

Lines in hypergraphs

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Abstract

One of the De Bruijn - Erdős theorems deals with finite hypergraphs where every two vertices belong to precisely one hyperedge. It asserts that, except in the perverse case where a single hyperedge equals the whole vertex set, the number of hyperedges is at least the number of vertices and the two numbers are equal if and only if the hypergraph belongs to one of simply described families, near-pencils and finite projective planes. Chen and Chvátal proposed to define the line uv in a 3-uniform hypergraph as the set of vertices that consists of u , v , and all w such that $\{u, v, w\}$ is a hyperedge. With this definition, the De Bruijn - Erdős theorem is easily seen to be equivalent to the following statement: If no four vertices in a 3-uniform hypergraph carry two or three hyperedges, then, except in the perverse case where one of the lines equals the whole vertex set, the number of lines is at least the number of vertices and the two numbers are equal if and only if the hypergraph belongs to one of two simply described families. Our main result generalizes this statement by allowing any four vertices to carry three hyperedges (but keeping two forbidden): the conclusion remains the same except that a third simply described family, complements of Steiner triple systems, appears in the extremal case.

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1 Introduction

Two distinct theorems are referred to as “the De Bruijn - Erdős theorem”. One of them [14] concerns the chromatic number of infinite graphs; the other [13] is our starting point:

Let m and n be positive integers such that $n \geq 2$; let V be a set of n points; let \mathcal{L} be a family of m subsets of V such that each member of \mathcal{L} contains at least two and at most $n - 1$ points of V and such that every two points of V belong to precisely one member of \mathcal{L} . Then $m \geq n$, with equality if and only if

one member of \mathcal{L} contains $n - 1$ points of V and each of the remaining $n - 1$ members of \mathcal{L} contains two points of V

or else $n = k(k - 1) + 1$, each member of \mathcal{L} contains k points of V , and each point of V is contained in k members of \mathcal{L} .

We study variations on this theme that are generated through the notion of lines in hypergraphs. A *hypergraph* (the term comes from Claude Berge [2]) is an ordered pair (V, \mathcal{E}) such that V is a set and \mathcal{E} is a set of subsets of V ; elements of V are the *vertices* of the hypergraph and elements of \mathcal{E} are its *hyperedges*; a hypergraph is called *k-uniform* if all its hyperedges have precisely k vertices. Given a 3-uniform hypergraph and its distinct vertices u, v , Chen and Chvátal [8] define the *line* \overline{uv} as the set of vertices that consists of u, v , and all w such that $\{u, v, w\}$ is a hyperedge. (When V is a subset of the Euclidean plane and \mathcal{E} consists of all collinear triples of vertices, \overline{uv} is the intersection of V and the Euclidean line passing through u and v .)

If, as in the hypothesis of the De Bruijn - Erdős theorem, (V, \mathcal{L}) is a hypergraph in which each hyperedge contains at least two vertices and every two vertices belong to precisely one hyperedge, then \mathcal{L} is the set of lines of a 3-uniform hypergraph (V, \mathcal{E}) : to see this, let \mathcal{E} consist of all the three-point subsets of all hyperedges in \mathcal{L} . As for the converse of this observation, if (V, \mathcal{E}) is a 3-uniform hypergraph, then each of its lines contains at least two vertices and every two vertices belong to at least one line, but they may belong to more than one line. For example, if V contains distinct vertices p, q, r, s such that $\{p, q, r\} \in \mathcal{E}$, $\{p, q, s\} \in \mathcal{E}$, $\{p, r, s\} \notin \mathcal{E}$, then lines \overline{pr} , \overline{ps} are distinct and p, q belong to both of them. Now we are going to show that this is the only example.

Theorem 1. *If, in a 3-uniform hypergraph (V, \mathcal{E}) , some two vertices belong to more than one line, then V contains distinct vertices p, q, r, s such that $\{p, q, r\} \in \mathcal{E}$, $\{p, q, s\} \in \mathcal{E}$, $\{p, r, s\} \notin \mathcal{E}$.*

Proof. Let (V, \mathcal{E}) be a 3-uniform hypergraph. Assuming that two of its vertices, u and v , belong not only to the line \overline{uv} , but also to some other line \overline{xy} , we will find distinct vertices p, q, r, s such that at least two but not all four of $\{p, q, r\}$, $\{p, q, s\}$, $\{p, r, s\}$, $\{q, r, s\}$ belong to \mathcal{E} .

CASE 1: *One of x, y is one of u, v .*

Symmetry lets us assume that $x = u$; now $\overline{uy} \neq \overline{uv}$ and $v \in \overline{uy}$, and so $y \neq v$ and $\{u, v, y\} \in \mathcal{E}$. Since $\overline{uy} \neq \overline{uv}$, some vertex z belongs to precisely one of these two lines; since $\{u, v, y\} \in \mathcal{E}$, the vertices u, v, y, z are all distinct. Since precisely one of $\{u, y, z\}$ and $\{u, v, z\}$ belongs to \mathcal{E} , we may take u, v, y, z for p, q, r, s .

CASE 2: *x, y, u, v are all distinct.*

Since $u \in \overline{xy}$ and $v \in \overline{xy}$, we have $\{u, x, y\} \in \mathcal{E}$ and $\{v, x, y\} \in \mathcal{E}$. If $\{u, v, x\} \notin \mathcal{E}$ or $\{u, v, y\} \notin \mathcal{E}$, then we may take u, v, x, y for p, q, r, s ; if $\{u, v, x\} \in \mathcal{E}$ and $\{u, v, y\} \in \mathcal{E}$, then we are back in Case 1 with (v, x) in place of (x, y) if $\overline{uv} \neq \overline{vx}$ and with (v, x) in place of (u, v) if $\overline{uv} = \overline{vx}$. \square

When $W \subseteq V$, the *sub-hypergraph of (V, \mathcal{E}) induced by W* is (W, \mathcal{F}) with \mathcal{F} consisting of all elements of \mathcal{E} that are subsets of W . In this terminology, Theorem 1 states that

*in a 3-uniform hypergraph, no sub-hypergraph induced
by four vertices has two or three hyperedges if and only if
every two vertices belong to precisely one line.* (1)

We will say that a 3-uniform hypergraph has the *De Bruijn - Erdős property* if it has at least as many distinct lines as it has vertices or else one of its lines consists of all its vertices. In this terminology, an immediate corollary of (1) and the De Bruijn - Erdős theorem states that

*if, in a 3-uniform hypergraph, no sub-hypergraph induced
by four vertices has two or three hyperedges,
then the hypergraph has the De Bruijn - Erdős property.* (2)

Not every 3-uniform hypergraph has the De Bruijn - Erdős property: here is a 3-uniform hypergraph (V, \mathcal{E}) with $|V| = 11$ that has precisely ten distinct

lines and none of these lines equals V . Its vertex set V is

$$\{1, 2\} \cup (\{a, b, c\} \times \{d, e, f\});$$

its hyperedges are the $\binom{9}{3}$ three-point subsets of $\{a, b, c\} \times \{d, e, f\}$ and the 18 three-point sets $\{i, (x_1, x_2), (y_1, y_2)\}$ with $x_i = y_i$; its ten lines are

$$\begin{aligned} &\{1, 2\}, \\ &\{1, (a, d), (a, e), (a, f)\}, \\ &\{1, (b, d), (b, e), (b, f)\}, \\ &\{1, (c, d), (c, e), (c, f)\}, \\ &\{2, (a, d), (b, d), (c, d)\}, \\ &\{2, (a, e), (b, e), (c, e)\}, \\ &\{2, (a, f), (b, f), (c, f)\}, \\ &\{1\} \cup (\{a, b, c\} \times \{d, e, f\}), \\ &\{2\} \cup (\{a, b, c\} \times \{d, e, f\}), \\ &\{a, b, c\} \times \{d, e, f\}. \end{aligned}$$

We will refer to this hypergraph as \mathcal{F}_0 . It comes from Section 2 of [8], which includes a construction of arbitrarily large 3-uniform hypergraphs on n vertices with only $\exp(O(\sqrt{\log n}))$ distinct lines and no line consisting of all n vertices. All of these hypergraphs contain induced sub-hypergraphs isomorphic to \mathcal{F}_0 .

2 A generalization of the De Bruijn - Erdős theorem

Various generalizations of the De Bruijn - Erdős theorem, or at least of its first part, can be found in [5, 22, 18, 16, 3, 25, 17, 24, 20, 10, 1] and elsewhere. We offer a generalization in a different spirit by strengthening (2): we will drop its assumption that no sub-hypergraph induced by four vertices has three hyperedges. (As shown by the hypergraph \mathcal{F}_0 of the preceding section, the assumption that no sub-hypergraph induced by four vertices has two hyperedges cannot be dropped.) This goes a long way towards generalizing the De Bruijn - Erdős theorem, but it does not quite get there: a description of the extremal hypergraphs is also required. To provide this description, we introduce additional notation and terminology.

We let $\binom{S}{3}$ denote the set of all three-point subsets of a set S . A *near-pencil* is a hypergraph (V, \mathcal{L}) such that

$$\mathcal{L} = \{V \setminus \{w\}\} \cup \{\{v, w\} : v \in V \setminus \{w\}\} \text{ for some vertex } w.$$

We say that a 3-uniform hypergraph (V, \mathcal{E}) *generates a near-pencil* if

$$\mathcal{E} = \binom{V \setminus \{w\}}{3} \text{ for some vertex } w.$$

Clearly, this is the case if and only if the set \mathcal{L} of lines of (V, \mathcal{E}) is such that (V, \mathcal{L}) is a near-pencil.

A *finite projective plane* is a hypergraph (V, \mathcal{L}) in which, for some integer k greater than one, every two vertices belong to precisely one hyperedge, $|V| = k(k-1) + 1$, and each hyperedge contains precisely k vertices. We say that a 3-uniform hypergraph (V, \mathcal{E}) *generates a finite projective plane* if, for some finite projective plane (V, \mathcal{L}) ,

$$\mathcal{E} = \bigcup_{L \in \mathcal{L}} \binom{L}{3}.$$

Clearly, this is the case if and only if the set of lines of (V, \mathcal{E}) is \mathcal{L} .

The two extremal hypergraphs (V, \mathcal{L}) in the De Bruijn - Erdős theorem are exactly the near-pencil and the finite projective plane.

We say that a 3-uniform hypergraph (V, \mathcal{E}) is the *complement of a Steiner triple system* if every two of its vertices belong to precisely one member of $\binom{V}{3} \setminus \mathcal{E}$. Clearly, this is the case if and only if the set of lines of (V, \mathcal{E}) is $\{V \setminus \{x\} : x \in V\}$.

We use the graph-theoretic terminology and notation of Bondy and Murty [11]. In particular,

- P_n denotes the chordless path graph with n vertices,
- $F + G$ denotes the disjoint union of graphs F and G ,
- $F \vee G$ denotes the *join* of graphs F and G (defined as $F + G$ with additional edges that join every vertex of F to every vertex of G).

As usual, we call a graph F -free if it has no induced subgraph isomorphic to graph F and, when talking about sets, we use the qualifier ‘maximal’ as ‘maximal with respect to set-inclusion’ rather than as ‘largest’.

Theorem 2. *Let (V, \mathcal{E}) be a 3-uniform hypergraph on at least two vertices in which no four vertices induce two hyperedges and let \mathcal{L} be the set of lines of this hypergraph. If $V \notin \mathcal{L}$, then $|\mathcal{L}| \geq |V|$, with equality if and only if (V, \mathcal{E}) generates a near-pencil or a finite projective plane or is the complement of a Steiner triple system.*

Proof. Let \mathcal{H} denote the hypergraph and let n denote the number of its vertices. We will use induction on n . The induction basis, $n = 2$, is trivial; in the induction step, we distinguish between two cases. We may assume that $V \notin \mathcal{L}$.

CASE 1: *Every two vertices of \mathcal{H} belong to precisely one maximal line.*

Let \mathcal{L}^{\max} denote the set of maximal lines of \mathcal{H} . The De Bruijn - Erdős theorem guarantees that $|\mathcal{L}^{\max}| \geq n$, with equality if and only if (V, \mathcal{L}^{\max}) is a near-pencil or a finite projective plane. Since $\mathcal{L} \supseteq \mathcal{L}^{\max}$, we have $|\mathcal{L}| \geq |\mathcal{L}^{\max}| \geq n$; if $|\mathcal{L}| = n$, then $\mathcal{L} = \mathcal{L}^{\max}$, and so (V, \mathcal{L}) is a near-pencil or a finite projective plane.

CASE 2: *Some two vertices of \mathcal{H} belong to more than one maximal line.*

Let p denote one of these two vertices and let Σ denote the graph with vertex set $V \setminus \{p\}$, where vertices u, v are adjacent if and only if $\{p, u, v\} \in \mathcal{E}$. Since p, u, v, w do not induce two hyperedges,

- (i) u, v, w induce two edges in $\Sigma \Rightarrow \{u, v, w\} \in \mathcal{E}$,
- u, v, w induce one edge in $\Sigma \Rightarrow \{u, v, w\} \notin \mathcal{E}$.

A theorem of Seinsche [23] states that every connected P_4 -free graph with more than one vertex has a disconnected complement; property (i). of Σ guarantees that it is P_4 -free (if it contained an induced P_4 , then the four vertices of this P_4 would induce two hyperedges, a contradiction); it follows that

- (ii) every connected induced subgraph of Σ with more than one vertex has a disconnected complement.

Having established (i) and (ii), we distinguish between two subcases.

SUBCASE 2.1: Σ is disconnected.

In this subcase, we will prove that $|\mathcal{L}| > n$. To begin, the assumption of this subcase means that

$$\Sigma = \Sigma_1 + \Sigma_2 + \dots + \Sigma_k,$$

where $k \geq 2$ and each Σ_i is connected; it follows from (ii) that each Σ_i is either a single vertex or has a disconnected complement. For each $i = 1, \dots, k$, let V_i denote the vertex set of Σ_i and let W_i denote $V_i \cup \{p\}$. We claim that

$$(iii) \quad x, y \in W_i, x \neq y, z \in V \setminus W_i \Rightarrow \{x, y, z\} \notin \mathcal{E}.$$

When one of x, y is p , the conclusion follows from the fact that all vertices in V_i are nonadjacent in Σ to all vertices in $V \setminus W_i$. When x, y are adjacent vertices of Σ_i , the conclusion follows from the same fact, combined with (i). When x, y are nonadjacent vertices of Σ_i , consider a shortest path P from x to y in Σ_i . Since x and y are nonadjacent and Σ is P_4 -free, P has exactly three vertices. Let w be the unique interior vertex of P . Now (i) implies that $\{x, y, w\} \in \mathcal{E}$, $\{x, z, w\} \notin \mathcal{E}$, $\{y, z, w\} \notin \mathcal{E}$; in turn, the fact that x, y, z, w do not induce two hyperedges implies that $\{x, y, z\} \notin \mathcal{E}$.

Let \mathcal{H}_i denote the sub-hypergraph of \mathcal{H} induced by W_i . In the inductive argument, we shall use the following restatement of (iii):

$$(iv) \quad u, v \in W_i, u \neq v \Rightarrow \text{the line } \overline{uv} \text{ in } \mathcal{H}_i \text{ equals the line } \overline{uv} \text{ in } \mathcal{H}.$$

Another way of stating (iii) is

$$(v) \quad u \in V_i, v \in V_j, i \neq j \Rightarrow \overline{uv} \cap (W_i \cup W_j) = \{u, v\}.$$

(The conclusion of (v) can be strengthened to $|\overline{uv} \cap W_s| \leq 1$ for all s , but this is irrelevant to our argument.)

Next, let us show that

$$(vi) \quad p \in \overline{uv} \Rightarrow \overline{uv} \subseteq W_i \text{ for some } i.$$

Since u and v are distinct, we may assume that $u \neq p$, and so $u \in V_i$ for some i . We claim that $v \in W_i$. If $v = p$, then this is trivial; if $v \neq p$, then $p \in \overline{uv}$ implies that u and v are adjacent in Σ , and so $v \in V_i$. Now $u, v \in W_i$, and so $\overline{uv} \subseteq W_i$ by (iv).

From (vi), we will deduce that

(vii) $W_r \notin \mathcal{L}$ for some r .

By assumption, there is a vertex y other than p such that p and y belong to at least two maximal lines of \mathcal{H} ; this vertex y belongs to some V_r ; by (vi), every line containing both p and y must be a subset of W_r ; since at least two maximal lines contain both p and y , it follows that $W_r \notin \mathcal{L}$.

With S standing for the set of subscripts i such that W_i is a line of \mathcal{H}_i , facts (iv) and (vii) together show that $|S| \leq k - 1$, and so we may distinguish between the following three subcases:

SUBCASE 2.1.1: $|S| = 0$.

By the induction hypothesis, each \mathcal{H}_i has at least $|W_i|$ distinct lines; by (iv), each of these lines is a line of \mathcal{H} ; since $|W_i \cap W_j| = 1$ whenever $i \neq j$, all of these lines with $i = 1, \dots, k$ are distinct; it follows that $|\mathcal{L}| \geq \sum_{i=1}^k |W_i| = n + k - 1 > n$.

SUBCASE 2.1.2: $|S| = 1$.

We may assume that $S = \{1\}$. Now W_1 is a line of \mathcal{H} . By (v), the $|V_1| \cdot |V_2|$ lines \overline{uv} of \mathcal{H} with $u \in V_1, v \in V_2$ are all distinct; since they have nonempty intersections with V_2 , they are distinct from W_1 . By the induction hypothesis, each \mathcal{H}_i with $i \geq 2$ has at least $|W_i|$ distinct lines; by (iv), each of these lines is a line of \mathcal{H} ; since $|W_i \cap W_j| = 1$ whenever $i \neq j$, all of these lines with $i = 2, \dots, k$ are distinct; since they are disjoint from V_1 , they are distinct from W_1 and from all \overline{uv} with $u \in V_1, v \in V_2$. It follows that $|\mathcal{L}| \geq 1 + \sum_{i=2}^k |W_i| + |V_1| \cdot |V_2| \geq 1 + \sum_{i=2}^k |W_i| + |V_1| = n + k - 1 > n$.

SUBCASE 2.1.3: $2 \leq |S| \leq k - 1$.

Let W^* denote $\bigcup_{i \in S} W_i$ and let \mathcal{H}^* denote the sub-hypergraph of \mathcal{H} induced by W^* . From (vi) and the assumption $|S| \geq 2$, we deduce that no line of \mathcal{H} contains W^* . By the induction hypothesis, \mathcal{H}^* has at least $1 + \sum_{i \in S} |V_i|$ distinct lines; it follows that \mathcal{H} has at least $1 + \sum_{i \in S} |V_i|$ distinct lines \overline{uv} with $u, v \in W^*$. By the induction hypothesis, each \mathcal{H}_i with $i \notin S$ has at least $|W_i|$ distinct lines; by (iv), each of these lines is a line of \mathcal{H} ; since $|W_i \cap W_j| = 1$ whenever $i \neq j$, all of these lines with $i \notin S$ are distinct; since they are disjoint from $W^* \setminus \{p\}$, they are distinct from all \overline{uv} with $u, v \in W^*$. It follows that $|\mathcal{L}| \geq 1 + \sum_{i \in S} |V_i| + \sum_{i \notin S} |W_i| = n + (k - |S|) > n$.

SUBCASE 2.2: Σ is connected.

By (ii), the assumption of this subcase implies that Σ has a disconnected

complement. This means that

$$\Sigma = \Sigma_1 \vee \Sigma_2 \vee \dots \vee \Sigma_k,$$

where $k \geq 2$ and each Σ_i has a connected complement; it follows from (ii) that each Σ_i is either a single vertex or a disconnected graph. For each $i = 1, \dots, k$, let V_i denote the vertex set of Σ_i and let W_i denote $V_i \cup \{p\}$. We claim that

$$(viii) \quad x, y \in W_i, x \neq y, z \in V \setminus W_i \Rightarrow \{x, y, z\} \in \mathcal{E}.$$

When one of x, y is p , the conclusion follows from the fact that all vertices in V_i are adjacent in Σ to all vertices in $V \setminus W_i$. When x, y are nonadjacent vertices of Σ_i , the conclusion follows from the same fact, combined with (i). When x, y are adjacent vertices of Σ_i , consider a shortest path P from x to y in the complement of Σ_i . Since Σ_i is P_4 -free, its complement is P_4 -free; it follows that P has exactly three vertices. Let w be the unique interior vertex of P . Now (i) implies that $\{x, y, w\} \notin \mathcal{E}$, $\{x, z, w\} \in \mathcal{E}$, $\{y, z, w\} \in \mathcal{E}$; in turn, the fact that x, y, z, w do not induce two hyperedges implies that $\{x, y, z\} \in \mathcal{E}$.

Let \mathcal{H}_i denote the sub-hypergraph of \mathcal{H} induced by W_i . In the inductive argument, we shall use the following restatement of (viii):

$$(ix) \quad u, v \in W_i, u \neq v \Rightarrow \text{the line } \overline{uv} \text{ in } \mathcal{H} \text{ equals } Z \cup (V \setminus W_i), \text{ where } Z \text{ is the line } \overline{uv} \text{ in } \mathcal{H}_i.$$

Fact (ix) implies that

$$(x) \quad \text{no line of } \mathcal{H}_i \text{ equals } W_i;$$

in turn, the induction hypothesis applied to \mathcal{H}_i guarantees that it has at least $|W_i|$ distinct lines; now (ix) implies that

$$(xi) \quad \mathcal{H} \text{ has at least } |W_i| \text{ distinct lines } \overline{uv} \text{ with } u, v \in W_i.$$

In addition, (ix) implies that

$$(xii) \quad u, v \in W_i, x, y \in W_j, i \neq j, \overline{uv} = \overline{xy} \Rightarrow \overline{uv} = \overline{xy} = V \setminus \{p\}.$$

SUBCASE 2.2.1: $V \setminus \{p\} \notin \mathcal{L}$. In this subcase, (xii) guarantees that

$$u, v \in W_i, x, y \in W_j, i \neq j \Rightarrow \overline{uv} \neq \overline{xy},$$

and so (xi) implies that $|\mathcal{L}| \geq \sum_{i=1}^k |W_i| = n + k - 1 > n$.

SUBCASE 2.2.2: $V \setminus \{p\} \in \mathcal{L}$. Fact (xi) guarantees that \mathcal{H} has at least $|W_i| - 1$ distinct lines \overline{uv} such that $u, v \in W_i$ and $\overline{uv} \neq V \setminus \{p\}$, and so (xii), combined with the assumption of this subcase, implies that $|\mathcal{L}| \geq \sum_{i=1}^k (|W_i| - 1) + 1 = n$.

To complete the analysis of this subcase, let us consider its extremal hypergraphs, those with $|\mathcal{L}| = n$. Here,

- (xiii) each \mathcal{H}_i has precisely $|W_i|$ distinct lines and V_i is one of these lines;
 \mathcal{L} consists of all the sets $Z \cup (V \setminus W_i)$ such that Z is a line of some \mathcal{H}_i .

We are going to prove that

- (xiv) the hyperedge set \mathcal{E}_i of each \mathcal{H}_i is $\binom{V_i}{3}$

Since V_i is a line of \mathcal{H}_i , it has at least two vertices. If $|V_i| = 2$, then both (xiv) and (x) amount to saying that \mathcal{H}_i has no hyperedges. Now we will assume that $|V_i| \geq 3$. The induction hypothesis, combined with (x), guarantees that \mathcal{H}_i generates a near-pencil or a finite projective plane or is the complement of a Steiner triple system; since V_i is one of the lines of \mathcal{H}_i , proving (xiv) amounts to proving that \mathcal{H}_i generates a near-pencil. The possibility of \mathcal{H}_i generating a finite projective plane is excluded by the fact that one of the lines of \mathcal{H}_i (namely, V_i) includes all the vertices but one. The possibility of \mathcal{H}_i being the complement of a Steiner triple system is excluded by the fact that Σ_i is disconnected, and so it includes vertices u, v, w such that u is nonadjacent to both v, w : now u and p belong to at least two members of $\binom{W_i}{3} \setminus \mathcal{E}_i$ (namely, $\{u, v, p\}$ and $\{u, w, p\}$). This completes our proof of (xiv).

Next, let us prove that

- (xv) for every $i = 1, 2, \dots, k$ and every x in V_i , there is an L in \mathcal{L} such that $V_i \setminus L = \{x\}$.

Choose any vertex z in $V \setminus W_i$. Since $\overline{xz} \neq V$, there is a w in V such that $w \neq x$, $w \neq z$, and $\{x, z, w\} \notin \mathcal{E}$; fact (viii) implies that $w \notin W_i$.

Next, consider an arbitrary vertex y in $V_i \setminus \{x\}$. Fact (viii) guarantees that $\{x, y, z\} \in \mathcal{E}$ and $\{x, y, w\} \in \mathcal{E}$; in turn, the fact that x, y, z, w do not induce two hyperedges implies that $\{y, z, w\} \in \mathcal{E}$. We conclude that $y \in \overline{zw}$, and so $V_i \setminus \overline{zw} = \{x\}$, which completes our proof of (xv).

Finally, let us prove that

(xvi) $V \setminus \{x\} \in \mathcal{L}$ for all x in V .

Since $V \setminus \{p\} \in \mathcal{L}$ by assumption of this subcase, we may restrict our argument to vertices x distinct from p . Every such x belongs to some V_i and, by (xv), there is an L in \mathcal{L} such that $V_i \setminus L = \{x\}$; by (xiii), there are a subscript j and a line Z of \mathcal{H}_j such that $L = Z \cup (V \setminus W_j)$. Now $V_i \not\subseteq L$ and $V_r \subseteq L$ whenever $r \neq j$, and so $j = i$. By (xiv), every line of \mathcal{H}_i either equals V_i or includes p ; since $V_i \setminus Z = V_i \setminus L = \{x\}$, it follows that $p \in Z$. Since $V_i \setminus Z = \{x\}$ and $p \in Z$ together imply that $Z = W_i \setminus \{x\}$, we conclude that $L = V \setminus \{x\}$. This completes our proof of (xvi).

Since $|\mathcal{L}| = n$, fact (xvi) guarantees that \mathcal{L} consists of the n sets $V \setminus \{x\}$ with x ranging over V . This means that for every two vertices u and v , there is a unique vertex in $V \setminus \overline{uv}$, which is just another way of saying that \mathcal{H} is the complement of a Steiner triple system. \square

3 Metric and pseudometric hypergraphs

We say that a 3-uniform hypergraph (V, \mathcal{E}) is *metric* if there is a metric space $(V, dist)$ such that

$$\mathcal{E} = \{\{u, v, w\} : u, v, w \text{ are all distinct and } dist(u, v) + dist(v, w) = dist(u, w)\}.$$

Chen and Chvátal [8] asked whether or not all metric hypergraphs have the De Bruijn - Erdős property; this question was investigated further by Chiniforooshan and Chvátal [9].

All induced sub-hypergraphs of metric hypergraphs are metric, and so metric hypergraphs can be characterized as hypergraphs without certain induced sub-hypergraphs, namely, the minimal non-metric ones. If there are only finitely many minimal non-metric hypergraphs, then metric hypergraphs can be recognized in polynomial time. However, it is conceivable that there are infinitely many minimal non-metric hypergraphs and it is not clear whether metric hypergraphs can be recognized in polynomial time.

In this section, we will list three minimal non-metric hypergraphs. To begin, we will prove that the hypergraphs without the De Bruijn - Erdős property mentioned in Section 1 cannot provide a negative answer to the Chen-Chvátal question. All of these hypergraphs contain the 11-vertex hypergraph denoted \mathcal{F}_0 in Section 1. We will prove that \mathcal{F}_0 is not metric. In fact, we will prove that it contains an 8-vertex induced sub-hypergraph \mathcal{F}_1 , which is minimal non-metric. The vertex set of \mathcal{F}_1 is $\{1, 2\} \cup (\{a, b, c\} \times \{d, e\})$; its hyperedges are the $\binom{6}{3}$ three-point subsets of $\{a, b, c\} \times \{d, e\}$ and the nine three-point sets $\{i, (x_1, x_2), (y_1, y_2)\}$ with $x_i = y_i$.

We will also prove that no complement of a Steiner triple system with more than three vertices is metric. In fact, we will exhibit 6-vertex minimal non-metric hypergraphs \mathcal{F}_2 and \mathcal{F}_3 such that every complement of a Steiner triple system with more than three vertices contains at least one of \mathcal{F}_2 and \mathcal{F}_3 .

A ternary relation \mathcal{B} on a set V is called a *metric betweenness* if there is a metric $dist$ on V such that $(u, v, w) \in \mathcal{B}$ if and only if

$$u, v, w \text{ are all distinct and } dist(u, v) + dist(v, w) = dist(u, w).$$

Menger [19] seems to have been the first to study this relation. He proved that, in addition to the obvious properties

- (M0) if $(u, v, w) \in \mathcal{B}$, then u, v, w are three points,
- (M1) if $(u, v, w) \in \mathcal{B}$, then $(w, v, u) \in \mathcal{B}$,
- (M2) if $(u, v, w) \in \mathcal{B}$, then $(u, w, v) \notin \mathcal{B}$,

every metric betweenness \mathcal{B} has the property

- (M3) if $(u, v, w), (u, w, x) \in \mathcal{B}$, then $(u, v, x), (v, w, x) \in \mathcal{B}$.

We will call a ternary relation \mathcal{B} on a set V a *pseudometric betweenness* if it has properties (M0), (M1), (M2), (M3). Not every pseudometric betweenness is a metric betweenness: see [12] for more on this subject.

Every ternary relation \mathcal{B} on a set V that has property (M0) gives rise to a hypergraph $(V, \mathcal{E}(\mathcal{B}))$ by discarding the order on each triple in \mathcal{B} :

$$\mathcal{E}(\mathcal{B}) = \{\{u, v, w\} : (u, v, w) \in \mathcal{B}\}.$$

We will say that a 3-uniform hypergraph (V, \mathcal{E}) is *pseudometric* if there is a pseudometric betweenness \mathcal{B} on V such that $\mathcal{E} = \mathcal{E}(\mathcal{B})$. Every metric hypergraph is pseudometric, but the converse is false: the *Fano hypergraph* is

pseudometric but it is not metric. (This hypergraph has seven vertices and seven hyperedges, every two of which share a single vertex; like all 3-uniform hypergraphs in which no two hyperedges share two vertices, it is pseudometric; it has been proved [12, 7] that it is not metric, but neither of the two proofs is very short.) We will prove that $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ are not even pseudometric. (There are many other minimal non-pseudometric hypergraphs: our computer search revealed 113 non-isomorphic ones on six vertices.)

Question 1. *True or false? All pseudometric hypergraphs have the De Bruijn - Erdős property.*

In proving that \mathcal{F}_1 is not pseudometric, we shall rely on the following fact.

Lemma 1. *If \mathcal{B} is a pseudometric betweenness on a set V such that $\mathcal{E}(\mathcal{B}) = \binom{V}{3}$ and $|V| \geq 5$, then there is an injection $f : V \rightarrow \mathbf{R}$ such that $(x, y, z) \in \mathcal{B}$ if and only if $f(y)$ is between $f(x)$ and $f(z)$.*

Proof. We will use induction on $|V|$. To begin, we claim that

- (i) for some element p of V , the elements of $V \setminus \{p\}$ can be enumerated as v_1, v_2, \dots, v_{n-1} , in such a way that $(v_i, v_j, v_k) \in \mathcal{B}$ if and only if j is between i and k .

To justify this claim, we consider the case of $|V| = 5$ separately from the rest. Here, note that $\binom{5}{3}$ is not a multiple of 3, and so some a and b appear in one or two triples of the form (a, x, b) in \mathcal{B} . This means that there are a, b, c, d such that $(a, c, b) \in \mathcal{B}$ and $(a, d, b) \notin \mathcal{B}$. Since $\{a, d, b\} \in \mathcal{E}(\mathcal{B})$, we must have $(a, b, d) \in \mathcal{B}$ or $(b, a, d) \in \mathcal{B}$. Setting $v_1 = a, v_2 = c, v_3 = b, v_4 = d$ if $(a, b, d) \in \mathcal{B}$ and $v_1 = b, v_2 = c, v_3 = a, v_4 = d$ if $(b, a, d) \in \mathcal{B}$, we get $(v_1, v_2, v_3), (v_1, v_3, v_4) \in \mathcal{B}$; now (M3) with $u = v_1, v = v_2, w = v_3, x = v_4$ guarantees that $(v_1, v_2, v_4), (v_2, v_3, v_4) \in \mathcal{B}$. In the case of $|V| \geq 6$, claim (i) is just the induction hypothesis.

With (i) justified, we distinguish between two cases.

CASE 1: $(v_i, p, v_{i+1}) \in \mathcal{B}$ for some i .

In this case, we claim that the proof can be completed by setting $f(p) = i+0.5$ and $f(v_j) = j$ for all j . To justify this claim, we first use induction on j , with the basis at $j = i+1$ and (M3) applied to $(v_i, p, v_j), (v_i, v_j, v_{j+1})$ in the induction step, to show that $(v_i, p, v_j) \in \mathcal{B}$ for all $j = i+1, i+2, \dots, n-1$. In turn, (M3) applied to $(v_i, p, v_j), (v_i, v_j, v_k)$ shows that $(p, v_j, v_k) \in \mathcal{B}$ whenever

$i+1 \leq j < k \leq n-1$. Appealing to the flip symmetry of the sequence v_1, v_2, \dots, v_{n-1} , we also note that $(v_r, v_s, p) \in \mathcal{B}$ whenever $1 \leq r < s \leq i$. Finally, given any r and j such that $1 \leq r < i$ and $i+1 \leq j \leq n-1$, we apply (M3) to $(v_i, p, v_j), (v_r, v_i, v_j)$ in order to check that $(v_r, p, v_j) \in \mathcal{B}$. This completes our analysis of Case 1.

CASE 2: For each $i = 1, \dots, n-2$, we have $(p, v_i, v_{i+1}) \in \mathcal{B}$ or $(v_i, v_{i+1}, p) \in \mathcal{B}$.

In this case, we claim that the proof can be completed by setting $f(v_j) = j$ for all j and either $f(p) = 0$ or $f(p) = n$. To justify this claim, we will first prove that

- (ii) there is no i such that $2 \leq i \leq n-2$ and $(v_{i-1}, v_i, p), (p, v_i, v_{i+1}) \in \mathcal{B}$,
- (iii) there is no i such that $2 \leq i \leq n-2$ and $(p, v_{i-1}, v_i), (v_i, v_{i+1}, p) \in \mathcal{B}$:

To justify (ii), assume the contrary. Since $\{v_{i-1}, p, v_{i+1}\}$ belongs to $\mathcal{E}(\mathcal{B})$, we may label its elements as x_1, x_2, x_3 in such a way that $(x_1, x_2, x_3) \in \mathcal{B}$. Since

$$(v_{i-1}, v_i, p), (p, v_i, v_{i+1}), (v_{i+1}, v_i, v_{i-1}) \in \mathcal{B},$$

we have $(x_1, v_i, x_2), (x_2, v_i, x_3), (x_3, v_i, x_1) \in \mathcal{B}$; now (M3) with $u = x_1, v = v_i, w = x_2, x = x_3$ implies $(v_i, x_2, x_3) \in \mathcal{B}$, which, together with $(x_2, v_i, x_3) \in \mathcal{B}$, contradicts (M2).

To justify (iii), assume the contrary. Since $n > 4$, we have $i > 2$ or $i < n-2$ or both; symmetry lets us assume that $i < n-2$. Now (ii) with $i+1$ in place of i guarantees that $(p, v_{i+1}, v_{i+2}) \notin \mathcal{B}$; the assumption of this case guarantees that $(v_{i+1}, p, v_{i+2}) \notin \mathcal{B}$; it follows that $(v_{i+1}, v_{i+2}, p) \in \mathcal{B}$. There are three ways of including $\{p, v_{i-1}, v_{i+1}\}$ in $\mathcal{E}(\mathcal{B})$; we will show that each of them leads to a contradiction. If $(p, v_{i-1}, v_{i+1}) \in \mathcal{B}$, then (M3) with (p, v_{i+1}, v_i) implies $(v_{i-1}, v_{i+1}, v_i) \in \mathcal{B}$, contradicting $(v_{i-1}, v_i, v_{i+1}) \in \mathcal{B}$. If $(v_{i+1}, p, v_{i-1}) \in \mathcal{B}$, then (M3) with (v_{i+1}, v_{i+2}, p) implies $(v_{i+1}, v_{i+2}, v_{i-1}) \in \mathcal{B}$, contradicting $(v_{i-1}, v_{i+1}, v_{i+2}) \in \mathcal{B}$. If $(p, v_{i+1}, v_{i-1}) \in \mathcal{B}$, then (M3) with (p, v_{i-1}, v_i) implies $(v_{i+1}, v_{i-1}, v_i) \in \mathcal{B}$, contradicting $(v_{i-1}, v_i, v_{i+1}) \in \mathcal{B}$.

Claims (ii) and (iii), combined with the assumption of this case, imply that we have either $(p, v_i, v_{i+1}) \in \mathcal{B}$ for all $i = 1, \dots, n-2$ or else $(v_{i-1}, v_i, p) \in \mathcal{B}$ for all $i = 2, \dots, n-1$. In the first case, induction on d with the basis at $d = 1$ and (M3) applied to $(p, v_i, v_{i+d}), (p, v_{i+d}, v_{i+d+1})$ in the induction step shows that $(p, v_i, v_{i+d}) \in \mathcal{B}$ for all $d = 1, \dots, n-1-i$; it follows that we

may set $f(p) = 0$ and $f(v_j) = j$ for all j . In the second case, induction on d with the basis at $d = 1$ and (M3) applied to (v_{i-d}, v_i, p) , $(v_{i-(d+1)}, v_{i-d}, p)$ in the induction step shows that $(v_{i-d}, v_i, p) \in \mathcal{B}$ for all $d = 1, \dots, i - 1$; it follows that we may set $f(v_j) = j$ for all j and $f(p) = n$. This completes our analysis of Case 2. \square

A weaker version of Lemma 1 was proved by Richmond and Richmond [21] and later also by Dovgoshei and Dordovskii [15]: there, the assumption that \mathcal{B} is pseudometric is replaced by the stronger assumption that \mathcal{B} is metric. As noted in [15], this weaker version of Lemma 1 implies a special case ($d = 1$ and finite spaces) of the following result of Menger ([19], Satz 1): *If every $(d + 3)$ -point subspace of a metric space admits an isometric embedding into \mathbf{R}^d , then the whole space admits an isometric embedding into \mathbf{R}^d .*

The conclusion of Lemma 1 may fail when $|V| = 4$: here, if \mathcal{B} includes two triples of the form (u, v, w) , (u, w, x) , then (M3) implies that it is isomorphic to

$$\{(a, b, c), (c, b, a), (a, b, d), (d, b, a), (a, c, d), (d, c, a), (b, c, d), (d, c, b)\}$$

as in the lemma's conclusion, but \mathcal{B} may include no such triples, in which case it is isomorphic to

$$\{(a, b, c), (c, b, a), (b, c, d), (d, c, b), (c, d, a), (a, d, c), (d, a, b), (b, a, d)\}.$$

This has been also pointed out (again, with “pseudometric” replaced by “metric”) by Richmond and Richmond [21] and by Dovgoshei and Dordovskii [15].

Theorem 3. \mathcal{F}_1 is a minimal non-metric hypergraph.

Proof. We will prove that \mathcal{F}_1 is not pseudometric and that all its proper induced sub-hypergraphs are metric.

Recall that the vertex set of \mathcal{F}_1 is $\{1, 2\} \cup (\{a, b, c\} \times \{d, e\})$; its hyperedges are the $\binom{6}{3}$ three-point subsets of $\{a, b, c\} \times \{d, e\}$ and the nine three-point sets $\{i, (x_1, x_2), (y_1, y_2)\}$ with $x_i = y_i$. For each r in $\{a, b, c\} \times \{d, e\}$ and for each $i = 1, 2$, let r_i denote the i -th component of r . We claim that

(\star) For every injection $f : \{a, b, c\} \times \{d, e\} \rightarrow \mathbf{R}$,
there exist r, s, t, u in $\{a, b, c\} \times \{d, e\}$ such that

r, s, t, u are four vertices,
 $f(s)$ is between $f(r)$ and $f(t)$,
 $f(u)$ is not between $f(r)$ and $f(t)$, and
either $r_1 = t_1$ or else $r_2 = t_2, s_2 = u_2$.

To verify this claim, we may assume without loss of generality that the range of f is $\{0, 1, 2, 3, 4, 5\}$. If there are distinct r, t with $r_1 = t_1$ and $2 \leq |f(r) - f(t)| \leq 4$, then (\star) can be satisfied by these r, t and a suitable choice of s, u . If there are distinct r, t with $r_2 = t_2$ and $|f(r) - f(t)| = 3$, then (\star) can be satisfied by these r, t and a suitable choice of s, u . If neither of these two conditions is met, then

$$\begin{aligned} r \neq t, r_1 = t_1 &\Rightarrow |f(r) - f(t)| \in \{1, 5\}, \\ r \neq t, r_2 = t_2 &\Rightarrow |f(r) - f(t)| \in \{1, 2, 4, 5\}, \end{aligned}$$

in which case $x_2 = y_2 \Rightarrow f(x) \equiv f(y) \pmod{2}$. But then (\star) can be satisfied by any choice of distinct r, t with $r_2 = t_2$ and a suitable choice of s, u .

To prove that \mathcal{F}_1 is not pseudometric, assume the contrary: there is a pseudometric betweenness \mathcal{B} such that $\mathcal{E}(\mathcal{B})$ is the hyperedge set of \mathcal{F}_1 . Now all 3-point subsets of $\{a, b, c\} \times \{d, e\}$ belong to $\mathcal{E}(\mathcal{B})$, and so Lemma 1 guarantees the existence of an injection $f : \{a, b, c\} \times \{d, e\} \rightarrow \mathbf{R}$ such that $(x, y, z) \in \mathcal{B}$ if and only if $f(y)$ is between $f(x)$ and $f(z)$. Next, (\star) implies that there are distinct r, s, t, u in $\{a, b, c\} \times \{d, e\}$ such that

$$(r, s, t) \in \mathcal{B},$$

$(r, u, t) \notin \mathcal{B}$, and either $r_1 = t_1$ or else $r_2 = t_2, s_2 = u_2$. Since $(r, u, t) \notin \mathcal{B}$ and $\{r, u, t\} \in \mathcal{E}(\mathcal{B})$, we have $(r, t, u) \in \mathcal{B}$ or $(t, r, u) \in \mathcal{B}$; after switching r and t if necessary, we may assume that

$$(r, t, u) \in \mathcal{B}.$$

Writing $x = 1$ if $r_1 = t_1$ and $x = 2$ if $r_2 = t_2$, note that

$$\{x, r, t\} \in \mathcal{E}(\mathcal{B}), \{x, r, s\} \notin \mathcal{E}(\mathcal{B}), \{x, r, u\} \notin \mathcal{E}(\mathcal{B}).$$

Since $\{x, r, t\} \in \mathcal{E}(\mathcal{B})$, we may distinguish between three cases.

In case $(x, r, t) \in \mathcal{B}$, property (M3) and $(r, s, t) \in \mathcal{B}$ imply $(x, r, s) \in \mathcal{B}$, contradicting $\{x, r, s\} \notin \mathcal{E}(\mathcal{B})$.

In case $(r, t, x) \in \mathcal{B}$, property (M3) and $(r, s, t) \in \mathcal{B}$ imply $(r, s, x) \in \mathcal{B}$, contradicting $\{x, r, s\} \notin \mathcal{E}(\mathcal{B})$.

In case $(r, x, t) \in \mathcal{B}$, property (M3) and $(r, t, u) \in \mathcal{B}$ imply $(r, x, u) \in \mathcal{B}$, contradicting $\{x, r, u\} \notin \mathcal{E}(\mathcal{B})$.

Symmetry of \mathcal{F}_1 reduces checking that all its proper induced sub-hypergraphs are metric to checking just three cases: vertex 1 removed, vertex 2 removed, and a vertex in $\{a, b, c\} \times \{d, e\}$ removed. Here are the distance functions of the corresponding three metric spaces:

$\mathcal{F}_1 \setminus 1 :$

	(a, d)	(b, d)	(c, d)	(a, e)	(b, e)	(c, e)	2
(a, d)	0	1	2	3	4	5	3
(b, d)	1	0	1	2	3	4	2
(c, d)	2	1	0	1	2	3	1
(a, e)	3	2	1	0	1	2	1
(b, e)	4	3	2	1	0	1	2
(c, e)	5	4	3	2	1	0	3
2	3	2	1	1	2	3	0

$\mathcal{F}_1 \setminus 2 :$

	(a, d)	(a, e)	(b, d)	(b, e)	(c, d)	(c, e)	1
(a, d)	0	2	4	6	8	10	6
(a, e)	2	0	2	4	6	8	4
(b, d)	4	2	0	2	4	6	5
(b, e)	6	4	2	0	2	4	3
(c, d)	8	6	4	2	0	2	4
(c, e)	10	8	6	4	2	0	6
1	6	4	5	3	4	6	0

$$\mathcal{F}_1 \setminus (c, d) :$$

	(a, d)	(b, d)	(b, e)	(c, e)	(a, e)	1	2
(a, d)	0	2	4	6	8	5	4
(b, d)	2	0	2	4	6	5	2
(b, e)	4	2	0	2	4	3	2
(c, e)	6	4	2	0	2	3	4
(a, e)	8	6	4	2	0	3	6
1	5	5	3	3	3	0	4
2	4	2	2	4	6	4	0

□

Next, we will consider the hypergraphs \mathcal{F}_2 and \mathcal{F}_3 defined by $\mathcal{F}_2 = (V, \mathcal{E}_2)$ and $\mathcal{F}_3 = (V, \mathcal{E}_3)$, where

$$\begin{aligned} V &= \{a_1, b_1, a_2, b_2, a_3, b_3\}, \\ \mathcal{E}_2 &= \binom{V}{3} \setminus \{\{a_1, b_2, b_3\}, \{a_2, b_1, b_3\}, \{a_3, b_1, b_2\}, \{a_1, a_2, a_3\}\}, \\ \mathcal{E}_3 &= \binom{V}{3} \setminus \{\{a_1, b_2, b_3\}, \{a_2, b_1, b_3\}, \{a_3, b_1, b_2\}\}. \end{aligned}$$

Theorem 4. \mathcal{F}_2 and \mathcal{F}_3 are minimal non-metric hypergraphs.

Proof. We will prove that neither of \mathcal{F}_2 and \mathcal{F}_3 is pseudometric and that all their proper induced sub-hypergraphs are metric.

To prove that neither of \mathcal{F}_2 and \mathcal{F}_3 is pseudometric, assume the contrary: some pseudometric betweenness \mathcal{B} on $\{a_1, b_1, a_2, b_2, a_3, b_3\}$ has

$$\begin{aligned} \binom{V}{3} \setminus \{\{a_1, b_2, b_3\}, \{a_2, b_1, b_3\}, \{a_3, b_1, b_2\}, \{a_1, a_2, a_3\}\} &\subseteq \mathcal{E}(\mathcal{B}) \\ &\subseteq \binom{V}{3} \setminus \{\{a_1, b_2, b_3\}, \{a_2, b_1, b_3\}, \{a_3, b_1, b_2\}\}. \end{aligned}$$

Since $\{b_1, b_2, b_3\} \in \mathcal{E}(\mathcal{B})$, we may assume (after permuting the subscripts if necessary) that

$$(i) \quad (b_1, b_2, b_3) \in \mathcal{B}.$$

Now, since $\{a_1, b_2, b_3\} \notin \mathcal{E}(\mathcal{B})$, property (M3) implies that $(b_1, b_3, a_1) \notin \mathcal{B}$ and $(b_3, b_1, a_1) \notin \mathcal{B}$; since $\{a_1, b_1, b_3\} \in \mathcal{E}(\mathcal{B})$, it follows that

$$(b_1, a_1, b_3) \in \mathcal{B}.$$

Next, since $\{a_1, b_2, b_3\} \notin \mathcal{E}(\mathcal{B})$, property (M3) implies that $(b_1, a_1, b_2) \notin \mathcal{B}$ and $(b_1, b_2, a_1) \notin \mathcal{B}$; since $\{a_1, b_1, b_2\} \in \mathcal{E}(\mathcal{B})$, it follows that

$$(a_1, b_1, b_2) \in \mathcal{B}.$$

Finally, since $\{a_3, b_1, b_2\} \notin \mathcal{E}(\mathcal{B})$, property (M3) implies that $(a_1, b_2, a_3) \notin \mathcal{B}$ and $(b_2, a_1, a_3) \notin \mathcal{B}$; since $\{a_1, b_2, b_3\} \in \mathcal{E}(\mathcal{B})$, it follows that

$$(ii) (b_2, a_3, a_1) \in \mathcal{B}.$$

Switching subscripts 1 and 3 in this derivation of (ii) from (i), we observe that (i) also implies

$$(iii) (b_2, a_1, a_3) \in \mathcal{B}.$$

But (ii) and (iii) together contradict property (M2).

Symmetry reduces checking that all proper induced sub-hypergraphs of \mathcal{F}_2 and \mathcal{F}_3 are metric to checking that two hypergraphs are metric: $\mathcal{F}_3 \setminus a_1$ (isomorphic to $\mathcal{F}_3 \setminus a_2$, $\mathcal{F}_3 \setminus a_3$, and to all five-point induced sub-hypergraphs of \mathcal{F}_2) and $\mathcal{F}_3 \setminus b_1$ (isomorphic to $\mathcal{F}_3 \setminus b_2$ and $\mathcal{F}_3 \setminus b_3$). Here are distance functions certifying that these two hypergraphs are metric:

$$\mathcal{F}_3 \setminus a_1 :$$

	a_2	a_3	b_3	b_2	b_1
a_2	0	1	2	1	2
a_3	1	0	1	2	3
b_3	2	1	0	1	2
b_2	1	2	1	0	3
b_1	2	3	2	3	0

$$\mathcal{F}_3 \setminus b_1 :$$

	a_2	a_1	a_3	b_3	b_2
a_2	0	1	2	1	1
a_1	1	0	1	2	2
a_3	2	1	0	1	1
b_3	1	2	1	0	2
b_2	1	2	1	2	0

□

Corollary 1. *No complement of a Steiner triple system with more than three vertices is metric.*

Proof. We will point out that every complement (V, \mathcal{E}) of a Steiner triple system with more than three vertices contains at least one of \mathcal{F}_2 and \mathcal{F}_3 .

To do this, note that, since V includes more than three vertices, it includes pairwise distinct vertices b_1, b_2, b_3 such that $\{b_1, b_2, b_3\} \in \mathcal{E}$. Since every two vertices in V belong to precisely one member of $\binom{V}{3} \setminus \mathcal{E}$, it follows first that there are vertices a_1, a_2, a_3 such that $\{a_1, b_2, b_3\}, \{a_2, b_1, b_3\}, \{a_3, b_1, b_2\} \notin \mathcal{E}$, then that $a_1, a_2, a_3, b_1, b_2, b_3$ are six distinct vertices, and finally that these six vertices induce in (V, \mathcal{E}) one of \mathcal{F}_2 and \mathcal{F}_3 . \square

4 Variations

In this section, we prove two variations on Theorem 2.

Theorem 5. *If, in a 3-uniform hypergraph, no sub-hypergraph induced by four vertices has one or three hyperedges, then the hypergraph has the De Bruijn - Erdős property.*

Proof. Let (V, \mathcal{E}) be a 3-uniform hypergraph in which no four vertices induce one or three hyperedges. We claim that

$$(\star) \quad \overline{uv} = \overline{uw} \text{ and } v \neq w \Rightarrow \overline{vw} = V.$$

To justify this claim, consider an arbitrary vertex x other than u, v, w : we propose to show that $x \in \overline{vw}$. Since $\overline{uv} = \overline{uw}$, we have $\{u, v, w\} \in \mathcal{E}$, and so the four vertices u, v, w, x induce one or three hyperedges in addition to $\{u, v, w\}$; since $\{u, v, x\} \in \mathcal{E} \Leftrightarrow \{u, w, x\} \in \mathcal{E}$, it follows that $\{v, w, x\} \in \mathcal{E}$.

To prove that (V, \mathcal{E}) has the De Bruijn - Erdős property, we may assume that none of its lines equals V . Now take any line L and any vertex v in $V \setminus L$. All the lines \overline{uv} with $u \neq v$ are pairwise distinct by (\star) and L is distinct from all of them since it does not contain v . \square

Theorem 6. *If, in a 3-uniform hypergraph, no sub-hypergraph induced by four vertices has four hyperedges, then the hypergraph has the De Bruijn - Erdős property.*

Proof. Let (V, \mathcal{E}) be a 3-uniform hypergraph in which no four vertices induce four hyperedges; let \mathcal{H} denote this hypergraph and let n stand for the number of its vertices. Assuming that $n \geq 4$, we propose to prove by induction on n that \mathcal{H} has at least n distinct lines. The induction basis, $n = 4$, can be verified routinely.

In the induction step, we may assume that some \overline{pq} has at least four vertices: otherwise the De Bruijn - Erdős theorem guarantees right away that \mathcal{H} has at least n distinct lines. Enumerate the vertices of \mathcal{H} as p, v_2, v_3, \dots, v_n with $v_2 = q$. By the induction hypothesis, at least $n - 1$ of the lines $\overline{v_r v_s}$ are distinct. We will complete the induction step by showing that at least one of the lines $\overline{pv_2}, \overline{pv_3}, \dots, \overline{pv_n}$ is distinct from all of them.

For this purpose, assume the contrary: each $\overline{pv_i}$ equals some $\overline{v_{r(i)} v_{s(i)}}$. Under this assumption, we are going to find four vertices inducing four hyperedges. To begin with, we may assume that one of $r(i), s(i)$ must be i : otherwise $p, v_i, v_{r(i)}, v_{s(i)}$ are four vertices inducing four hyperedges and we are done. It follows that we may set $r(i) = i$, and so $\overline{pv_i} = \overline{v_i v_{s(i)}}$ for all i . Let us write j for $s(2)$.

CASE 1: $s(j) = 2$.

In this case, $\overline{pv_2} = \overline{v_2 v_j} = \overline{pv_j}$; by assumption, this line has at least four vertices; p, v_2, v_j , and any one of its other vertices induce four hyperedges, a contradiction.

CASE 2: $s(j) \neq 2$.

In this case, $p, v_2, v_j, v_{s(j)}$ are four vertices. We have $\{p, v_2, v_j\} \in \mathcal{E}$ since $\overline{pv_2} = \overline{v_2 v_j}$ and we have $\{p, v_j, v_{s(j)}\} \in \mathcal{E}$ since $\overline{pv_j} = \overline{v_j v_{s(j)}}$; now $v_2 \in \overline{pv_j}$, and so $\overline{pv_j} = \overline{v_j v_{s(j)}}$ implies $\{v_2, v_j, v_{s(j)}\} \in \mathcal{E}$; next, $v_{s(j)} \in \overline{v_2 v_j}$, and so $\overline{pv_2} = \overline{v_2 v_j}$ implies $\{p, v_2, v_{s(j)}\} \in \mathcal{E}$. But then $p, v_2, v_j, v_{s(j)}$ induce four hyperedges, a contradiction. \square

For all sufficiently large n (certainly for all n at least 27 and possibly for all n), the conclusion of Theorem 6 can be strengthened: the hypergraph has at least as many distinct lines as it has vertices whether or not one of its lines consists of all its vertices. In fact, the number of distinct lines grows much faster with the number of vertices:

Theorem 7. *If, in a 3-uniform hypergraph with n vertices, no sub-hypergraph induced by four vertices has four hyperedges, then the hypergraph has at least $(n/3)^{3/2}$ distinct lines.*

Proof. Let (V, \mathcal{E}) denote the hypergraph and let m denote the number of its distinct lines. We will proceed by induction on n . For the induction basis, we choose the range $n \leq 3$, where the inequality $m \geq (n/3)^{3/2}$ holds trivially. In the induction step, consider a largest set S of unordered pairs of distinct

vertices such that all the lines \overline{vw} with $\{v, w\} \in S$ are identical and write $s = |S|$.

CASE 1: $s \leq n^{1/2} + 1$.

By assumption of this case, we have

$$m \geq \frac{\binom{n}{2}}{n^{1/2} + 1} = \frac{1}{2}n^{3/2} \cdot \frac{n^{1/2}}{n^{1/2} + 1} \cdot \frac{n-1}{n};$$

since $n \geq 4$, we have

$$\frac{1}{2}n^{3/2} \cdot \frac{n^{1/2}}{n^{1/2} + 1} \cdot \frac{n-1}{n} \geq \frac{1}{2}n^{3/2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{4}n^{3/2} > \left(\frac{n}{3}\right)^{3/2}.$$

CASE 2: $s > n^{1/2} + 1$.

By assumption of this case and since $n \geq 4$, we have $s > 3$. Every two pairs in S must share a vertex (else the four vertices would induce four hyperedges); since $s > 3$, it follows that there is a vertex common to all the pairs in S , and so these pairs can be enumerated as $\{u, v_1\}, \{u, v_2\}, \dots, \{u, v_s\}$. We are going to prove that

(\star) each of the lines $\overline{v_i v_j}$ with $1 \leq i < j \leq s$ is uniquely defined

in the sense that $\overline{v_i v_j} = \overline{xy} \Rightarrow \{x, y\} = \{v_i, v_j\}$. To do this, consider vertices v_i, v_j, x, y such that $\overline{v_i v_j} = \overline{xy}$. These vertices cannot be all distinct (else they would induce four hyperedges), and so symmetry lets us assume that $x = v_i$; we will derive a contradiction from the assumption that $y \neq v_j$. Since $y \in \overline{v_i y} = \overline{v_i v_j}$, we have $\{y, v_i, v_j\} \in \mathcal{E}$; since $\overline{v_i y} = \overline{v_i v_j} \neq \overline{uv_i}$, we have $y \neq u$; since $\overline{uv_i} = \overline{uv_j}$, we have $\{u, v_i, v_j\} \in \mathcal{E}$; now $u \in \overline{v_i v_j} = \overline{v_i y}$, and so $\{u, v_i, y\} \in \mathcal{E}$; finally, $y \in \overline{uv_i} = \overline{uv_j}$ implies $\{y, u, v_j\} \in \mathcal{E}$. But then the four vertices u, v_i, v_j, y induce four hyperedges; this contradiction completes our proof of (\star).

Let \mathcal{L}_1 denote the set of all lines $\overline{v_i v_j}$, let \mathcal{L}_2 denote the set of all lines \overline{xy} with $x, y \notin \{v_1, v_2, \dots, v_s\}$, and let us set $c = 3^{-3/2}$. By (\star), we have $|\mathcal{L}_1| = \binom{s}{2}$ and $\mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset$; by the induction hypothesis, we have $|\mathcal{L}_2| \geq c(n-s)^{3/2}$; it follows that

$$m \geq \binom{s}{2} + c(n-s)^{3/2} > \binom{s}{2} + cn^{3/2} - \frac{3}{2}cn^{1/2}s > cn^{3/2} + \frac{1}{2}n^{1/2}s(1-3c) > cn^{3/2}.$$

□

For large n , the constant $3^{-3/2}$ in the lower bound of Theorem 7 can be improved by more careful analysis, but the magnitude of this lower bound, $n^{3/2}$, is the best possible. To see this, consider the hypergraph $(V_1 \cup \dots \cup V_k, \mathcal{E})$, where V_1, \dots, V_k are pairwise disjoint, $\{u, v, w\} \in \mathcal{E}$ if and only if $u, v \in V_i, w \in V_j, i < j$, and $u \neq v$. Here, no four vertices induce four hyperedges; the lines are all the sets $\{u, v\} \cup V_{i+1} \cup \dots \cup V_k$ and all the sets $V_i \cup \{w\}$ such that $w \in V_{i+1} \cup \dots \cup V_k$; when $|V_i| = k$ for all i , their total number is $k^2(k-1)$.

In a sense, Theorem 7 is the only theorem of its kind: in the hypergraph $(V, \binom{V}{3})$, every sub-hypergraph induced by four vertices has four hyperedges and the hypergraph has only one line.

Combining Theorems 2, 5, 6 suggests the following questions:

Question 2. *True or false? If, in a 3-uniform hypergraph, every sub-hypergraph induced by four vertices has at least two hyperedges, then the hypergraph has the De Bruijn - Erdős property.*

Question 3. *True or false? If, in a 3-uniform hypergraph, every sub-hypergraph induced by four vertices has one or two or four hyperedges, then the hypergraph has the De Bruijn - Erdős property.*

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References

- [1] N. Alon, K.E. Mellinger, D. Mubayi, J. Verstraëte, The de Bruijn-Erdős theorem for hypergraphs, [arXiv:1007.4150v1 \[math.CO\]](#).
- [2] C. Berge, *Graphes et hypergraphes*. Monographies Universitaires de Mathématiques, No. 37. Dunod, Paris, 1970.

- [3] L. Babai, On the nonuniform Fisher inequality, *Discrete Mathematics* **66** (1987), 303–307.
- [4] L.M. Blumenthal, *Theory and Applications of Distance Geometry*. Oxford University Press, Oxford, 1953.
- [5] T.C. Bose, A note on Fisher’s inequality for balanced incomplete block designs, *Ann. Math. Statistics* **20** (1949), 619 – 620.
- [6] H. Busemann, *The Geometry of Geodesics*. Academic Press, New York, 1955.
- [7] X. Chen, The Sylvester-Chvátal theorem, *Discrete & Computational Geometry* **35** (2006), 193 – 199.
- [8] X. Chen and V. Chvátal, Problems related to a de Bruijn - Erdős theorem, *Discrete Applied Mathematics* **156** (2008), 2101 – 2108.
- [9] E. Chiniforooshan and V. Chvátal, A De Bruijn - Erdős theorem and metric spaces, *Discrete Mathematics & Theoretical Computer Science* **13** (2011), 67 – 74.
- [10] A. Chowdhury, On a conjecture of Frankl and Füredi. *The Electronic Journal of Combinatorics* **18** (2011), Paper 56, 16 pages.
- [11] J.A. Bondy, and U.S.R. Murty, *Graph theory*. Springer, New York, 2008.
- [12] V. Chvátal, Sylvester-Gallai theorem and metric betweenness, *Discrete & Computational Geometry* **31** (2004), 175 – 195.
- [13] N.G. De Bruijn and P. Erdős, On a combinatorial problem, *Indagationes Mathematicae* **10** (1948), 421 – 423.
- [14] N.G. De Bruijn and P. Erdős, A colour problem for infinite graphs and a problem in the theory of relations, *Indagationes Mathematicae* **13** (1951), 369 – 373.
- [15] A.A. Dovgoshei and D.V. Dordovskii, Betweenness relation and isometric imbeddings of metric spaces, *Ukrainian Mathematical Journal* **61** (2009), 1556 – 1567.

- [16] P. Frankl, and Z. Füredi, Families of finite sets with missing intersections, *Finite and infinite sets, (Eger, 1981), Colloq. Math. Soc. János Bolyai* **37**, North-Holland, Amsterdam, 1984, pp. 305 – 318,
- [17] P. Frankl, and Z. Füredi, A sharpening of Fisher’s inequality, *Discrete Mathematics* **90** (1991), 103 – 107.
- [18] D.J. Houck and M.E. Paul, On a theorem of de Bruijn and Erdős, *Linear Algebra Appl.* **23** (1979), 157–165.
- [19] K. Menger, Untersuchungen über allgemeine metrik, *Mathematische Annalen* **100** (1928), 75 – 163.
- [20] G.V. Ramanan, Proof of a conjecture of Frankl and Füredi, *Journal of Combinatorial Theory Series A* **79** (1997), 53 – 67.
- [21] B. Richmond and T. Richmond, Metric spaces in which all triangles are degenerate, *American Mathematical Monthly* **104** (1997), 713 – 719.
- [22] H.J. Ryser, An extension of a theorem of de Bruijn and Erdős on combinatorial designs, *Journal of Algebra* **10** (1968), 246 – 261.
- [23] D. Seinsche, On a property of the class of n -colorable graphs. *Journal of Combinatorial Theory Series B* **16** (1974), 191 – 193.
- [24] H.S. Snevily, On generalizations of the de Bruijn-Erdős theorem, *Journal of Combinatorial Theory Series A* **68** (1994), 232 – 238.
- [25] L.E. Varga, Geometric matrices and an inequality for $(0, 1)$ -matrices, *Discrete Mathematics* **82** (1990), 303 – 315.