

Claim 1. *Let $x \in V$ and $\alpha \in L(x)$ be arbitrary. Assume we give color α to x and update exhaustively in the graph $(G|(V_2 \cup \{x\}), L)$. Let c be partial coloring thus obtained. For each $y \in V_1 \setminus \{x\}$, let $c(y)$ be the unique color in $L(y)$. Then there is an edge uv such that $c(u) = c(v)$.*

Proof. Let us give color α to x and update exhaustively, but only considering vertices and edges in the graph $(G|(V_2 \cup \{x\}), L)$. We denote this partial coloring by c. For each $y \in V_1 \setminus \{x\}$, let $c(y)$ be the unique color in $L(y)$. For a contradiction, suppose that this partial coloring c is proper.

Since G is an obstruction, c is not a coloring of G, meaning there are still vertices with two colors left on their list. We denote the set of these vertices by U. By minimality of G, we know that both graphs $(G', L') := (G \setminus U, L)$ and $(G'', L'') := (G|(U \cup V_1) \setminus x, L)$ are colorable and have at least one vertex.

Let c' be the coloring of G' such that $c'(u) = c(u)$ for all $u \in V(G')$, and let c'' be a coloring of G''. It is clear that c' and c'' agree on the vertices in $V(G') \cap V(G'') = V_1 \setminus \{x\}$. Moreover, if $v \in (V_2 \setminus U) \cup \{x\}$ and $u \in U$ such that $uv \in E(G)$, then $c(v) \notin L(u)$. Since $c'(v) = c(v)$ for every $v \in V(G')$, we deduce that $c'(v) \neq c''(u)$ for every $uv \in E(G)$ with $u \in U$ and $v \in (V_2 \setminus U) \cup \{x\}$. Consequently, we found a coloring of (G, L), a contradiction. □

Claim 2. *It holds that $|V_1| \leq 2$.*

Proof. Suppose that $|V_1| \geq 3$, and let $x \in V_1$ and $\alpha \in L(x)$. Let us give color α to x and update exhaustively, but only considering vertices and edges in the graph $(G|(V_2 \cup \{x\}), L)$. We denote this partial coloring by c.

Since G is minimal, there is no edge uv with $u, v \in V_2 \cup \{x\}$ and $c(u) = c(v)$. Since $(G|(V_1 \setminus \{x\}), L)$ is colorable by the minimality of G, and since $|L(v)| = 1$ for every $v \in V_1 \setminus \{x\}$, it follows that there is an edge uv with $u \in V_2 \cup \{x\}$ and $v \notin V_2 \cup \{x\}$ such that $L(v) = \{c(u)\}$. It follows that $(G|(V_2 \cup \{x, v\}), L)$ is not colorable, and so by the minimality of G, $V_1 = \{x, v\}$, as required. This proves the first claim. □

Next we prove that, loosely speaking, G is the union of at most two paths, starting at a common vertex, updating along each of which yields an improper partial coloring. Depending on the cardinality of V_1, we arrive at three different situations which are described by the following three claims.

Claim 3. *Assume that $|V_1| = 0$, and pick $x \in V$ arbitrarily. Let us say that $L(x) = \{1, 2\}$. For $\alpha = 1, 2$ there is a path $P^\alpha = v_1^\alpha\text{-}\dots\text{-}v_{k_\alpha}^\alpha$, not necessarily induced, with the following properties.*

(a) If we give color α to x and update along P^α, then all vertices of P^α will be colored.

(b) Assume that v_i^α gets colored in color γ_i, $i = 1, \dots, k_\alpha$. Then there is an edge of the form $v_i^\alpha v_j^\alpha$ with $\gamma_i = \gamma_j$.

(c) $V = V(P^1) \cup V(P^2)$.

Proof. We give color α to x and update exhaustively from x. According to Claim 1, after some round of updating an edge appears whose end vertices receive the same color. We then stop the updating procedure. During the whole updating procedure we record an auxiliary digraph $D = (W, A)$ as follows. Initially, $W = \{x\}$ and $A = \emptyset$. Whenever we update a vertex u from a vertex v, we add the vertex u to W and the edge (v, u) to A. This gives a directed tree whose root is x.

We can find directed paths R and S in T both starting in x and ending in vertices y and z, say, such that y and z are adjacent in G and they receive the same color during the updating procedure. We may assume that $R = u_1\text{-}\dots\text{-}u_k\text{-}v_1\text{-}\dots\text{-}v_r$ and $S = u_1\text{-}\dots\text{-}u_k\text{-}w_1\text{-}\dots\text{-}w_s$, where R and S share only the vertices $u_1, \dots, u_k$. For each vertex $v \in V(R) \cup V(S)$, let $c(v)$ be the color received by v in the updating procedure. Moreover, let $c'(v)$ be the unique color in $L(v) \setminus \{c(v)\}$. Observe that, setting $w_0 = v_0 = u_k$, we have that $c'(w_i) = c(w_{i-1})$ for every $i \in \{1, \dots, s\}$, and $c'(v_i) = c(v_{i-1})$ for every $i \in \{1, \dots, k\}$.

Consider the following, different updating with respect to x. We again give color α to x, and then update along R. Now we update w_s from v_r, thus giving it color $c'(w_s)$. This, in turn, means we can update w_{s-1} from w_s, giving it color $c'(w_{s-1})$, and so on. Finally, when we update w_1 and it receives color $c'(w_1)$, an edge appears whose end vertices are colored in the same color. Indeed, $u_k w_1$ is such an edge since $c(u_k) = c'(w_1)$. Summing up, the path

$$P^\alpha = u_1\text{-}\dots\text{-}u_k\text{-}v_1\text{-}\dots\text{-}v_r\text{-}w_s\text{-}w_{s-1}\text{-}\dots\text{-}w_1$$

starts in x and, when we give x the color α and update along P^α, we obtain an improper partial coloring. As $\alpha \in \{1, 2\}$ was arbitrary, the assertions (a) and (b) follow.

To see (c), just note that the graph $G|(V(P_1) \cup V(P_2))$ is an obstruction: giving either color of $L(x)$ to x and updating exhaustively yields a monochromatic edge. By the minimality of G, $G = G|(V(P_1) \cup V(P_2))$ and so (c) holds. $\square$

Claim 4. *Assume that $|V_1| = 1$, say $V_1 = \{x\}$ with $L(x) = \{\alpha\}$. Then the following holds:*

(a) there is a Hamiltonian path $P = v_1$-…-v_k of G with $x = v_1$;

(b) updating from $x = v_1$ along P assigns a color γ_i to v_i, $i = 1, \ldots, k$; and

(c) there is an edge of the form $v_i v_j$ with $\gamma_i = \gamma_j$.

Proof. We assign color α to x and update exhaustively from x. Let c be the obtained partial coloring. According to Claim 1, there is an edge uv of G with $c(u) = c(v)$. Since (G, L) is a minimal obstruction, every vertex of G received a color in the updating process: otherwise, we could simply remove such a vertex and still have an obstruction.

Repeating the argument from the proof of Claim 3, we obtain a path P that starts in x and, when we give x color α and update along P, we obtain an improper partial coloring. This proves (b) and (c). Due to the minimality of (G, L), P is a Hamiltonian path, which proves (a). □

Claim 5. *Assume $|V_1| = 2$, say $V_1 = \{x, y\}$ with $L(x) = \{\alpha\}$ and $L(y) = \{\beta\}$. Then the following holds:*

(a) there is a Hamiltonian path $P = v_1$-…-v_k of G with $x = v_1$ and $y = v_k$; and

(b) updating from v_1 along P assigns the color β to v_{k-1}.

Proof. We color x with color α and update exhaustively from x, but only considering vertices and edges of the graph $G \setminus y$. Let c be the obtained partial coloring. By minimality, c is proper. According to Claim 1, there is a neighbor u of y in G with $c(u) = \beta$.

Like in the proof of Claim 3 and Claim 4, we see that there is a path P from x to y whose last edge is uy such that giving color α to x and then updating along P implies that u is colored with color β, which implies (b). Due to the minimality of (G, L), P is a Hamiltonian path , and thus (a) holds. □

We can now prove our main lemma.

Proof of Lemma 4. Recall from Claim 2 that $|V_1| \leq 2$.

Case $|V_1| = 0$. For this case Claim 3 applies and we obtain x, P^1 and P^2 as in the statement of the claim. We may assume that, among all possible choices of x, P^1 and P^2, the value $\max\{|V(P^1)|, |V(P^2)|\}$ is minimum and, subject to this, $\min\{|V(P^1)|, |V(P^2)|\}$ is minimum.

Let us say that $P^1 = v_1$-v_2-…-v_s, where $v_1 = x$. Consider v_1 to be colored in color 1, and update along P^1, but only up to v_{s-1}. Due to the choice of P^1 and P^2 being of minimum length, the coloring so far is proper. Now when we update from v_{s-1} to v_s, two adjacent vertices receive the same color. Let the partial coloring obtained so far be denoted c. Let X be the set of neighbors w of v_s in $V(P^1)$ with $c(w) = c(v_s)$, and let r be minimum such that $v_r \in X$.

We claim that $s - r \leq \lambda$. To see this, let $c'(v_j)$ be the unique color in $L(v_j) \setminus \{c(v_j)\}$, for all $j = 1, \ldots, s$. We claim that the following assertions hold.

(a) $c(v_j) = c'(v_{j+1})$ for all $j = r, \ldots, s-1$.

(b) For every edge v_iv_j with $r \leq i, j \leq s-1$ it holds that $c(v_i) \neq c(v_j)$.

(c) For every edge v_iv_j with $r \leq i, j \leq s$ and $j - i \geq 2$ it holds that $c(v_i) \neq c'(v_j)$.

(d) For every edge v_iv_j with $r+2 \leq i, j \leq s$ it holds that $c'(v_i) \neq c'(v_j)$.

Assertion (a) follows from the fact that P obeys the assertions of Claim 3. For (b), note that the choice of P_1 to be of minimum length implies that until we updated v_s, the partial coloring is proper.

To see (c), suppose there is an edge v_iv_j with $r \leq i, j \leq s$ and $j - i \geq 2$ such that $c(v_i) = c'(v_j)$. Then the path P^1 can be shortened to the path v_1-…-v_i-v_j-…-v_s, which is a contradiction.

Now we turn to (d), and consider the following coloring. We color P^1 as before up to v_r. Now we update from v_r to v_s, giving color $c'(v_s)$ to v_s. Then we color v_{s-1} with color $c'(v_{s-1})$, then v_{s-2} with color $c'(v_{s-2})$, and so on, until we reach v_{r+1}. But $c'(v_{r+1}) = c(v_r)$ due to (a), which means that the path $Q^1 = v_1$-v_2-…-v_r-v_s-v_{s-1}-v_{s-2}-…-v_{r+1} is a choice equivalent to P^1. In particular, due to the choice of P^1 and P^2, the constructed coloring of Q^1 is proper if we leave out v_{r+1}. Hence, there is no edge v_iv_j with $r+2 \leq i, j \leq s$ such that $c'(v_i) = c'(v_j)$. This yields (d).

From (a)-(d) it follows that every edge v_iv_j with $r+2 \leq i < j \leq s$ and $i \leq j-2$ is such that

$$S(v_i) = \alpha\beta \text{ and } S(v_j) = \beta\gamma, \tag{2}$$

where $\{1,2,3\} = \{\alpha, \beta, \gamma\}$. Consequently, the path v_r-…-v_{s-1} is a propagation path. By assumption, $|\{v_r, \ldots, v_{s-1}\}| \leq \lambda$ and so $s - r \leq \lambda$.

A symmetric consideration holds for P^2. Let us now assume that $|V(P^1)| \geq |V(P^2)|$. It remains to show that r is bounded by some constant. To this end, recall that $\lambda \geq 20$.

Suppose that there is an edge v_iv_j with $3 \leq i \leq j \leq r$ such that $c'(v_i) = c'(v_j)$. We then put $x' = v_j$, $Q^1 = v_j$-…-v_s, and $Q^2 = v_j$-…-v_i. But this is a contradiction to the choice of x, P^1, and P^2, as $\max\{|V(Q^1)|, |V(Q^2)|\} < \max\{|V(P^1)|, |V(P^2)|\}$. In addition to the assertion we just proved, which corresponds to assertion (d) above, the assumptions (a)-(c) from above also hold here, where we replace r by 1 and s by r. Hence, using the same argumentation as above, we see that $r \leq \lambda + 1$. Summing up, we have $|V| \leq |V(P^1) \cup V(P^2)| \leq 2|V(P^1)| \leq 4\lambda + 2$, as desired.

Case $|V_1| = 1$. Now Claim 4 applies and we obtain the promised path, say $P = v_1$-…-v_s, with $|L(v_1)| = 1$. Let us say $L(v_1) = \{1\}$. Consider v_1 to be colored in color 1, and update along P, but only up to v_{s-1}. Due to the choice of P, the coloring so far is proper. Now when we update from v_{s-1} to v_s, two adjacent vertices receive the same color. Let the partial coloring obtained so far be denoted c. Let X be the set of neighbors w of v_s on P with $c(w) = c(v_s)$, and let r be minimum such that $v_r \in X$. Moreover, let $c'(v_j)$ be the unique color in $L(v_j) \setminus \{c(v_j)\}$, for all $j = 2, \ldots, s$. Just like in the case $|V_1| = 0$, we obtain the assertions (a)-(d) from above and this implies $s - r \leq \lambda$.

It remains to show that $r \leq \lambda + 1$. To see this, suppose that there is an edge $v_i v_j$ with $2 \leq i \leq j \leq r$ such that $c'(v_i) = c'(v_j)$. We then put $P^1 = v_j$-…-v_s and $P^2 = v_j$-…-v_i. Now, if we give color $c(v_j)$ to v_j and update along P^1 we obtain an improper coloring. Moreover, if we give color $c'(v_j)$ to v_j and update along P^2 we also obtain an improper coloring. This means that the pair $(G|(V(P^1) \cup V(P^2)), L)$ is not colorable, in contradiction to the minimality of (G, L).

The assertion we just proved corresponds to assertion (d) above, and the assumptions (a)-(c) also hold here, where we replace r by 1 and s by r. Hence, we know $r \leq \lambda + 1$ and obtain $|V| = |V(P)| \leq 2\lambda + 1$.

Case $|V_1| = 2$**.** Claim 5 applies and we obtain the promised path, say $P = v_1$-…-v_s, with $|L(v_1)| = |L(v_s)| = 1$. Let us say $L(v_1) = \{\alpha\}$ and $L(v_s) = \{\beta\}$. Consider v_1 to be colored in color α, and update along P, but only up to v_{s-1}. Due to the choice of P, the partial coloring so far is proper. Let the partial coloring obtained so far be denoted c, and put $c(v_s) = \beta$. We now have $c(v_{s-1}) = c(v_s)$, and this is the unique pair of adjacent vertices of G that receive the same color.

For each $j = 2, \ldots, s-1$, we denote by $c'(v_j)$ the unique color in $L(v_j) \setminus \{c(v_j)\}$. We will show that $s \leq \lambda + 1$. Just like in the cases above the following assertions apply.

(a) $c(v_j) = c'(v_{j+1})$ for all $j = 1, \ldots, s-2$.

(b) For every edge $v_i v_j$ with $1 \leq i, j \leq s-1$ it holds that $c(v_i) \neq c(v_j)$.

(c) For every edge $v_i v_j$ with $1 \leq i, j \leq s-1$ and $j - i \geq 2$ it holds that $c(v_i) \neq c'(v_j)$.

(d) For every edge $v_i v_j$ with $3 \leq i, j \leq s-1$ it holds that $c'(v_i) \neq c'(v_j)$.

Let $r' = 1$ and $s' = s - 1$. As above we see that $s' - r' \leq \lambda - 1$. Hence, $s \leq \lambda + 1$. From the fact that P is a Hamiltonian path in G we obtain the desired bound $|V| = |V(P)| \leq \lambda + 1$. This completes the proof. □

4 P_6-free minimal list-obstructions

The aim of this section is to prove that there are only finitely many P_6-free minimal list-obstructions. In Section 4.1 we prove the following lemma which says that there are only finitely many P_6-free minimal list-obstructions with lists of size at most two.

Lemma 5. *Let (G, L) be a P_6-free minimal list-obstruction for which $|L(v)| \leq 2$ and $L(v) \subseteq \{1, 2, 3\}$ holds for all $v \in V(G)$. Then $|V(G)| \leq 100$.*

Our proof of this lemma is computer-aided. We also have a computer-free proof, but it is tedious and complicated, and gives a significantly worse bound on the size of the obstructions, so we will only sketch the idea of the computer-free proof.

In Section 4.3 we solve the general case, where each list may have up to three entries, making extensive use of Lemma 5. We prove the following lemma.

Lemma 6. *There exists an integer C such that the following holds. Let G be a P_6-free graph, and let L be a list system such that $L(v) \subseteq \{1,2,3\}$ for every $v \in V(G)$. Suppose that (G,L) is a minimal list obstruction. Then $|V(G)| \le C$. Consequently, there are only finitely many P_6-free minimal list-obstructions.*

The main technique used in the proof of Lemma 6 is to guess the coloring on a small set S of vertices of the minimal list-obstruction at hand, (G,L) say. After several transformations, we arrive at a list-obstruction (G,L') where each list has size at most two, and so we may apply Lemma 5 to show that there is a minimal list-obstruction (H,L') with a bounded number of vertices induced by (G,L'). We can prove that G is essentially the union of these graphs H (one for each coloring of S), and so the number of vertices of G is bounded by a function of the number of guesses we took in the beginning. Since we precolor only a (carefully chosen) small part of the graph, we can derive that the number of vertices of G is bounded by a constant.

To find the right vertex set to guess colors for, we use a structure theorem for P_t-free graphs [3] that implies the existence of a well-structured connected dominating subgraph of a minimal list-obstruction.

4.1 Proof of Lemma 5

Let (G,L) be a P_6-free minimal list-obstruction such that every list contains at most two colors. Suppose that $P = v_1\text{-}\dots\text{-}v_k$ is a propagation path in (G,L). We show that if G is P_6-free, then $k \le 24$. In view of Lemma 4, this proves that G has at most 100 vertices.

Our proof is computer-aided, but conceptually very simple. The program generates the paths v_1, $v_1\text{-}v_2$, $v_1\text{-}v_2\text{-}v_3$, and so on, lists for each v_i, as in the definition of a propagation path, and edges among the vertices in the path. Whenever a P_6 or an edge violating condition (1) of the definition of a propagation path is found, the respective branch of the search tree is closed. Since the program does not find such a path on 25 vertices (cf. Table 1), our claim is proved.

The pseudocode of the algorithm is shown in Algorithm 1 and 2. Our implementation of this algorithm can be downloaded from [7]. Table 1 lists the number of configurations generated by our program.

Next we sketch the idea of the computer-free proof. Let (G,L) be a minimal list obstruction with all lists of size at most two, and suppose for a contradiction that there is a (very) long propagation path P in G. We may assume that G does not contain a clique with four vertices. It follows from the main result of [6] that $G|V(P)$ contains a large induced subgraph H, which is a complete

Vertices	1	2	3	4	5	6	7	8	
Propagation paths	1	2	6	22	86	350	1 220	2 656	
Vertices	9	10	11	12	13	14	15	16	
Propagation paths	4 208	5 360	5 864	5 604	5 686	5 004	4 120	3 400	
Vertices	17	18	19	20	21	22	23	24	25
Propagation paths	2 454	1 688	1 064	516	202	72	18	2	0

Table 1: Counts of all P_6-free propagation paths with lists of size 2 meeting condition (1) generated by Algorithm 1.

Algorithm 1 Generate propagation paths and lists

$H \leftarrow (\{v_2\}, \emptyset)$
$c(v_1) \leftarrow 1$
$L(v_1) \leftarrow \{1\}$
Construct(H, c, L)
    // We may assume $c(v_1) = 1$ and $L(v_1) = \{1\}$.

Algorithm 2 Construct(Graph H, coloring c, list system L)

$j \leftarrow |V(H)|$
$V(H) \leftarrow V(H) \cup \{v_{j+1}\}$
$E(H) \leftarrow E(H) \cup \{v_j v_{j+1}\}$
    // This extends the path by the next vertex v_{j+1}.
for all $\alpha \in \{1,2,3\} \setminus \{c(v_j)\}$ and all $I \subseteq \{1, 2, \ldots, j-1\}$ **do**
    $H' \leftarrow H$
    $E(H') \leftarrow E(H') \cup \{v_i v_{j+1} : i \in I\}$
        // This adds edges from v_{j+1} to earlier vertices in all possible ways.
    $c(v_{j+1}) \leftarrow \alpha$
    $L(v_{j+1}) \leftarrow \{\alpha, c(v_j)\}$
    if (H', c, L) is P_6-free and satisfies condition (1) **then**
        Construct(H', c, L)
            // If the propagation path is not pruned, we extend it further.
    end if
end for

bipartite graph; let (A, B) be a bipartition of H. Using Ramsey's Theorem [21] we may assume that all vertices of A have the same shape, and all vertices of B have the same shape (by "coloring" the edges of H by the shapes of their ends). We can now choose a large subset A' of A all of whose members are pairwise far apart in P, and such that A' is far in P from some subset B' of B. We analyze the structure of short subpaths of P containing each $a \in A$, and the edges between such subpaths, and to B'. We can again use Ramsey's Theorem to assume that the structure is the same for every member of A' and every member of B'. Finally, we accumulate enough structural knowledge to find a P_6 in G, thus reaching a contradiction. If instead of P_6 we wanted to use the same method to produce P_5, the argument becomes much shorter, and it was carried out in [25].

4.2 Reducing obstructions

In this section we prove three lemmas which help us reduce the size of the obstructions. These lemmas will be used in the proofs of Sections 4.3 and 5.2.

Let (G, L) be a list-obstruction and let R be an induced subgraph of G. Let $\mathcal{L}$ be a set of subsystems of L satisfying the following assertions.

1. For every $L' \in \mathcal{L}$ there exists an induced subgraph $R(L')$ of R such that $|L'(v)| = 1$ for every $v \in V(R(L'))$.
2. For each $L' \in \mathcal{L}$, $L'(v) = L(v)$ for $v \in V(G) \setminus R(L')$ and $L'(v) \subseteq L(v)$ for $v \in R(L')$.
3. For every L-coloring c of R there exists a list system $L' \in \mathcal{L}$ such that $c(v) \in L'(v)$ for every $v \in R(L')$.

We call $\mathcal{L}$ a *refinement of L with respect to R*. Observe that $\{L\}$ with $R(L)$ being the empty graph is a refinement of L with respect to G, though this is not a useful refinement. For each list system $L' \in \mathcal{L}$ it is clear that (G, L') is again a list-obstruction, though not necessarily a minimal one, even if (G, L) is minimal.

Lemma 7. *Assume that (G, L) is a minimal list-obstruction. Let R be an induced subgraph of G, and let $\mathcal{L} = \{L_1, L_2, \ldots, L_m\}$ be a refinement of L with respect to R. For every $L_i \in \mathcal{L}$, let (G_{L_i}, L_i) be a minimal obstruction induced by (G, L_i).*

Then $V(G) = R \cup \bigcup_{L_i \in \mathcal{L}} V(G_{L_i})$. Moreover, if each G_{L_i} can be chosen such that $|V(G_{L_i} \setminus R| \leq k$, then G has at most $|V(R)| + km$ vertices.

Proof. Let $(G_i, L_i|_{G_i})$ be a minimal list-obstruction induced by (G, L_i) such that $|V(G_i) \setminus V(R)| \leq k$, $i = 1, \ldots, m$. Suppose for a contradiction that there exists a vertex v in $V(G) \setminus R$ such that v is contained in none of the G_i, $i = 1, \ldots, m$. By the minimality of (G, L), $G \setminus \{v\}$ is L-colorable. Let c be an L-coloring of $G \setminus \{v\}$. We may assume that $c(r) \in L_1(r)$ for every $r \in V(R(L_1)) \cap V(G_1)$. Then c is a coloring of (G_1, L_1), which is a contradiction. This proves the first assertion of Lemma 7. Consequently,

$$|V(G)| \leq |\bigcup_{i=1}^{m} V(G_i)| \leq |V(R)| + |\bigcup_{i=1}^{m} V(G_i \setminus R)|$$

and the second assertion follows.

□

Next we prove a lemma which allows us to update three times with respect to a set of vertices with lists of size 1.

Lemma 8. *Let (G, L) be a list-obstruction, and let $X \subseteq V(G)$ be a vertex subset such that $|L(x)| = 1$ for every $x \in X$. Let L' be the list obtained by updating with respect to X three times. Let (G', L') be a minimal list-obstruction induced by (G, L'). Then there exists a minimal list-obstruction induced by (G, L), say (G'', L), with $|V(G'')| \leq 36|V(G')|$.*

Proof. Let $Y = X_1$ and $Z = X_2$, as in the definition of updating i times. We choose sets R, S, and T as follows.

- For every $v \in V(G') \setminus (X \cup Y \cup Z)$, define $R(v)$ to be a minimum subset of $(X \cup Y \cup Z) \cap N(v)$ such that $\bigcup_{s \in R(v)} L'(s) = L(v) \setminus L'(v)$, and let $R = \bigcup_{v \in V(G') \setminus (X \cup Y \cup Z)} R(v)$.
- For every $v \in (V(G') \cup R) \cap Z$, define $S(v)$ to be a minimum subset of $(X \cup Y) \cap N(v)$ such that $\bigcup_{s \in S(v)} L'(s) = L(v) \setminus L'(v)$, and let $S = \bigcup_{v \in (V(G') \cup R) \cap Z} S(v)$.
- For every $v \in (V(G') \cup R \cup S) \cap Y$, define $T(v)$ to be a minimum subset of $X \cap N(v)$ such that $\bigcup_{s \in T(v)} L'(s) = L(v) \setminus L'(v)$, and let $T = \bigcup_{v \in (V(G') \cup R \cup S) \cap Y} T(v)$.

Clearly, $|R(v)| \leq 3$ for every $v \in V(G') \setminus (X \cup Y \cup Z)$, $|S(v)| \leq 2$ for every $v \in (V(G') \cup R) \cap Z$, and $|T(v)| \leq 2$ for every $v \in (V(G') \cup R \cup S) \cap Y$. Let $P = R \cup S \cup T \cup V(G')$, and observe that $|P| \leq (1+3+8+24)|V(G')| = 36|V(G')|$. It remains to prove that $(G|P, L)$ is not colorable. Suppose there exists a coloring c of $(G|P, L)$. Then c is not a coloring of (G', L'), and since $V(G') \subseteq P$, it follows that there exists $w \in V(G')$ such that $c(w) \notin L'(w)$. Therefore $c(w) \in L(w) \setminus L'(w)$.

We discuss the case when $v \in V(G') \setminus (X \cup Y \cup Z)$, as the cases of $v \in (V(G') \cup R) \cap Z$ and $v \in (V(G') \cup R \cup S) \cap Y$ are similar. We can choose $m \in R(w)$ such that $L'(m) = \{c(w)\}$ and one of the following holds.

- $m \in X$, and thus $L(m) = L'(m) = \{c(w)\}$.
- $m \in Y$, and thus for any $i \in L(m) \setminus L'(m)$ there exists $n_i \in T(m)$ such that $L(n_i) = \{i\}$.
- $m \in Z$, and thus for any $i \in L(m) \setminus L'(m)$ there exists $n_i \in S(m)$ such that either $L(n_i) = \{i\}$ or, for any $j \in L(n_i) \setminus \{i\}$, there exists $l_j \in T(n_i)$ with $L(l_j) = \{j\}$.

In all cases it follows that $c(m) = c(w)$, in contradiction to the fact that m and w are adjacent. This completes the proof. $\square$

Let A be a subset of $V(G)$ and L be a list system; let c be an L-coloring of $G|A$, and let L_c be the list system obtained by setting $L_c(v) = \{c(v)\}$ for every $v \in A$ and updating with respect to A three times; we say that L_c is *obtained from L by precoloring A (with c) and updating three times.* If for every c, $|L_c(v)| \leq 2$ for every $v \in V(G)$, we call A a *semi-dominating set* of (G, L).

If $L(v) = \{1, 2, 3\}$ for every $v \in G$ and A is a semi-dominating set of (G, L), we say that A is a *semi-dominating set* of G. Note that a dominating set is always a semi-dominating set. Last we prove a lemma for the case when G has a bounded size semi-dominating set.

Lemma 9. *Let (G, L) be a minimal list-obstruction, and assume that G has a semi-dominating set A with $|A| \leq t$. Assume also that if (G', L') is a minimal obstruction where G' is an induced subgraph of G, and L' is a subsystem of L with $|L'(v)| \leq 2$ for every v, then $|V(G')| \leq m$. Then $|V(G)| \leq 36 \cdot 3^t \cdot m + t$.*

Proof. Consider all possible L-colorings $c_1, \ldots, c_s$ of A; then $s \leq 3^t$. For each i, let L_i be the list system obtained by updating with respect to A three times. Then $|L_i(v)| \leq 2$ for every $v \in V(G)$ and for every $i \in \{1, \ldots, s\}$. Now Lemma 7 together with Lemma 8 imply that $|V(G)| \leq 36 \cdot 3^t \cdot m + t$. This completes the proof. □

4.3 Proof of Lemma 6

We start with several claims that deal with vertices that have a special structure in their neighborhood.

Claim 6. *Let G be a graph, and let $X \subseteq V(G)$ be connected. If $v \in V(G) \setminus X$ is mixed on X, then there exist adjacent $x_1, x_2 \in X$ such that v is adjacent to x_1 and non-adjacent to x_2.*

Proof. Since v is mixed on X, both the sets $N(v) \cap X$ and $X \setminus N(v)$ are non-empty, and since X is connected, there exist $x_1 \in N(v) \cap X$ and $x_2 \in X \setminus N(v)$ such that x_1 is adjacent to x_2. This proves Claim 6. □

Claim 7. *Let G be a P_6-free graph and let $v \in V(G)$. Suppose that $G|N(v)$ is a connected bipartite graph with bipartition (A, B). Let G' be obtained from $G \setminus (A \cup B)$ by adding two new vertices a, b with $N_{G'}(a) = \{b\} \cup \bigcup_{u \in A}(N_G(u) \cap V(G'))$ and $N_{G'}(b) = \{a\} \cup \bigcup_{u \in B}(N_G(u) \cap V(G'))$. Then G' is P_6-free.*

Proof. Suppose Q is a P_6 in G. Then $V(Q) \cap \{a, b\} \neq \emptyset$. Observe that if both a and b are in $V(Q)$, then $v \notin V(Q)$. If only one vertex of Q, say q, has a neighbor in $\{a, b\}$, say a, then we get a P_6 in G by replacing a with a neighbor of q in A, and, if $b \in V(Q)$, replacing b with v. Thus we may assume that two vertices q, q' of Q have a neighbor in $\{a, b\}$. If q and q' have a common neighbor $u \in A \cup B$, then $G|((V(Q) \setminus \{a, b\} \cup \{u\}))$ is a P_6 in G, a contradiction. So no such u exists, and in particular $v \notin V(Q)$. Let Q' be an induced path from q to q' with $V(Q') \setminus \{q, q'\} \subseteq A \cup B \cup \{v\}$, meeting only one of the sets A, B if possible. Then $G|((V(Q) \setminus \{a, b\}) \cup V(Q'))$ is a P_6 in G, a contradiction. This proves Claim 7. □

In the remainder of this section G is a P_6-free graph.

Claim 8. *Let (G, L) be a minimal list-obstruction. Let A be a stable set in G. Let U be the set of vertices of $V(G) \setminus A$ that are not mixed on A, and let $k = |V(G) \setminus (A \cup U)|$. Then $|A| \leq 7 \times 2^k$.*

Proof. Partition A by the adjacency in $V(G)\setminus(A\cup U)$ and by lists; more precisely let $A = A_1 \cup A_2 \cup \ldots \cup A_{7\times 2^k}$ such that for every i and for every $x, y \in A_i$, $N(x) \setminus (A \cup U) = N(y) \setminus (A \cup U)$ and $L(x) = L(y)$. Since A is stable and no vertex of U is mixed on A, it follows that for every $x, y \in A_i$, $N(x) = N(y)$. We claim that $A_i \leq 1$ for every i. Suppose for a contradiction that there exist $x, y \in A_i$ with $x \neq y$. By the minimality of (G, L), $G \setminus x$ is L-colorable. Since $N(x) = N(y)$ and $L(x) = L(y)$, we deduce that G is L-colorable by giving x the same color as y, a contradiction. This proves Claim 8. □

Claim 9. *Let (G, L) be a minimal list-obstruction. Let H be an induced subgraph of G such that H is connected and bipartite. Let (A, B) be the bipartition of H, and let $u \in V(G)$ be complete to $A \cup B$. Let U be the set of all vertices in $V(H) \setminus (A \cup B)$ that are not mixed on A. Let $K = V(G) \setminus (A \cup B \cup U)$ and $k = |K|$. Then $|A| \leq 7 \cdot 2^{7\cdot 2^k + k}$.*

Proof. We partition A according to the adjacency in K and the lists of the vertices. More precisely, let $A = A_1 \cup A_2 \cup \ldots \cup A_{7\cdot 2^k}$ such that for any i and for any $x, y \in A_i$, $N(x) \cap K = N(y) \cap K$ and $L(x) = L(y)$. Analogously, let $B = B_1 \cup B_2 \cup \ldots \cup B_{7\cdot 2^k}$.

Next, for $i = 1, \ldots, 7 \cdot 2^k$, we partition $A_i = A_i^1 \cup \ldots \cup A_i^{7\cdot 2^k}$ according to the sets $B_1, B_{7\cdot 2^k}$ in which they have neighbors. More precisely, for any t and any $x, y \in A_i^t$, $N(x) \cap B_j \neq \emptyset$ if and only if $N(y) \cap B_j \neq \emptyset$, $j = 1, \ldots, 7 \cdot 2^k$. We partition the sets in B analogously. We claim that $A_i^j \leq 1$. Suppose that there exist $x, y \in A_i^j$ with $x \neq y$. Let $N = N(x) \cap B$. Let C be the component of $H \setminus x$ with $y \in C$. Let N_1 be the set of vertices of N whose unique neighbor in H is x, and let $N_2 = N \setminus N_1$.

Suppose first that there is a vertex $s \in N_2 \setminus C$. Since y is not dominated by x, there exists $t \in (B \cap N(y)) \setminus N(x)$. Since $s \notin C$, it follows that s and t have no common neighbor in H, and so, since G is P_6-free, there is a 5-vertex path Q in H with ends s and t; let the vertices of Q be q_1-…-q_5, where $s = q_1$ and $t = q_5$. Since $s \notin C$, it follows that $q_2 = x$. Since y-t-q_4-q_3-x-s is not a P_6 in G, it follows that y is adjacent to q_3, and so we may assume that $q_4 = y$. Let $p \in A \setminus \{x\}$ be a neighbor of s. Since p-s-x-q_3-y-t is not a P_6, it follows that p has a neighbor in $\{t, q_3\}$, contrary to the fact that $s \notin C$. This proves that $N_2 \subseteq C$.

Observe that $C \cap N_1 = \emptyset$. By the minimality of (G, L), there is a coloring c of $(G \setminus (N_1 \cup \{x\}), L)$. Since u is complete to $V(C)$, it follows that $C \cap A$ and $C \cap B$ are both monochromatic. We now describe a coloring of G. Color x with $c(y)$. Let $n_1 \in N_1$ be arbitrary. Since $x, y \in A_i^j$, there exists n_1' in B such that n_1' is adjacent to y, $L(n_1) = L(n_1')$, and n_1, n_1' have the same neighbors in K. Now color n_1 with $c(n_1')$. Repeating this for every vertex of N_1 produces a coloring of (G, L) a contradiction. This proves Claim 9. □

Claim 10. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let (G, L) be a minimal list-obstruction. Let $D_1, \ldots, D_t$ be connected subsets of $V(G)$ with the following properties.*

- $|D_i| > 1$ *for every* i,
- $D_1, \dots, D_t$ *are pairwise disjoint and anticomplete to each other,*
- *for each* i *there is a set* $U_i \subseteq V(G)$ *such that* U_i *is complete to* D_i,
- D_i *is anticomplete to* $V(G) \setminus (U_i \cup D_i)$,
- *for every* $i \in \{1, \dots, t\}$ *there is* $c_i \in \{1, 2, 3\}$ *such that* $c(u) \neq c_i$ *for every coloring* c *of* (G, L) *and for every* $u \in U_i$, *and*
- $V(G) \neq D_1 \cup U_1$.

Then there is an induced subgraph F *of* G *such that* $V(G) \setminus \bigcup_{i=1}^{t} D_i \subseteq V(F) \subseteq V(G)$ *and a list system* L' *such that*

- $|L'(v)| \leq 2$ *for every* $v \in V(F) \cap \bigcup_{i=1}^{t} D_i$,
- $L'(v) = L(v)$ *for every* $v \in V(F) \setminus \bigcup_{i=1}^{t} D_i$, *and*
- (F, L') *is a minimal list-obstruction, and* $|V(G)| \leq q(|V(F)|)$.

Proof. Write $D = \bigcup_{i=1}^{t} D_i$. Since $V(G) \neq U_1 \cup D_1$, it follows from the minimality of (G, L) that $(G|(D_i \cup U_i), L)$ is colorable for every i, and so each $G|D_i$ is bipartite. Let D_i^1, D_i^2 be the bipartition of $G|D_i$. Then for every i there is a coloring of $(G|D_i, L)$ in which each of the sets D_i^1, D_i^2 is monochromatic, and in particular $\bigcap_{d \in D_i^j} L(d) \neq \emptyset$ for every $i \leq t$ and $j \in \{1, 2\}$.

For every $i \leq t$ and $j \in \{1, 2\}$, let $d_i^j \in D_i^j$ such that d_i^1 is adjacent to d_i^2. Set $L''(d_i^j) = \bigcap_{d \in D_i^j} L(d)$. Let F'' be the graph obtained from G by deleting $D \setminus (\bigcup_{i=1}^{m} \{d_i^1, d_i^2\})$. Set $L''(v) = L(v)$ for every $v \in V(F'') \setminus D$.

We may assume that there exists $s \in \{0, 1, \dots, t\}$ such that $|L''(d_i^1)| = 3$ for every $i \leq s$, and that for $i \in \{s+1, \dots, t\}$ and $j \in \{1, 2\}$, $|L''(d_i^j)| \leq 2$.

Let F' be obtained from F'' by deleting $\{d_1^1, \dots, d_s^1\}$. For $i \in \{1, \dots, s\}$, let $L'(d_i^2) = L''(d_i^1) \setminus \{c_i\}$. Let $L'(v) = L''(v)$ for every other vertex of F'. Then $V(G) \setminus D \subseteq V(F')$, $L'(v) = L(v)$ for every $v \in V(F') \setminus D$, and $|L'(v)| \leq 2$ for $v \in V(F') \cap D$.

We claim that (F', L') is not colorable. Suppose c' is a coloring of (F', L'). We construct a coloring of (G, L). Set $c(v) = c'(v)$ for every $v \in V(G) \setminus D$. For $i \in \{1, \dots, t\}$ and for every $d \in D_i^2$, set $c(d) = c'(d_i^2)$. Then $c(d) \neq c_i$ if $i \leq s$. For $i \in \{1, \dots, s\}$, set $c(d) = c_i$ for every $d \in D_i^1$, and for $i \in \{s+1, \dots, t\}$, set $c(d) = c'(d_i^1)$ for every $d \in D_i^1$. Now c is a coloring of (G, L), a contradiction.

Let (F, L') be a minimal obstruction induced by (F', L'). We claim that $V(G) \setminus D \subseteq V(F)$. Suppose not, let $v \in V(G) \setminus (D \cup V(F))$. It follows from the minimality of G that $(G \setminus v, L)$ is colorable. Let c be such a coloring. Set $c'(v) = c(v)$ for every $v \in V(F) \setminus D$. Let $i \in \{1, \dots, t\}$. If $U_i \neq \{v\}$, then each of these sets U_i, D_i^1, D_i^2 is monochromatic in c. If $U_i = \{v\}$, then D_i is anticomplete to $V(G) \setminus (D_i \cup \{v\})$, and we may assume that each of D_i^1, D_i^2 is monochromatic in c. Now set $c'(d_i^2)$ to be the unique color that appears in D_i^2, and for $i \in \{s+1, \dots t\}$, set $c'(d_i^1)$ to be the unique color that appears in D_i^1. Then c' is a coloring of $(F \setminus v, L)$, a contradiction. Now Claim 10 follows from at most $t \leq |V(F)|$ applications of Lemma 9. □

Claim 11. *Let $x \in V(G)$ such that $L(x) = \{1,2,3\}$ and let $U, W \subseteq V(G)$ be disjoint non-empty sets such that $N(x) = U \cup W$ and U is complete to W. Then (possibly exchanging the roles of U and W) either*

1. *there is a path P with $|V(P)| = 4$, such that the ends of P are in U, no internal vertex of P is in U, and $V(P) \cap W = \emptyset$, or*
2. *there exist distinct $i, j \in \{1,2,3\}$ and vertices $u_i, u_j \in U$ and w_i, w_j such that for every $k \in \{i, j\}$*
    - *$k \in L(u_k)$,*
    - *$|L(w_k) \cap \{i, j\}| = 1$,*
    - *there is a path P_k from u_k to w_k,*
    - *if $|V(P_k)|$ is even, then $k \notin L(w_k)$,*
    - *if $|V(P_k)|$ is odd, then $k \in L(w_k)$*

    and $V(P_i)$ is anticomplete to $V(P_j)$.

Proof. Since we may assume that $G \neq K_4$, it follows that U and W are both stable sets. By the minimality of G, there is an L-coloring of $G \setminus x$, say c. Since G does not have an L-coloring, we may assume that there exist $u_1, u_2 \in U$ such that $c(u_i) = i$. Then $c(w) = 3$ for every $w \in W$, and $c(u) \in \{1,2\}$ for every $u \in U$. For $i = 1,2$ let $U_i = \{u \in U : c(u) = i\}$. Let $V_{12} = \{v \in V(G) : c(v) \in \{1,2\}\}$, and let $G_{12} = G|V_{12}$. Suppose first that some component of G_{12} meets both U_1 and U_2. Let P be a shortest path from U_1 to U_2 in G_{12}. Then $|V(P)| = 4$ since G is P_6-free. Moreover, $V(P) \cap W = \emptyset$, and no interior vertex of P is in U, and 11.1 holds.

So we may assume that no component of G_{12} meets both U_1 and U_2. For $i = 1,2$ let V_i be the union of the components of G_{12} that meet U_i. Then V_1 is anticomplete to V_2. If we can exchange the colors 1 and 2 on every component that meets U_1, then doing so produces a coloring of $G \setminus x$ where every vertex of U is colored 2 and every vertex of W is colored 3; this coloring can then be extended to G, which is a contradiction. So there is a component D that meets U_1 and where such an exchange is not possible. Let (D_1, D_2) be a bipartition of D. We may assume that $U_1 \cap D_2 = \emptyset$. Let us say that $d \in D$ is *deficient* if either $d \in D_1$ and $2 \notin L(d)$, or $d \in D_2$ and $1 \notin L(d)$. Then there is a deficient vertex in D. Let P_1 be a shortest path in D from some vertex $u_1 \in U_1$ to a deficient vertex w_1 of D. Then u_1, w_1, P_1 satisfy the conditions of 11.2. It follows from symmetry that there exist $u_2, w_2 \in V_2$ and a path P_2 from u_2 to w_2 satisfying the condition of 11.2. Since V_1 is anticomplete to V_2, it follows that $V(P_1)$ is anticomplete to $V(P_2)$ and 11.2 holds. □

We also make use of the following result.

Theorem 10 (Camby and Schaudt [3]). *For all $t \geq 3$, any connected P_t-free graph H contains a connected dominating set whose induced subgraph is either P_{t-2}-free, or isomorphic to P_{t-2}.*

Our strategy from now on is as follows. Let (G, L) be a minimal list-obstruction, where G is P_6-free. At every step, we find a subgraph R of G, and consider all possible partial precolorings of R. For each precoloring, we update three times with respect to R, and possibly modify the lists further, to produce a minimal list-obstruction (G', L') where G' is an induced subgraph of G, and $|L(v)| \leq 2$ for every $v \in V(G')$, and such that $|V(G)|$ is bounded from above by a function of $|V(G')|$ (the function does not depend on G, it works for all P_6-free graphs G). Since by Lemma 5 $V(G')$ has bounded size, it follows from Lemma 7 and Lemma 8 that $V(G) \setminus R$ has bounded size. Now we use the minimality of G and the internal structure of R to show that R also has a bounded number of vertices, and so $|V(G)|$ is bounded. Next we present the details of the proof.

Claim 12. *There exists an integer C such that the following holds. Let (G, L) is a minimal list-obstruction, where G is C_5-free. Then $|V(G)| \leq C$.*

Proof. We may assume that $|V(G)| > 8$, and therefore there is no K_4 in G. Let H be as in Theorem 10. Then H is either P_4 or P_4-free. If $|V(H)| \leq 4$, the result follows from Lemma 9, so we may assume that H is P_4-free. Now by a result of [23] $V(H) = A \cup B$, where A is complete to B, and both A and B are non-empty. Choose $a \in A, b \in B$. Define the set S_0 as follows. If there is a vertex c complete to $\{a, b\}$, let $S_0 = \{a, b, c\}$; if no such c exists, let $S_0 = \{a, b\}$. Then no vertex of $V(G)$ is complete to S_0. Let X_0 be the set of vertices of G with a neighbor in S_0, and let $Y_0 = V(G) \setminus (S_0 \cup X_0)$. By Theorem 10 every vertex of Y_0 has a neighbor in X_0. By Lemma 7 and Lemma 8 we may assume that the vertices of S_0 are precolored; let L_1 be the list system obtained from L by updating three times with respect to S_0. Then $|L_1(x)| \leq 2$ for every $v \in X_0$. Let $S'_1 = \{v \in V(G) : |L(v_1)| = 1\}$, and let S_1 be the connected component of S'_1 such that $S_0 \subseteq S_1$. Let X_1 be the set of vertices of $V(G) \setminus S_1$ with a neighbor in S_1, and let $Y_1 = V(G) \setminus (S_1 \cup X_1)$. Then $X_1 \cap S'_1 = \emptyset$, and every vertex of Y_1 has a neighbor in X_1. Since $S_0 \subseteq S_1$, no vertex of G is complete to S_1. For $i, j \in \{1, 2, 3\}$ let $X^1_{ij} = \{x \in X_1 : L_1(x_1) = \{i, j\}\}$.

We now construct the sets S_2, X_2, Y_2. For every $i, j \in \{1, 2, 3\}$ let $U_{i,j}$ be defined as follows. If there is a vertex $u \in X^1_{ij}$ such that there exist $y, z, w \in Y_1$ where $\{y, z, w\}$ is a clique and u has exactly one neighbor in $\{y, z, w\}$, choose such a vertex u with $N(u) \cap Y_1$ maximal, and let $U_{i,j} = \{u\}$. If no such vertex u exists, let $U_{i,j} = \emptyset$. Let ${X^1}'_{i,j}$ be the set of vertices of $X^1_{i,j}$ that are anticomplete to $U_{i,j}$. Let Y'_1 be the set of vertices in Y_1 that are anticomplete to $U_{1,2} \cup U_{1,3} \cup U_{2,3}$.

Next we define $V_{i,j}$ for every $i, j \in \{1, 2, 3\}$. If there is a vertex $v \in {X^1}'_{ij}$ such that there exist adjacent $y, z \in Y'_1$ where v has exactly one neighbor in $\{y, z\}$, choose such a vertex v with $N(v) \cap Y'_1$ maximal, and let $V_{i,j} = \{v\}$. If no such vertex v exists, let $V_{i,j} = \emptyset$.

Now let $S_2 = S_1 \cup \bigcup_{i,j \in \{1,2,3\}} (U_{ij} \cup V_{ij})$. Precolor the vertices of S_2. Observe that S_2 is connected. Let L_3 be the list system obtained from L_1 by updating three times. By Lemma 7 and Lemma 8 we may assume that (G, L_3) is a minimal list-obstruction. Let $S'_3 = \{v \in V(G) : |L(v_1)| = 1\}$, and let S_3 be the connected component of S'_3 such that $S_2 \subseteq S_3$. Let X_3 be the set of vertices of $V(G) \setminus S_3$ with a neighbor in S_3, and let $Y_3 = V(G) \setminus (S_3 \cup X_3)$. Then every vertex of Y_3 has a neighbor in X_3. Since $S_1 \subseteq S_3$, no vertex of G is complete to S_3, and for every $x \in X_3$, $|L(x)| = 2$. For $i, j \in \{1, 2, 3\}$ let $X^3_{ij} = \{x \in X_1 : L_1(x_1) = \{i, j\}\}$. Then X^3_{ij} is anticomplete to $U_{ij} \cup V_{ij}$.

No vertex of X_3 is mixed on an edge of Y_3. (3)

Suppose that there exist $x \in X_3$ and $y, z \in Y_3$ such that y is adjacent to z, and x is adjacent to y and not to z. We may assume that $x \in X^3_{12}$. Then $x \in X^1_{12} \cup Y_1$, and $y, z \in Y_1$. Suppose first that $x \in Y_1$. Then there is $s_3 \in S_3 \setminus S_1$ such that x is adjacent to s_3. Since $y, z \in Y_3$, it follows that s_3 is anticomplete to $\{y, z\}$. Since $s_3 \in S_3 \setminus S_1$, there is a path P from s_3 to some vertex $s'_3 \in S_3$, such s'_3 has a neighbor in S_1, and $V(P) \setminus \{s'_3\}$ is anticomplete to S_1. Then s'_3 is not complete to S_1, and since S_1 is connected, it follows from Claim 6 that there exist $s_1, s'_1 \in S_1$ such that s'_3-s_1-s'_1 is a path. But now z-y-x-s_3-P-s'_3-s_1-s'_1 is a P_6, a contradiction. This proves that $x \notin Y_1$, and therefore $x \in X^1_{12}$.

Since $y, z \in Y_1 \cap Y_3$, it follows that $V_{12} \neq \emptyset$. Let v be the unique element of V_{12}. Then v is non-adjacent to x, y, z. Since x is adjacent to y, and v is non-adjacent to y, it follows from the choice of v that there exists $y' \in Y_1$ such that y' is adjacent to v and not to x. Since v has a neighbor in S_1, and v is not complete to S_1, and since S_1 is connected, Claim 6 implies that there exist $s, s' \in S_1$ such that v-s-s' is a path. Since neither of s'-s-v-y'-y-z and s'-s-v-y'-z-y is a P_6, it follows that y' is either complete or anticomplete to $\{y, z\}$. Suppose first that y' is anticomplete to $\{y, z\}$. Let P be a path from v to x with interior in S_1. Then $|V(P)| \geq 3$. Now y'-v-P-x-y-z is a P_6, a contradiction. This proves that y' is complete to $\{y, z\}$.

Now $\{y', y, z\}$ is a clique in Y_1, and v has exactly one neighbor in it. This implies that $U_{12} \neq \emptyset$. Let u be the unique element of U_{12}. Then u is anticomplete to $\{v, y', y, z\}$. It follows from the maximality of u that there exists $y'' \in Y_1$ such that y'' is adjacent to u and non-adjacent to v. Since u has a neighbor in S_1, and u is not complete to S_1, and since S_1 is connected, Claim 6 implies that there exist $t, t' \in S_1$ such that $u - t - t'$ is a path. Suppose y'' has a neighbor in $\{y', y, z\}$. Since $G \neq K_4$, there exist $q, q' \in \{y', y, z\}$ such that y'' is adjacent to q and not to q'. But now t'-t-u-y''-q-q' is a P_6, a contradiction. This proves that y'' anticomplete to $\{y', y, z\}$. Let P be a path from u to v with interior in S_1. Then $|V(P)| \geq 3$. Now y''-v-P-u-y'-y is a P_6, a contradiction. This proves (3).

For $i, j \in \{1, 2, 3\}$ let X_{ij} be the set of vertices in X^3_{ij} with a neighbor in Y_3.

The sets X_{12}, X_{13}, X_{23} are pairwise complete to each other. (4)

Suppose $x_1 \in X_{12}$ is non-adjacent to $x_2 \in X_{13}$. Since S_3 is connected and both x_1, x_2 have neighbors in S_3, there is a path P from x_1 to x_2 with $V(P) \setminus \{x_1, x_2\} \subseteq S_3$. Since $L_3(x_1) = \{1,2\}$ and $L_3(x_2) = \{1,3\}$, it follows that no vertex of S_3 is adjacent to both x_1 and x_2, and so $|V(P)| \geq 4$. Let $y_i \in Y_3$ be adjacent to x_i. If $|V(P)| > 4$ or $y_1 \neq y_2$, then y_1-x_1-P-x_2-y_2 contains a path with at least six vertices, a contradiction. So $|V(P)| = 4$ and $y_1 = y_2$. But now y_1-x_1-P-x_2-y_1 is a C_5 in G, again a contradiction. This proves (4).

Let $D_1, \ldots, D_t$ be the components of Y_3 that have size at least two. Moreover, let $Y' = \bigcup_{i=1}^t D_i$.

> *There is an induced subgraph F of G with $V(G) \setminus Y' \subseteq V(F)$ and a list system L' such that*
>
> - $|L'(v)| \leq 2$ *for every* $v \in V(F) \cap Y'$,
> - $L'(v) = L(v)$ *for every* $v \in V(F) \setminus Y'$, *and*
> - (F, L') *is a minimal list-obstruction, and* $|V(G)| \leq q(|V(F)|)$ *for a function* $q : \mathbb{N} \to \mathbb{N}$.
>
> (5)

For $in \in \{1, \ldots, t\}$ let U_i be the set of vertices of X_3 with a neighbor in D_i. It follows from Claim 6 and (3) that U_i is complete to D_i. Since each D_i contains an edge, (4) implies that each U_i is a subset of one of $X^3_{1,3}, X^3_{1,3}, X^3_{2,3}$. Therefore there exists $c_i \in \{1,2,3\}$ such that for every $u \in U_i$, $c_i \notin L(u)$. Now (5) follows from Claim 10. This proves (5).

Let (F, L') be as in (5). Since our goal is to prove that (G, L_3) induces a minimal obstruction of bounded size, it is enough to show that $|V(F)|$ has bounded size (where the bound is independent of G). Therefore we may assume that $G = F$ and $L_3 = L'$, and in particular that $|L_3(v)| \leq 2$ for every $v \in Y'$.

Let $Y = Y_3 \setminus Y'$. Then the set Y is stable, $N(y) \subseteq X_{12} \cup X_{13} \cup X_{23}$ for every $y \in Y$, and for $v \in V(G)$, if $|L_3(v)| = 3$, then $v \in Y$. Moreover, if $y \in Y$ has $|L_3(y)| = 3$ and $N(y) \subseteq X_{ij}$ for some $i, j \in \{1,2,3\}$, then (G, L_3) is colorable if and only if $(G \setminus y, L_3)$ is colorable, contrary to the fact that (G, L_3) is a minimal list obstruction. Thus for every $y \in Y$ with $|L_3(y)| = 3$, $N(y)$ meets at least two of X_{12}, X_{13}, X_{23}. By (4) it follows that the sets X_{12}, X_{13}, X_{23} are pairwise complete to each other, and therefore no $v \in Y$ has neighbors in all three of X_{12}, X_{13}, X_{23}.

Next we define a refinement $\mathcal{L}$ of L_3.

- If exactly one of X_{12}, X_{13}, X_{23} is non-empty, then $\mathcal{L} = \{L_3\}$.
- If at least two of X_{12}, X_{13}, X_{23} are non-empty and some X_{ij} contains two adjacent vertices a, b, let L' be the list obtained by precoloring $\{a, b\}$ and updating three times, and let $\mathcal{L} = \{L'\}$.

- Now assume that at least two of X_{ij} are non-empty, and each of X_{ij} is a stable set. Observe that in this case, in every coloring of G at least one of X_{ij} is monochromatic. For all i, j such that $X_{ij} \neq \emptyset$ and for all $k \in \{i, j\}$ add to $\mathcal{L}$ the list system L^i_{ij}, where $L^i_{ij}(x) = \{i\}$ for all $x \in X_{ij}$ and $L^i_{ij}(v) = L_3(v)$ for all $v \in V(G) \setminus X_{ij}$, and we updated three times with respect to X_{ij}.

Now $\mathcal{L}$ is a refinement of L and satisfies the hypotheses of Lemma 7. We claim that for every $L' \in \mathcal{L}$ there exist $i, j \in \{1, 2, 3\}$ such that after the first step of updating $|L'(x)| = 1$ for all $x \in (X_{12} \cup X_{13} \cup X_{23}) \setminus X_{ij}$,

In view of (4), this is clear if some X_{ij} is not stable or if only one of the sets X_{12}, X_{13}, X_{23} is non-empty. So we may assume that all X_{ij} are stable, and at least two are non-empty. Let $L' = L^i_{ij}$. Then $L'(x) = \{i\}$ for all $x \in X_{ij}$. Let $k \in \{1, 2, 3\} \setminus \{i, j\}$, then by (4) after the first step of updating $L'(x) = \{k\}$ for every $x \in X_{ik}$. Thus after the first step of updating only one of the sets X_{ij} may contain vertices with lists of size two.

Since every $y \in Y$ with $|L_3(y)| = 3$ has neighbors in at least two of X_{12}, X_{13}, X_{23}, it follows that after the second step of updating all vertices of Y have lists of size at most two, and so for all $L' \in \mathcal{L}$ we have that $|L'(v)| \leq 2$ for all $v \in V(G)$. By Lemma 5, each of (G, L') induces a minimal obstruction with at most 100 vertices. Applying the Lemma 7 and Lemma 8, we deduce that $|V(G) \setminus (X_{12} \cup X_{13} \cup X_{23})|$ depends only on the sizes of the minimal obstructions induced by (G, L'), and therefore does not depend on G. Now, since each of X_{ij} is a stable set, Claim 8 implies that $|V(G)|$ is bounded, and Claim 12 follows. □

In the remainder of this proof we deal with minimal list-obstructions (G, L) containing a C_5, by taking advantage of the structure that it imposes. Let C be a C_5 in G, say $C = c_1\text{-}c_2\text{-}c_3\text{-}c_4\text{-}c_5\text{-}c_1$. Let $X(C)$ be the set of vertices of $V(G) \setminus V(C)$ that have a neighbor in C, let $Y(C)$ be the set of vertices of $V(G) \setminus (V(C) \cup X(C))$ that have a neighbor in X, and let $Z(C) = V(G) \setminus (V(C) \cup X(C) \cup Y(C))$.

Claim 13. *Assume that $|V(G)| \geq 7$. Then the following assertions hold.*

1. *For every $x \in X(C)$ there exist indices $i, j \in \{1, \dots, 5\}$ such that $x\text{-}c_i\text{-}c_j$ is an induced path.*
2. *No vertex of $Y(C)$ is mixed on an edge of $G|Z(C)$.*
3. *If $v \in X(C)$ is mixed on an edge of $G|(Y(C) \cup Z(C))$, then the set of neighbors of v in C is not contained in a 3-vertex path of C.*
4. *If $v \in X(C)$ has a neighbor in $Y(C)$, then the set of neighbors of v in C is not contained in a 2-vertex path of C.*
5. *If $z \in Z(C)$ and $u, t \in N(z) \cap Y(C)$ are non-adjacent, then no vertex of $X(C)$ is mixed on $\{u, t\}$.*

6. *Let D be a component of $Z(C)$ with $|D| = 1$, and let N be the set of vertices of $Y(C)$ with a neighbor in D. Then either N is anticonnected, or $N = U \cup W$ where U and W are stable sets, and U is complete to W.*

7. *Let D be a component of $Z(C)$ with $|D| > 1$, and let N be the set of vertices of $Y(C)$ with a neighbor in D. Then D is bipartite, and N is a stable set complete to D.*

Proof. Since $|V(G)| \geq 7$, no vertex is complete to $V(C)$, as that would lead to a list-obstruction on 6 vertices. Thus, the first assertion follows from the fact that G is connected and Claim 6.

Next we prove the second assertion. Suppose that $u \in Y(C)$ is mixed on the edge st with $s, t \in Z(C)$, namely u is adjacent to s and not to t. Let $b \in N(u) \cap X$ and $i, j \in \{1, \ldots, 5\}$ be such that b-c_i-c_j is an induced path (as in Claim 13.1). Then t-s-u-b-c_i-c_j is a P_6, a contradiction.

To see the third assertion, suppose that $x \in X$ is adjacent to $t \in Y$ and non-adjacent to $s \in Y \cup Z$, where t is adjacent to s, and suppose that $N(x) \cap V(C) \subseteq \{c_1, c_2, c_3\}$. We may assume that x is adjacent to c_3. Then c_5-c_4-c_3-x-t-s is a P_6 in G, a contradiction.

To prove the fourth statement, we may assume that $x \in X$ is adjacent to c_1 and to $y \in Y$, and non-adjacent to c_2, c_3 and c_4. Now y-x-c_1-c_2-c_3-c_4 is a P_6 in G, a contradiction.

To prove the fifth statement, suppose that $w \in X(C)$ is adjacent to u and non-adjacent to t. By Claim 13.1 there exists i, j such that w-c_i-c_j is an induced path. Then t-z-u-w-c_i-c_{i+1} is a P_6, a contradiction.

Next let $D = \{v\}$ be a component of Z, then $N(v) \subseteq Y$, and Claim 13.6 follows immediately from the fact that there is no K_4 in G.

Finally let D be a component of $Z(C)$ with $|D| > 1$. By Claim 13.2 N is complete to D. Since there is no K_4 in G, it follows that D is bipartite and N is a stable set. This proves Claim 13.7. □

By Lemma 7 and Lemma 8 we may assume that in (G, L) the vertices of C are precolored, and that we have updated three times with respect to $V(C)$. We may assume that $|V(G)| > 8$.

Claim 14. *There is an induced subgraph F of G with $V(G) \setminus Z(C) \subseteq V(F)$, a list system L' and a function $q : \mathbb{N} \to \mathbb{N}$ such that such that*

- $|L'(v)| \leq 2$ *for every* $v \in V(F) \cap Z(C)$,
- $L'(v) = L(v)$ *for every* $v \in V(F) \setminus Z(C)$, *and*
- (F, L') *is a minimal list-obstruction, and* $|V(G)| \leq q(|V(F)|)$.

Proof. We write $X = X(C)$, $Y = Y(C)$ and $Z = Z(C)$. Let $D_1, \ldots, D_t$ be components of Z with $|D_i| \geq 2$. Write $D = \bigcup_{i=1}^{t} D_i$. For every i let U_i be the set of vertices of Y with a neighbor in D_i. By Claim 13.7 for every i, U_i is a stable set complete to D_i. By Claim 13.3 every $x \in X$ with a neighbor in U_i has neighbors of two different colors in $V(C)$, and so every such x has list of size one after the first step of updating. Now by Claim 13.5 and since we have updated three times, it follows that for every i there exists $c_i \in \{1,2,3\}$ such that for every $u \in U_i$, $c_i \notin L(u)$. By Claim 10 there exist an induced subgraph F of G with $V(G) \setminus Z(C) \subseteq V(F)$ and a list system L' such that

- $|L'(v)| \leq 2$ for every $v \in V(F) \cap D$,
- $L'(v) = L(v)$ for every $v \in V(F) \setminus Z(C)$, and
- (F, L') is a minimal list-obstruction, and $|V(G)| \leq q(|V(F)|)$ for a function $q : \mathbb{N} \to \mathbb{N}$.

It remains to show that that $L'(v) \leq 2$ for every $v \in V(F) \cap Z$. Suppose there is $v \in V(F) \cap Z$ with $|L(v)| = 3$. Let D be the component of Z containing v. Then $D = \{v\}$. If $N(v)$ is anticonnected, then by Claim 13.5 every $x \in X$ with a neighbor in $N(v)$ dominates v, contrary to the fact that (F, L') is a minimal list-obstruction. So by Claim 13.6 $N(v) = U \cup W$, both U and W are stable sets, and U is complete to W.

We now apply Claim 11. We may assume that if Claim 11.1 holds then $p_1, p_4 \in U$, and if Claim 11.2 holds, then $u_1, u_2 \in U$. We show that in both cases some vertex $t \in V(G) \setminus U$ is mixed on U. If Claim 11.1 holds, we can take $t = p_1$, so we may assume that Claim 11.2 holds. We may assume that $u_1 = w_1$ and $u_2 = w_2$, for otherwise some vertex of $V(P_1) \cup V(P_2)$ is mixed on U. By Claim 13.3 and Claim 13.5, and since we have updated, it follows that there exists $i \in \{1,2,3\}$ such that for every $u \in U$, $i \notin L(u)$. Since $|L(v)| = 3$, and we have updated three times, it follows that after the second step of updating all $u \in U$ have exactly the same list, and this list has size two. Since $u_1 = w_1$ and $u_2 = w_2$, it follows that the lists of u_1 and u_2 changed and became different in the third step of updating, and so some vertex $V(G) \setminus U$ is mixed on U, as required. This proves the claim. Let t be a vertex of $V(G) \setminus U$ that is mixed on U. By Claim 13.5, it follows that $t \in Y \cup Z$.

First we show that if $y \in Y \setminus (U \cup W)$ has a neighbor $u \in U$, and $x \in X$ is adjacent to y, then x is complete to U. Suppose not, let i be such that $x - c_i - c_{i+1}$ is a path (such i exists by Claim 13.1). By Claim 13.5, x is anticomplete to U. Then v-u-y-x-c_i-c_{i+1} is a P_6, a contradiction. This proves the claim.

Now we claim that $Y \setminus (U \cup W)$ is anticomplete to $U \cup W$. Suppose $y \in Y$ has a neighbor $u \in U$, and let $x \in X$ be adjacent to y. Then x is complete to U. Since x does not dominate v, it follows that x has a non-neighbor $w \in W$, and again by the previous claim, y is anticomplete to W. Let $x_1 \in X$ be adjacent to w. By Claim 13.5 x_1 is complete to W. Since x_1 does not dominate v, it follows that x_1 has a non-neighbor in U, and so by Claim 13.5 x_1 is anticomplete to U. By the previous claim, x_1 is non-adjacent to y. Let i be such that x_1-c_i-c_{i+1} is a path (such i exists by Claim 13.1). Now c_{i+1}-c_i-x_1-w-u-y is a P_6, a contradiction. This proves the claim, and in particular we deduce that $t \in Z$.

Since t is mixed on U, there exists an edge a, b with one end in U and the other in W, such that t is adjacent to b and not to a. Let $x \in X$ be adjacent to a. Since x does not dominate v, we deduce that x is not complete to $U \cup W$, and so by Claim 13.5 x is non-adjacent to b. Let i be such that x-c_i-c_{i+1} is a path (as in 13.1). Now t-b-a-x-c_i-c_{i+1} is a P_6, a contradiction. This proves Claim 14. □

Let (F, L') be as in Claim 14. Since our goal is to prove that $|V(G)|$ is bounded, it is enough to prove that $|V(F)|$ is bounded, and so we may assume that $G = F$, $L = L'$, and in particular $|L(v)| \leq 2$ for every $v \in Z(C)$.

Claim 15. *Assume that in the precoloring of C, c_2 and c_5 receive the same color, say j. Let $A = \{a \in X(C) : N(a) \cap V(C) = \{c_2, c_5\}\}$ and $W = \{y \in Y : N(y) \cap X \subseteq A\}$. Let D be a component of W such that there exists a vertex with list of size 3 in D, and let N be the set of vertices of A with a neighbor in D. Then D is complete to N, and either*

- *D is anticomplete to $V(G) \setminus (D \cup N)$, or*
- *there exists vertices $d \in D$ and $v \in N(d)$ such that precoloring d, v with distinct colors and updating with respect to the set $\{d, v\}$ three times reduces the list size of all vertices in W to at most two.*

Proof. By Claim 13.3 no vertex of $X(C)$ is mixed on D, and so N is complete to D. Also by Claim 13.3 W is anticomplete to $Z(C)$.

Let $d \in D$, and let $v \in N(d) \setminus (N \cup D)$. Since $D \subseteq W$, it follows that $v \in Y(C)$. By Claim 13.3, $N(v) \cap A = N$. Let $x \in N(v) \cap (X(C) \setminus A)$. Then $N(x) \cap V(C)$ are not contained in a 3-vertex path of C, and therefore $|L(x)| = 1$.

First suppose that $N(x) \cap \{c_2, c_5\} \neq \emptyset$. Then $j \notin L(x)$. We precolor $\{v, d\}$ and update with respect to the set $\{v, d\}$ three times. We may assume that the precoloring of $G|(V(C) \cup \{v, d\})$ is proper. Since $\{v, d\}$ is complete to N and not both v, d are precolored j, it follows that $|L(n)| = 1$ for every $n \in N$, and $|L(u)| \leq 2$ for every $u \in W$ such that u has a neighbor N. Suppose there is $t \in W$ with $|L(t)| = 3$. Then t is anticomplete to $\{v, d\}$. Since $t \in W$, there exists $s \in A$ adjacent to t, and so s is not complete to $\{v, d\}$. Since $s \in A$, it follows from Claim 13.3 that s is not mixed on the edge vd, and so s is anticomplete to $\{v, d\}$. Since $|L(t)| = 3$, it follows that $L(s) = \{1, 2, 3\} \setminus \{j\}$, and so s is non-adjacent to x (since we have updated three times with respect to $V(C)$). Assume by symmetry that c_2 is adjacent to x, then t-s-c_2-x-v-d is a P_6, a contradiction.

Therefore we may assume that $N(x) \cap \{c_2, c_5\} = \emptyset$, and so $N(x) = \{c_1, c_3, c_4\}$. It follows that $L(x) = \{j\}$, and consequently $L(v) \subseteq \{1, 2, 3\} \setminus \{j\}$. If $D = \{d\}$, then $|L(d)| = 3$; but $L(u) \subseteq \{1, 2, 3\} \setminus \{j\}$ for all $u \in N(d)$, which contradicts the fact that (G, L) is a minimal list obstruction. Therefore we may assume there exists $d' \in N(d) \cap D$. Since G is not a K_4, d' is not adjacent to v. But now c_5-c_1-x-v-d-d' is a P_6, a contradiction. □

Claim 16. *Assume that there is a vertex $c_1' \in V(G)$ adjacent to c_1, c_2, c_5 and non-adjacent to c_3, c_4. Then $|V(G)|$ is bounded from above (and the bound does not depend on G).*

Proof. By Lemma 7 and Lemma 8, we can precolor the vertices of $V(C) \cup \{c_1'\}$ and update with respect to $V(C) \cup \{c_1'\}$ three times. By symmetry, we may assume that $L(c_1) = \{1\}$, $L(c_1') = L(c_3) = \{2\}$, $L(c_2) = L(c_5) = \{3\}$ and $L(c_4) = \{1\}$. Let $C' = c_1'\text{-}c_2\text{-}c_3\text{-}c_4\text{-}c_5\text{-}c_1'$. We write $X = X(C)$, $X' = X(C')$, and define the sets Y, Y', Z, and Z' in a similar manner. We abuse notation an denote the list system thus obtained by L. Recall that (G, L) is a minimal list-obstruction.

Let A be the set of all vertices $a \in X \cup X'$ for which $N(a) \cap \{c_1, c_1', c_2, c_3, c_4, c_5\} = \{c_2, c_5\}$. Let W be the set of vertices $y \in Y \cap Y'$ such that $N(y) \cap (X \cup X') \subseteq A$. Since we have updated $|L(x)| \leq 2$ for every $x \in X \cup X'$. By Claim 14 applied to C', we may assume that $|L(z)| \leq 2$ for every $z \in Z \cup Z'$. Thus if $|L(v)| = 3$ then $v \in Y \cap Y'$, and an easy case analysis shows that $v \in W$. By Lemma 5 we may assume that $W \neq \emptyset$. Let $D_1, \ldots, D_t$ be the components of W that contain vertices with lists of size three. Suppose first that $|D_i| = \{d\}$ for some i. Then, letting c be a coloring of $G \setminus d$, we observe that no vertex of $N(d)$ is colored 3, and so we can get a coloring of G by setting $c(d) = 3$, a contradiction. This proves that $|D_i| \geq 2$ for every i.

Let $i \in \{1, \ldots, t\}$. Let U_i be the set of vertices of A with a neighbor in D_i. By Claim 15 , D_i is complete to U_i and anticomplete to $V(G) \setminus (D_i \cup U_i)$. Since $U_i \subseteq A$, it follows that $3 \notin L(u)$ for every $u \in U_i$. Let (F, L') be as in Claim 10. Since $|L'(v)| \leq 2$ for every $v \in V(F)$, Lemma 5 implies that $|V(F)| \leq 100$. Since $|V(G)|$ depends only on $|V(F)|$, Claim 16 follows. This completes the proof of Claim 16. □

We can now prove the following claim, which is the last step of our argument. We may assume that C is precolored in such a way that the precoloring is proper, and the set $\{c_2, c_4\}$ is monochromatic and the set $\{c_3, c_4\}$ is monochromatic.

Claim 17. *$|V(G)|$ is bounded from above (and the bound does not depend on G).*

Proof. We may assume that $L(c_1) = 1$, $L(c_2) = L(c_4) = 2$ and $L(c_3) = L(c_5) = 3$. Write $X = X(C), Y = Y(C)$ and $Z = Z(C)$. Let $A' = \{v \in X : N(v) \cap C = \{c_2, c_4\}\}$ and $B' = \{v \in X : N(v) \cap C = \{c_3, c_5\}\}$.

It follows from Claim 13.4 that after the first step of updating every $v \in X \setminus (A' \cup B')$ with a neighbor in Y has list of size one. Let Y' be the set of vertices that have lists of size 3 after the third step of updating. Since $L(z) \leq 2$ for every $z \in Z$, it follows that $Y' \subseteq Y$, and $N(y) \cap X \subseteq A \cup B$ for every $y \in Y'$.

Let A, B be the subsets of A', B' respectively consisting of all vertices with a neighbor in Y'. Then after the second step of updating, the list of every vertex in A is $\{1, 3\}$ and the list of every vertex in B is $\{1, 2\}$. If one of A', B' is not a stable set, Claim 16 completes the proof. So, we may assume that each of A', B' is a stable set.

Let H be the graph obtained from $G|(A \cup B)$ by making each of A, B a clique. Let $C_1, \ldots, C_t$ be the anticomponents of H such that both $A_i = C_i \cap A$ and $B_i = C_i \cap B$ are nonempty. Let $A'' = A \setminus \bigcup_{i=1}^t C_i$ and let $B'' = B_i \setminus \bigcup_{i=1}^t C_i$.

Let $v \in Y'$. Then $N(v) \cap A$ is complete to $B' \setminus N(v)$, and $N(v) \cap B$ is complete to $A' \setminus N(v)$. In particular, A is complete to $B' \setminus B$, B is complete to $A' \setminus A$, and v is not mixed on C_i for any i. (6)

Suppose this is false. By symmetry, we may assume there exists $w \in A$ non-adjacent to $k \in B'$ such that v is adjacent to w but not to k. Then v-w-c_2-c_1-c_5-k is a P_6 in G, a contradiction. This proves (6).

Suppose $v \in Y'$ is adjacent to $y \in V(G) \setminus (A \cup B \cup Y')$. Then precoloring y and v and updating three times reduces the list size of all vertices in Y' to at most two. (7)

Since $v \in Y'$, it follows that $N(v) \cap X \subseteq A \cup B$, and therefore $y \notin X$. It follows from Claim 13.3 that $N(v) \cap X$ is complete to $N(v) \setminus X$.

By Claim 15 v has both a neighbor in A and a neighbor in B. We precolor v and y and update three times; denote the new list system by L''. If v and y have the same color, or one of v, y is colored 1, then $L''(u) = \emptyset$ for some vertex $u \in N(v) \cap (A \cup B)$, and (7) holds. Thus we may assume that one of v, y is precolored 2, and the other one 3. We claim that, after updating, $|L(x)| = 1$ for every $x \in X$. Recall that even before we precolored v and y we had that $|L(x)| = 1$ for every $x \in X \setminus (A' \cup B')$. Since v and y are colored $2, 3$, and $\{v, y\}$ is complete to $N(v) \cap X$, it follows that $|L(x)| = 1$ for every $x \in N(v) \cap X$. Since both $N(v) \cap A$ and $N(v) \cap B$ are nonempty, $L(x) = \{1\}$ for every $x \in N(v) \cap X$. By (6), $N(v) \cap A$ is complete to $B' \setminus N(v)$, and $N(v) \cap B$ is complete to $A' \setminus N(v)$. Since we have updated, $L(a) = \{3\}$ for every $a \in A' \setminus N(v)$ and $L(b) = \{2\}$ for every $b \in B' \setminus N(v)$. Consequently $|L(w)| \leq 2$ for every $w \in Y$. This proves 7.

In view of (7), Lemma 7 and Lemma 8, we may assume that $|L(v)| \leq 2$ for every $v \in Y(C)$ for which $N(v) \setminus (A \cup B) \neq \emptyset$.

Let $T = \{y \in Y : N(y) \subseteq A'' \cup B''\}$. There is collection $\mathcal{L}$ of list systems such that for every $L' \in \mathcal{L}$

- *$|L'(v)| \leq 2$ for every $v \in T$, and*
- *$L'(v) = L(v)$ for every $v \in V(G) \setminus T$,* (8)

For every $L' \in \mathcal{L}$, let $(G_{L'}, L')$ be a minimal list obstruction induced by (G, L'). Then $|V(G)|$ depends only on $|\bigcup_{L' \in \mathcal{L}} V(G_{L'})|$.

Let $y \in T \cap Y'$. First we show that y has a neighbor in A and a neighbor in B. Suppose $N(y) \cap B = \emptyset$. Then, by the remark following (7), $N(y) \subseteq A$. But now a coloring of $G \setminus y$ can be extended to a coloring of G by assigning color 2 to y, contrary to the fact (G, L) is a minimal obstruction. This proves that y has a neighbor in A and a neighbor in B. In particular both A'' and B'' are non-empty.

Observe that in every coloring of G either A'' or B'' is monochromatic (since they are complete to each other). Let $\mathcal{L}$ be the following collection of list systems. For each $i \in \bigcap_{a \in A''} L(a)$ we add to $\mathcal{L}$ the list system L', where $L'(a) = \{i\}$ for all $a \in A''$ and $L'(v) = L(v)$ for all $v \in V(G) \setminus A''$; and we update three times with respect to A''. Moreover, for each $j \in \bigcap_{b \in B''} L(b)$ we add to $\mathcal{L}$ the list system L', where $L'(b) = \{j\}$ for all $b \in B''$ and $L'(v) = L(v)$ for all $v \in V(G) \setminus B''$, and we update three times with respect to B''.

Now $\mathcal{L}$ is a refinement of L and satisfies the hypotheses of Lemma 7 with $R = G|(A'' \cup B'')$. Let $L' \in \mathcal{L}$. Since either $|L(a)| = 1$ for every $a \in A''$, or $|L(b)| = 1$ for every $b \in B''$, and since we have updated three times, we have that $|L'(y)| \leq 2$ for every $y \in T$. Let $(G_{L'}, L')$ be a minimal list-obstruction induced by (G, L').

By Lemma 7 and Lemma 8,

$$V(G) = A \cup \bigcup_{L' \in \mathcal{L}} V(G_{L'}).$$

Since A is a stable set, Claim 8 implies that $|A|$ only depends on $|\bigcup_{L' \in \mathcal{L}} V(G_{L'})|$, and (8) follows. This proves (8).

Let $\mathcal{L}$ be as in (8). Since our goal is to prove that G has bounded size, it is enough to show that (G, L') induces a minimal obstruction of bounded size for every $L' \in \mathcal{L}$. Therefore we may assume that for every $y \in Y'$ there exists an index i such that y is complete to C_i.

Let $y_1 \in Y'$ and let $C_1 \subseteq N(y_1)$. Then we may assume that no vertex of $V(G) \setminus C_1$ is mixed on A_1 (and similarly on B_1). (9)

Suppose $x \in V(G) \setminus C_1$ is mixed on A_1. Since x is mixed on C_1, and C_1 is an anticomponent of H, there exist $a_1 \in A_1$ and $b_1 \in B_1$ such that a_1b_1 is a non-edge, and x is mixed on this non-edge. Let $a_1' \in A_1$ be such that x is mixed on $\{a_1, a_1'\}$. By Lemma 7 and Lemma 8 we can precolor $T = \{x, a_1, a_1', b_1, y_1\}$, and update three times with respect to T. Let Y'' be the set of vertices with lists of size 3 after updating. We claim that $Y'' = \emptyset$. Suppose not and let $v \in Y''$. By the remark following (8) there exists an index i such that v is complete to C_i. Then $i \neq 1$. Since $v \in Y''$, $\{a_1, a_1'\}$ is complete to B_i and b_1 is complete to A_i, and we have updated three times with respect to T, it follows that $L(a_1) = L(a_1') = \{3\}$ and $L(b_1) = \{2\}$. Since x has a neighbor in $\{a_1, a_1'\}$ we may assume that $L(x) \neq \{3\}$.

First consider the case that x is adjacent to a_1 and not to b_1. Then x is non-adjacent to a_1'. Choose $a_i \in A_i$. Since x-a_1-y_1-b_1-a_i-v is not a P_6 in G, it follows that x is adjacent to a_i. Since $v \in Y''$, it follows that $L(x) = \{2\}$, and therefore x is anticomplete to B_i. Choose b_i such that a_ib_i is a non-edge, then x-a_i-v-b_i-a_1'-y_1 is a P_6 in G, a contradiction. Therefore x is adjacent to b_1 and not to a_1. Since x-b_1-y_1-a_1-b-v is not a P_6 in G for any $b \in B_i$, it follows that x

is complete to B_i, which is a contradiction since $L(x) \neq \{3\}$ and $v \in Y''$. This proves (9).

Let $v \in Y'$ and let $C_i \in N(v)$. Then we may assume $|A_i| = |B_i| = 1$. (10)

Suppose this is false. We may assume that $i = 1$. By (9), no vertex of $G \setminus C_1$ is mixed on A_1 and no vertex of $G \setminus C_1$ is mixed on B_1. Choose $a_1 \in A_1$ and $b_1 \in B_1$ such that a_1b_1 is an edge if possible. Then $(G \setminus (A_1 \cup B_1)) \cup \{a_1, b_1\}$ is not L-colorable, since otherwise we can color A_1 in the color of a_1 and B_1 in the color of b_1. Since (G, L) is a minimal list-obstruction, (10) follows.

Let $Y_1 = \{y \in Y' : N(y) \subseteq (A \setminus A'') \cup (B \setminus B'')\}$, and let $Y_2 = Y' \setminus Y_1$. By (10) and since (G, L) is a minimal list-obstruction, every $y \in Y_1$ is complete to more than one of $C_1, \ldots, C_t$. We may assume that each of $C_1, \ldots, C_s$ is complete to some vertex of Y_1, and $C_{s+1} \cup \ldots \cup C_t$ is anticomplete to Y_1. Let F be the graph with vertex set $V(F) = \{1, \ldots, s\}$ where i is adjacent to j if and only if there is a vertex $y \in Y_1$ complete to $C_i \cup C_j$. We will refer to the vertices of F as $1, \ldots, s$ and $C_1, \ldots, C_s$ interchangeably.

Let $F_1, \ldots, F_k$ be the components of F, let $A(F_i) = \bigcup_{C_j \in F_i} A_j$, and let $B(F_i) = \bigcup_{C_j \in F_i} B_j$. Moreover, let $Y(F_i) = \{y \in Y_1 : N(y) \subseteq A(F_i) \cup B(F_i)\}$.

Let $i \in \{1, \ldots, k\}$ and let $T \subseteq V(G)$ be such that $A(F_i) \cup B(F_i) \cup Y_1 \subseteq T$. Then for every L-coloring of $G|T$, both of the sets $A(F_i)$ and $B(F_i)$ are monochromatic, and the color of $A(F_i)$ is different from the color of $B(F_i)$. (11)

Let c be a coloring of $G|T$. Let $y \in Y(F_i)$. We may assume that y is complete to C_1, and $C_1 \in F_i$. Let $\alpha = c(A_1)$ and $\beta = c(B_1)$, where $c(A_i)$ and $c(B_i)$ denote the color given to the unique vertices in the sets A_i and B_i respectively. Since y is complete to at least two of $C_1, \ldots, C_s$, the sets $N(y) \cap A$ and $N(y) \cap B$ are monochromatic, and $\alpha \neq \beta$. Pick any $t \in F_i$, and let P be a shortest path in F from C_1 to t. Let s be the neighbor of t in P. We may assume that $s = C_2$ and $t = C_3$. We proceed by induction and assume that $c(A_2) = \alpha$, and $c(B_2) = \beta$. Since s is adjacent to t in F, there is $y' \in Y_1$ such that y' is complete to $C_2 \cup C_3$. Then $c(y') \in \{1, 2, 3\} \setminus \{\alpha, \beta\}$. Moreover, A_2 is complete to B_3, and A_3 is complete to B_2, and so $c(A_3) \notin \{c(y'), \beta\}$ and $c(B_3) \notin \{c(y'), \alpha\}$. It follows that $c(A_3) = \alpha$ and $c(B_3) = \beta$, as required. This proves (11).

We now construct a new graph G' where we replace each F_i by a representative in A and a representative in B, as follows. Let G' be the graph obtained from $G \setminus (C_1 \cup \ldots \cup C_s \cup Y_1)$ by adding $2s$ new vertices $a_1, \ldots, a_s, b_1, \ldots, b_s$, where

$$N_{G'}(a_i) = \{b_i\} \cup \bigcup_{a \in A(F_i)} (N_G(a) \cap V(G'))$$

and

$$N_{G'}(b_i) = \{a_i\} \cup \bigcup_{b \in B(F_i)} (N_G(b) \cap V(G')),$$

for all $i \in \{1, \dots, s\}$. Note that, in G', the set $\{a_1, \dots, a_s\}$ is complete to the set $\{b_1, \dots, b_s\}$. Let $L(a_i) = \{1, 3\}$ and $L(b_i) = \{1, 2\}$ for every i. By repeated applications of Claim 7, we deduce that G' is P_6-free.

Let $A^* = (A \setminus (A'' \cup A_1 \ldots \cup A_s)) \cup \{a_1, \dots, a_s\}$ and $B^* = (B \setminus (B'' \cup B_1 \ldots \cup B_s)) \cup \{b_1, \dots, b_s\}$. Note that A^* is complete to B'', and B^* is complete to A''.

Let $R = G|(A^* \cup B^* \cup A'' \cup B'')$.

We may assume that $|A^*| \geq 2$, and define the list systems L_1, L_2, and L_3 as follows.

$$L_1(v) = \begin{cases} \{3\} & \text{if } v \in A'' \\ \{2\} & \text{if } v \in B'' \\ L(v) & \text{if } v \notin A'' \cup B'' \end{cases}$$

$$L_2(v) = \begin{cases} \{3\} & \text{if } v \in A^* \\ L(v) & \text{if } v \notin A^* \end{cases}$$

$$L_3(v) = \begin{cases} \{2\} & \text{if } v \in B^* \\ L(v) & \text{if } v \notin B^* \end{cases}$$

Let $\mathcal{L} = \{L_1, L_2, L_3\}$. It is clear that, for every L-coloring c of G', there exists a list system $L' \in \mathcal{L}$ such that c is also an L'-coloring of G'. Recall that by the remark following (8) every vertex of Y_2 has a neighbor in A^*, a neighbor in B^*, and a neighbor in $A'' \cup B''$. Therefore, for every $L' \in \mathcal{L}$, every vertex in Y_2 is adjacent to some vertex v with $|L'(v)| = 1$. Now by Lemma 5, Lemma 7, and Lemma 8, for every $L' \in \mathcal{L}$, G' contains an induced subgraph G'' such that (G'', L) is not colorable, and $|V(G'') \setminus V(R)| \leq 3 \cdot 36 \cdot 100$. We may assume that for every index i, $a_i \in G''$ or $b_i \in G''$, for otherwise we can just delete F_i from G contradicting the minimality of (G, L).

We claim that the subgraph induced by G on the vertex set

$$S = (V(G) \cap V(G'')) \cup Y_1 \cup \bigcup_{i=1}^{s} (A(F_i) \cup B(F_i))$$

is not L-colorable. Suppose this is false and let c be such a coloring. By (11), for every $i \in \{1, \dots, k\}$ the sets $A(F_i)$ and $B(F_i)$ are both monochromatic, and c can be converted to a coloring of G'' by giving a_i the unique color that appears in $A(F_i)$ and b_i the unique color that appears in $B(F_i)$, a contradiction. Thus $V(G) = S$, and it is sufficient to show that $|Y_1 \cup \bigcup_{i=1}^{s} (A(F_i) \cup B(F_i))|$ has bounded size. To see this, let $T = S \setminus (A \cup B \cup Y_1)$, then $|T| < |V(G'') \setminus R| \leq 3 \cdot 36 \cdot 100$.

First we bound s. Partition the set of pairs $\{(a_1, b_1), \dots, (a_s, b_s)\}$ according to the adjacency of each (a_i, b_i) in T; let $H_1, \dots, H_l$ be the blocks of this partition. Then $l \leq 2^{2|T|}$.

We claim that $|H_i| = 1$ for every i. Suppose for a contradiction that $(a_i, b_i), (a_j, b_j) \in H_1$. Let c be an L-coloring of $G'' \setminus \{a_i, b_i\}$. Note that, since $N(a_i) = N(a_j)$ and $N(b_i) = N(b_j)$, setting $c(a_i) = c(a_j)$ and $c(b_i) = c(b_j)$ gives an L-coloring of G'', a contradiction. This proves that $s \leq 2^{2|T|}$.

Next we bound $|F_i|$ for each i. Let $i \in \{1, \dots, s\}$. Partition the set $\{C_j : j \in F_i\}$ according to the adjacency of C_j in T. Let $C_1^i, \dots, C_{q_i}^i$ be the blocks of the partition. Then $q_i \leq 2^{|T|}$. Let $C_l \in C_1^i$. For each $j \in \{2, \dots, q_i\}$ let Q_j^i be a shortest path from C_l to C_j^i in F. In G, Q_j^i yields a path ${Q_j^i}' = a_1'\text{-}y_1'\text{-}a_2'\text{-}y_2'\text{-}\dots\text{-}y_m'\text{-}a_m'$ where $a_1' \in C_l$, $a_m' \in A \cap C_j^i$, $a_2', \dots, a_{m-1}' \in \bigcup_{l \in \{1,\dots,q\} \setminus \{1,j\}} A \cap C_l^i$ and $y_1', \dots, y_m' \in Y_1$. Let $Y(Q_j^i) = \{y_1', \dots, y_m'\}$. Since ${Q_j^i}'$ does not contain a P_6, it follows that $|Y(Q_j)| \leq 2$. Let $Y_1^i = \bigcup_{j=2}^{q_i} Y(Q_j^i)$, and note that $|Y_1^i| \leq 2q_i - 2 \leq 2(2^{|T|} - 1)$. Moreover, let $\hat{Y} = \bigcup_{i=1}^{s} Y_1^i$, and note that $|\hat{Y}| \leq 2(2^{|T|} - 1)s$.

Next we claim that $\hat{Y} = Y_1$. To see this, suppose that there exists a vertex $y \in Y_1 \setminus \hat{Y}$. Note that y is critical, and let c be a coloring of $G \setminus y$. We may assume that $N(y) \subseteq \bigcup_{i \in F_1} C_i$. We will construct a coloring of G'' and obtain a contradiction. By (11), for every $i \in \{2, \dots, s\}$ both of the sets $A(F_i)$ and $B(F_i)$ are monochromatic and so we can color a_i and b_i with the corresponding colors.

Let F' be the graph with vertex set F_1 and such that i is adjacent to j if and only if there is a vertex $y' \in \hat{Y}$ (and therefore $y' \in Y_1^1$) complete to $C_i \cup C_j$. Recall the partition $C_1^1, \dots, C_{q_1}^1$. By the definition of Y_1^1, there exists $C_1' \in C_1^1$ such that for every $i \in \{2, \dots, q_1\}$ there is a path in F' from C_1' to a member C_i' of C_i^1. Write $\{a_1'\} = C_1' \cap A$ and $\{b_1'\} = C_1' \cap B$, and let $\alpha = c(a_1')$ and $\beta = c(b_i')$. Following the outline of the proof of (11) we deduce that $\alpha \neq \beta$, and that for each $i \in \{1, \dots, q\}$ some vertex of $\bigcup_{C \in C_i^1} C \cap A$ is colored with color α, and some vertex of $\bigcup_{C \in C_i^1} C \cap B$ is colored with color β. Observe that for every index i only vertices of $Y_1 \cup \bigcup_{C \in C_i^1} (C \cap B)$ are mixed on $\bigcup_{C \in C_i^1} (C \cap A)$, and only vertices of $Y_1 \cup \bigcup_{C \in C_i^1} (C \cap A)$ are mixed on $\bigcup_{C \in C_i^1} (C \cap B)$. Thus we can color a_1 with color α and b_1 with color β, obtaining a coloring of G'', a contradiction. This proves that $|Y_1| \leq 2(2^{|T|} - 1)s$. Now applying Claim 9 $|Y_1|$ times implies that there is a function q that does not depend on G, such that $|\bigcup_{i=1}^{s} (A(F_i) \cup B(F_i))| \leq q(|T|)$. Consequently, $|V(G)| \leq |T| + |Y_1| + q(|T|) \leq |T| + 2(2^{|T|} - 1)s + q(T)$. This completes the proof. $\square$

Now Lemma 6 follows from Claim 17.

5 $2P_3$-free 4-vertex critical graphs

The aim of this section is to show that there are only finitely many $2P_3$-free 4-vertex critical graphs. The proof follows the same outline as the proof of the previous section. Lemma 11 deals with $2P_3$-free minimal list-obstructions where every list has size at most two. In view of Lemma 16 the exact analogue of Lemma 5 does not hold in this case, however if we add the additional assumption that the minimal list-obstruction is contained in a $2P_3$-free 4-vertex-critical graph that was obtained by updating with respect to a set of precolored vertices, then we can show that the size of the obstruction is bounded.

Lemma 11. *There is a an integer $C > 0$ such that the following holds. Let (G, L) be a list-obstruction. Assume that G is $2P_3$-free and the following holds.*

(a) $|L(v)| \leq 2$ for every vertex v of G.

(b) Every vertex v of G with $|L(v)| = 2$ has a neighbor u with $|L(u)| = 1$ such that for all $w \in V(G)$ with $|L(w)| = 2$, $uw \in E(G)$ implies $L(w) = L(v)$.

Then (G, L) contains a minimal list-obstruction with at most C vertices.

Like in the case of P_6-free list-obstructions, we can use the precoloring technique to prove that the lemma above implies our main lemma.

Lemma 12. *There is an integer $C > 0$ such that every $2P_3$-free 4-vertex-critical graph has at most C vertices. Consequently, there are only finitely many $2P_3$-free 4-vertex-critical graphs.*

5.1 Proof of Lemma 11

Let G' be an induced subgraph of G such that (G', L) is a minimal list-obstruction. By Lemma 4, it suffices to prove that the length of any propagation path of (G', L) is bounded by a constant. To see this, let $P = v_1\text{-}v_2\text{-}\ldots\text{-}v_n$ be a propagation path of (G', L) starting with color α, say. Consider v_1 to be colored with α, and update along P until every vertex of P is colored. Let this coloring of P be denoted by c. Recall condition (1) from the definition of propagation path: every edge v_iv_j with $3 \leq i < j \leq n$ and $i \leq j - 2$ is such that

$$S(v_i) = \alpha\beta \text{ and } S(v_j) = \beta\gamma,$$

where $\{1, 2, 3\} = \{\alpha, \beta, \gamma\}$.

First we prove that there is a constant δ such that there is a subpath $Q = v_m\text{-}v_{m+1}\text{-}\ldots\text{-}v_{m'}$ of P of length at least $\lfloor \delta n \rfloor$ with the following property. After permuting colors if necessary, it holds for all $i \in \{m, \ldots, m'\}$ that

$$S(v_i) = \begin{cases} 32, & \text{if } i \equiv 0 \mod 3 \\ 13, & \text{if } i \equiv 1 \mod 3 \\ 21, & \text{if } i \equiv 2 \mod 3 \end{cases}.$$

To see this, suppose there are two indices $i, j \in \{3, \ldots, n-3\}$ such that $i + 2 \leq j$ and $c(v_i) = c(v_{i+2}) = c(v_j) = c(v_{j+2})$. Moreover, suppose that $c(v_i) = c(v_{i+2}) = c(v_j) = c(v_{j+2}) = \alpha$ and $c(v_{i+1}) = c(v_{j+1}) = \beta$ for some α, β with $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$. Thus, $L(v_{i+1}) = L(v_{i+2}) = L(v_{j+1}) = L(v_{j+2}) = \{\alpha, \beta\}$, $\alpha \in L(v_{j+3})$, and $\alpha \neq c(v_{j+3})$. But now $v_i\text{-}v_{i+1}\text{-}v_{i+2}$ and $v_{j+1}\text{-}v_{j+2}\text{-}v_{j+3}$ are both induced P_3's, according to (1), and there cannot be any edge between them. This is a contradiction to the assumption that G is $2P_3$-free. The same conclusion holds if $c(v_{i+1}) = c(v_{j+1}) = \gamma$. Hence, there cannot be three indices $i, j, k \in \{3, \ldots, n-3\}$ such that $i + 2 \leq j$, $j + 2 \leq k$, and

$$c(v_i) = c(v_{i+2}) = c(v_j) = c(v_{j+2}) = c(v_k) = c(v_{k+2}) = \alpha.$$

Consider the following procedure. Pick the smallest index $i \in \{3, \dots, n-3\}$ such that $c(v_i) = c(v_{i+2}) = 1$, if possible, and remove the vertices v_i, v_{i+1}, and v_{i+2} from P. Let P' be the longer of the two paths v_1-v_2-…-v_{i-1} and v_{i+3}-v_2-…-v_n. Repeat the deletion process and let $P'' = v_r$-v_{r+1}-…-$v_{r'}$ be the path obtained. As shown above, we now know that there is no index $j \in \{r+2, \dots, r'-3\}$ with $c(v_j) = c(v_{j+2}) = 1$.

Repeating this process for colors 2 and 3 shows that there is some $\delta > 0$ such that there is a path $Q = v_m$-v_{m+1}-…-$v_{m'}$ of length $\lfloor \delta n \rfloor$ where $c(v_i) \neq c(v_{i+2})$ for all $i \in \{m-1, \dots, m'-2\}$. Thus, after swapping colors if necessary we have the desired property defined above.

From now on we assume that G has sufficiently many vertices and hence $m'-m$ is sufficiently large. Since G is $2P_3$-free and hence P_7-free, the diameter of every connected induced subgraph of G is bounded by a constant. In particular, the diameter of the graph $G|(\{v_m, \dots, v_{m'}\})$ is bounded, and so we may assume that there is a vertex v_i with $m \leq i \leq m'$ with at least 20 neighbors in the path Q. We may assume that $c(v_i) = 1$ and, thus, $S(v_i) = 13$.

We discuss the case when $|N(v_i) \cap \{v_m, \dots, v_{i-1}\}| \geq 10$. The case of $|N(v_i) \cap \{v_{i+1}, \dots, v_{m'}\}| \geq 10$ can be dealt with in complete analogy.

We pick distinct vertices $v_{i_1}, \dots, v_{i_{10}} \in N(v_i) \cap \{v_m, \dots, v_{i-1}\}$ where $i_1 < i_2 < \dots < i_{10}$. Note that (1) implies that $S(v_{i_j}) = 21$ for all $j \in \{1, \dots, 10\}$.

We can pick three indices j_1, j_2, j_3 with $r' < j_1 < j_2 < j_3 < m'$ such that

- $S(v_{j_1}) = S(v_{j_2}) = S(v_{j_3}) = 32$, and
- $i_2 + 5 = j_1$, $j_1 + 6 = j_2$, $j_2 + 4 \leq i_7$, $i_8 + 5 = j_3$, and $j_3 + 4 = i$.

Recall that assumption (b) of the lemma we are proving implies the following. Since $L(v_{j_u}) = \{2, 3\}$, v_{j_u} has a neighbor x_{j_u} with $L(x_{j_u}) = \{1\}$, $u = 1, 2, 3$, such that x_{j_u} is not adjacent to any vertex v_j with $m \leq j \leq m'$ and $j \equiv 1 \mod 3$ or $j \equiv 2 \mod 3$.

Suppose that $x_{j_u} = x_{j_{u'}}$ for some $v_{j_{u'}}$ with $u' \in \{1, 2, 3\} \setminus \{u\}$. Now the path v_{j_u}-x_{j_u}-$v_{j_{u'}}$ is an induced P_3, and so is the path v_{i_1}-v_i-v_{i_2}, both according to condition (1). Moreover, there is no edge between those two paths, due to (1), which is a contradiction. Hence, the three vertices x_{j_u}, x_{j_u}, and x_{j_u} are mutually distinct and, due to the minimality of (G, L), mutually non-adjacent.

Consider the induced P_3's v_{j_1+1}-v_{j_1}-x_{j_1} and v_{j_3+1}-v_{j_3}-x_{j_3}. Since G is $2P_3$-free, there must be an edge between these two paths. According to (1), it must be the edge $v_{j_1+1}v_{j_3}$. For similar reasons, the edge $v_{j_2+1}v_{j_3}$ must be present. Now the path v_{j_1+1}-v_{j_3}-v_{j_2+1} is an induced P_3, and so is the path v_{i_7}-v_i-v_{i_8}. Moreover, there is no edge between those two paths, due to (1), which is a contradiction. This completes the proof.

5.2 Proof of Lemma 12

We start with two statements that allow us to precolor sets of vertices with certain properties. In this subsection G is always a $2P_3$-free graph, and all lists are subsets of $\{1, 2, 3\}$.

Claim 18. *Assume that (G, L) is a list-obstruction. Let $X \subseteq V(G)$ be such that there exists a coloring c of $G|X$ with the following property: for each $x \in X$ there exists a set $N_x \subseteq V(G)$ with $|N_x| \leq k$ such that x is colored $c(x)$ in every coloring of $(G|(\{x\} \cup N_x), L)$. Let L' be a list system such that*

$$L'(v) = \begin{cases} L(v), & \text{if } v \in V(G) \setminus X \\ \{c(x)\}, & \text{if } v \in X \end{cases}.$$

Then the following holds.

(a) (G, L') is a list-obstruction.

(b) If $K \subseteq V(G)$ is such that $(G|K, L')$ is a minimal list-obstruction induced by (G, L'), then (G, L) contains a minimal list-obstruction of size at most $(k+1)|K|$.

Proof. Since $L'(v) \subseteq L(v)$ for all $v \in V(G)$, (G, L') is also a list-obstruction. This proves (a).

Let $A = G|(K \cup \bigcup_{x \in K \cap X} N_x)$, then $|V(A)| \leq (k+1)|K|$. Suppose that there exists a coloring, c' of (A, L). Note that for every $x \in V(A)$, $N_x \subseteq A$. Hence by the definition of X, $c'(x) = c(x)$ for every $x \in V(A)$. This implies that c' is also a coloring of (A, L'), which gives a coloring of $(G|K, L')$, a contradiction. Therefore (A, L) is a list-obstruction induced by (G, L). Since $|V(A)| \leq (k+1)|K|$, (b) holds. This completes the proof. □

Claim 19. *Let (G, L) be a list-obstruction, and let $X \subseteq V(G)$ be a vertex subset such that $|L(x)| = 1$ for every $x \in X$. Let $Y = N(X)$, and let $Y' \subseteq Y$ be such that for every $v \in Y'$, $|L(v)| = 3$. For every $v \in Y'$, pick $x_v \in N(v) \cap X$. Let L' be the list defined as follows.*

$$L'(v) = \begin{cases} L(v), & \text{if } v \in V(G) \setminus Y' \\ L(v) \setminus L(x_v), & \text{if } v \in Y' \end{cases}.$$

Let (G', L') be a minimal list-obstruction induced by (G, L'). Then there exists a minimal list-obstruction induced by (G, L), say (G'', L), with $|V(G'')| \leq 2|V(G')|$.

Proof. Let $R = \{x_v : v \in V(G') \cap Y'\}$ and let $P = R \cup V(G')$. It follows that $|V(P)| \leq 2|V(G')|$. It remains to prove that $(G|P, L)$ is not colorable. Suppose there exists a coloring c of $(G|P, L)$. Note that c is not a coloring of (G', L') and G' is an induced subgraph of $G|P$. Hence there exists $w \in V(G')$ such that $c(w) \notin L'(w)$. By the construction of L', it follows that $w \in Y'$ and that $c(w) \in L(w) \setminus L'(w) = \{c(x_w)\}$, which is a contradiction. This completes the proof. □

Let G be a $2P_3$-free 4-vertex-critical graph such that $|V(G)| \geq 5$, then the following claim holds.

Claim 20. *At least one of the following holds*

1. *There exists $S_0 \subseteq V(G)$ such that $|S_0| \leq 5$, $G|S_0$ contains a copy of P_3 and $S_0 \cup N(S_0) \cup N(N(S_0)) = V(G)$, or*

2. *G has a semi-dominating set of size at most 5.*

Proof. Since G is $2P_3$-free and thus also P_7-free, Theorem 10 states that G has a dominating induced P_5 or a dominating P_5-free connected induced subgraph, denoted by D_f. Recall that a dominating set is always a semi-dominating set; so we may assume that the latter case holds and $|V(D_f)| \geq 6$. By applying Theorem 10 to D_f again, we deduce that D_f has a dominating induced subgraph T, which is isomorphic to P_3 or a connected P_3-free graph.

If T is isomorphic to P_3, then we are done by setting $S_0 = V(T)$. Hence we may assume T is a connected P_3-free graph. Therefore T is a complete graph, and so $V(T) \leq 3$. If there exists a vertex $s' \in V(G \setminus T)$ mixed on T, we are done by setting $S_0 = V(T) \cup \{s'\}$. Hence we may assume that for every $v \in V(D_f \setminus T)$, v is complete to T. Since $|V(G)| \geq 5$, it follows that D_f is K_4-free. Therefore there exist $v, w \in V(D_f \setminus T)$ such that v is non-adjacent to w and we are done by setting $S_0 = V(T) \cup \{v, w\}$. □

If G has a semi-dominating set of size at most 5, we are done by Lemma 9 and Lemma 11. Hence we may assume there exists S_0 defined as in Claim 20.

For a list system L' of G, we say that (X_1, X_2, B, S) is the *partition with respect to L'* by setting:

(a) $S = \{v \in V(G) : |L'(v)| = 1\}$.

(b) $B = N(S)$; assume that $|L'(v)| = 2$ for every $v \in B$.

(c) Let $X = V(G) \setminus (S \cup B)$. We say that C is a *good component* of X if there exist $x \in C$ and $\{i, j\} \subseteq \{1, 2, 3\}$ so that x has two adjacent neighbors $a, b \in B_{ij}$, where $B_{ij} = \{b \in B$ such that $L'(b) = \{i, j\}\}$. Let X_1 be the union of all good components of X and let $X_2 = X \setminus X_1$.

Let (X_1, X_2, B, S) be the partition with respect to L'. Define $X = X_1 \cup X_2$. For every $1 \leq i \leq j \leq 3$, define $B_{ij} = \{b \in B$ such that $L'(b) = \{i, j\}\}$ and $X_{ij} = \{x \in X_2$ such that $|N(x) \cap B_{ij}| \geq 2\}$. For $\{i, j, k\} = \{1, 2, 3\}$, let us say that a component C of X_2 is *i-wide* if there exist a_j in B_{ik} and a_k in B_{ij} such that C is complete to $\{a_j, a_k\}$. We call a_j and a_k *i-anchors* of C. Note that a component can be i-wide for several values of i. Let L'' be a subsystem of L' and let (X'_1, X'_2, B', S') be the partition with respect to L''. Then $S \subseteq S'$, $B' \setminus B \subseteq X_1 \cup X_2$ and $X'_2 \subseteq X_2$.

Next we define a sequence of new lists $L_0, \ldots, L_5$. Let $\{i, j, k\} = \{1, 2, 3\}$. Let S_0 be as in Claim 20, and let $L_0 = L$.

1. Let L_1 be the list system obtained by precoloring S_0 and updating three times. Let (X^1_1, X^1_2, B^1, S^1) be the partition with respect to L_1.

2. For each $k \in \{1, 2, 3\}$, choose $x_k \in X^1_{ij}$ such that $|N(x_k) \cap B^1_{ij}|$ is minimum. Let $a_k, b_k \in N(x_k) \cap B^1_{ij}$. Let L_2 be the list system obtained from L_1 by precoloring $\bigcup_{i=1}^{3} \{a_i, b_i, x_i\}$ and updating the lists of vertices three times. Let (X^2_1, X^2_2, B^2, S^2) be the partition with respect to L_2 .

3. For each $k \in \{1, 2, 3\}$, let $\hat{B}_k \subseteq B^2_{ij}$ with $|\hat{B}_k| \le 1$ be defined as follows. If there does not exist a vertex $v \in B^2_{ij}$ that starts a path v-u-w where $u, w \in X^2_2$, then $\hat{B}_k = \emptyset$. Otherwise choose $b_k \in B^2_{ij}$ maximizing the number of pairs (u, w) where b_k-u-w is a path and let $\hat{B}_k = \{b_k\}$. Let L_3 be the list system from L_2 obtained by precoloring $\hat{B}_1 \cup \hat{B}_2 \cup \hat{B}_3$ and updating three times. Let (X^3_1, X^3_2, B^3, S^3) be the partition with respect to L_3.

4. Apply step 2 to (X^3_1, X^3_2, B^3, S^3) with list system L_3; let L_4 be the list system obtained and let (X^4_1, X^4_2, B^4, S^4) be the partition with respect to L_4.

5. For every component C_t of X^4_2 with size 2, if C_t is i-wide with i-anchors a^t, b^t, set $L_5(a^t) = L_5(b^t) = \{i\}$; then let L_5 be the list system after updating with respect to $\bigcup_t \{a^t, b^t\}$ three times. Let (X^5_1, X^5_2, B^5, S^5) be a partition with respect to L_5.

By Lemma 7 and Lemma 8, it is enough to prove that (G, L_4) induces a bounded size list-obstruction. To do that, we prove the same for (G, L_5), and then use Claim 18 and Lemma 8, as we explain in the remainder of this section.

We start with a few technical statements.

Claim 21. *Let $1 \le m \le l \le 5$. Then the following holds.*

1. *For every vertex in $x \in X^m$, $|L_m(x)| = 3$, and every component of X^m is a clique with size at most 3.*

2. *If no vertex of B^m_{ij} is mixed on an edge in $G|X^m_2$, then no vertex of B^l_{ij} is mixed on an edge in $G|X^l_2$.*

3. *If no vertex of X^m_2 has two neighbors in B^m_{ij} and no vertex of B^m is mixed on an edge in $G|X^m_2$, then no vertex of X^l_2 has two neighbors in B^l_{ij}.*

Proof. By construction, for every vertex in $x \in X^m$, $|L_m(x)| = 3$. Observe that $S_0 \subseteq S^m$. Recall that G is $2P_3$-free and that S_0 contains a P_3. Hence X^m does not contain a P_3, and so every component of X^m is a clique. Since $|V(G)| \ge 5$, it follows that every component of X^m has size at most 3. This proves the first statement.

Let $b \in B^l_{ij}$ be mixed on the edge uv such that $u, v \in X^l_2$. Recall that $X^l_2 \subseteq X^m_2$; thus $\{u, v\} \subseteq X^m_2$. By assumption $b \notin B^m_{ij}$ and hence $b \in X^m$. But now b-u-v is a P_3 in X^m, a contradiction. This proves the second statement.

To prove the last statement, suppose that there exists $y \in X_2^l$ with two neighbors $u, v \in B_{23}^l$. Since $y \in X_2^l$, it follows that u, v are non-adjacent. Note that $y \in X_2^m$, hence by assumption and symmetry, we may assume that $v \notin B^m$. Therefore $v \in X^m$. Since X_1^m is the union of components of X^m, and $y \in X_2^m$ is adjacent to v, it follows that $v \in X^m \setminus X_1^m$, and consequently $v \in X_2^m$. If $u \notin B^m$, then u-y-v is a P_3 in $G|X^m$, contrary to the first statement. Hence $u \in B^m$ and then u is mixed on the edge vy of $G|X_2^m$, a contradiction. This completes the proof. □

Claim 22. $X_{12}^1 \cup X_{23}^1 \cup X_{13}^1 \subseteq B^2 \cup S^2$.

Proof. Suppose that there exists $x' \in X_{ij}^1 \setminus (B^2 \cup S^2)$ for some $1 \leq i \leq j \leq 3$; then $|L_2(x')| = 3$. Let $x_k \in X_{ij}^1$ and $a_k, b_k \in N(x_k) \cap B_{ij}^1$ be the vertices chosen to be precolored in the step creating L_2. Then x' is non-adjacent to $\{x_k, a_k, b_k\}$. The minimality of $|N(x_k) \cap B_{ij}^1|$ implies that there exist $a', b' \in (N(x') \cup B_{ij}^1) \setminus N(x_k)$. Since G is $2P_3$-free, there exists an edge between $\{a_k, b_k, x_k\}$ and $\{a', b', x'\}$. Specifically, there exists an edge between $\{a_k, b_k\}$ and $\{a', b'\}$. We may assume that $L_2(a_k) = \{i\}$ and a_k is adjacent to at least one of a', b'. Recall that L_1 is obtained by precoloring $\bigcup\limits_{i=1}^{3} \{a_i, b_i, x_i\}$ and updating three times. It follows that $j \notin L_2(x')$, a contradiction. □

Claim 23. *No vertex of B^3 is mixed on an edge of X_2.*

Proof. Suppose that there exists a path b'-x_1'-x_2' such that $b' \in B_{ij}^3$ and $x_1', x_2' \in X_2^3$. Note that $x_1', x_2' \in X_2^2$ since L_3 is a subsystem of L_2. By Claim 21.1, X^2 is P_3-free. Hence $b' \in B_{ij}^2$. By Claim 21.3, there exists $b \in B_{ij}^2$ such that $b - x - y$ is a path where $x, y \in X_2^2$. Then in step 3, $\hat{B}_k \neq \emptyset$ and let $b \in \hat{B}_k$. By the construction of L_3 and since $x_1', x_2' \in X_2^3$, b is anticomplete to $\{b', x_1', x_2'\}$. By the construction of $\hat{B}_k$, there exist $x_1, x_2 \in X_2^2$ such that b-x_1-x_2 is a path and b' is not mixed on x_1x_2. If $\{x_1, x_2\}$ is not anticomplete to $\{x_1', x_2'\}$, then by Claim 21.1 $G|\{x_1, x_2, x_1', x_2'\}$ is a K_4, a contradiction to the fact that $|V(G)| \geq 5$. Hence $\{x_1, x_2\}$ is anticomplete to $\{x_1', x_2'\}$. Since G is $2P_3$-free, there exists an edge between b' and $\{x_1, x_2\}$. Consequently, b' is complete to $\{x_1, x_2\}$. Now x_1 has two neighbors in B_{ij}^2, namely b and b'. By Claim 22, $x_1 \notin B_{ij}^1$. It follows that either $b \in X^1$ or $b' \in X^1$. If $b \in X^1$, then $b - x - y$ is a P_3 in X^1, contrary to Claim 21.1. Hence $b' \in X^1$. It follows that b'-x_1'-x_2' is a P_3 in X^1, again contrary to Claim 21.1. This completes the proof. □

We are now ready to prove that it suffices to show that (G, L_5) induces a mininal list-obstruction of bounded size. Let C_t be an i-wide component of X_1^4 with $C_t = \{x_t, y_t\}$, and let a_t, b_t be the i-anchors of C_t that were chosen in step 5. By the definition of i-anchors, $L_4(a_t) \cap L_4(b_t) = \{i\}$ and $\{a_t, b_t\}$ is complete to C_t; therefore $c(a_t) = c(b_t) = i$ for every coloring c of $(G|\{x_t, y_t, a_t, b_t\}, L_4)$. Hence we can apply Claim 18 to L_4. By Claim 18 and Lemma 8, it is enough to show that (G, L_5) induces a bounded size list-obstruction.

Claim 24. *X_2^5 is stable.*

Proof. Since $|V(G)| \geq 5$ and since no vertex of B^5 is mixed on an edge of $G|X_2^5$, by Claim 21.1 every component of X_2^5 has size at most 2. We may assume some component C of X_2^5 has size exactly 2, for otherwise the claim holds. Then C is a component of X_2^4. By Claim 21 and Claim 22, no vertex of X_2^4 has two neighbors in B_{ij}^4. Since every vertex in G has degree at least 3, every vertex of C has a neighbor in at least two of $B_{12}^4, B_{23}^4, B_{13}^4$. It follows that C is i-wide for some i and therefore $C \subseteq S^5 \cup B^5$, a contradiction. □

By Claim 21 and Claim 23, no vertex of X_2^5 has two neighbors in B_{ij}^5. Since every vertex in G has degree at least 3, it follows that every vertex of X_2^5 has exactly one neighbor in each of B_{ij}^5. Let $Y_0, Y_1, \ldots, Y_6$ be a partition of X_2^5 as follows. Let $x \in X_2^5$ and $a_k = N(x) \cap B_{ij}^5$ for $\{i,j,k\} = \{1,2,3\}$. If $\{a_1, a_2, a_3\}$ is a stable set, then $x \in Y_0$; if $E(G|\{a_1, a_2, a_3\}) = \{a_i a_j\}$, then $x \in Y_k$; and if $E(G|\{a_1, a_2, a_3\}) = \{a_i a_j, a_i a_k\}$, then $x \in Y_{i+3}$. Note that $G|\{a_1, a_2, a_3\}$ cannot be a clique since $V(G) \geq 5$. For each non-empty Y_s, pick $x_s \in Y_s$, and let $a_{sk} \in N(x_s) \cap B_{ij}^5$ for $\{i,j,k\} = \{1,2,3\}$. Let L_6 be the list system obtained by precoloring $\bigcup_{i=0}^{6} \{x_i, a_{i1}, a_{i2}, a_{i3}\}$ with c and updating three times.

Claim 25. *For every $x \in X_2^5$, $|L_6(x)| \leq 2$*

Proof. Suppose there exists $y \in Y_i$ such that $|L_6(y)| = 3$; let $b_1 = N(y) \cap B_{23}^5$, $b_2 = N(y) \cap B_{13}^5$, and $b_3 = N(y) \cap B_{12}^5$. Then $\{a_{i1}, a_{i2}, a_{i3}, x_i\}$ and $\{b_1, b_2, b_3, y\}$ are disjoint sets. Note that $c(a_{i1}), c(a_{i2}), c(a_{i3})$ can not all be pairwise different, and so by symmetry we may assume that $c(a_{i1}) = c(a_{i2}) = 3$ and $c(a_{i3}) = 2$. Thus, $a_{i1}a_{i2}$ is a non-edge. By the construction of L_6 and since $|L_6(y)| = 3$, the only possible edges between the sets $\{a_{i1}, a_{i2}, a_{i3}\}$ and $\{b_1, b_2, b_3\}$ are $a_{i3}b_2$, $a_{i1}b_3$ and $a_{i2}b_3$. Recall that every vertex of X_2^5 has exactly three neighbors in B^5, and so $N(y) \cap B^5 = \{b_1, b_2, b_3\}$ and $N(x_i) \cap B^5 = \{a_{i1}, a_{i2}, a_{i3}\}$. Since $G|\{a_{i1}, x_i, a_{i2}, b_1, y, b_2\}$ is not a $2P_3$, it follows that b_1 is adjacent to b_2. But this contradicts to the fact that both x_i and y belong to Y_i. □

Let (X_6^1, X_6^2, B^6, S^6) be the partition with respect to L_6. For every component $C_s \subseteq X_1^6$, let $\{i,j,k\} = \{1,2,3\}$ be such that there exists $x_s^k \in C_s$ with two adjacent neighbors in B_{ij}^6. Define $L_6'(x_k^s) = \{k\}$; let P be the set of all such vertices x_k^s, and let $L_6'(v) = L_6(v)$ for every $v \notin P$. Let L^* be the list system obtained from L_6' by updating with respect to P three times. Pick $x \in P$, then there exist $i,j \in \{1,2,3\}$ for which some $a, b \in N(x) \cap B_{ij}^6$ are adjacent. Then $L_6(a) = L_6(b) = \{i,j\}$. As a result, for every coloring c of $(G|\{x,a,b\}, L_6)$, $c(x) = k$. This implies that we can apply Claim 18 to L_6. By Lemma 8 and Claim 18, it is enough to prove that (G, L^*) induces a bounded size list-obstruction. Let (X_1^*, X_2^*, B^*, S^*) be the partition with respect to L^*. Then by Claim 21.1 X_1^*, X_2^* are empty. Now (G, L^*) satisfies the hypotheses of Lemma 11, and this finishes the proof of Lemma 12.

6 $P_4 + kP_1$-free minimal list-obstructions

In this section we prove that there are only finitely many $P_4 + kP_1$-free minimal list-obstructions. This also implies that there are only finitely many $P_4 + kP_1$-free 4-vertex-critical graphs.

Lemma 13. *Let (G, L) be a minimal list-obstruction such that*

$|L(v)| \leq 2$ for every vertex v of G. Moreover, let G be $(P_4 + kP_1)$-free, for some $k \in \mathbb{N}$. Then $V(G)$ is bounded from above by a constant depending only on k.

Proof. By Lemma 4, it suffices to prove that every propagation path in (G, L) has a bounded number of vertices. To see this, let $P = v_1\text{-}\ldots\text{-}v_n$ be a propagation path in (G, L) starting with color α, say. Consider v_1 to be colored with α, and update along P until every vertex is colored. Call this coloring c. Suppose that $n \geq 100k^2 + 100$. Our aim is to show that this assumption is contradictory. Recall condition (1) from the definition of propagation path: every edge $v_i v_j$ with $3 \leq i < j \leq n$ and $i \leq j - 2$ is such that

$$S(v_i) = \alpha\beta \text{ and } S(v_j) = \beta\gamma,$$

where $\{1, 2, 3\} = \{\alpha, \beta, \gamma\}$.

First we suppose that there is a sequence $v_i, v_{i+1}, \ldots, v_j$ with $2 \leq i \leq j \leq n$ and $j - i \geq 5 + 2k$ such that $c(v_{i'}) = c(v_{i'+2})$ for all i' with $i \leq i' \leq j - 2$. But then (1) implies that $v_{i+1}\text{-}v_{i+2}\text{-}\ldots\text{-}v_j$ is an induced path, and thus G is not $P_4 + kP_1$-free, a contradiction.

Suppose now that there is an index i with $2 \leq i \leq \lceil n/2 \rceil - 3$ such that $c(v_i) = c(v_{i+2}) = \alpha$ and $c(v_{i+1}) = c(v_{i+3}) = \beta$. In particular, $L(v_{i+3}) = \{\alpha, \beta\}$. Now condition (1) of the definition of a propagation path implies that there cannot be an edge between v_i and v_{i+3}, and so $v_i\text{-}v_{i+1}\text{-}v_{i+2}\text{-}v_{i+3}$ is an induced P_4. Therefore no such sequence exists.

We now pick k disjoint intervals of the form $\{j, \ldots, j + 7 + 2k\} \subseteq \{\lceil n/2 \rceil + 1, \ldots, n\}$. As shown above, each of these intervals contains an index i' in its interior with $c(v_{i'}) = \alpha$. These $v_{i'}$ form a stable set and (1) implies that the induced path $v_i\text{-}v_{i+1}\text{-}v_{i+2}\text{-}v_{i+3}$ is anticomplete to each $v_{i'}$, a contradiction to the fact that G is $P_4 + kP_1$-free.

Now suppose that there is an index i with $r + 1 \leq i \leq \lceil (r + s)/2 \rceil - 3$ such that $c(v_i) = c(v_{i+2}) = \alpha$ and $c(v_{i+1}) = \beta$. From what we have shown above we know that $c(v_{i-1}) = c(v_{i+3}) = \gamma$, where $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$. Thus, we have $S(v_i) = \alpha\gamma$, $S(v_{i+1}) = \beta\alpha$, $S(v_{i+2}) = \alpha\beta$, and $S(v_{i+3}) = \gamma\alpha$. According to (1), the path $v_i\text{-}v_{i+1}\text{-}v_{i+2}\text{-}v_{i+3}$ is induced.

Pick a vertex v_j with $\lceil n/2 \rceil + 1 \leq j \leq n$. According to (1), v_j is anticomplete to the path $v_i\text{-}v_{i+1}\text{-}v_{i+2}\text{-}v_{i+3}$ unless one of the following holds.

(a) $S(v_j) = \alpha\beta$,

(b) $S(v_j) = \alpha\gamma$,

(c) $S(v_j) = \beta\gamma$, or

(d) $S(v_j) = \gamma\beta$.

Let us say that v_j is of *type A* if it satisfies one of the above conditions. If v_j is not of type A, we say it is of *type B*.

We claim that there are at most $3k-3$ vertices of type B. To see this, suppose there are at least $3k-2$ vertices of type B. By definition, each vertex of type B is anticomplete to the set the path v_i-v_{i+1}-v_{i+2}-v_{i+3}. Since (G, L) is a minimal obstruction and not every vertex is of type B, the graph induced by the vertices of type B is 3-colorable. Picking the vertices of the majority color yields a set S of k independent vertices of type B. But now the set $\{v_i, \ldots, v_{i+3}\} \cup S$ induces a $P_4 + kP_1$ in G, a contradiction.

So, there are at most $3k-3$ vertices of type B. Suppose there are more than $(3k-2)(7+2k)$ many vertices of type A. Then there is an index $t \geq \lceil n/2 \rceil + 1$ such that $v_t + j'$ is of type A for all $j' \in \{0, \ldots, 6+2k\}$. Suppose that there is an index $j' \in \{0, \ldots, 5+2k\}$ such that $c(v_{t+j'}) = \alpha$. Then $S(v_{t+j'+1}) = \cdot\, \alpha$, in contradiction to the fact $v_{t+j'+1}$ is of Type A. So, for all $j' \in \{0, \ldots, 5+2k\}$ we have that $c(v_{t+j'}) \neq \alpha$, in contradiction to what we have shown above. Summing up, n is bounded by $2(3k-2)(7+2k)+1$ if there is an index i with $2 \leq i \leq \lceil n/2 \rceil - 3$ such that $c(v_i) = c(v_{i+2}) = \alpha$ and $c(v_{i+1}) = \beta$.

Hence, our assumption $n \geq 100k^2 + 100$ implies that $c(v_i) \neq c(v_{i+2})$ for all i with $2 \leq i \leq \lceil n/2 \rceil - 3$. This means that, without loss of generality,

$$c(v_i) = \begin{cases} 1, & i = 1\ (3) \\ 2, & i = 2\ (3) \\ 3, & i = 0\ (3) \end{cases} \tag{12}$$

for all i with $2 \leq i \leq \lceil n/2 \rceil - 3$.

Consider the path v_4-v_5-$\ldots$-v_{7+2k}. Since G is $P_4 + kP_1$-free, this is not an induced path. Hence, there is an edge of the form v_iv_j with $i < j$. If $S(v_i) = \alpha\beta$, we must have $S(v_j) = \beta\gamma$, due to (1). Consequently, $S(v_{i-1}) = \beta\gamma$, and $S(v_{j+1}) = \alpha\beta$. In particular, (1) implies that v_{i-1} is non-adjacent to v_{j+1}, and so v_{i-1}-v_i-v_j-v_{j+1} is an induced path.

Like above, we now pick k disjoint intervals of the form $\{j, \ldots, j+7+2k\} \subseteq \{\lceil n/2 \rceil + 1, \ldots, n\}$. Each of these intervals contains an index i' in it's interior with $c(v_{i'}) = \alpha$. These $v_{i'}$ form a stable set and (1) implies that the induced path v_{i-1}-v_i-v_j-v_{j+1} is anticomplete to each $v_{i'}$, a contradiction. This completes the proof. □

Using the above statement, we can now derive our main lemma.

Lemma 14. *There are only finitely many $P_4 + kP_1$-free minimal list-obstructions, for all $k \in \mathbb{N}$.*

Proof. Let (G, L) be a $P_4 + kP_1$-free minimal list-obstruction. If G is P_4-free, we are done, since there is only a finite number of P_6-free minimal obstructions. So, we may assume that G contains an induced P_4, say v_1-v_2-v_3-v_4. Let $R = V(G) \setminus N(\{v_1, v_2, v_3, v_4\})$. Let S be a maximal stable set in R; then every vertex of $V(R) \setminus S$ has a neighbor in S. Since G is $P_4 + kP_1$-free, it follows that $|S| \leq k - 1$, and so $\{v_1, v_2, v_3, v_4\} \cup S$ is a dominating set of size at most $k + 3$ in G. Now Lemma 14 follows from Lemma 9 and Lemma 13. □

7 Necessity

The aim of this section is to prove the following two statements.

Lemma 15. *There are infinitely many H-free 4-vertex-critical graphs if H is a claw, a cycle, or $2P_2 + P_1$.*

Here, a *claw* is the graph consisting of a central vertex plus three pairwise non-adjacent pendant vertices attached to it. In the list-case, the following variant of this statement holds.

Lemma 16. *There are infinitely many H-free minimal list-obstructions if H is a claw, a cycle, $2P_2 + P_1$, or $2P_3$.*

We remark that Lemma 15 implies the following. Whenever H is a graph containing a claw, a cycle, or $2P_2+P_1$ as an induced subgraph, there are infinitely many H-free 4-vertex-critical graphs. A similar statement is true with respect to Lemma 16 and minimal list-obstructions.

7.1 Proof of Lemma 15

Recall that there are infinitely many 4-vertex-critical claw-free graphs. For example, this follows from the existence of 4-regular bipartite graphs of arbitrarily large girth (cf. [17] for an explicit construction of these) whose line graphs are necessarily 4-chromatic. Moreover, there are 4-chromatic graphs of arbitrarily large girth, which follows from a classical result of Erdős [5]. This, in turn, implies that there exist 4-vertex-critical graphs of arbitrary large girth. Putting these two remarks together, we see that if H is the claw or a cycle, then there are infinitely many 4-vertex-critical graphs.

We now recall a construction due to Pokrovskiy [20] which gives an infinite family of 4-vertex-critical P_7-free graphs. It is presented in more detail in our earlier work [4].

For each $r \geq 1$, let G_r be the graph defined on the vertex set $v_0, \ldots, v_{3r}$ with edges as follows. For all $i \in \{0, 1, \ldots, 3r\}$ and $j \in \{0, 1, \ldots, r-1\}$, the vertex v_i is adjacent to v_{i-1}, v_{i+1}, and v_{i+3j+2}. Here, we consider the indices to be taken modulo $3r + 1$. The graph G_5 is shown in Figure 1.

Up to permuting the colors, there is exactly one 3-coloring of $G_r \setminus v_0$. Indeed, we may assume that v_i receives color i, for $i = 1, 2, 3$, since $\{v_1, v_2, v_3\}$ forms a triangle in G_r. Similarly, v_4 receives color 1, v_5 receives color 2 and so on. Finally, v_{3r} receives color 3. It follows that G_r is not 3-colorable, since v_0 is adjacent to all of v_1, v_2, v_{3r}.

As the choice of v_0 was arbitrary, we know that G_r is 4-vertex-critical. The graph G_r is $2P_2 + P_1$-free which can be seen as follows.

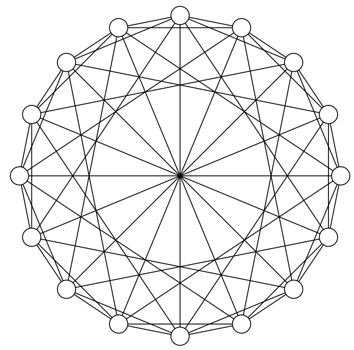

Figure 1: A circular drawing of G_5

Claim 26. *For all r the graph G_r is $2P_2 + P_1$-free.*

Proof. Suppose there is some r such that G_r is not $2P_2+P_1$-free. Let $v_{i_1}, \dots, v_{i_5}$ be such that $G_r[\{v_{i_1}, \dots, v_{i_5}\}]$ is a $2P_2 + P_1$. Since G_r is vertex-transitive, we may assume that $i_1 = 1$ and $N(v_{i_1}) \cap \{v_{i_2}, \dots, v_{i_5}\} = \emptyset$. In particular, $i_2, \dots, i_5 \neq 0$.

Consider $G_r \setminus v_0$ to be colored by the coloring c proposed above, where each v_i receives the color $i \mod 3$. Due to the definition of G_r, v_{i_1} is adjacent to every vertex of color 3, and thus $c(v_j) \neq 3$ for all $j \in \{i_2, \dots, i_5\}$.

We may assume that $c(v_{i_2}) = c(v_{i_4}) = 1$, $c(v_{i_3}) = c(v_{i_5}) = 2$, and both $v_{i_2}v_{i_3}$ and $v_{i_4}v_{i_5}$ are edges of $E(G_r)$. For symmetry, we may further assume that $i_2 < i_4$. Due to the definition of G_r, v_{i_2} and v_{i_4} are adjacent to every vertex of color 2 with a smaller index, and thus $i_4 < i_3$. But now $i_2 < i_4 < i_3$, a contradiction to the fact that $v_{i_2}v_{i_3} \in E(G_r)$. This completes the proof. □

Consequently, there are infinitely many $2P_2+P_1$-free 4-vertex-critical graphs, as desired.

7.2 Proof of Lemma 16

In view of Lemma 15, it remains to prove that there are infinitely many $2P_3$-free minimal list-obstructions.

For all $r \in \mathbb{N}$, let H_r be the graph defined as follows. The vertex set of H_r is $V(H_r) = \{v_i : 1 \leq i \leq 3r - 1\}$. There is an edge from v_1 to v_2, from v_2 to v_3 and so on. Thus, $P := v_1$-v_2-$\dots$-v_{3r-1} is a path. Moreover, there is an edge between a vertex v_i and a vertex v_j if $i \leq j - 2$, $i \equiv 2 \mod 3$, and $j \equiv 1 \mod 3$. There are no further edges. The graph H_5 is shown in Figure 2.

The list system L is defined by $L(v_1) = L(v_{3r-1}) = \{1\}$ and, assuming

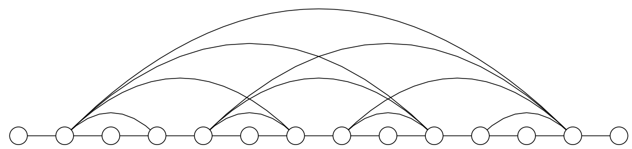

Figure 2: A drawing of H_5. The vertices v_1 to v_{14} are shown from left to right.

$2 \le i \le 3r-2$,

$$L(v_i) = \begin{cases} \{2,3\}, & \text{if } i \equiv 0 \mod 3 \\ \{1,3\}, & \text{if } i \equiv 1 \mod 3 \\ \{1,2\}, & \text{if } i \equiv 2 \mod 3 \end{cases}.$$

Next we show that the above construction has the desired properties.

Claim 27. *The pair (H_r, L) is a minimal $2P_3$-free list-obstruction for all r.*

Proof. Let us first show that, for any r, H_r is not colorable. Consider the partial coloring c that assigns color 1 to v_1. Since $L(v_2) = \{1,2\}$, the coloring can be updated from v_1 to v_2 by putting $c(v_2) = 2$. Now we can update the coloring from v_2 to v_3 by putting $c(v_3) = 3$. Like this we update the coloring along P until v_{3r-2} is colored. However, we have to put $c(v_{3r-2}) = 1$, in contradiction to the fact that $L(v_{3r-1}) = \{1\}$. Thus, H_r is not colorable.

Next we verify that (H_r, L) is a minimal list-obstruction. If we delete v_1 or v_{3r-1}, the graph becomes colorable. So let us delete a vertex v_i with $2 \le i \le 3r-2$. We can color $(H_r \setminus v_i, L)$ as follows. Give color 1 to v_1 and update along P up to v_{i-1}. Moreover, give color 1 to v_{3r-1} and update along P backwards up to v_{i+1}. Call this coloring c.

To check that c is indeed a coloring, we may focus on the non-path edges for obvious reasons. Pick an edge between a vertex v_j and a vertex v_k with $j \le k-2$, if any. By definition, $j \equiv 2 \mod 3$ and $k \equiv 1 \mod 3$. If $j < i < k$, $c(v_j) = 2$ and $c(v_k) = 3$. Moreover, if $j < k < i$, $c(v_j) = 2$ and $c(v_k) = 1$. Finally, if $i < j < k$, $c(v_j) = 1$ and $c(v_k) = 3$. So, c is indeed a coloring of $H_r \setminus v_i$ and it remains to prove that H_r is $2P_3$-free.

Suppose this is false, and let r be minimum such that H_r contains an induced $2P_3$. Let F be a copy of such a $2P_3$ in H_r. It is clear that $r \ge 2$. Note that $H_r \setminus N(v_2)$ is the disjoint union of complete graphs of order 1 and 2, and so $v_2 \notin V(F)$. Since $N(v_1) = \{v_2\}$, we know that $v_1 \notin V(F)$. Moreover, as $F \setminus (N(v_5) \cup \{v_1, v_2\})$ is the disjoint union of complete graphs of order 1 and 2, we deduce that $v_5 \notin V(F)$. But $F' := F \setminus \{v_1, v_2, v_3\}$ is isomorphic to H_{r-1}, and thus the choice of r implies that F' is $2P_3$-free. Consequently, $v_3 \in V(F)$. Since $N(v_3) = \{v_2, v_4\}$ and $v_2 \notin V(F)$, we know that $v_4 \in V(F)$. Finally, the fact that $N(v_4) = \{v_2, v_3, v_5\}$ implies that v_3 and v_4 both have degree one in F, and they are adjacent, a contradiction. □

8 Proof of Theorem 2 and Theorem 3

We now prove our main results. We start with a lemma.

Lemma 17. *For every graph H, one of the following holds.*

1. *H contains a cycle, a claw or $2P_2 + P_1$.*
2. *$H = 2P_3$.*
3. *H is contained in P_6.*
4. *There exists $k > 1$ such that H is contained in $P_4 + kP_1$.*

Proof. We may assume that H does not contain $2P_2 + P_1$, a cycle, or a claw. It follows that every component of H induces a path. Let $H_1, H_2, \dots H_k$ be the components of H, ordered so that $|H_1| \geq |H_2| \geq \dots \geq |H_k|$.

If $|H_2| \geq 2$, then, since H is $2P_2 + P_1$-free, it follows that $k = 2$, $|H_1| \leq 3$, and $|H_2| \leq 3$, and so either H is contained in P_6 or $H = 2P_3$. This proves that $|H_2| = \dots = |H_k| = 1$.

If $|H_1| \geq 5$, then since H is $2P_2 + P_1$-free, it follows that $k = 1$, and H is contained in P_6. This proves that $|H_1| \leq 4$, and so H is contained in $P_4 + (k-1)P_1$. This proves Lemma 17. □

Next we prove Theorem 2, which we restate:

Theorem 2. *Let H be a graph. There are only finitely many H-free 4-vertex-critical graphs if and only if H is an induced subgraph of P_6, $2P_3$, or $P_4 + kP_1$ for some $k \in \mathbb{N}$.*

Proof. If H contains a cycle, a claw or $2P_2 + P_1$, then there is an infinite list of 4-vertex-critical graphs by Lemma 15. Otherwise, by Lemma 17, $H = 2P_3$, H is contained in P_6, or for some $k > 1$, H is contained in $P_4 + kP_1$, and Lemmas 12, 6 and 14, respectively, imply that there are only finitely many H-free 4-vertex-critical graphs. □

Finally, we prove the list version of the result, Theorem 3, which we restate:

Theorem 3. *Let H be a graph. There are only finitely many H-free minimal list-obstructions if and only if H is an induced subgraph of P_6, or of $P_4 + kP_1$ for some $k \in \mathbb{N}$.*

Proof. If H contains a cycle, a claw, $2P_2+P_1$ or $2P_3$, then there is an infinite list of obstructions by Lemma 16. Otherwise, by Lemma 17, H is contained in P_6, or for some $k > 1$, H is contained in P_4+kP_1. Now Lemmas 6 and 14, respectively, imply that there are only finitely many H-free list-obstructions. □

Acknowledgments

We thank Alexey Pokrovskiy for suggesting the construction of the graph G_r, and Frédéric Maffray for many useful discussions and suggestions. Several of the computations for this work were carried out using the Stevin Supercomputer Infrastructure at Ghent University. This material is based upon work supported in part by the U. S. Army Research Laboratory and the U. S. Army Research Office under grant number W911NF-16-1-0404.

References

[1] F. Bonomo, M. Chudnovsky, P. Maceli, O. Schaudt, M. Stein, and M. Zhong, *Three-coloring and list three-coloring graphs without induced paths on seven vertices*, Combinatorica **38(4)** (2018), 779–801.

[2] D. Bruce, C.T. Hoàng, and J. Sawada, *A certifying algorithm for 3-colorability of P_5-free graphs*, Proceedings of the 20th International Symposium on Algorithms and Computation, Springer-Verlag, 2009, pp. 594–604.

[3] E. Camby and O. Schaudt, *A new characterization of P_k-free graphs*, Algorithmica **75** (2016), no. 1, 205–217.

[4] M. Chudnovsky, J. Goedgebeur, O. Schaudt, and M. Zhong, *Obstructions for three-coloring graphs with one forbidden induced subgraph*, Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2016, Arlington, VA, USA, 2016, pp. 1774–1783.

[5] P. Erdős, *Graph theory and probability*, Canadian Journal of Mathematics **11** (1959), 34–38.

[6] F. Galvin, I. Rival, and Sands. B., *A ramsey-type theorem for traceable graphs*, Journal of Combinatorial Theory, Series B **33(1)** (1982), 7–16.

[7] J. Goedgebeur, Homepage of generator for obstructions against list-3 colorability: `http://caagt.ugent.be/listcriticalpfree/`.

[8] J. Goedgebeur and O. Schaudt, *Exhaustive generation of k-critical $\mathcal{H}$-free graphs*, Journal of Graph Theory **87(2)** (2018), 188–207.

[9] P.A. Golovach, M. Johnson, D. Paulusma, and J. Song, *A survey on the computational complexity of coloring graphs with forbidden subgraphs*, Journal of Graph Theory **84(4)** (2017), 331–363.

[10] P. Hell and S. Huang, *Complexity of coloring graphs without paths and cycles*, LATIN 2014: Theoretical Informatics, Springer, 2014, pp. 538–549.

[11] P. Hell and S. Huang, *Complexity of coloring graphs without paths and cycles*, Discrete Applied Mathematics **216, Part 1** (2017), 211–232.

[12] C.T. Hoàng, B. Moore, D. Recoskie, J. Sawada, and M. Vatshelle, *Constructions of k-critical P_5-free graphs*, Discrete Applied Mathematics **182** (2015), 91–98.

[13] I. Holyer, *The NP-completeness of edge-coloring*, SIAM Journal on Computing **10** (1981), 718–720.

[14] B.M.P. Jansen and S. Kratsch, *Data reduction for graph coloring problems*, Information and Computation **231** (2013), 70–88.

[15] M. Kamiński and V.V. Lozin, *Coloring edges and vertices of graphs without short or long cycles*, Contributions to Discrete Mathematics **2** (2007), 61–66.

[16] D. Král', J. Kratochvíl, Zs. Tuza, and G.J. Woeginger, *Complexity of coloring graphs without forbidden induced subgraphs*, Proceedings of the International Workshop on Graph-Theoretic Concepts in Computer Science 2001 (A. Brandstadt and V.B. Le, eds.), Lecture Notes in Computer Science, vol. 2204, 2001, pp. 254–262.

[17] F. Lazebnik and V.A. Ustimenko, *Explicit construction of graphs with an arbitrary large girth and of large size*, Discrete Applied Mathematics **60** (1995), 275–284.

[18] D. Leven and Z. Galil, *NP-completeness of finding the chromatic index of regular graphs*, Journal of Algorithms **4** (1983), 35–44.

[19] F. Maffray and G. Morel, *On 3-colorable P_5-free graphs*, SIAM Journal on Discrete Mathematics **26** (2012), 1682–1708.

[20] A. Pokrovskiy, *private communication.*

[21] F.P. Ramsey, *On a problem of formal logic*, Proceedings of the London Mathematical Society **30** (1930), 264–286.

[22] B. Randerath and I. Schiermeyer, *3-Colorability $\in$ P for P_6-free graphs*, Discrete Applied Mathematics **136(2)** (2004), 299–313.

[23] D. Scheische, *On a property of the class of n-colorable graphs*, Journal of Combinatorial Theory, Series B **16(2)** (1974), 191–193.

[24] P. Seymour, *Barbados workshop on graph coloring and structure, 2014.*

[25] E. Zhang, *Bounded obstructions for three-coloring graphs with lists of size two*, 2017, Senior Thesis, Princeton University.