Excluding the fork and antifork

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Abstract

The *fork* is the tree obtained from the claw $K_{1,3}$ by subdividing one of its edges once, and the *antifork* is its complement graph. We give a complete description of all graphs that do not contain the fork or antifork as induced subgraphs.

1 Introduction

Graphs in this paper are finite, and without loops or parallel edges. The *fork* is the tree obtained from a four-vertex path by adding a vertex adjacent to the second vertex of the path. The *antifork* is its complement graph; thus, the antifork is obtained from a four-vertex path by adding one more vertex adjacent to the first three vertices of the path. (See figure 1.)

Figure 1: The fork and the antifork.

Let us say G is *uncluttered* if no induced subgraph of G is a fork or antifork. Our goal in this paper is to give a complete description of all uncluttered graphs. The line graph of a triangle-free graph is uncluttered, and so is its complement; and our main theorem says that every uncluttered graph can be obtained by piecing together line graphs of triangle-free graphs and their complements. Before we can make a precise statement we need to explain the "piecing together" process, and that is the content of the next section. We state our main result in 2.1.

This question was motivated by discussions with T. Karthick, who told us several results about colouring graphs not containing forks and/or antiforks [1, 2, 3] (note, however, that in [2] "fork" has a different meaning from its meaning here). In particular, he asked for the best possible "χ-bounding function" for uncluttered graphs. The answer follows from our main result 2.1, and we give a proof in the final section.

1.1 *For every uncluttered graph, its chromatic number is at most twice its clique number.*

This is asymptotically best possible, since if H is a triangle-free graph with largest stable set of cardinality k, then the complement of the line graph of H is uncluttered, with clique number at most $|V(H)|/2$ and with chromatic number $|V(H)| - k$; and we can choose H and k with $(|V(H)| - k)/|V(H)|$ arbitrarily close to 1.

2 Some safe operations

There are several ways to make larger uncluttered graphs from smaller ones. The most obvious is: if G_1, G_2 are both uncluttered, then so is their disjoint union. The *complete join* of G_1, G_2 is obtained from their disjoint union by adding edges between every vertex of G_1 and every vertex of G_2. Since the complement of an uncluttered graph is also uncluttered, and the complete join of G_1, G_2 is the complement of the disjoint union of $\overline{G_1}, \overline{G_2}$, it follows that if G_1, G_2 are both uncluttered, then so is their complete join. Let us say G is *anticonnected* if its complement graph $\overline{G}$ is connected.

If $X, Y \subseteq V(G)$ are disjoint, we say X is *complete* to Y if every vertex in X is adjacent to every vertex in Y; and X is *anticomplete* to Y if there are no edges between X and Y. If $v \in V(G)$, we say "v is complete to Y" meaning that $\{v\}$ is complete to Y, and so on.

If $X \subseteq V(G)$ we denote by $G[X]$ the subgraph induced on X. Two (distinct) vertices u, v of G are *twins* if u, v have the same neighbours in $V(G) \setminus \{u, v\}$; (u, v may or may not be adjacent). A *homogeneous set* in G means a set $X \subseteq V(G)$ such that every vertex of G not in X is either complete or anticomplete to X; and the homogeneous set is *nontrivial* if $|X| \geq 2$ and $X \neq V(G)$. If X is a homogeneous set in G, let Y be the set of vertices in $V(G) \setminus X$ that are complete to X. Take a new vertex v, and let H be the graph formed from $G \setminus X$ by adding the vertex v and making v adjacent to the vertices in Y. Then we say G is obtained from H by *substituting* $G[X]$ *for* v.

Say a vertex v is *simplicial* if the set of its neighbours is a clique. If v is a simplicial vertex in an uncluttered graph G, then we may substitute a complete graph for v, and the new graph we obtain is also uncluttered. Consequently, if G has adjacent simplicial twins, then G can be obtained from a smaller graph by the operation just described. A vertex v is *antisimplicial* if it is simplicial in the complement graph, that is, if the set of all vertices nonadjacent to v is a stable set.

Let G be a graph, let $k \geq 1$ be an integer, and let $Y_1, \ldots, Y_k, Z_1, \ldots, Z_k$ be disjoint nonempty sets of $V(G)$ with union $V(G)$, such that

- $Y_1, \ldots, Y_k$ are cliques, and $Z_1, \ldots, Z_k$ are stable sets;
- for $1 \leq i < j \leq k$, Y_i is anticomplete to Y_j, and Z_i is complete to Z_j;
- for $1 \leq i, j \leq k$, Y_i is complete to Z_j if $i = j$, and otherwise Y_i is anticomplete to Z_j.

We call such a graph G a *candelabrum*, with *base* $Z_1 \cup \cdots \cup Z_k$. (See figure 2.)

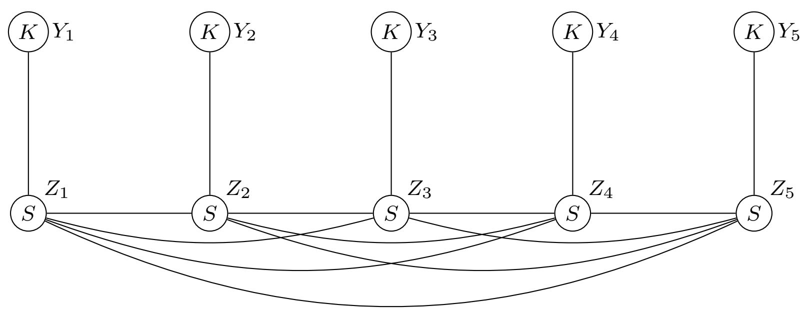

Figure 2: A candelabrum. (Lines indicate complete pairs, and K, S mark cliques and stable sets.)

Candelabra are uncluttered, but they can also be used to make larger uncluttered graphs. Let H_1 be an uncluttered graph, and let H_2 be a candelabrum with base Z. Take the disjoint union of H_1, H_2, and add edges to make $V(H_1)$ complete to Z. Let G be the graph we produce. Then G is uncluttered (we leave checking this to the reader), and $V(H_1)$ is a homogeneous set of G. We say G is *candled* if it can be constructed by this process; that is, if G has an induced subgraph H that is a candelabrum, with base Z say, and Z is complete to $V(G) \setminus V(H)$, and $V(H) \setminus Z$ is anticomplete to $V(G) \setminus V(H)$. (Thus $V(G) \setminus V(H)$ is a homogeneous set, but it might not be nontrivial; it might even be empty.)

Now we can state our main theorem:

2.1 *Let G be an uncluttered graph. Then either*

- *one of $G, \overline{G}$ is disconnected; or*
- *one of $G, \overline{G}$ has adjacent simplicial twins; or*
- *one of $G, \overline{G}$ is candled; or*
- *one of $G, \overline{G}$ is the line graph of a triangle-free graph.*

The proof of 2.1 will occupy the remainder of the paper, and will be completed in 4.7.

3 Homogeneous sets

In this section we will prove:

3.1 *Let G be an uncluttered graph. Suppose that*

- *G is connected and anticonnected;*
- *G has no adjacent simplicial twins, and no nonadjacent antisimplicial twins; and*
- *$G, \overline{G}$ are not candled.*

Then G has no nontrivial homogeneous set.

The proof requires several steps, that we carry out in this section. Let us say an *anticomponent* of a graph G is a induced subgraph whose complement graph is a component of $\overline{G}$. We begin with:

3.2 *Let G be uncluttered, and connected and anticonnected, with a nontrivial homogeneous set that is not a clique or stable set. Then one of $G, \overline{G}$ is candled.*

Proof. Let X be a nontrivial homogeneous set that is not a clique or stable set. By replacing X by a superset if necessary, we may assume that no proper superset of X is a nontrivial homogeneous set. Let Z be the set of vertices in $V(G) \setminus X$ that are complete to X, and let Y be the set of vertices in $V(G) \setminus X$ that are anticomplete to X. Let $Y_1, \ldots, Y_k$ be the vertex sets of the components of $G[Y]$, and let $Z_1, \ldots, Z_\ell$ be the vertex sets of the anticomponents of $G[Z]$.

(1) *$Y_1, \ldots, Y_k$ and $Z_1, \ldots, Z_\ell$ are homogeneous sets of G.*

Suppose that Y_1 is not a homogeneous set; then there exists $v \in V(G) \setminus Y_1$ that has a neighbour and a non-neighbour in Y_1, and since $G[Y_1]$ is connected, there is an edge yy' of $G[Y_1]$ such that v is adjacent to y and not to y'. Since X is anticomplete to Y_1, it follows that $v \notin X$, and similarly $v \notin Y_2, \ldots, Y_k$; so $v \in Z$. Now X is not a clique, so there exist nonadjacent $x_1, x_2 \in X$. But then $G[\{x_1, x_2, v, y, y'\}]$ is a fork, a contradiction. Thus Y_1 is a homogeneous set, and similarly so are $Y_2, \ldots, Y_k$; and so are $Z_1, \ldots, Z_\ell$, by applying the same argument in the complement graph. This proves (1).

It follows that for each Y_i and each Z_j, Y_i is either complete or anticomplete to Z_j.

(2) *Y, Z are both nonempty. Moreover, for $1 \le i \le k$, there exist $j, j' \in \{1, \ldots, \ell\}$ such that Y_i*

is complete to Z_j and anticomplete to $Z_{j'}$, and for $1 \le j \le \ell$, there exist $i, i' \in \{1, \dots, k\}$ such that Z_j is complete to Y_i and anticomplete to $Y_{i'}$.

Since X is a nontrivial homogeneous set, $X \neq V(G)$ and so $Y \cup Z \neq \emptyset$. Since G is connected, it follows that $Y \neq \emptyset$, and similarly $Z \neq \emptyset$ since G is anticonnected. Since G is connected, for $1 \le i \le k$ there exists $j \in \{1, \dots, \ell\}$ such that Y_i is complete to Z_j; and also, since $X \cup \{Y_i\}$ is not a homogeneous set from the maximality of X, it follows that there exists $j' \in \{1, \dots, \ell\}$ such that Y_i is anticomplete to $Z_{j'}$. The same argument in the complement shows the final statement. This proves (2).

Let $i_1, \dots, i_t \in \{1, \dots, k\}$ be distinct, and let $j_1, \dots, j_t \in \{1, \dots, \ell\}$ be distinct. We say the pairs $(i_1, j_1), \dots, (i_t, j_t)$ form a *matching of order t* if for $1 \le r, s \le t$, Y_{i_r} is complete to Z_{j_s} if $r = s$ and otherwise Y_{i_r} is anticomplete to Z_{j_s}. Similarly the pairs form an *antimatching of order t* if for $1 \le r, s \le t$, Y_{i_r} is anticomplete to Z_{j_s} if $r = s$ and otherwise Y_{i_r} is complete to Z_{j_s}.

(3) *There exist pairs $(i, j), (i', j')$ that form a matching of order two.*

Choose $i \in \{1, \dots, k\}$ such that Y_i is complete to Z_j for as many $j \in \{1, \dots, \ell\}$ as possible; by (2), we can choose $j' \in \{1, \dots, \ell\}$ such that Y_i is anticomplete to $Z_{j'}$; by (2) we can choose $i' \in \{1, \dots, k\}$ such that $Y_{i'}$ is complete to $Z_{j'}$; and then from the choice of i, there exists $j \in \{1, \dots, \ell\}$ such that Z_j is complete to Y_i and not to $Y_{i'}$. Then $(i, j), (i', j')$ forms a matching of order two. This proves (3).

Choose $t \ge 2$ maximum such that there are t pairs (i, j) (with $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, \ell\}$) that form a matching or antimatching; and by taking complements if necessary, we may assume the pairs form a matching. By renumbering, we may assume that $(1, 1), \dots, (t, t)$ form a matching. For $1 \le i \le t$, choose $y_i \in Y_i$ and $z_i \in Z_i$.

(4) *Every vertex in $Y_{t+1} \cup \dots \cup Y_k$ is complete or anticomplete to $Z_1 \cup \dots \cup Z_t$, and every vertex in $Z_{t+1} \cup \dots \cup Z_\ell$ is complete or anticomplete to $Y_1 \cup \dots \cup Y_t$.*

If $v \in Y_{t+1} \cup \dots \cup Y_k$ has a neighbour in Z_1 and a nonneighbour in Z_2 say, then $G[\{v, z_1, z_2, y_1, y_2\}]$ is a fork, a contradiction. Similarly, if $v \in Z_{t+1} \cup \dots \cup Z_\ell$ has a neighbour in Y_1 and a nonneighbour in Y_2, then $G[\{v, z_1, z_2, y_1, y_2\}]$ is an antifork, a contradiction. This proves (4).

Let P be the set of $v \in Y_{t+1} \cup \dots \cup Y_k$ such that v is complete to $Z_1 \cup \dots \cup Z_t$, and let P' be the set of $v \in Y_{t+1} \cup \dots \cup Y_k$ such that v is anticomplete to $Z_1 \cup \dots \cup Z_t$. Let Q be the set of $v \in Z_{t+1} \cup \dots \cup Z_\ell$ such that v is complete to $Y_1 \cup \dots \cup Y_t$, and let Q' be the set of $v \in Z_{t+1} \cup \dots \cup Z_\ell$ such that v is anticomplete to $Y_1 \cup \dots \cup Y_t$.

(5) *$X \cup Y_1 \cup \dots \cup Y_t \cup Z_1 \cup \dots \cup Z_t \cup P \cup Q'$ is a homogeneous set.*

Let us call this set A. We will show that P' is anticomplete to A, and Q is complete to A. Let $v \in P'$; then v is anticomplete to X since $v \in Y$; v is anticomplete to $Y_1 \cup \dots \cup Y_t$ since $Y_1, \dots, Y_t$ are vertex sets of components of $G[Y]$; and v is anticomplete to $Z_1 \cup \dots \cup Z_t$ by definition of P'.

Since v belongs to some Y_i, and Y_i is a homogeneous set by (1), and so $Y_i \subseteq P'$, it follows that v is anticomplete to P. Also from the maximality of t, v is anticomplete to Q'. This proves that P' is anticomplete to A.

Now let $v \in Q$. We must show that v is complete to A. Certainly v is complete to X and to $Z_1 \cup \cdots \cup Z_t$, and to $Y_1 \cup \cdots \cup Y_t$ from the definition of Q. Since v belongs to an anticomponent of $G[Z]$ with vertex set in Q, it follows that v is complete to Q'. It remains to show that v is complete to P. Suppose not, and let $u \in P$ be nonadjacent to v. For $1 \le i \le t$, choose $y_i \in Y_i$ and choose $z_i \in Z_i$. If $t \ge 3$, $G[\{v, y_1, y_2, u, z_3\}]$ is a fork, a contradiction, and so $t = 2$. Let $u \in Y_3$ and $v \in Z_3$ say; then Y_3 is anticomplete to Z_3 by (1), and so the three pairs $(1,2), (2,1), (3,3)$ form an antimatching, contrary to the maximality of t. This proves (5).

From the maximality of X, (5) implies that the set of (5) is not a nontrivial homogeneous set, and so $P', Q = \emptyset$. But then $X \cup P \cup Q'$ is a nontrivial homogeneous set, and the maximality of X implies that $P, Q' = \emptyset$. If there exist nonadjacent $y, y' \in Y_1$, then $G[\{y, y', z_1, z_2, y_2\}]$ is a fork, a contradiction; so $Y_1, \ldots, Y_t$ are cliques. If there exist adjacent $z, z' \in Z_1$, then $G[\{z, z', y_1, z_2, y_2\}]$ is an antifork, a contradiction. Thus $Z_1, \ldots, Z_t$ are stable sets. But then G is candled, as required. This proves 3.2. ∎

Let P be a four-vertex induced path in a graph G. A *centre* for P means a vertex of $V(G) \setminus V(P)$ that is complete to $V(P)$, and an *anticentre* for P is a vertex of $V(G) \setminus V(P)$ that is anticomplete to $V(P)$.

3.3 *Let G be uncluttered, and let P be a four-vertex induced path in G. If there is a centre and an anticentre for P then there is a nontrivial homogeneous set in G that is not a clique or stable set.*

Proof. Let A be the set of all anticentres for P, and let C be the set of all centres for P. Thus $A, C \neq \emptyset$. Let B be the set of $v \in V(G) \setminus (V(P) \cup A \cup C)$; thus B is the set of all vertices not in $V(P)$ with a neighbour and a nonneighbour in $V(P)$. Now either every vertex in C has a neighbour in A, or every vertex in A has a nonneighbour in C; and by taking complements if necessary, we assume the first. Let P have vertices p_1-p_2-p_3-p_4 in order.

(1) *B is complete to C.*

Let $b \in B$ and $c \in C$, and suppose that b, c are nonadjacent. Choose $a \in A$ adjacent to c. Suppose first that a, b are nonadjacent. Since P is anticonnected, and $b \in B$, there exist nonadjacent $p, p' \in V(P)$ such that b is adjacent to p and not to p'; but then $G[\{p, p', a, b, c\}]$ is a fork, a contradiction. So a, b are adjacent. Let I be the set of $i \in \{1, \ldots, 4\}$ such that b, p_i are adjacent. Since $G[\{a, b, c, p_1, p_4\}]$ is not a fork, one of $1, 4 \in I$, and we may assume $1 \in I$ without loss of generality. If $4 \notin I$, then $2 \in I$ since $G[\{a, b, c, p_2, p_4\}]$ is not a fork; so $3 \notin I$ since the subgraph induced on $G[\{b, p_1, p_2, p_3, p_4\}]$ is not an antifork; but then $G[\{b, c, p_1, p_2, p_4\}]$ is an antifork, a contradiction. Thus $4 \in I$. Since b is not a centre, one of $2, 3 \notin I$, and we assume $3 \notin I$ without loss of generality. But then $G[\{a, b, p_1, p_3, p_4\}]$ is a fork, a contradiction. This proves (1).

Let A' be the union of the vertex sets of the components of $G[A]$ that are not anticomplete to B, and let $A'' = A \setminus A'$.

(2) *A' is complete to C.*

Let $c \in C$, and suppose that c is not complete to A'. From the definition of A', there is an induced path Q with one end in B, and all other vertices in A', such that some vertex of Q is not adjacent to c. Choose Q minimal, with vertices q_1-$\cdots$-q_k in order, where $q_k \in B$ and $q_1, \ldots, q_{k-1} \in A$. From (1), $k \geq 2$. From the minimality of Q, it follows that c is nonadjacent to q_1 and adjacent to all of $q_2, \ldots, q_k$. Choose adjacent q_{k+1}, q_{k+2} of P such that q_k is adjacent to q_{k+1} and not to q_{k+2}. (This is possible since $q_k \in B$.) Then $G[\{c, q_1, q_2, q_3, q_4\}]$ is an antifork, a contradiction. This proves (2).

Let $X = V(P) \cup B \cup A'$. From (1) and (2), C is complete to X; and from the definition of A'', A'' is anticomplete to X. Thus X is a nontrivial homogeneous set satisfying the theorem. This proves 3.3. ∎

3.4 *Let G be uncluttered and connected, and let A, B, C, D be disjoint subsets of $V(G)$, with union $V(G)$, and with the following properties:*

- *A is a clique and $A \neq \emptyset$;*
- *B is a stable set and $|B| \geq 2$;*
- *A is complete to B and anticomplete to C, D;*
- *B is complete to C and anticomplete to D.*

Then G is candled.

Proof. Let $C_1, \ldots, C_k$ be the vertex sets of the anticomponents of $G[C]$, and let $D_1, \ldots, D_\ell$ be the vertex sets of the components of $G[D]$.

(1) *$C_1, \ldots, C_k$ and $D_1, \ldots, D_\ell$ are homogeneous sets.*

Suppose C_1 is not a homogeneous set; then there exist $d \in V(G) \setminus C_1$ and nonadjacent $c, c' \in C_1$ with d adjacent to c and not to c'. Choose $a \in A$ and $b \in B$; then $G[\{a, b, c, c', d\}]$ is a fork, a contradiction. Thus $C_1, \ldots, C_k$ are all homogeneous sets.

Now suppose D_1 is not a homogeneous set; then similarly there exists $c \in C$ and adjacent $d, d' \in D_1$ such that c is adjacent to d and not to d'. Since $|B| \geq 2$ and B is stable, there exist nonadjacent $b, b' \in B$; but then $G[\{b, b', c, d, d'\}]$ is a fork, a contradiction. This proves (1).

It follows that for each C_i and each D_j, C_i is either complete or anticomplete to D_j. For $1 \leq i \leq k$ choose $c_i \in C_i$, and for $1 \leq j \leq \ell$ choose $d_j \in D_j$. Choose $a \in A$ and $b \in B$.

(2) *For $1 \leq j \leq \ell$, there is a unique value of $i \in \{1, \ldots, k\}$ such that D_j is complete to C_i.*

We assume $j = 1$ without loss of generality. Since G is connected and $A \cup B \cup D_2 \cup \cdots \cup D_\ell$ is anticomplete to D_1, it follows from (1) that there exists $i \in \{1, \ldots, k\}$ such that D_1 is complete to

C_i. Suppose there are two such values of i, say $i = 1, 2$; then $G[\{a, b, c_1, c_2, d_1\}$ is an antifork, a contradiction. This proves (2).

(3) *For $1 \le i \le k$ there is at most one value of $j \in \{1, \ldots, \ell\}$ such that C_i is complete to D_j.*

For let $i = 1$ say, and suppose that C_1 is complete to D_1, D_2 say. Then $G[\{a, b, c_1, d_1, d_2\}]$ is a fork, a contradiction. This proves (3).

From (2) and (3) it follows that $k \ge \ell$, and we may renumber such that D_i is complete to C_i for $1 \le i \le \ell$. For $1 \le i \le k$, D_i is a clique, since if $d, d' \in D_i$ are nonadjacent then $G[\{a, b, c_i, d, d'\}]$ is a fork; and also C_i is a stable set, since if $c, c' \in C_i$ are adjacent then $G[\{a, b, c, c', d_i\}]$ is an antifork. Thus the restriction of G to

$$A \cup B \cup C_1 \cup \cdots \cup C_\ell \cup D_1 \cup \cdots \cup D_\ell$$

is a candelabrum with base $B \cup C_1 \cup \cdots \cup C_\ell$. Since $C_{\ell+1} \cup \cdots \cup C_k$ is complete to $B \cup C_1 \cup \cdots \cup C_\ell$ and anticomplete to $A \cup D_1 \cup \cdots \cup D_\ell$, it follows G is candled. This proves 3.4. ∎

Now we can prove the main result of this section, which we restate:

3.5 *Let G be an uncluttered graph. Suppose that*

- *G is connected and anticonnected;*
- *G has no adjacent simplicial twins, and no nonadjacent antisimplicial twins;*
- *$G, \overline{G}$ are not candled.*

Then G has no nontrivial homogeneous set.

Proof. For each nontrivial homogeneous set X, we define its "score" as follows. By 3.2, X is either a clique or a stable set. If X is stable, its *score* is the number of components of $G[Y]$, where Y is the set of all vertices in $V(G) \setminus X$ that are anticomplete to X. If X is a clique, its *score* is the number of anticomponents of $G[Z]$, where Z is the set of vertices in $V(G) \setminus X$ that are complete to X.

Suppose that there is a nontrivial homogeneous set X, and choose X with minimum score. Let Z be the set of vertices in $V(G) \setminus X$ that are complete to X, and Y the set that are anticomplete to X. Let $Y_1, \ldots, Y_k$ be the vertex sets of the components of $G[Y]$, and let $Z_1, \ldots, Z_\ell$ be the vertex sets of the anticomponents of $G[Z]$. By replacing G by its complement if necessary, we may assume that X is a stable set.

(1) *$Y_1, \ldots, Y_k$ are homogeneous sets, and cliques.*

If Y_1 is not a homogeneous set, then there exists $z \in Z$ and $y, y' \in Y_1$, such that z-y-y' is an induced path (because $G[Y_1]$ is connected). Let $x, x' \in X$ be nonadjacent; then $G[\{x, x', z, y, y'\}]$ is a fork, a contradiction. This proves that Y_1 is a homogeneous set. By 3.2, Y_1 is either a stable set or a clique; but $G[Y_1]$ is connected, and so Y_1 is a clique. This proves (1).

By hypothesis, there are no nonadjacent antisimplicial twins in G; and in particular, vertices in X are not antisimplicial. Thus Y is not stable, and so we may assume that $|Y_1| \ge 2$. Let N be the set of vertices in Z that are complete to Y_1. By hypothesis, there are no adjacent simplicial twins, and in particular the vertices in Y_1 are not simplicial; so N is not a clique. Hence there is an anticomponent D of N with at least two vertices. Let $N' = N \setminus D$, and $Y' = Y_2 \cup \cdots \cup Y_k$. Thus the six sets $X, D, N', Z \setminus N, Y_1, Y'$ are pairwise disjoint and have union $V(G)$.

(2) *D is a homogeneous set, and D is stable.*

Since $G[D]$ is anticonnected, if D is not a homogeneous set then there exists $v \in V(G) \setminus D$ and nonadjacent $d, d' \in D$, such that v is adjacent to d and not to d'. Since $X \cup N' \cup Y_1$ is complete to D, it follows that $v \in (Z \setminus N) \cup Y'$. Choose $y, y' \in Y_1$, adjacent; then $G[\{v, d, d', y, y'\}]$ is an antifork, a contradiction. Thus D is homogeneous. By 3.2 is either stable or a clique, and it is not a clique since $G[D]$ is anticonnected and $|D| \geq 2$. This proves (2).

Let U be the set of vertices in $Z \setminus N$ that are complete to D, and let W be the set of vertices in $Z \setminus N$ that are anticomplete to D. Thus $U \cup W = Z \setminus N$ by (2).

(3) *W is a clique.*

If $w, w' \in W$ are nonadjacent, choose $y \in Y_1$, $d \in D$ and $x \in X$; then $G[\{w, w', y_1, d, x\}]$ is a fork, a contradiction. This proves (3).

(4) *Y' is anticomplete to $D \cup W$, and N' is complete to W, and U is anticomplete to W.*

From the minimality of the score of X, it follows that the score of D is at least that of X, that is, at least k. For $2 \leq i \leq k$, Y_i is a homogeneous set by (1), and since D is also a homogeneous set by (2), it follows that Y_i is complete or anticomplete to D. Let C denote the set of vertices not in D and anticomplete to D. Then C consists of W and some of $Y_2, \ldots, Y_k$, those Y_i that are anticomplete to D. In particular, since W is a clique by (3), the score of D is one more (for W) than the number of $i \in \{2, \ldots, k\}$ such that Y_i is anticomplete to $D \cup W$. Since the score of D is at least k, it follows that $Y_2, \ldots, Y_k$ are all anticomplete to $D \cup W$. This proves the first assertion of (4).

For the second assertion of (4), let $y \in Y_1, d \in D, x \in X$ and $w \in W$; then for $n \in N'$, if n is nonadjacent to w then $G[\{y, d, x, w, n\}]$ is an antifork. This proves that N' is complete to W.

For the third assertion, let $u \in U$, and let y, d, x, w be as before; if u, w are adjacent then $G[\{y, d, x, u, w\}]$ is an antifork, a contradiction. This proves (4).

If $N' = \emptyset$, then the four sets $W, X, D \cup U, Y$ (in this order) satisfy the hypotheses of 3.4, a contradiction. Thus $N' \neq \emptyset$. If $Y' = \emptyset$, then the four sets $X, Y_1, W \cup U, N' \cup D$ satisfy the hypotheses of 3.4 applied in $\overline{G}$, a contradiction. So $Y' \neq \emptyset$. Choose $y \in Y_1$, $y' \in Y'$, $d \in D$, $x \in X$, $n \in N'$ and $w \in W$; then y-d-x-w is a four-vertex induced path, and n is a centre and y' is an anticentre, contrary to 3.3. This proves 3.5. ∎

4 Graphs without homogeneous sets

In view of 3.1, henceforth we can restrict our attention to uncluttered graphs with no nontrivial homogeneous set. First here is a useful lemma.

4.1 *Let G be a graph with no nontrivial homogeneous set, and let $A \subseteq V(G)$, not a clique, with $A \neq V(G)$. Then there exist $v \in V(G) \setminus A$ and nonadjacent $a, a' \in A$, such that v is adjacent to a and not to a'.*

Proof. Since A is not a clique, there is an anticomponent X of $G[A]$ with at least two vertices. Since $X \neq V(G)$ and X is not a nontrivial homogeneous set, there is a vertex $v \in V(G) \setminus X$ with a neighbour and a nonneighbour in X. Since X is an anticomponent of $G[A]$, it follows that $v \notin A$. Since $G[X]$ is anticonnected, and v has a neighbour and a nonneighbour in X, there exist nonadjacent $a, a' \in X$ such that v is adjacent to a and not to a'. This proves 4.1. ∎

If X is a subgraph of G, or a set of vertices of G, we say that X is *dominating* if every vertex not in X has a neighbour in X. The *diamond* is the graph with four vertices and five edges, and a *triangle* is a clique of cardinality three. We begin with:

4.2 *Let G be an uncluttered graph with no nontrivial homogeneous set. Then every diamond in G is dominating. Moreover, every triangle that is contained in a diamond of G is dominating.*

Proof. Suppose that there is a diamond D that is not dominating. Let v_1, v_2 be the two vertices of D that have degree three in D; so there is a set A of G, not a clique, such that $v_1, v_2 \notin A$ and A is complete to $\{v_1, v_2\}$, and $A \cup \{v_1, v_2\}$ is not dominating. Choose such a set A, maximal. Since A is not a nontrivial homogeneous set, 4.1 implies that there exist $v \in V(G) \setminus A$ and nonadjacent $a, a' \in A$ such that v is adjacent to a and not to a'. Thus $v \neq v_1, v_2$. If v is nonadjacent to both v_1, v_2, then $G[\{v, v_1, v_2, a, a'\}]$ is an antifork, a contradiction; so we may assume that v is adjacent to v_1. Since $A \cup \{v_1, v_2\}$ is not dominating, there is a vertex w that is anticomplete to $A \cup \{v_1, v_2\}$. Now there are four possibilities: w may or may not be adjacent to v, and v may or may not be adjacent to v_2. Suppose first that v is not adjacent to v_2, and so v-a-v_2-a' is an induced four-vertex path, P say. Now v_1 is a centre for P, so by 3.3, w is not an anticentre, and therefore w is adjacent to v. But then $G[\{w, v, v_1, a, v_2\}]$ is an antifork, a contradiction. So v is adjacent to v_2. If w, v are nonadjacent, then we can add v to A, contrary to the maximality of A. Thus v, w are adjacent; but then $G[\{w, v, v_1, v_2, a'\}]$ is an antifork, a contradiction. This proves that every diamond is dominating.

Now let T be a triangle, contained in a diamond; thus some vertex $v \in V(G) \setminus T$ has exactly two neighbours in T. Suppose that T is not dominating, and let w be anticomplete to T. Since the diamond $G[T \cup \{v\}]$ is dominating, w is adjacent to v; but then $G[T \cup \{v, w\}]$ is an antifork, a contradiction. This proves 4.2. ∎

4.3 *Let G be an uncluttered graph with no nontrivial homogeneous set, such that G is not the line graph of a bipartite graph. Then for every nondominating triangle T in G, there is a unique maximal clique including T, say C, and it is not dominating. Moreover, for each $v \in C$, if C is not the only maximal clique containing v, then there is exactly one other maximal clique containing v, say C_v, and $C_v \cap C = \{v\}$, and C_v is not dominating.*

Proof. Let C be a maximal nondominating clique including T; and let S be the set of vertices that are anticomplete to C. Thus $S \neq \emptyset$. Let A be the set of vertices in $V(G) \setminus C$ with exactly one neighbour in C, and let B be the set of vertices in $V(G) \setminus C$ with at least two neighbours in C. Thus A, B, C, S are pairwise disjoint and have union $V(G)$.

(1) *B is complete to $C \cup S$.*

No triangle included in C is dominating, and hence by 4.2, no triangle included in C is contained in a diamond. Consequently every vertex in B is complete to C. From the maximality of C, B is complete to S. This proves (1).

Every vertex in A has a unique neighbour in C. For each $c \in C$, let A_c be the set of vertices in A that are adjacent to c. Thus $A = \bigcup_{c \in C} A_c$.

(2) *There is at most one $c \in C$ with $A_c = \emptyset$.*

For otherwise the set of $c \in C$ with $A_c = \emptyset$ is a nontrivial homogeneous set, which is impossible. This proves (2).

(3) *$|B| \leq 1$, and B is anticomplete to A.*

Suppose first that B is not a clique. By 4.1, there is a vertex $v \in V(G) \setminus B$ and nonadjacent $b, b' \in B'$ such that v is adjacent to b and not to b'. By (1), $v \in A_c$ for some $c \in C$. Choose $c_1, c_2 \in C$ different from c; then $G[\{v, b, b', c_1, c_2\}]$ is an antifork, a contradiction. This proves that B is a clique.

Suppose that some $b \in B$ has a neighbour in A. Since G is anticonnected (because it has no nontrivial homogeneous set), it follows that b has a nonneighbour; and since b is complete to $C \cup S$ and B is a clique, it follows that b has a nonneighbour in A. By (2), there are at least two vertices $c \in C$ such that $A_c \neq \emptyset$; and so there exist distinct $c_1, c_2 \in C$, and $a_i \in A_{c_i}$ for $i = 1, 2$, such that b is adjacent to a_1 and nonadjacent to a_2. (To see this, choose a neighbour $a \in A$ of b and a nonneighbour $a' \in A$ of b. If a, a' both belong to A_{c_1} say, choose $a'' \in A_{c_2}$, and replace one of a, a' by a''.) Choose $c_3 \in C \setminus \{c_1, c_2\}$. Since $G[\{b, c_1, c_3, a_1\}]$ is a diamond, the triangle $G[\{b, c_1, c_3\}]$ is dominating by 4.2, and yet a_2 has no neighbour in this triangle, a contradiction.

Thus B is anticomplete to A, and so B is a homogeneous set; and hence $|B| \leq 1$. This proves (3).

(4) *For each $c \in C$, A_c is a clique; and if $B \neq \emptyset$ then S is a clique.*

Suppose that $c \in C$ and A_c is not a clique; then by 4.1, there exists $v \notin A_c$ and nonadjacent $a, a' \in A_c$ such that v is adjacent to a and not to a'. Thus $v \in A \cup S$, by (3); and so there exists $c' \in C \setminus \{c\}$ nonadjacent to v. But then $G[\{c, c', a, a', v\}]$ is a fork, a contradiction. Thus each A_c is a clique. Now suppose that $B = \{b\}$ say, and S is not a clique. Then by 4.1, there exists $v \notin S$ and nonadjacent $s, s' \in S$ such that v is adjacent to s and not to s'. Thus $v \in A_c$ for some $c \in C$. Choose $c' \in C \setminus \{c\}$. Then $G[\{b, s, s', v, c'\}]$ is a fork, a contradiction. This proves (4).

(5) *$B = \emptyset$.*

Suppose that $B \neq \emptyset$, and so $|B| = 1$ by (3). Let $B = \{b\}$ say. Define $A_b = S$; thus $C \cup \{b\}$ is a clique D say, and every vertex not in D belongs to one of the sets A_d ($d \in D$), and has a unique neighbour in D; and each set A_d ($d \in D$) is a clique by (4). We claim:

- for all distinct $d_1, d_2 \in D$, each vertex in A_{d_1} has at most one neighbour in A_{d_2}; and
- for all distinct $d_1, d_2, d_3 \in D$, if $a_i \in A_{d_i}$ for $i = 1, 2, 3$, and a_1 is adjacent to both a_2, a_3, then a_2, a_3 are adjacent.

To see the first claim, suppose that $v \in A_{d_1}$ has two neighbours $a, a' \in A_{d_2}$. By (4), a, a' are adjacent. Choose $d_3 \in D \setminus \{d_1, d_2\}$; then $G[\{v, a, a', d_2, d_3\}]$ is an antifork, a contradiction. This proves the first claim.

For the second claim, suppose that $d_1, d_2, d_3 \in D$ are distinct, and $a_i \in A_{d_i}$ for $i = 1, 2, 3$, and a_1 is adjacent to a_2, a_3, and a_2, a_3 are not adjacent. Since $|C| \geq 3$ it follows that $|D| \geq 4$; choose $d_4 \in D \setminus \{d_1, d_2, d_3\}$. Then $G[\{a_1, a_2, a_3, d_1, d_4\}]$ is a fork, a contradiction. This proves the second claim.

Let H be the subgraph of G obtained by deleting the edges of the cliques $A_d \cup \{d\}$ ($d \in D$). From the two bullet claims above, it follows that each component of H is a clique. (D itself is one such component.) Thus we have found two sets of cliques of G; the sets $A_d \cup \{d\}$ ($d \in D$), and the components of H. Each vertex of G belongs to exactly one clique in the first set, and exactly one in the second; and every edge of G belongs to one of the cliques in one of the sets. Consequently G is the line graph of a bipartite graph, a contradiction. This proves that $B = \emptyset$, and so proves (5).

It follows that C is a maximal clique of G, and is nondominating. Moreover, every edge of C is not contained in any other maximal clique; and every vertex $c \in C$ is contained in at most two maximal cliques, namely C and $A_c \cup \{c\}$ if $A_c \neq \emptyset$. To complete the proof of the theorem, we only need to show that the cliques $A_c \cup \{c\}$ are not dominating. Suppose then that $c \in C$, and $A_c \cup \{c\}$ is dominating. We have already shown that every clique including a nondominating triangle is itself nondominating; and consequently all triangles included in $A_c \cup \{c\}$ are dominating. But A_c is not dominating (because there is a vertex in C anticomplete to A_c), so $|A_c| \leq 2$.

(6) $|A_c| = 2$.

Suppose that $|A_c| \leq 1$; and hence $|A_c| = 1$, since $A_c \cup \{c\}$ is dominating and so is not anticomplete to S, since $S \neq \emptyset$. Let $A_c = \{a\}$; then for the same reason, a is complete to S. Also $A_c \cup \{c\}$ is not anticomplete to any vertex in $A_{c'}$ for $c' \in C \setminus \{c\}$; and so a is complete to $A_{c'}$ for all $c' \in C \setminus \{c\}$. Let $c' \in C \setminus \{c\}$, and let $a' \in A_{c'}$, and $s \in S$. Since a', s are both adjacent to a, and there exists $c'' \in C \setminus \{c, c'\}$, and $G[\{s, a, a', c, c''\}]$ is not a fork, it follows that a', s are adjacent; and so $A_{c'}$ is complete to S for all $c' \in C \setminus \{c\}$. If $c_1, c_2 \in C \setminus \{c\}$ are distinct, and there exist $a_1 \in A_{c_1}$ and $a_2 \in A_{c_2}$ nonadjacent, then $G[\{a_1, a_2, a, s, c_1\}]$ is an antifork, a contradiction; so all the sets $A_{c'}$ ($c' \in C$) are complete to each other. Hence they are all homogeneous sets, and so is S; so they all have cardinality at most one. But then G is the line graph of a bipartite graph, a contradiction. This proves (6).

Let $A_c = \{p, q\}$ say. For each $d \in C \setminus \{c\}$, let P_d be the set of vertices in A_d adjacent to p, and define Q_d similarly.

(7) *For each $d \in C \setminus \{c\}$, P_d, Q_d are disjoint subsets of A_d with union A_d.*

Since $A_c \cup \{c\}$ is dominating, it follows that $P_d \cup Q_d = A_d$ for each $d \in C \setminus \{c\}$. If some $v \in A_d$ is adjacent to both p, q, choose $c' \in C \setminus \{c, d\}$; then $G[\{v, p, q, c, c'\}]$ is an antifork, a contradiction. This proves (7).

Let S_p, S_q be the sets of vertices in S adjacent to p, q respectively.

(8) *S_p, S_q are disjoint subsets of S with union S; and P_d is complete to S_p, and Q_d is complete to S_q, for each $d \in C \setminus \{c\}$.*

Certainly $S_p \cup S_q = S$ since $A_c \cup \{c\}$ is dominating. If $v \in S$ is adjacent to both p, q, choose $d \in C \setminus \{c\}$, and then $G[\{v, p, q, c, d\}]$ is an antifork, a contradiction. This proves the first assertion.

Let $d \in C \setminus \{c\}$, and suppose $v \in P_d$ and $s \in S_p$ are nonadjacent. Choose $c' \in C \setminus \{c, d\}$, and then $G[\{v, s, p, c, c'\}]$ is a fork, a contradiction. Thus P_d is complete to S_p, and similarly Q_d is complete to S_q, for each $d \in C \setminus \{c\}$. This proves (8).

Let $P = \{p\} \cup S_p \cup \bigcup_{d \in C \setminus \{c\}} P_d$, and define Q similarly.

(9) *P, Q are cliques.*

Suppose P is not a clique, say, and choose $p_1, p_2 \in P$, nonadjacent. Thus $p_1, p_2 \neq p$. If $p_1 \in S$, then by (8), $p_2 \in S$, and $G[\{p_1, p_2, p, c, d\}]$ is a fork where $d \in C \setminus \{c\}$. Thus $p_1, p_2 \notin S$, and so we may assume that $p_i \in P_{c_i}$ for $i = 1, 2$, where $c_1, c_2, c \in C$ are distinct. But then $G[\{p_1, p_2, p, q, c_1\}]$ is a fork, a contradiction. This proves (9).

(10) *For each $d \in C \setminus \{c\}$, P_d is anticomplete to S_q, and Q_d is anticomplete to S_p.*

Let $v \in P_d$, and suppose v is adjacent to $u \in S_q$. Choose $d' \in C \setminus \{c, d\}$; then $G[\{u, v, p, d, d'\}]$ is a fork, a contradiction. This proves (10).

(11) *For each $d \in C \setminus \{c\}$, P_d is anticomplete to $Q \setminus Q_d$.*

Suppose that $v \in P_d$ and $u \in Q_{d'}$ are adjacent, where $d' \in C \setminus \{c, d\}$. Choose $s \in S$; then s is adjacent to exactly one of p, q. If s is adjacent to p, then $s \in P$, and by (9) s is adjacent to v; and by (10), s is nonadjacent to u. But then $G[\{u, v, s, q, d'\}]$ is a fork, a contradiction. Thus s is adjacent to q and not to p, and hence adjacent to u and not to v. But then $G[\{u, v, s, p, d\}]$ is a fork, a contradiction. This proves (11).

(12) *Every vertex of S_p has at most one neighbour in S_q and vice versa.*

Suppose $v \in S_p$ is adjacent to $u, w \in S_q$. Then $G[\{v, u, w, q, c\}]$ is an antifork, a contradiction. This proves (12).

From (11), since P_d is complete to Q_d, it follows that P_d is a homogeneous set, and so $|P_d| \leq 1$, and similarly $|Q_d| \leq 1$. Let H be the subgraph obtained from G by deleting the edges of the three cliques C, P, Q. By (11), every edge of H either has both ends in $A_d \cup \{d\}$ for some $d \in C$, or has one end in S_p and the other in S_q; and hence by (12), every component of H is a clique, and has at most one vertex in common with C, P or Q. Consequently G is the line graph of a bipartite graph, a contradiction. This proves 4.3. ∎

4.4 *Let G be an uncluttered graph with no nontrivial homogeneous set, such that G is not the line graph of a bipartite graph. Then either every triangle is dominating, or no clique is dominating.*

Proof. Let A be union of the vertex sets of all dominating cliques, and let B be the union of the vertex sets of all nondominating triangles. By 4.3, every vertex of a nondominating triangle only belongs to nondominating cliques, so $A \cap B = \emptyset$. Suppose there is a nondominating triangle T, and a dominating clique C. Thus $C \subseteq A$, and $T \subseteq B$. Every triangle is either dominating or nondominating, and so is a subset of one of A, B. Consequently, every vertex of T has at most one neighbour in C; and hence exactly one since C is dominating; and since no vertex in C has more than one neighbour in T, it follows that there are three vertices c_1, c_2, c_3 such that t_i is adjacent to c_i for $i = 1, 2, 3$, where $T = \{t_1, t_2, t_3\}$. By 4.3, $\{c_1, c_2, c_3\}$ is dominating. Since T is nondominating, there is a vertex y that is anticomplete to T; and since $\{c_1, c_2, c_3\}$ is dominating, we may assume that y is adjacent to c_1. If y is nonadjacent to c_2 then $G[\{y, c_1, t_1, t_3, c_2\}]$ is a fork, a contradiction. But if y is adjacent to c_2, then since the triangle $\{y, c_1, c_2\}$ has a vertex in A, it follows that $y, c_1, c_2 \in A$, and in particular $\{y, c_1, c_2\}$ is dominating; and this is impossible since t_3 has no neighbour in $\{y, c_1, c_2\}$. This proves 4.4. ∎

The *claw* is the complete bipartite graph $K_{1,3}$, and its *centre* is its vertex of degree three. Thus if T is a nondominating triangle in G, and v has no neighbour in T, then v is a claw centre in $\overline{G}$, and vice versa. Let us say an *anticlaw* is a four-vertex graph whose complement is a claw.

Figure 3: The claw and the anticlaw.

We need one more lemma.

4.5 *Let G be a graph with no claw or anticlaw. Then for one of $G, \overline{G}$, say H, either*

- *each component of H is a path or cycle, and hence H is the line graph of a triangle-free graph, or*
- *$|V(H)| \leq 9$, and H is the line graph of a bipartite graph.*

Proof. The *net* is the graph on six vertices consisting of three pairwise adjacent vertices t_1, t_2, t_3, and three more vertices s_1, s_2, s_3, where for $1 \leq i \leq 3$ s_i has degree one and t_i is its unique neighbour. The *antinet* is the complement graph of the net.

Figure 4: The net and the antinet.

We begin with:

(1) *If G contains a net as an induced subgraph, then G is a net, and so G is the line graph of a bipartite graph.*

Suppose that $s_1, s_2, s_3, t_1, t_2, t_3$ are distinct vertices of G forming a net in the notation above. Let $W = \{s_1, s_2, s_3, t_1, t_2, t_3\}$. If $W = V(G)$ the claim holds, so we assume there exists $v \in V(G) \setminus W$. Since $G[\{v, t_1, t_2, t_3\}]$ is not an anticlaw, v is adjacent to one of t_1, t_2, t_3, say t_1. Since $G[\{v, t_1, s_2, s_3\}]$ is not a claw, v is nonadjacent to one of s_2, s_3, say s_2. Since $G[\{v, s_1, s_2, t_1\}]$ is not an anticlaw, v is nonadjacent to s_1. Since $G[\{v, s_1, t_1, t_3\}]$ is not a claw, v is adjacent to t_3. But then $G[\{v, t_1, s_2, t_3\}]$ is an anticlaw, a contradiction. This proves (1).

From (1) we may assume G contains no net as an induced subgraph, and by taking complements we may also assume that G contains no antinet.

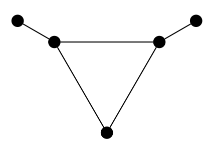

Figure 5: The bull.

The *bull* is the graph with five vertices $t_1, \ldots, t_5$, where t_1-t_2-t_3-t_4 is an induced path and t_5 is adjacent to t_2, t_3 and nonadjacent to t_1, t_4. Note that the complement of a bull is a bull.

(2) *We may assume that G contains a bull as an induced subgraph.*

Suppose first that G has no triangle. Then G has maximum degree at most two, since it has no claw or triangle, and the theorem holds. So we may assume that G has a triangle, and (by taking complements) G has a stable set of cardinality three. Choose a triangle T, and a set S of three pairwise nonadjacent vertices, with $S \cup T$ minimal. Let $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$. Certainly $|S \cup T| \leq 6$; suppose that equality holds. If t_1 is adjacent to at most one of s_1, s_2, s_3, say not to s_2, s_3, then $\{t_1, s_2, s_3\}$ is a stable set, contradicting the minimality of $S \cup T$. So each t_i is adjacent to at least two of s_1, s_2, s_3. By the same argument in the complement, each s_j is nonadjacent to at least two of t_1, t_2, t_3; but this is impossible. Thus $|S \cup T| \leq 5$; and so equality holds, since $|S \cap T| \leq 1$. We may assume that $s_3 = t_3$. Now each of s_1, s_2 is adjacent to one of t_1, t_2, since it is not an anticlaw centre; but each of t_1, t_2 is adjacent to at most one of s_1, s_2, since it is not a claw centre. Consequently $G[S \cup T]$ is a bull. This proves (2).

Let $W = \{t_1, \ldots, t_5\}$, and let $G[W]$ be a bull in G, with notation as in the definition of a bull. For each vertex $v \in V(G) \setminus W$, let $W(v)$ denote the set of neighbours of v in W. Define A_1, A_2, B_1, B_2 by:

- A_1 is the set of $v \in V(G) \setminus W$ with $W(v) = \{t_1, t_5\}$;
- A_2 is the set of $v \in V(G) \setminus W$ with $W(v) = \{t_4, t_5\}$;
- B_1 is the set of $v \in V(G) \setminus W$ with $W(v) = \{t_1, t_2, t_4\}$;
- B_2 is the set of $v \in V(G) \setminus W$ with $W(v) = \{t_1, t_3, t_4\}$.

(See figure 6.)

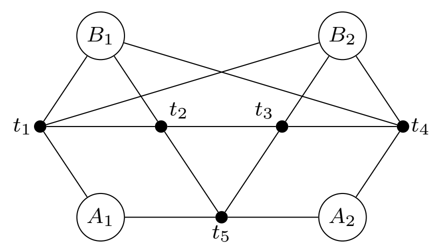

Figure 6: For step (3) of the proof of 4.5.

We claim:

(3) *$A_1 \cup A_2 \cup B_1 \cup B_2 = V(G) \setminus W$.*

Let $v \in V(G) \setminus W$. Assume first that $t_5 \in W(v)$. Since $G[\{v, t_5, t_1, t_4\}]$ is not a claw, one of $t_1, t_4 \notin W(v)$, say t_1. Since $G[\{v, t_5, t_3, t_1\}]$ is not an anticlaw, $t_3 \notin W(v)$. Since $G[\{v, t_1, t_2, t_3\}]$ is not a claw, $t_2 \notin W(v)$. Since $G[W \cup \{v\}]$ is not a net, $t_4 \in W(v)$. But then $v \in A_2$. Now assume that $t_5 \notin W(v)$. We apply the same argument in the complement to deduce that $v \in B_1 \cup B_2$. This proves (3).

(4) *Each of A_1, A_2, B_1, B_2 has cardinality at most one.*

If there exist $u, v \in A_1$, nonadjacent, then $G[\{u, v, t_1, t_2\}]$ is a claw, a contradiction; and if there exist $u, v \in A_1$ adjacent, then $G[\{u, v, t_1, t_4\}]$ is an anticlaw, a contradiction. Similarly $|A_2| \leq 1$; and by taking complements it follows that $|B_1|, |B_2| \leq 1$. This proves (4).

(5) *The pairs $(A_1, A_2), (A_1, B_2), (A_2, B_1)$ are complete, and the pairs $(B_1, B_2), (A_1, B_1), (A_2, B_2)$ are anticomplete.*

If there exist $u \in A_1$ and $v \in A_2$, nonadjacent, then $G[\{u, v, t_5, t_2\}]$ is a claw; so A_1 is complete to A_2. By taking complement it follows that B_1 is anticomplete to B_2. If there exists $u \in A_1$ and $v \in B_1$, adjacent, then $G[\{u, v, t_1, t_3\}]$ is an anticlaw, a contradiction; so A_1 is anticomplete to B_1, and from the symmetry A_2 is anticomplete to B_2. By taking complements it follows that A_1 is complete to B_2, and A_2 is complete to B_1. This proves (5).

From (4) and (5) it follows that $|V(G)| \leq 9$, and G is an induced subgraph of the line graph of $K_{3,3}$. This proves 4.5. ∎

We use 4.5 to prove the next result:

4.6 *Let G be an uncluttered graph with no nontrivial homogeneous set. Then one of $G, \overline{G}$ is the line graph of a triangle-free graph.*

Proof. Suppose that neither of $G, \overline{G}$ is the line graph of a triangle-free graph. Suppose first that G contains a claw and an anticlaw. Then since there is a nondominating triangle, it follows from 4.4 that no clique is dominating. If $a \in V(G)$ is a claw centre, then it is in at least three maximal cliques, and so by 4.3 it is in no triangle (since all triangles are nondominating), and hence the set of neighbours of a is stable. By taking complements, if $b \in V(G)$ is an anticlaw centre, then the set of vertices nonadjacent to b is a clique.

(1) *There do not exist a claw centre a and an anticlaw centre b with $a \neq b$.*

Suppose there exist such a, b. By taking complements if necessary, we may assume that a, b are nonadjacent. Since b is an anticlaw centre, the set of vertices of G nonadjacent to b is a clique C say. Now $|C| \geq 3$ since b is an anticlaw centre; and $a \in C$. Thus a belongs to a triangle, and so belongs to at most two maximal cliques by 4.3, contradicting that a is a claw centre. This proves (1).

(2) *There do not exist both a claw and an anticlaw in G.*

Suppose there is both a claw and an anticlaw. Then by (1), there is a vertex c that is the unique claw centre and the unique anticlaw centre. Let A be its set of neighbours and let $B = V(G) \setminus (A \cup \{c\})$. Since c is not in a triangle, A is stable, and similarly B is a clique. Since c is a claw centre, $|A| \geq 3$, and similarly $|B| \geq 3$. Let $b \in B$. If b has no neighbour in A, then the stable set $\{b, c\}$ is a dominating clique of $\overline{G}$, a contradiction to 4.4 applied to $\overline{G}$. If b has at least two nonneighbours in A, say a_1, a_2, let b be adjacent to $a_3 \in A$ and then $G[\{b, a_1, a_2, a_3, c\}]$ is a fork, a contradiction. But b has at most two neighbours in A since b is not a claw centre; so $|A| = 3$ and b has exactly two neighbours in A. By the same argument applied in the complement, $|B| = 3$ and every vertex in A has exactly two nonneighbours in B, which is impossible by counting edges between A and B. This proves (2).

From (2), and taking complements if necessary, we may assume there is no claw in G. By 4.5 we may assume that there is an anticlaw in G, that is, there is a nondominating triangle. Consequently every clique is nondominating, by 4.4. By 4.3 every vertex that is in a triangle is in at most two maximal cliques. But if a vertex v is not in any triangle, then since there is no claw it follows that v has degree at most two, and so v is in at most two maximal cliques. This proves that every vertex is in at most two maximal cliques. Let $C_1, \ldots, C_t$ be the maximal cliques of G, and make a graph H with vertex set $\{1, \ldots, t\}$, where distinct i, j are adjacent if $C_i \cap C_j \neq \emptyset$. If $|C_i \cap C_j| \geq 2$ for some distinct i, j, then $C_i \cap C_j$ is a nontrivial homogeneous set of G, which is impossible; so G is the line graph of H. It remains to show that H is triangle-free. Suppose not; then we may assume that

C_1, C_2, C_3 pairwise intersect. Choose $v_1 \in C_2 \cap C_3$, and define v_2, v_3 similarly. Then $v_1 \notin C_1$, since every vertex belongs to at most two of C_1, C_2, C_3, and similarly $v_2 \notin C_2$ and $v_3 \notin C_3$. But v_1, v_2 are adjacent, because they both belong to C_3, and similarly v_1, v_2, v_3 are pairwise adjacent; and hence there is a maximal clique containing all three of v_1, v_2, v_3. It is different from C_1, C_2, C_3, and so v_1 belongs to three different maximal cliques, a contradiction. This proves that H is triangle-free, and so proves 4.6. ∎

By combining 4.6 and 3.1, we deduce our main result, which we restate:

4.7 *Let G be an uncluttered graph. Then either*

- *one of $G, \overline{G}$ is disconnected; or*
- *one of $G, \overline{G}$ has adjacent simplicial twins; or*
- *one of $G, \overline{G}$ is candled; or*
- *one of $G, \overline{G}$ is the line graph of a triangle-free graph.*

Proof. If G has a nontrivial homogeneous set, then by 3.1 either one of $G, \overline{G}$ is disconnected, or one of $G, \overline{G}$ has adjacent simplicial twins, or one of $G, \overline{G}$ is candled, and in each case the theorem holds. If G has no nontrivial homogeneous set, then by 4.6, one of $G, \overline{G}$ is the line graph of a triangle-free graph, and again the theorem holds. This proves 4.7. ∎

5 Karthick's question

We denote the chromatic number of a graph G by $\chi(G)$, and the cardinality of its largest clique by $\omega(G)$. Let us deduce from 2.1 a result we stated earlier, that answers a question of Karthick. We restate it:

5.1 *For every uncluttered graph G, $\chi(G) \leq 2\omega(G)$.*

Proof. We proceed by induction of $|V(G)|$. We may apply 2.1. If G is the disjoint union of two graphs G_1, G_2, then

$$\chi(G) = \max(\chi(G_1), \chi(G_2)) \leq \max(2\omega(G_1), 2\omega(G_2)) \leq 2\omega(G)$$

as required. If $\overline{G}$ is the disjoint union of $\overline{G_1}, \overline{G_2}$, then

$$\chi(G) = \chi(G_1) + \chi(G_2) \leq 2\omega(G_1) + 2\omega(G_2) \leq 2\omega(G)$$

as required.

If G has a simplicial vertex v, then we can extend any colouring of $G \setminus \{v\}$ to a colouring of G if we have at least $\omega(G)$ colours. Consequently

$$\chi(G) \leq \max(\chi(G \setminus \{v\}), \omega(G)) \leq \max(2\omega(G \setminus \{v\}, \omega(G)) \leq 2\omega(G)$$

as required. If G has nonadjacent twins u, v, then

$$\chi(G) = \chi(G \setminus \{u\}) \leq 2\omega(G \setminus \{u\} = 2\omega(G)$$

as required. So we may assume that G has no simplicial vertex and no two nonadjacent twins. Consequently neither G nor $\overline{G}$ has adjacent simplicial twins.

If G is candled, let $Y_1, \ldots, Y_k, Z_1, \ldots, Z_k$ be as in the definition of "candled"; then any two vertices in Z_i are nonadjacent twins, and so each Z_i has cardinality one. But then the vertices in each Y_i are simplicial, a contradiction. If $\overline{G}$ is candled, again let $Y_1, \ldots, Y_k, Z_1, \ldots, Z_k$ be as in the definition; then any two vertices in Y_i are nonadjacent twins in G, so each Y_i has cardinality one; but then the vertices in each Z_i are simplicial in G, a contradiction.

If G is the line graph of a triangle-free graph H, then $\chi(G)$ is the edge-chromatic number $\chi'(H)$ of H, and $\omega(G)$ is the maximum degree $\delta(H)$ of H. By Vizing's theorem, $\chi'(H) \leq \Delta(H) + 1$, so $\chi(G) \leq \omega(G) + 1 \leq 2\omega(G)$ (because we can assume that $\omega(G) > 0$). Finally, if $\overline{G}$ is the line graph of a triangle-free graph H, then $\chi(G)$ is the size $\tau(H)$ of the smallest set of vertices of H that meets every edge of H, and $\omega(G)$ is the size $\mu(H)$ of the largest matching in H. But $\tau(H) \leq 2\mu(H)$, and so again $\chi(G) \leq 2\omega(G)$. This proves 5.1. ∎

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