Stable sets in flag spheres

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Abstract

We provide lower and upper bounds on the minimum size of a maximum stable set over graphs of flag spheres, as a function of the dimension of the sphere and the number of vertices. Further, we use stable sets together with graph rigidity to obtain an improved Lower Bound Theorem for the face numbers of flag spheres. We propose a graph rigidity approach to settle the Lower Bound Conjecture for flag spheres in full.

1 Introduction

Given a graph G, a set $X \subseteq V(G)$ is *stable* (or *independent*) if no edge of G has both ends in X. Equivalently, X is a clique in the complementary graph of G. We denote by $\alpha(G)$ the size of a largest stable set in G; a stable set of size $\alpha(G)$ is called a *maximum stable set* of G. Stable sets are a basic concept in graph theory, but it is in general very difficult to understand what the structure of maximum stable sets is (this is related to the fact that the problem of computing $\alpha(G)$ is NP-complete). In this paper we study maximum stable sets in graphs whose clique complex is topologically a sphere of fixed dimension (these are called *graphs of flag spheres*). These graphs possess a beautiful recursive structure, since the neighborhood of every vertex is a graph of the same type but of lower dimension. They are also of great interest in topological combinatorics and beyond, e.g. in the study of manifolds with nonpositive sectional curvature, via the Charney-Davis conjecture [3, 5].

Our main objective is the following natural invariant: the minimum size over maximum stable sets in n-vertex graphs of flag $(d-1)$-dimensional spheres,

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namely

$\alpha(d,n) = \min(\alpha(G) : |V(G)| = n,\ cl(G)$ triangulates the $(d-1)$-dimensional sphere$)$.

(Here $cl(G)$ is the complex of cliques of G.) For fixed d we are interested in the growth of $\alpha(d,n)$ as $n \to \infty$.

Conjecture 1.1. *For every $d \geq 2$ and $n \geq 2d$, $\alpha(d,n) = \lceil \frac{n+d-3}{2(d-1)} \rceil$.*

This conjecture holds for $d = 2$ (easy) and $d = 3$ (see Theorem 2.3, using the 4-color theorem (4CT) for the lower bound). For $d = 4$ we prove that the conjectured upper bound holds. For general $d \geq 4$ we show:

Theorem 1.2. *Let $d \geq 4$ and $n \geq 2d$. Then*

$$\frac{1}{4} n^{\frac{1}{d-2}} \leq \alpha(d,n) \leq \left\lceil \frac{\lceil \frac{n}{\lfloor d/4 \rfloor} \rceil + 1}{6} \right\rceil.$$

The lower bound slightly improves on the Ramsey bound ($\Omega(n^{\frac{1}{d}})$) by using the 4CT within the base case $d = 4$, see Theorem 2.7. The upper bound, which is roughly $\frac{2n}{3d}$ for large d, is obtained by taking the join of copies of the best flag 3-spheres constructed in Theorem 2.4 for the upper bound, and taking up to 3 extra suspensions to reach dimension $d-1$. Indeed, a maximum stable set in the join is a maximum stable set in a component of the join – now, ignoring rounding, such component is a 3-sphere on a $4/d$ fraction of the n vertices, and a $1/6$ fraction of its vertices form a maximum stable set.

Our second result is an improved lower bound theorem on the number of edges for the class of flag spheres; the proof relies on the existence of a large stable set in such graphs. Deducing from this bound lower bounds on the number of higher dimensional k-faces appeared in the proof of [14, Prop.3.2], following the MPW-reduction.

Theorem 1.3. *(i) Fix $\delta > 0$. There exists $d(\delta)$ such that for all $d \geq d(\delta)$ and n large enough (compared to δ and d), each n-vertex flag $(d-1)$-sphere has at least $(d + \frac{1-\delta}{2d+1})n$ edges.*

(ii) For all $d \geq 6$, and n large enough (compared to d), each n-vertex flag $(d-1)$-sphere has at least $(d + \frac{0.987}{2d+1})n$ edges.

Note that the Lower Bound Theorem for simplicial spheres [2, 8] guarantees in (i) for simplicial spheres at least $dn - \binom{d+1}{2}$ edges, hence more than $(d-\delta)n$ edges, and Gal's conjecture [5], which, if true, is tight, guarantees at least $(2d-3)n - 2d(d-2)$ edges, hence more than $(2d-3-\delta)n$ edges (it does hold for $d \leq 5$). For $d \geq 6$ the lower bound in Theorem 1.3(ii) appears to be new. If Conjecture 1.1 holds then this lower bound would further improve to at least $(d + \frac{1}{2d-2})n$ edges, for all $d \geq 6$, for large enough n.

Outline: In Section 2 we construct low dimensional flag spheres whose maximum independent sets are small, proving Conjecture 1.1 for $d = 3$ and the upper bound there for $d = 4$, and deducing both bounds in Theorem 1.2.

In Section 3 we prove Theorem 1.3 by combining stable sets with framework rigidity. We propose in Problem 3.5 a rigidity statement that would settle both Gal's lower bound conjecture for flag spheres [5], and a conjecture of Lutz and Nevo on characterizing the extremal cases [11]. In Section 4 we give some results and conjectures regarding the corresponding invariant for the other extreme:

$$\alpha_M(d,n) = \max(\alpha(G) :\ |V(G)| = n,\ cl(G) \text{ triangulates the } (d-1)\text{-dimensional sphere}).$$

2 The construction

We construct graphs, denoted $W_{d,k}$. First we analyze their α, and next we analyze their clique complex. Figure 1(middle) illustrates $W_{3,3}$.

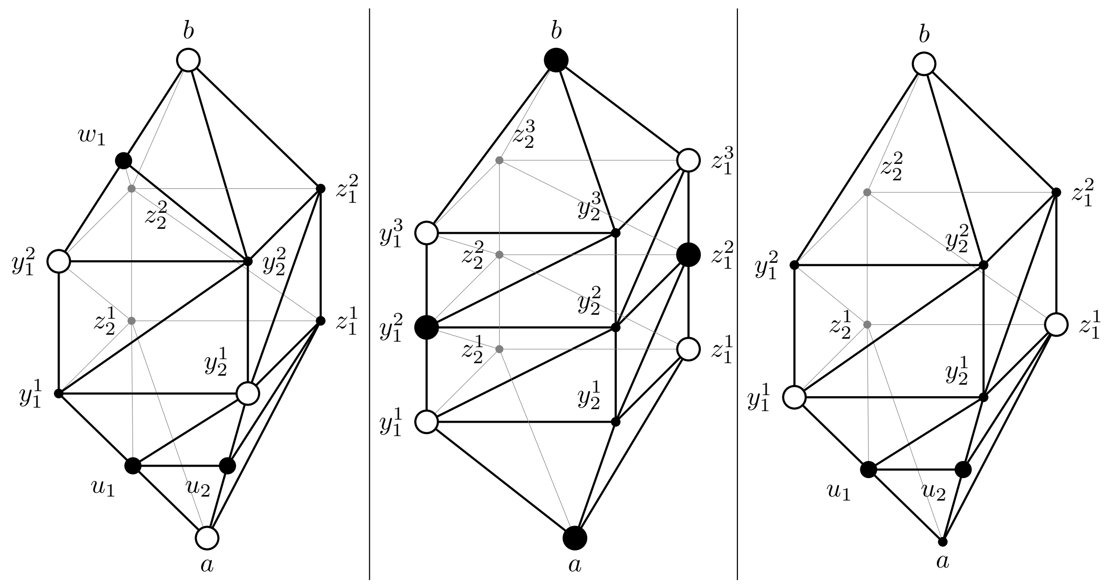

Figure 1: **Middle**: The graph $W_{3,3}$ is depicted. The bold black and bold white vertices indicate stable sets of size $\alpha(W_{3,3}) = 4$. The shaded edges indicate edges that are not visible from a front view of the depicted realization of the flag 2-sphere $cl(W_{3,3})$ in 3-space. Similarly, **Right**: the graph $X(3,2,2)$ is depicted. The bold white vertices indicate a stable set of size $\alpha(X(3,2,2)) = 3$; **Left**: the graph $Y(3,2,1)$ is depicted. The bold white vertices indicate a stable set of size $\alpha(Y(3,2,1)) = 4$.

Fix an integer $d \geq 2$. For $k \geq 1$ let $W_{d,k}$ be the following graph. $V(W_{d,k}) = \{a,b\} \cup X_1 \cup \ldots \cup X_k$ where the sets $X_1, \ldots, X_k, \{a,b\}$ are pairwise disjoint and $|X_i| = 2d-2$ for every $i \in \{1,\ldots,k\}$. Denote $X_i = \{y_1^i, \ldots, y_{d-1}^i, z_1^{i'}, \ldots, z_{d-1}^i\}$. Next we list the edges of $W_{d,k}$.

- a is complete to X_1 and b is complete to X_k and there are no other edges incident with a, b.
- For every i, the induced graph $W_{d,k}[X_i]$ is the 1-skeleton of the $(d-1)$-dimensional crosspolytope, a.k.a. the graph of the octahedral $(d-2)$-sphere, with non-edges $y_1^i z_1^i, \dots, y_{d-1}^i z_{d-1}^i$.
- X_i is anticomplete to X_j if $|i-j| > 1$.
- For $1 \leq i \leq k-1$, the set of edges of $W_{d,k}$ with one endpoint in X_i and the other endpoint in X_{i+1} is $E_i^+ \cup E_i^-$, where $E_i^+ = \{y_s^i y_t^{i+1}, z_s^i z_t^{i+1} : \ 1 \leq s \leq t \leq d-1\}$ and $E_i^+ = \{y_s^i z_t^{i+1}, z_s^i y_t^{i+1} : 1 \leq t < s \leq d-1\}$. Denote $E^+ := \cup_{1 \leq i \leq d-1} E_i^+$ and $E^- := \cup_{1 \leq i \leq d-1} E_i^-$.
- All pairs of vertices of $W_{d,k}$ that are not mentioned above are non-edges.

Now we define certain "edge subdivisions" on $W_{d,k}$, corresponding to stellar subdivisions at edges of the complex $cl(W_{d,k})$.

Define an *edge subdivision* of a graph $G = (V, E)$ at an edge $xy \in E$ as the process producing the new graph $G(xy) = (V', E')$ where $V' = V \cup \{v_{xy}\}$ ($v_{xy} \notin V$) and $E' = (E \setminus \{xy\}) \cup \{xv_{xy}, yv_{xy}\} \cup \{uv_{xy} : \ ux, uy \in E\}$. Note that the stellar subdivision of $cl(G)$ at the edge xy produces $cl(G(xy))$; in particular, $cl(G)$ and $cl(G(xy))$ are homeomorphic.

Now, consider a maximal clique among the neighbors of a (resp. b) in $W_{d,k}$, say $y_1^1 y_2^1 y_3^1 \dots y_{d-1}^1$ (resp. $y_1^k y_2^k y_3^k \dots y_{d-1}^k$). Make the following sequence of $2d-2$ edge subdivisions: $X(d,k,0) := W_{d,k}$, and for $j \in \{1, \dots, d-1\}$, having defined $X(d,k,j-1)$ and u_{j-1} (for $j > 1$), let $X(d,k,j) := X(d,k,j-1)(ay_j^1)$ and $u_j := v_{ay_j^1}$. For example, Figure 1(right) illustrates $X(3,2,2)$.

Next let $Y(d,k,0) := X(d,k,d-1)$, and for $j \in \{1, \dots, d-1\}$, having defined $Y(d,k,j-1)$ and w_{j-1} (for $j > 1$), let $Y(d,k,j) := Y''(d,k,j-1)(by_j^k)$ and $w_j := v_{by_j^k}$. For example, Figure 1(left) illustrates $Y(3,2,1)$.

Theorem 2.1. *For every $d \geq 2, k \geq 1, d-1 \geq j \geq 0$,*
$\alpha(X(d,k,j)) = k+1 = \frac{|V(X(d,k,j))|-2-j}{2d-2} + 1.$
For every $d \geq 3, k \geq 1, d-1 \geq j \geq 1$,
$\alpha(Y(d,k,j)) = k+2 = \frac{|V(Y(d,k,j))|-2+(d-1-j)}{2d-2} + 1.$

Proof. Let G be one of the graphs $X(d,k,j)$ or $Y(d,k,j)$. Let U be the set of vertices of the form u_j in G, and let W be the set of vertices of the form w_j in G. Then $W \neq \emptyset$ only if $|U| = d-1$. Moreover $U \cup a$ and $W \cup b$ are both cliques in G. Denote by $N_G(v)$ the neighbors of v in G. Then, $X_1 \setminus N_G(a) \subseteq \{y_1^1, \dots, y_{d-1}^1\}$, and for every j we have that $X_1 \setminus N_G(u_j) = \{y_1^1, \dots, y_{j-1}^1, z_j^1\}$. In particular, $\alpha(G[X_1 \setminus N_G(v)]) \leq 1$ for every $v \in U \cup \{a\}$. Similarly, $\alpha(G[X_k \setminus N_G(v)]) \leq 1$ for every $v \in W \cup \{b\}$.

Let S be a stable set of G. First we prove an upper bound on $|S|$. Clearly for every i we have that $\alpha(G[X_i]) = 2$. Moreover every vertex of X_{i+1} has a neighbor in every non-edge of $G[X_i]$, and every vertex of X_i has a neighbor in every non-edge of $G[X_{i+1}]$. Consequently, $|S \cap (X_i \cup X_{i+1})| \leq 2$.

Hence $|S \setminus (U \cup W \cup \{a, b\})| \leq k+1$. Suppose $|S \setminus (U \cup W \cup \{a, b\})| = k+1$. Then k is odd, and $|S \cap X_1| = |S \cap X_k| = 2$. It follows that $S \cap (U \cup W \cup \{a, b\}) = \emptyset$ and $|S| = k+1$.

Next suppose that $|S \setminus (U \cup W \cup \{a, b\})| = k$. Since $U \cup \{a\}$ and $W \cup \{b\}$ are both cliques, it follows that $|S| \leq k+2$, and so we may assume that $G = X(d, k, j)$ for some j (for otherwise $G = Y(d, k, j)$ and the upper bound on $\alpha(G)$ holds).

In particular $W = \emptyset$ and b is adjacent to every vertex of X_k. Since $|S \setminus (U \cup W \cup \{a, b\})| = k$, it follows that $|S \cap (X_1 \cup X_k)| = 2$. If $|S \cap X_k| \neq \emptyset$, then $b \notin S$, and, since $U \cup \{a\}$ is a clique, we have that $|S| \leq k+1$. Thus we may assume that $S \cap X_k = \emptyset$, and so $|S \cap X_1| = 2$. Since $\alpha(G[X_1 \setminus N(v)]) \leq 1$ for every $v \in U \cup \{a\}$, we deduce that $S \cap (U \cup \{a\}) = \emptyset$, and so $|S| = k$ if $b \notin S$ and $|S| = k+1$ if $b \in S$.

Clearly if $|S \setminus \{a, b\}| < k$ then, since $U \cup \{a\}$ and $W \cup \{b\}$ are both cliques, we have that $|S| \leq k+1$. Thus in all cases the upper bound on $|S|$ holds.

Next we show that if $G = X(d, k, j)$ for some $j \geq 0$ then $\alpha(G) = k+1$. Let $S' = \bigcup_{i \in 1, \ldots, k;\ i \text{ odd}} \{y_1^i, z_1^i\}$. If k is odd let $S = S'$. If k is even, let $S = S' \cup \{b\}$. In both cases $|S| = k+1$.

Finally we show that if $G = Y(d, k, j)$ for some $j \geq 1$ then $\alpha(G) = k+2$. Since $j \geq 1$, we have that a is anticomplete to $\{y_1^1, \ldots, y_{d-1}^1\}$ and $w_1 \in W$. Let

$$S = \{a, w_1\} \cup \bigcup_{i \in \{1, \ldots, k\};\ k-i \text{ odd}} \{y_1^i\} \cup \bigcup_{i \in \{1, \ldots, k\};\ k-i \text{ even}} \{z_1^i\}.$$

Then S a stable set of size $k+2$ in G.

So far we have proved that $\alpha(X(d, k, j)) = k+1$ for every $d \geq 2, k \geq 1$ and $d-1 \geq j \geq 0$, and that $\alpha(Y(d, k, j)) = k+2$ for every $d \geq 3, k \geq 1$ and $d-1 \geq j \geq 1$. The remaining equalities follow by a direct computation. $\square$

Observe that $W_{d,1}$ is the 1-skeleton of the d-dimensional crosspolytope. Further,

Observation 2.2. *The clique complex of $W_{3,k}$ is a flag 2-sphere for every $k \geq 1$.*

Proof. For each i, $W_{3,k}[X_i]$ is a 4-cycle. Consider $W_{3,k}[X_i \cup X_{i+1}]$: adding to the two disjoint 4-cycles $W_{3,k}[X_i] \cup W_{3,k}[X_{i+1}]$ the "vertical" edges $y_s^i y_s^{i+1}$ and $z_s^i z_s^{i+1}$ makes a cylinder subdivided into 4 squares; adding the other edges in $W_{3,k}$ that cross from X_i to X_{i+1} subdivides each of the four squares into two triangles. Thus, the clique complex of $W_{3,k}[X_1 \cup \ldots \cup X_k]$ is a triangulated cylinder, and adding a, b with their edges makes the clique complex a flag 2-sphere. $\square$

Next we show:

Theorem 2.3. *For every $n \geq 6$, $\alpha(3, n) = \lceil \frac{n}{4} \rceil$.*

Proof. Observe that $|V(X(3, k, j))| \equiv_4 2 + j$, and $|V(Y(3, k, j))| \equiv_4 j$, and thus for every $n \geq 6$ there exist integers $k \geq 1$ and $j \geq 0$ and a graph $G \in$

$\{X(3,k,j), Y(3,k,j)\}$ such that $|V(G)| = n$. Now by Theorem 2.1 for every $k \geq 1$ and $j \geq 0$ we have that $\alpha(X(3,k,j)) = \lceil \frac{|V(X(3,k,j))|}{4} \rceil$, and for every $k \geq 1$ and $j \geq 1$ we have that $\alpha(Y(3,k,j)) = \lceil \frac{|V(Y(3,k,j))|}{4} \rceil$. Finally, since $X''(3,k,j)$ and $Y''(3,k,j)$ are obtained from $cl(W_{3,k})$ by stellar edge subdivisions, it follows from Observation 2.2 that their clique complexes are flag 2-spheres. We have shown that for every $n \geq 6$, $\alpha(3,n) \leq \lceil \frac{n}{4} \rceil$. Since by the 4CT every n-vertex triangulation of the 2-dimensional sphere has a stable set of size $\lceil \frac{n}{4} \rceil$, $\alpha(3,n) \geq \lceil \frac{n}{4} \rceil$. □

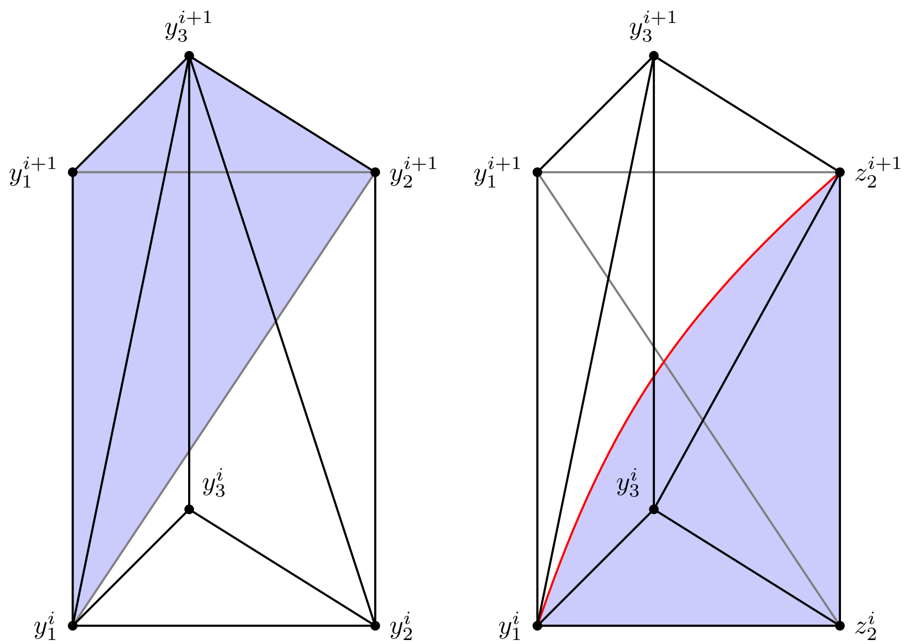

Figure 2: Two triangular prisms with the induced graphs in $W'_{4,k}$ on their vertices. Left: prism of Case 1. Right: prism of Case 2. The grey edges indicate edges not visible from a front view of the depicted realization embeded in 3-space. The red edge is bent inside the right prism. In purple are sample induced tetrahedra. Note that in each prism, its clique complex triangulates it.

For $d = 4$, the graph $W_{4,k}$, with all "non vertical" edges in $E^+ \cup E^-$ deleted – namely delete the edges in $E^+ \cup E^- \setminus \{y_s^i y_s^{i+1}, z_s^i z_s^{i+1} : 1 \leq i \leq k-1, 1 \leq s \leq 3\}$, induces a cell structure on the 3-sphere, consisting of tetrahedra with a vertex a or b and of triangular prisms consisting of a triangle on X_i and the corresponding triangle on X_{i+1} (the corresponding vertices differ only in the superscript). Adding the non-vertical edges in $E^+ \cup E^-$ adds exactly one diagonal in each square face. For each triangular prism T, exactly one of two cases hold:

Case 1. The 3 diagonals on the square faces of T form a path. In this case, the clique complex on the induced subgraph $W_{4,k}[V(T)]$ triangulates T, by three tetrahedra. ($V(T)$ stands for the vertices of T.) For example, if $V(T) = \{y_1^i, y_2^i, y_3^i, y_1^{i+1}, y_2^{i+1}, y_3^{i+1}\}$ then the three tetrahedra are given by the vertex sets $\{y_1^i, y_1^{i+1}, y_2^{i+1}, y_3^{i+1}\}$, $\{y_1^i, y_2^i, y_2^{i+1}, y_3^{i+1}\}$ and $\{y_1^i, y_2^i, y_3^i, y_3^{i+1}\}$.

Case 2. The 3 diagonals on the square faces of T form a matching. In this case the cliques of the graph $G(T) := W_{4,k}[V(T)]$ make the octahedral 2-sphere, triangulating the boundary of the prism T. Note that adding any edge e (of the three missing edges in $G(T)$), $cl(G(T) \cup \{e\})$ triangulates T by four tetrahedra. There are exactly two such prisms (for a fixed $1 \leq i \leq k-1$), one has vertices $y_1^i, z_2^i, y_3^i; y_1^{i+1}, z_2^{i+1}, y_3^{i+1}$ and its "antipodal prism" has vertices $z_1^i, y_2^i, z_3^i; z_1^{i+1}, y_2^{i+1}, z_3^{i+1}$.

For every $1 \leq i \leq k-1$ we add the edge $y_1^i z_2^{i+1}$ to triangulate the first, and the edge $z_1^i y_2^{i+1}$ to triangulate the second (such added edge is "bent" inside the prism, the resulted triangulation of the prism is topological, not geometric); denote the resulting graph by $W'_{4,k}$. Let $X'(4,k,j)$ and $Y'(4,k,j)$ be the graphs obtained from $X(4,k,j)$ and $Y(4,k,j)$, respectively, by adding these same edges. See Figure 2 for an illustration of how the triangular prisms are triangulated.

Theorem 2.4. *The clique complex of $W'_{4,k}$ is a flag 3-sphere for every $k \geq 1$.*

Proof. By the discussion above on the triangulation of the triangular prisms, for every prism T in Case 1, $cl(W_{4,k}[V(T)])$ triangulates T, and for every prism T in Case 2, $cl(W'_{4,k}[V(T)])$ triangulates T. Thus, there are only two claims to verify, showing no "extra cliques" appear, namely: for every $1 \leq i \leq k-1$,

(i) all the triangles containing the edge $y_1^i z_2^{i+1}$ are contained in $V(T(i)) := \{y_1^i, z_2^i, y_3^i; y_1^{i+1}, z_2^{i+1}, y_3^{i+1}\}$, and

(ii) all the triangles containing the edge $z_1^i y_2^{i+1}$ are contained in $V(T'(i)) := \{z_1^i, y_2^i, z_3^i; z_1^{i+1}, y_2^{i+1}, z_3^{i+1}\}$.

We verify that (i) holds. (For (ii), one either verifies similarly, or uses symmetry.) Indeed, the potential "extra" triangles could occur only in the unique prism $T''(i)$ that intersects $T(i)$ in the unique square containing both y_1^i and z_2^{i+1}. Note that $V(T''(i)) \setminus V(T(i)) = \{z_3^i, z_3^{i+1}\}$, however, neither $y_1^i z_3^{i+1}$ nor $z_3^i z_2^{i+1}$ are edges in $W'_{4,k}$, hence no extra triangles occur, as desired. □

Next we show:

Theorem 2.5. *For all $n \geq 8$, $\alpha(4,n) \leq \lceil \frac{n+1}{6} \rceil$.*

Proof. Observe that $|V(X(4,k,j))| \equiv_6 2+j$, and $|V(Y(4,k,j))| \equiv_6 j-1$ (here $0 \leq j \leq 3$), and thus for every $n \geq 8$ there exist integers $k \geq 1$ and $j \geq 0$ and a graph $G \in \{X(4,k,j), Y(4,k,j)\}$ such that $|V(G)| = n$. Now by Theorem 2.1 for every $k \geq 1$ and $j \geq 0$ we have that $\alpha(X(4,k,j)) = \lceil \frac{|V(X(4,k,j))|+1}{6} \rceil$, and for every $k \geq 1$ and $j \geq 1$ we have that $\alpha(Y(4,k,j)) = \lceil \frac{|V(Y(4,k,j))+1|}{6} \rceil$. Since $X'(4,k,j)$ and $Y'(4,k,j)$ are obtained from $X(4,k,j)$ and $Y(4,k,j)$ by adding edges, we deduce that $\alpha(X'(4,k,j)) \leq \lceil \frac{|V(X(4,k,j))|+1}{6} \rceil = \lceil \frac{|V(X'(4,k,j))|+1}{6} \rceil$ for every $k \geq 1$ and $j \geq 0$, and $\alpha(Y'(4,k,j)) \leq \lceil \frac{|V(Y(4,k,j))|+1}{6} \rceil = \lceil \frac{|V(Y'(4,k,j))|+1}{6} \rceil$ and for every $k \geq 1$ and $j \geq 1$.

Finally, since $cl(X'(4,k,j))$ and $cl(Y'(4,k,j))$ are obtained from $cl(W'(4,k))$ by stellar edge subdivisions, it follows from Theorem 2.4 that their clique complexes are flag 3-spheres. This completes the proof. $\square$

Remark 2.6. *In fact,* $\alpha(X'(4,k,j)) = \lceil \frac{|V(X(4,k,j))|+1}{6} \rceil$ *for every* $k \geq 1$ *and* $j \geq 0$*, and* $\alpha(Y'(4,k,j)) = \lceil \frac{|V(Y(4,k,j))|+1}{6} \rceil$ *for every* $k \geq 1$ *and* $j \geq 1$*.*

Indeed, for $d = 4$ the sets S constructed in the proof of Theorem 2.1 are also independent in $X'(4,k,j)$ and $Y'(4,k,j)$ resp.

Finally we prove the lower bound of Theorem 1.2.

Theorem 2.7. *Let* $d \geq 4$*. Then for all* $n \geq 2d$*,*

$$\alpha(d,n) \geq \frac{1}{4} n^{\frac{1}{d-2}}$$

Proof. The proof is by induction on d, and we work in the generality of flag *homology* spheres, over some fixed field. Let Δ be a $(d-1)$-flag homology sphere. Recall Δ has at least $2d$ vertices [13], say it has n vertices.

For the base case let $d = 4$. Then the link of v in Δ, denoted $lk_v(\Delta)$, is a planar triangulation for every vertex v of Δ, and therefore, by the 4CT, $lk_v(\Delta)$ contains a stable set of size $\lceil \frac{|V(lk_v(\Delta))|}{4} \rceil$. Thus if for some vertex v of Δ we have that $|V(lk_v(\Delta))| \geq n^{\frac{1}{2}}$, then the theorem holds. If $|V(lk_v(\Delta))| < n^{\frac{1}{2}}$ for every v, then a stable set of size $\frac{n}{n^{\frac{1}{2}}} = n^{\frac{1}{2}} > \frac{1}{4} n^{\frac{1}{2}}$ can be obtained greedily. Indeed, set $I = \emptyset$. At each iteration, starting with the graph G, add some vertex v from the remaining graph to I and delete v and its neighbors from the graph; repeat as long as the remaining graph is nonempty. Then the resulted I is stable in G and its cardinality equals the number of iterations, which is at least $n/\sqrt{n}$. This finishes the case when $d = 4$.

Now we turn to general d. In this case $lk_v(\Delta)$ is a $(d-2)$-flag homology sphere for every vertex v of Δ, and therefore, inductively, $lk_v(\Delta)$ contains a stable set of size $\frac{1}{4}|V(lk_v(\Delta))|^{\frac{1}{d-3}}$. Thus if for some vertex v of Δ we have that $|V(lk_v(\Delta))| \geq n^{\frac{d-3}{d-2}}$, then the theorem holds. If $|V(lk_v(\Delta))| < n^{\frac{d-3}{d-2}}$ for every v, then a stable set of size $\frac{n}{n^{\frac{d-3}{d-2}}} = n^{\frac{1}{d-2}} > \frac{1}{4} n^{\frac{1}{d-2}}$ can be obtained greedily. This completes the proof. $\square$

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3 Lower bounds on f_1

The goal of this section is to prove Theorem 1.3.

Proof. Let $\Delta = cl(G)$ be a flag $(d-1)$-sphere on $n = f_0(\Delta)$ vertices and $f_1 = f_1(\Delta)$ edges. Let $\epsilon > 0$, and assume $f_1 < (d+\epsilon)n$. We look for the largest $\epsilon = \epsilon(d)$ for which we reach a contradiction (when d is chosen large enough, and then n is chosen large enough w.r.t. d).

By an easy restatement of Turán's theorem from [12] there is a stable set I of G with $|I| \geq \frac{n}{2(d+\epsilon)+1}$.

We may assume $d \geq 4$. Then, we use the following well known facts: (i) G is generically d-rigid [8, Thm.1.2], hence its space of stresses (see e.g. [10, Eq.(1)] for a definition, a.k.a. affine 2-stresses [10, Def.2]) has dimension $g_2(\Delta) := f_1 - dn + \binom{d+1}{2}$, see e.g. the discussion above Theorem 6.3 in Kalai [8]. (ii) For every vertex link, its graph is generically $(d-1)$-rigid (see Gluck [6] for $d = 4$ and Kalai [8, Thm.1.2] for $d \geq 5$), and is not stacked (by flagness), hence, by the Cone Lemma, see e.g. [16, Cor.1.5], for every vertex $v \in \Delta$ there exists a stress supported in the closed star of v (namely in the induced graph of G on v and its neighbors) such that some edge containing v has a nonzero weight.

Now, as I is independent, the stresses mentioned above, one per $v \in I$, are linearly independent (indeed, the stress for v has at least one edge containing v with a nonzero weight, while the weights on the edges containing v is zero for the stresses of all $u \in I \setminus \{v\}$), and hence

$$f_1 - dn + \binom{d+1}{2} \geq |I| \geq \frac{n}{2(d+\epsilon)+1},$$

Thus, for n large enough w.r.t. d, we can ignore the $\binom{d+1}{2}$ term and get: $\epsilon n > \frac{n}{2(d+\epsilon)+1}$, namely $\epsilon > \frac{1}{2(d+\epsilon)+1}$.

Solving the quadric for ϵ we get a contradiction if $\epsilon < \frac{-(2d+1)+\sqrt{(2d+1)^2+8}}{4}$.

Hence for arbitrarily small $\delta > 0$, if d is large enough we reach a contradiction for $\epsilon = \frac{1-\delta}{2d+1}$, proving part (i). For part (ii), note that $\sqrt{x^2+8} - x > \frac{3.95}{x}$ for $x \geq 13 = 2 \cdot 6 + 1$, thus for all $d \geq 6$ (and large enough n) we will reach a contradiction if $\epsilon \leq \frac{3.95}{4(2d+1)} = \frac{0.987}{2d+1}$. □

Note that if Conjecture 1.1 holds then plugging the larger value for $|I|$ yields $f_1 \geq (d + \frac{1}{2d-2})n$ for all $d \geq 6$ and large enough n.

Conjecture 3.1. *For all $d \geq 5$, the graph of every flag $(d-1)$-sphere is generically $(d+1)$-rigid.*

If true, this conjecture would imply $f_1 \geq (d+1)f_0 - \binom{d+2}{2}$ for flag spheres of dimension $d - 1 \geq 4$. A standard use of the Cone and Gluing Lemmas, see Kalai [8], reduces Conjecture 3.1 to the case $d = 5$. For $d < 5$ its assertion is false.

Conjecture 3.1 holds for flag spheres with few vertices:

Lemma 3.2. *Let $d \geq 5$. The graph of every flag $(d-1)$-sphere on at most $2d+3$ vertices is generically $(d+1)$-rigid.*

Proof. First, note that for $d \geq 4$, if Δ is a flag $(d-1)$-sphere with a generically $(d+1)$-rigid graph, and Δ' is obtained from Δ by Whiteley's vertex split, then Δ' has a generically $(d+1)$-rigid graph, as the two new vertices in Δ' have at least $2(d-2)$ common neighbors by [13] (see also [5, Lem.2.1.14]) and $2(d-2) \geq d$;

see Whiteley [18, Prop.1] for the case $d = 3$, however, his proof easily extends to $d > 3$.

Second, note that for $d \geq 5$, the graph of the $(d-1)$-dimensional octahedral sphere is $(d+1)$-rigid, by induction on d. Indeed, one checks that the graph of the 4-dimensional octahedral sphere is 6-rigid. Decompose the $(d-1)$-dimensional octahedral sphere as a union of two cones over the same $(d-2)$-dimensional octahedral sphere. By the Cone Lemma [16, Cor.1.5] and induction, each of the two cones has a generically $(d+1)$-rigid graph, and by the Gluing Lemma [1] the union of these two graphs is generically $(d+1)$-rigid, as the intersection of the cones has $2(d-1) \geq d+1$ vertices.

Third, the proofs of [15, Prop.5.4, 5.5, 5.6] show that every flag $(d-1)$-sphere with at most $2d+3$ vertices can be obtained from the $(d-1)$-dimensional octahedral sphere by a sequence of (at most three) vertex splits, and hence, by the previous two items, has a generically $(d+1)$-rigid graph. □

Remark 3.3. *Alan Lew suggested to us a balder conjecture, based on computer experiments, that for $d \geq 3$ the graph of every flag $(d-1)$-sphere is generically $(2d-1-\lfloor\sqrt{2d}+1/2\rfloor)$-rigid, which, by Jordán [7, Lem.2.1(2)], would be tight for the octahedral $(d-1)$-sphere. If true, this conjecture would imply $f_1 \geq (2d - O(\sqrt{d}))n$ for n-vertex flag $(d-1)$-spheres.*

Let us single out the case of the octahedral spheres, which may be of independent interest.

Conjecture 3.4. *(Alan Lew) For every $d \geq 3$, the graph of the octahedral $(d-1)$-sphere, namely the complete graph on $2d$ vertices minus a perfect matching, is generically $(2d-1-\lfloor\sqrt{2d}+1/2\rfloor)$-rigid.*

Inspired by Alan Lew's conjectures we propose the following problem. For $d \geq 4$, call a flag $(d-1)$-sphere *irreducible* if it is not a suspension and if every edge is contained in an induced 4-cycle. Edge contractions show that is it enough to prove Gal's conjecture on γ-nonnegativity [5] for irreducible triangulations in order to confirm his conjecture for all flag spheres.

Problem 3.5. *Let $d \geq 4$. Then the graph of every irreducible flag $(d-1)$-sphere is generically $(2d-3)$-rigid.*

If true, it would imply both Gal's lower bound conjecture for flag spheres, namely, $\gamma_2 := f_1 - (2d-3)n + 2d(d-2) \geq 0$ (where n is the number of vertices), and a conjecture of Lutz and Nevo [11, Conj.6.1] characterizing the cases of equality $\gamma_2 = 0$. Further, it would imply $\gamma_2 \geq d-3$ for irreducible flag $(d-1)$-spheres.

Explicit irreducible flag spheres are not so easy to come by. Alan Lew verified by computer that the assertion of Problem 3.5 holds for the irreducible flag 3-sphere on 12 vertices recently constructed by Venturello [17].

4 $\alpha_M(d,n)$

Fix $d \geq 4$ and let $n \to \infty$. Then there exist simplicial $(d-1)$-spheres on n vertices where the proportion of vertices in an independent set is arbitrarily close to 1. To see this, start with the boundary complex Δ of a cyclic d-polytope with $m > d$ vertices, and note that Δ is a neighborly $(d-1)$-sphere, i.e. all $\binom{m}{\lfloor \frac{d}{2} \rfloor}$ subsets consisting of $\lfloor \frac{d}{2} \rfloor$ vertices are faces in Δ. It is easy to check that Δ has $\Theta(m^{\lfloor \frac{d}{2} \rfloor})$ facets. Perform stellar subdivisions on all facets. Then the set I of the newly added vertices is stable and of size $\Theta(m^{\lfloor \frac{d}{2} \rfloor})$, while only the original m vertices are not in I.

In contrast, for flag spheres we conjecture that the proportion of vertices in an independent set can not exceed 1/2.

Conjecture 4.1. *For all* $d \geq 2$*,* $\alpha_M(d,n) = \lfloor \frac{n-2(d-2)}{2} \rfloor$.

This conjecture clearly holds for $d = 2$ and we prove it for $d = 3$. The lower bound holds for all $d \geq 2$ by the following construction: consider the $(d-2)$-fold suspension over the $(n-2(d-2))$-gon. A maximum stable set is obtained by taking every second vertex along the $(n-2(d-2))$-gon.

Theorem 4.2. *For all* $n \geq 6$*,* $\alpha_M(3,n) = \lfloor \frac{n-2}{2} \rfloor$.

Proof. The construction above proves the lower bound $\alpha_M(3,n) \geq \lfloor \frac{n-2}{2} \rfloor$. To show $\alpha_M(3,n) \leq \lfloor \frac{n-2}{2} \rfloor$, let I be a maximum stable set in the graph $G = (V,E)$ of a flag 2-sphere on n vertices (it forces $n \geq 6$). Let $G' = (V,B)$ be the subgraph of G whose edges are those with exactly one vertex in I. Then G' is bipartite and planar. Further, G' has at least two vertices in I (as each vertex in G has a non-neighbor) and at least two (in fact 4) vertices in the complement of I (as each vertex in I has degree at least 4 by flagness). Thus, G' has at most $2n-4$ edges (this is known, see e.g. [9, Lemmas 4.2, 4.3] for a proof). On the other hand,

$$|B| = \sum_{v \in I} \deg(v) \geq 4|I|,$$

as each vertex in G has degree at least 4, and for all $v \in I$ the degree is preserved when passing to G'. Thus $4|I| \leq 2n-4$, hence $|I| \leq \lfloor \frac{n-2}{2} \rfloor$. □

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References

[1] L. Asimow and B. Roth. The rigidity of graphs. II. *J. Math. Anal. Appl.*, 68(1):171–190, 1979.

[2] David W. Barnette. A proof of the lower bound conjecture for convex polytopes. *Pac. J. Math.*, 46:349–354, 1973.

[3] Ruth Charney and Michael Davis. The Euler characteristic of a nonpositively curved, piecewise Euclidean manifold. *Pacific J. Math.*, 171(1):117–137, 1995.

[4] Maria Chudnovsky and Eran Nevo. Stable sets in flag spheres. *FPSAC2022, to appear.*

[5] Światosław R. Gal. Real root conjecture fails for five- and higher-dimensional spheres. *Discrete Comput. Geom.*, 34(2):269–284, 2005.

[6] Herman Gluck. Almost all simply connected closed surfaces are rigid. In *Geometric topology (Proc. Conf., Park City, Utah, 1974)*, pages 225–239. Lecture Notes in Math., Vol. 438. Springer, Berlin, 1975.

[7] Tibor Jordán. A note on generic rigidity of graphs in higher dimension. *Discrete Appl. Math.*, 297:97–101, 2021.

[8] Gil Kalai. Rigidity and the lower bound theorem. I. *Invent. Math.*, 88(1):125–151, 1987.

[9] Gil Kalai, Eran Nevo, and Isabella Novik. Bipartite rigidity. *Trans. Amer. Math. Soc.*, 368(8):5515–5545, 2016.

[10] C. W. Lee. P.L.-spheres, convex polytopes, and stress. *Discrete Comput. Geom.*, 15(4):389–421, 1996.

[11] Frank H. Lutz and Eran Nevo. Stellar theory for flag complexes. *Math. Scand.*, 118(1):70–82, 2016.

[12] J. Komlös M. Ajtai, P. Erdös and E. Szemeredi. On turán's theorem for sparse graphs. *Combinatorica*, 1:313–317, 1981.

[13] Roy Meshulam. Domination numbers and homology. *J. Combin. Theory Ser. A*, 102(2):321–330, 2003.

[14] Eran Nevo. Remarks on missing faces and generalized lower bounds on face numbers. *Electron. J. Combin.*, 16(2, Special volume in honor of Anders Bjorner):Research Paper 8, 11, 2009.

[15] Eran Nevo and T. Kyle Petersen. On γ-vectors satisfying the Kruskal-Katona inequalities. *Discrete Comput. Geom.*, 45(3):503–521, 2011.

[16] Tiong-Seng Tay, Neil White, and Walter Whiteley. Skeletal rigidity of simplicial complexes. I. *European J. Combin.*, 16(4):381–403, 1995.

[17] Lorenzo Venturello. On flag spheres with few equators. *arXiv:2203.10003*, 2022.

[18] Walter Whiteley. Vertex splitting in isostatic frameworks. *Struc. Top.*, 16:23–30, 1989.