Immersion in four-edge-connected graphs

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Abstract

Fix $g > 1$. Every graph of large enough tree-width contains a $g \times g$ grid as a minor; but here we prove that every four-edge-connected graph of large enough tree-width contains a $g \times g$ grid as an immersion (and hence contains any fixed graph with maximum degree at most four as an immersion). This result has a number of applications.

1 Introduction

Let G, H be graphs. (All graphs in this paper are finite, possibly with loops or parallel edges.) A *weak immersion* of H in G is a map η, with domain $V(H) \cup E(H)$, mapping each vertex of H to a vertex of G, and each edge of H to a path or cycle of G, satisfying the following:

- $\eta(u) \neq \eta(v)$ for all distinct $u, v \in V(H)$
- for each $e \in E(H)$ with distinct ends u and v, $\eta(e)$ is a path of G with ends $\eta(u), \eta(v)$;
- for each loop in H with end v, $\eta(e)$ is a cycle of G passing through $\eta(v)$; and
- for all distinct $e, f \in E(H)$, $E(\eta(e) \cap \eta(f)) = \emptyset$.

If in addition we have

- for all $v \in V(H)$ and $e \in E(H)$, if e is not incident with v in H then $\eta(v) \notin V(\eta(e))$

then η is called a *strong immersion*. This paper is only concerned with strong immersion, and from now on we omit "strong", and just speak of "immersion". If there is an immersion of H in G, we say that "H can be immersed in G" and "G contains H as an immersion" (or just "G immerses H"). If in addition, for all distinct $e, f \in E(H)$, every vertex of $\eta(e) \cap \eta(f)$ is equal to $\eta(v)$ for some $v \in V(H)$ incident in H with both e and f, then η is called a *subdivision map* of H in G.

If $g > 1$ is an integer, the $g \times g$ *grid* is a graph with vertex set $\{v_{ij} : 1 \leq i, j \leq g\}$, where v_{ij} is adjacent to $v_{i'j'}$ if $|i - i'| + |j - j'| = 1$. We denote this graph by J_g.

A *tree-decomposition* of a graph G is a pair $(T, (W_t : t \in V(T)))$, such that

- T is a tree
- $W_t \subseteq V(G)$ for each $t \in V(T)$
- $V(G) = \bigcup(W_t : t \in V(T))$
- for every edge uv of G, there exists $t \in V(T)$ with $u, v \in W_t$
- for $t, t', t'' \in V(T)$, if t' belongs to the path of T between t and t'', then $W_t \cap W_{t''} \subseteq W_{t'}$.

We call $\max(|W_t| - 1 : t \in V(T))$ the *width* of the tree-decomposition, and say that G has *tree-width* k if k is minimum such that G admits a tree-decomposition of width k.

We say that H is a *minor* of G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. The following is well-known [3]:

1.1 *For all $g > 1$ there exists k such that every graph with tree-width at least k contains J_g as a minor.*

(Note that this is sharp in the sense that for all k there exists g such that no graph of tree-width less than k contains J_g as a minor.) In this paper we prove a similar result for immersion, the following. (Two versions of this result were found independently by two subsets of the authors, and one of these versions appears in [2].)

1.2 *For all $g > 1$ there exists $k \geq 0$ such that every four-edge-connected graph with tree-width at least k contains J_g as an immersion.*

This is not exactly an analogue of 1.1, because it is not sharp in the same sense. It is not true that for all k there exists g such that no four-edge-connected graph of tree-width less than k contains J_g as a minor. To see this, let G be obtained from a star with $g^2 + 1$ vertices by replacing each edge by four parallel edges. Then G has tree-width one, and yet immerses J_g. Thus large tree-width is too strong a requirement. One might think that we should measure the width by decomposing the graph with a tree-structure of edge-cutsets of bounded size, instead of vertex-cutsets (which is essentially what tree-width does); but this is now too *weak*; a two-vertex graph with many parallel edges has large width under this measure, and yet does not immerse J_2. So the correct concept is somewhere between the two, and getting it right is beyond the scope of this paper (see [1]). (Note that these problems arise since there is no bound on the maximum degree of G; if we bound the maximum degree, then the vertex-cut and edge-cut versions of tree-width become equivalent, and 1.2 becomes sharp in the desired sense.)

One nice application of 1.2 is the following. Let G be four-edge-connected, with large tree-width. Then by 1.2 it immerses a large grid. Take the union of the ith row and ith column of the grid, for each i; then these provide a set of edge-disjoint connected subgraphs of G, pairwise intersecting. Consequently, in the line graph of G, there are many pairwise vertex-disjoint connected subgraphs, every two joined by an edge; that is, the line graph of G contains a large clique as a minor.

This idea is used in [7] to give a simple algorithm for the "k edge-disjoint paths problem", when the input graph is four-edge-connected. We are given k pairs of vertices of a four-edge-connected graph G, and we want to test whether the pairs can be joined by edge-disjoint paths. If G has small tree-width we can use dynamic programming, so we can assume that G has large tree-width, and hence its line graph has a large clique minor. So by a result of [6] applied to the line graph of G, if we take the smallest edge-cutset that separates $\{s_1, t_1, \ldots, t_k\}$ from the bulk of this clique minor, we can contract everything on the clique-minor side of this cutset to a single vertex without changing whether the paths exist. And then repeat, until the tree-width becomes small.

A second application of 1.2 is the following. It is easy to see that every graph with maximum degree at most four can be immersed in a large enough grid (map the vertices far apart, and then route the edges so that no three pass through the same vertex, and any two that share a vertex cross there). Consequently 1.2 implies that for every graph H with maximum degree at most four, H can be immersed in every four-edge-connected graph with sufficiently large tree-width.

This is developed further in [1]. There the authors use 1.2 as the basic of an induction to give a rough characterization of when a k-edge-connected graph G contains an immersion of a fixed graph H with maximum degree at most k, for $k \geq 4$. (But for weak immersion; they also do strong immersion, but it is more complicated.)

Let $h \geq 2$ be even. An *elementary wall of height h* is a graph whose vertex set can be labeled

$$\{v_{ij} \ : 1 \leq i \leq h+1, 1 \leq j \leq 2h+2, (i,j) \neq (1, 2h+2), (h+1, 1)\}$$

where distinct vertices $v_{ij}, v_{i'j'}$ are adjacent if either

- $i = i'$ and $|j' - j| = 1$, or
- $j = j'$ and $|i' - i| = 1$ and $\min(i, i') + j$ is even.

A *wall of height h* is a subdivision of an elementary wall of height h, and a *wall contained in a graph* G (or just a "wall in G") means a subgraph of G that is a wall.

The advantage of walls is that they permit us to state a version of 1.1 using subgraphs instead of minors, the following (this is easy to see):

1.3 *For all even $h \geq 2$ there exists k such that every graph with tree-width at least k contains a wall of height h.*

Thus the following is equivalent to 1.2:

1.4 *For all $g > 1$ there exists $h \geq 2$, even, such that every four-edge-connected graph containing a wall of height h contains J_g as an immersion.*

This admits two strengthenings, which we need to describe next. First, with the usual labelling of the vertex set of a wall, the vertices $v_{i,2i}$ ($2 \leq i \leq h$) are called its *diagonal* vertices. We will be able to replace the "four-edge-connected" hypothesis with a weaker hypothesis that a large number of the diagonal vertices are pairwise four-edge-connected. Second we want to show that if we start with a large wall, we get a large grid immersion that is in some sense "close" to the wall. More precisely, we insist that the immersion map each vertex of the grid to one of the diagonal vertices in our 4-edge-connected set. If $S \subseteq V(G)$, and η is an immersion of H in G, we say η is *S-rooted* if $\eta(v) \in S$ for each $v \in V(H)$. The following version of 1.4 incorporates both the strengthenings just discussed, and is the main result of the paper.

1.5 *For all $g > 1$ there exists $b \geq 0$, with the following property. Let W be a wall contained in a graph G, and let S be a set of diagonal vertices of W, pairwise 4-edge-connected in G, and with $|S| \geq b$. Then there is an S-rooted immersion of J_g in G.*

2 Applying lemmas from graph minors

Let W be a wall of height h, and let W_0 be the elementary wall of which W is a subdivision. Label the vertices of W_0 as in the definition. Each edge of W_0 corresponds to a path of W, and we call such a path a *branch* of W. Choose h' even with $2 \leq h' \leq h$, and choose four integers i_1, i_2, j_1, j_2, satisfying

- i_1, j_1 are odd
- $1 \leq i_1 \leq i_2 \leq h+1$ and $1 \leq j_1 \leq j_2 \leq 2h$
- $i_2 - i_1 = h'$ and $j_2 - j_1 = 2h' - 1$.

Then the subgraph of W_0 induced on the vertex set

$$\{v_{i,j} : i_1 \leq i \leq i_2, j_1 \leq j \leq j_2, (i,j) \neq (i_1, j_2), (i_2, j_1)\}$$

is an elementary wall of height h', and we call such a wall an *elementary subwall* of W_0. The corresponding subgraph of W is called a *subwall* of W.

Let W be a wall, in a graph G, and let v be a diagonal vertex of W. The *surround* of v means the set of vertices u of W such that either $u = v$, or there is a path of W between u, v in which every vertex different from v has degree two in W. A *fin* for W means a triple (s, F, t) such that

- s is a diagonal vertex of W,
- $t \in V(W)$ does not belong to the surround of s, and
- F is a path in G with ends s, t, edge-disjoint from W.

We also call this a *fin at* s. We say that $(W, (s_i, F_i, t_i)_{1 \le i \le b})$ is a *fin system in* G if

- W is a wall in G
- $s_1, \dots, s_b$ are diagonal vertices of W, all different
- for $1 \le i \le b$, (s_i, F_i, t_i) is a fin for W, and
- for $1 \le i, j \le b$, if $i \ne j$ then $s_i \notin V(F_j)$.

In this section we prove the following.

2.1 *For all $g > 1$ there exists an integer $b \ge 0$, with the following property. Let*

$$(W, (s_i, F_i, t_i)_{1 \le i \le b})$$

be a fin system in G. Then there is an $\{s_1, \dots, s_b\}$-rooted immersion of J_g in G.

The proof is in several steps. Some of the proofs of the steps are merely sketches, because they are standard applications of methods of the graph minors papers (rather straightforward, since the underlying graph is a wall), and to import all the definitions and theorems of the corresponding graph minors papers and to apply them precisely would take a considerable amount of space (and we suspect would not improve clarity).

Let W be an elementary wall of height h; then there is a drawing of W in the plane so that all finite regions have boundary of length six (and such a drawing is unique up to homeomorphisms of the plane). We call this the *standard drawing*. A standard drawing of a wall is obtained by subdividing edges of the standard drawing of the corresponding elementary wall.

Let W be a wall with its standard drawing. If s, t are points of the plane, we define their *distance* $d(s, t)$ to be zero if $s = t$, and otherwise to be the minimum of the number of points of F in the drawing, taken over all subsets F in the plane homeomorphic to $[0, 1]$ with ends s and t. The *perimeter* of W is the cycle bounding the infinite region.

We begin with

2.2 *Let $g > 1$. Then there exist integers $a_1, b_1 > 0$ with the following property. Let G be a graph and $(W, (s_i, F_i, t_i)_{1 \le i \le b_1})$ be a fin system in G. Suppose in addition that:*

- *the vertices $s_1, \dots, s_{b_1}, t_1, \dots, t_{b_1}$ are all pairwise at distance at least a_1, and*
- *the paths $F_1, \dots, F_{b_1}$ are pairwise edge-disjoint.*

Then there is an $\{s_1, \dots, s_{b_1}\}$-rooted immersion of J_g in G.

Proof. We may assume that g is even (by replacing g by $g+1$ if necessary). Let a_1 be large (in terms of g). Let $n = g^2$ and $b_1 = n+1$. Let J_g have vertex set $\{j_1, \ldots, j_n\}$ say. Since g is even, there is a perfect matching M in J_g.

Now let G and $(W, (s_i, F_i, t_i)_{1 \le i \le b_1})$ be as in the theorem. Since $s_1, \ldots, s_{b_1}, t_1, \ldots, t_{b_1}$ are pairwise at distance at least a_1, at most one of them has distance less than $a_1/2$ of a vertex of the perimeter of W; so we may assume that none of $s_1, \ldots, s_n, t_1, \ldots, t_n$ is within distance $(a_1 - 1)/2$ of the perimeter.

Now each s_i has degree three in W; let the three neighbours of s_i in W be $x_{i,1}, x_{i,2}, x_{i,3}$, enumerated in clockwise order around s_i, with an arbitrary first vertex. For $1 \le i \le n$, let the edges of J_g incident with j_i be $e_{i,1}, \ldots, e_{i,k_i}$ (where k_i is the degree of j_i in J_g), enumerated in clockwise order around j_i, such that $e_{i,k_i} \in M$. If a_1 is large enough (in terms of g), then by theorem 7.5 of [4] applied to $W \setminus \{s_1, \ldots, s_n\}$, (and compare theorem 4.4 of [5] for a similar situation), we deduce that for each edge e of J_g there is a path Q_e of $W \setminus \{s_1, \ldots, s_n\}$ satisfying the following, where e has ends j_h, j_i say in J_g:

- if $e \in M$ then Q_e has ends t_h, t_i;
- if $e \notin M$, let $e = e_{h,p} = e_{i,q}$; then the ends of Q_e are $x_{h,p}$ and $x_{i,q}$;
- the paths Q_e ($e \in E(J_g)$) are pairwise vertex-disjoint.

For each $e \in M$ with ends j_h, j_i say, let $\eta(e)$ be a path between s_h, s_i in the the union of the paths F_h, Q_e and F_i. For each $e \in E(J_g) \setminus M$ with ends j_h, j_i say, let p, q satisfy $e = e_{h,p} = e_{i,q}$, and let $\eta(e)$ be the path between s_h, s_i formed by the union of $s_h x_{h,p}$, $x_{i,q} s_i$ to Q_e. Let $\eta(j_i) = s_i$ $(1 \le i \le n)$; then η is the desired immersion. This proves 2.2. ▮

Next we need:

2.3 *Let $g > 1$, and let a_1, b_1 be as in 2.2. Let G be a graph, and let W be a wall in G. Let X be a connected subgraph of G, edge-disjoint from W. Let S be a set of $2b_1$ diagonal vertices of W, pairwise at distance at least a_1, each in $V(X)$ and with degree one in X. Then there is an S-rooted immersion of J_g in G.*

Proof. Let G, W, X, S be as in the statement of the theorem. We may assume that $|S| = 2b_1$. Since X is connected, there is a spanning tree T of X. Since $|S|$ is even, it is possible to pair up the vertices in S such that there are pairwise edge-disjoint paths of T joining the pairs. This provides a fin system satisfying the hypotheses of 2.2, and the result follows. ▮

2.4 *Let $g > 1$. Then there exists integers a_2, b_2 with the following property. Let G be a graph, and let $(W, (s_i, F_i, t_i)_{1 \le i \le b_2})$ be a fin system in G. Suppose that*

- *$s_1, \ldots, s_{b_2}$ are pairwise at distance at least a_2,*
- *the vertices s_i, t_i are at distance at least a_2 for $1 \le i \le b_2$, and*
- *the paths $F_1, \ldots, F_{b_2}$ are pairwise edge-disjoint.*

Then there is an $\{s_1, \ldots, s_{b_2}\}$-rooted immersion of J_g in G.

Proof. Let a_2, b_2 satisfy $a_2 \geq 2a_1$ and $b_2 \geq 6(2b_1+1)b_1$, where a_1, b_1 satisfy 2.2. Let G and

$$(W, (s_i, F_i, t_i)_{1 \leq i \leq b_2})$$

be as in the theorem.

Since for each j there is at most one s_i with distance less than $a_2/2$ to t_j, we may assume that $d(s_i, t_j) \geq a_2/2$ for all $i, j \leq n_1$, where $n_1 \geq b_2/3 \geq 2(2b_1+2)b_1$. Suppose that at least b_1 of the t_i's pairwise have distance at least a_1; then the result follows from 2.2. So for some $j \leq n_1$, there are at least n_1/b_1 values of $i \leq n_1$ such that $d(t_i, t_j) < a_1$. Let $n_2 = 2(2b_1+2)$; then $n_1/b_1 \geq n_2$, and we may assume that $d(t_i, t_1) < a_1$ for $1 \leq i \leq n_2$.

Now W is a wall, and $s_1, \ldots, s_{n_2}$ are diagonal vertices of it. There are therefore two subwalls of W, say W_1, W_2, pairwise disjoint, such that for $i = 1, 2$, exactly $n_2/2$ of $s_1, \ldots, s_{n_2}$ are diagonal vertices of W_i. From the symmetry we may assume that $t_1 \notin V(W_1)$. Since $d(t_i, t_1) < a_1$ for $1 \leq i \leq n_2$, and the s_i's pairwise have distance at least a_2, there is a subwall W' of W_1 and hence of W (obtained from W_1 by removing an appropriate border) such that at least $n_2/2 - 2 \geq 2b_1$ of $s_1, \ldots, s_{n_2}$ are diagonal vertices of it, and such that none of the corresponding t_i's belong to W'. But $W \setminus V(W')$ is connected, and the result follows from 2.3, taking X to be the union of $W \setminus V(W')$ and all the paths F_i with $t_i \in V(W) \setminus V(W')$. ∎

2.5 *Let $g > 1, c \geq 0$. Then there exist integers $a_3, b_3 \geq 0$ with the following property. Let G be a graph, and let $(W, (s_i, F_i, t_i)_{1 \leq i \leq b_3})$ be a fin system in G. Suppose that*

- *$s_1, \ldots, s_{b_3}$ are pairwise at distance at least a_3*
- *s_i, t_i are at distance at most c for $1 \leq i \leq b_3$, and*
- *$F_1, \ldots, F_{b_3}$ are pairwise edge-disjoint.*

Then there is an $\{s_1, \ldots, s_{b_3}\}$-rooted immersion of J_g in G.

Proof. Let $n = g^2$. Let J_g have vertex set $\{j_1, \ldots, j_n\}$ say. For $1 \leq i \leq n$, let the edges of J_g incident with j_i be $e_{i,1}, \ldots, e_{i,k_i}$ (where k_i is the degree of j_i in J_g), enumerated in clockwise order around j_i.

Let a_3, b_3 be big (in terms of g, c). There is at most one i such that s_i has distance less than $a_3/2$ to some vertex of the perimeter, so we may assume that s_i has distance at least $a_3/2$ to the perimeter, for $1 \leq i \leq b_3 - 1$.

Let $1 \leq i \leq b_3 - 1$. Then there is a subwall W_i of W of height c containing both s_i, t_i. Since we choose a_3 much greater than c, these subwalls are pairwise disjoint. Let W' be obtained from W by deleting all vertices of W_i for $1 \leq i \leq b_3 - 1$, and all internal vertices of branches of W with an end in W_i. Let R_i be the region of W' in which W_i was drawn. For $1 \leq i \leq b_3 - 1$, choose four (distinct) vertices of the boundary of R_i, say $x_{i,1}, \ldots, x_{i,4}$ in clockwise order around R_i, such that there are four paths $B(i,1), \ldots, B(i,4)$ of G from s_i to $x_{i,1}, \ldots, x_{i,4}$ respectively, pairwise edge-disjoint, each with only its final vertex in the boundary of R_i, and such that for every edge e of each of these four paths, either $e \in E(F_i)$ or e is an edge of W drawn within R_i.

For $n + 1 \leq i \leq b_3 - 1$, let us add the edges $x_{i,1}x_{i,3}$ and $x_{i,2}x_{i,4}$ to W', forming W'' say. By theorem 4.5 of [5], if b_3 is sufficiently large (in terms of n), then in W'' there are $6n$ connected subgraphs $X_1, \ldots, X_{6n}$, pairwise disjoint, such that

- for $1 \le i < j \le 6n$ there is an edge of W'' between X_i and X_j
- there is no partition (A, B, C) of $V(W'')$ such that
  - $|C| < 6n$
  - no vertex in A has a neighbour in B
  - A contains at least $6n$ diagonal vertices of W, and
  - B contains one of $X_1, \dots, X_{6n}$.

But then, from theorem 5.4 of [6], it follows that for every edge e of J_g, with ends j_h, j_i say in J_g, there is a path Q_e of W'' satisfying the following:

- let $e = e_{h,p} = e_{i,q}$; then the ends of Q_e are $x_{h,p}$ and $x_{i,q}$;
- the paths Q_e $(e \in E(J_g))$ are pairwise vertex-disjoint.

For $1 \le i \le n$ let $\eta(j_i) = s_i$; and for each edge e of J_g with ends j_h, j_i say in J_g, let $e = e_{h,p} = e_{i,q}$, and let $\eta(e)$ be the union of the three paths $B_{h,p}, Q_e, B_{j,q}$. Thus η is an immersion of J_g in W'', which is itself immersed in G. Consequently there is an immersion of J_g in G as desired. This proves 2.5. ∎

Proof of 2.1. Let a_1, b_1 satisfy 2.2, let a_2, b_2 satisfy 2.4 and let a_3, b_3 satisfy 2.5, taking $c = a_2$. Let $a = \max(a_1, a_2, a_3)$ and $b = 2b_1(b_2 + b_3)a$. Now let $(W, (s_i, F_i, t_i)_{1 \le i \le b})$ be a fin system in G. Since $s_1, \dots, s_b$ are all diagonal vertices, we can choose $b/a = 2b_1(b_2 + b_3)$ of them pairwise with distance at least a, say $s_1, \dots, s_{n_1}$ where $n_1 = 2b_1(b_2 + b_3)$. If some F_j is such that at least $2b_1$ other F_i's contain an internal vertex of F_j, then the result follows from 2.3. Thus we may assume that there is no such F_j, and so we may assume that $F_1, \dots, F_{n_2}$ pairwise have no internal vertex in common, where $n_2 \ge n_1/(2b_1) = b_2 + b_3$. If there are at least b_2 values of $i \in \{1, \dots, n_2\}$ such that the distance from s_i to t_i is at least a_2, the result follows from 2.4, so we assume there do not exist b_2 such values. But then there are at least b_3 values of $i \in \{1, \dots, n_2\}$ such that the distance from s_i to t_i is at most a_2, and the result follows from 2.5. This proves 2.1. ∎

3 Four-edge-connectivity

In this section we use 2.1 to prove 1.5. We begin with a lemma. If η is an immersion of H in G, and W is a subgraph of H, we denote by $\eta(W)$ the subgraph of G formed by the union of the paths $\eta(e)(e \in E(W))$ and the vertices $\eta(v)(v \in V(W))$.

3.1 *For all $g > 1$ let b be as in 2.1. Let G be a graph, and let η_0 be an immersion in G of an elementary wall W_0, and let S_0 be a set of diagonal vertices of W_0, such that*

- *if e, f are distinct edges of W_0 and some internal vertex of $\eta_0(e)$ equals some internal vertex of $\eta_0(f)$, then there exists $s \in S_0$ incident with both e, f*
- $|S_0| = b$

- *for each $s \in S_0$ there is a path F of G with distinct ends $\eta_0(s), t$, where t is a vertex of $\eta_0(W_0 \setminus s)$ and no edge of F belongs to $E(\eta_0(W_0))$).*

Then there is an $\eta_0(S_0)$-rooted immersion of J_g in G.

Proof. Let $S_0 = \{s_1, \dots, s_b\}$, say, and for $1 \leq i \leq b$ let F_i be a path from $\eta_0(s_i)$ to some vertex t_i of $\eta_0(W_0 \setminus s_i)$ as in the theorem. We proceed by induction on the sum, over all pairs of distinct edges e, f of W_0, of the number of vertices in $\eta_0(e) \cap \eta_0(f)$. For this quantity fixed, we proceed by induction on the sum of the lengths of $F_1, \dots, F_b$. We may therefore assume that for $1 \leq i \leq b$, t_i is the only vertex of F_i that belongs to $\eta_0(W_0 \setminus s_i)$. If there do not exist distinct edges e, f of W_0 such that some internal vertex of $\eta_0(e)$ equals some internal vertex of $\eta_0(f)$, then $\eta_0(W_0)$ is a wall and the result follows from 2.1. Thus we assume that some v is an internal vertex of $\eta_0(e_1), \dots, \eta_0(e_k)$ say, where $e_1, \dots, e_k$ are distinct edges of W_0 and $k \geq 2$. From the hypothesis, every two of $e_1, \dots, e_k$ have a common end in S_0, and since every edge has at most one end in S_0, it follows that $e_1, \dots, e_k$ are all incident with some member of S_0, say s_1. It follows that $v \notin \eta_0(W_0 \setminus s_1)$, and so $v \neq t_1$.

Let d_1, d_2 be the two edges of $\eta_0(e_1)$ incident with v, with ends u_1, v and u_2, v respectively. Let G' be obtained from G by deleting d_1, d_2 and adding a new edge $d_0 = u_1u_2$. Let $\eta_0'(w) = \eta(w)$ for all $w \in V(W_0)$, and $\eta_0'(d) = \eta(d)$ for all $d \in E(W_0)$ with $d \neq e_1$; let $\eta_0'(e_1)$ be a path joining the ends of $\eta_0(e_1)$ with edge set in $E(\eta_0(e_1) \setminus \{d_1, d_2\}) \cup \{d_0\}$. Then η_0' is an immersion of W_0 in G'. Moreover, for $1 \leq i \leq b$, if $t_i = v$ then e_2 is not incident with s_i (because e_2 is incident with none of $s_2, \dots, s_{b_2}$, and $t_1 \neq v$). Consequently, $t_i \in \eta_0'(W_0 \setminus s_i)$ for $1 \leq i \leq b$. Thus from the inductive hypothesis, the theorem holds for G', and hence it also holds for G. This proves 3.1. ∎

Proof of 1.5. Let b be as in 2.1 (we may assume that $b \geq 2$). Let W be a wall in G, and let S be a set of diagonal vertices with $|S| = b$, pairwise four-edge-connected in G. Let W_0 be an elementary wall of the same height, and let S_0 be the set of diagonal vertices of W_0 mapped to S under the corresponding subdivision map. Choose an immersion η_0 of W_0 in G and a subset D of S_0, with the following properties:

- if e, f are distinct edges of W_0 and some internal vertex of $\eta_0(e)$ equals some internal vertex of $\eta_0(f)$, then there exists $s \in S_0$ incident with both e, f
- $\eta_0(s) \in S$ for each $s \in S_0$
- for each $s \in D$ there is a path F of G with distinct ends $\eta_0(s), t$, where t is a vertex of $\eta_0(W_0 \setminus s)$ and no edge of F belongs to $E(\eta_0(W_0))$
- subject to all these conditions, $|D|$ is maximum.

(Satisfying the first three conditions is trivially possible, taking $D = \emptyset$ and η_0 the subdivision map to W.)

Suppose that there exists $s' \in S_0 \setminus D$. Let the three neighbours of s' in W_0 be x_1, x_2, x_3, and let B_1, B_2, B_3 be the images under η_0 of the edges $s'x_1, s'x_2, s'x_3$. Now since $|S| \geq 2$ and $\eta_0(s')$ is four-edge-connected to the other members of S, it follows that there are four edge-disjoint paths $P_1, \dots, P_4$ of G from $\eta_0(s')$ to $V(\eta_0(W_0 \setminus s'))$, each with no internal vertex in $V(\eta_0(W_0 \setminus s'))$; and since the three branches of $\eta_0(W_0)$ incident with $\eta_0(s')$ provide three such paths, it follows from the theory of augmenting paths that $P_1, \dots, P_4$ can be chosen such that P_1, P_2, P_3 have final vertices

$\eta_0(x_1), \eta_0(x_2), \eta_0(x_3)$ respectively. Let P_4 have final vertex t' say. Now let $\eta'_0(v) = \eta_0(v)$ for each $v \in V(W_0)$, and $\eta'_0(e) = \eta_0(e)$ for every edge e of W_0 not incident with s'; let $\eta'_0(s'x_i) = P_i$ for $i = 1, 2, 3$. It follows that η'_0 is an immersion of W_0 in G, satisfying the second condition above. Moreover, the first condition above is satisfied; for if e, f are distinct edges of W_0 and some internal vertex of $\eta'_0(e)$ equals some internal vertex of $\eta'_0(f)$, we may assume that e is incident with s' and f is not; but then some internal vertex of one of P_1, P_2, P_3 belongs to $\eta_0(W_0 \setminus s')$, a contradiction. Let $D' = D \cup \{s'\}$; we claim that the third condition is satisfied. Certainly, the new member s' of D' satisfies the condition, since P_4 is a path of G with distinct ends $\eta'_0(s'), t'$, and t' is a vertex of $\eta'_0(W_0 \setminus s)$ and no edge of P_4 belongs to $E(\eta'_0(W_0)))$; but we must check that the other members of D' still satisfy the condition. Thus, let $s \in D$. There is a path F of G with distinct ends $\eta_0(s), t$, where t is a vertex of $\eta_0(W_0 \setminus s)$ and no edge of F belongs to $E(\eta_0(W_0))$. The same path F also works for η'_0 unless either

- t is not a vertex of $\eta'_0(W_0 \setminus s)$; this implies that t is an internal vertex of one of B_1, B_2, B_3, and then we can augment F to a path with the desired properties; or
- some edge or internal vertex of F belongs to $E(\eta'_0(W_0)))$; but this implies that some edge or internal vertex of F belongs to one of P_1, P_2, P_3, and then a subpath of F has the desired properties, since $s \neq s'$.

Thus the three conditions are still satisfied, contrary to the maximality of $|D|$. Thus there is no such s', and so $D = S_0$, and the result follows from 3.1. This completes the proof of 1.5. ∎

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