On treewidth and maximum cliques

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Abstract

We construct classes of graphs that are variants of the so-called *layered wheel*. One of their key properties is that while the treewidth is bounded by a function of the clique number, the construction can be adjusted to make the dependance grow arbitrarily. Some of these classes provide counter-examples to several conjectures. In particular, the construction includes hereditary classes of graphs whose treewidth is bounded by a function of the clique number while the tree-independence number is unbounded, thus disproving a conjecture of Dallard, Milanič and Štorgel [Treewidth versus clique number. II. Tree-independence number. *Journal of Combinatorial Theory, Series B*, 164:404–442, 2024.]. The construction can be further adjusted to provide, for any fixed integer c, graphs of arbitrarily large treewidth that contain no K_c-free graphs of high treewidth, thus disproving a conjecture of Hajebi [Chordal graphs, even-hole-free graphs and sparse obstructions to bounded treewidth, arXiv:2401.01299, 2024].

1 Introduction

Graphs in this paper are oriented and infinite (with neither loops nor multiple edges). However, this is only for technical reasons and most of our results will be about finite and simple graphs.

A *clique* in a graph is a set of pairwise adjacent vertices and an *independent set* is a set of pairwise non-adjacent vertices. The maximum number of vertices in a clique (resp. indepedendent set) in a graph G is denoted by $\omega(G)$ (resp. $\alpha(G)$). We denote by $\chi(G)$ the *chromatic number of* G, that is the minimum number of colors needed to color vertices of G in such a way that adjacent vertices receive different colors. A *hole* in a graph is a chordless cycle of length at least 4. It is *even* if it contains an even number of vertices. We denote by $N(v)$ the neighborhood of v and set $N[v] = N(v) \cup \{v\}$. A class of graphs is *hereditary* if it closed under taking induced subgraphs.

A *tree decomposition* of a graph G is a pair $\mathcal{T} = (T, (X_s)_{s \in V(T)})$ where T is a tree and every node $s \in T$ is assigned a set $X_s \subseteq V(G)$ called a *bag* such that the following conditions are satisfied: every vertex is in at least one bag, for every edge $uv \in E(G)$ there exists a bag X_s such that $\{u, v\} \subseteq X_s$, and for every vertex $u \in V(G)$, the set $\{s \in V(T) | u \in X_s\}$ induces a connected subgraph of T. The *width* of $\mathcal{T}$ is the maximum value of $|X_s| - 1$ over all $s \in V(T)$. The *independent width* of $\mathcal{T}$ is the maximum value of $\alpha(X_s)$ over all $s \in V(T)$. The *treewidth* of a graph G, denoted by $\mathrm{tw}(G)$, is the minimum width of a tree decomposition of G. The *tree-independence number* of a graph G, denoted by tree-$\alpha(G)$, is the minimum independent width of a tree decomposition of G. It was first defined by Yolov in [12] and rediscovered independently by Dallard, Milanič and Štorgel in [4]. The treewidth, and more recently the tree-independence number, attracted some attention, see for instance the introduction of [4].

The main contribution of this paper is a variant of the so-called *layered wheel*. It was first introduced by Sintiari and Trotignon in [11] to provide graphs of arbitrarily large treewidth that exclude several kinds of induced subgraphs, such as K_4 and even holes, or triangles and thetas (not worth defining here). Our variant may contain cliques of any size. For every integer $\ell \geq 4$, we construct a variant that contains only holes of length at least ℓ. Our construction provides answers to questions and counter-examples to conjectures due to different authors, all about the treewidth and the tree-independence number in hereditary classes of graphs, as we explain now.

Conjectures and questions

A hereditary class of graphs $\mathcal{C}$ is said to be (tw, ω)-bounded if there exists a function g such that the treewidth of any graph $G \in \mathcal{C}$ is at most $g(\omega(G))$. In [3] and [5], Dallard, Milanič and Štorgel asked whether every (tw, ω)-bounded class of graphs is in fact *polynomially* (tw, ω)*-bounded*. We rephrase this question formally as follows.

Question 1.1 (Dallard, Milanič and Štorgel, see [5, Question 8.4])**.** *For every* (tw, ω)*-bounded class of graphs* $\mathcal{C}$*, does there exist a polynomial* g *such that every graph* $G \in \mathcal{C}$ *satisfies* $\mathrm{tw}(G) \leq g(\omega(G))$ *?*

Our construction provides a negative answer to this question, see Theorem 5.1 below. In [4, Lemma 3.2] it is observed that the answer is affirmative for classes with bounded tree-independence number. It is also observed that hereditary classes of graphs with bounded tree-independence number are (tw, ω)-bounded (this is an easy consequence of Ramsey theorem, see [5]). The following conjecture is proposed by Dallard, Milanič and Štorgel.

Conjecture 1.2 (Dallard, Milanič and Štorgel, see [5, Conjecture 8.5])**.** *Let $\mathcal{C}$ be a hereditary graph class. Then G is (tw, ω)-bounded if and only if $\mathcal{C}$ has bounded tree-independence number.*

Our construction disproves this conjecture, even when the function that bounds the treewidth is assumed to be a polynomial, see Theorem 5.2 below.

Our construction also sheds light on certifying a large treewidth in a graph G by exhibiting some simpler substructure of G of large treewidth. The well-known Grid Theorem by Robertson and Seymour [9] gives a neat certificate when the substructure under consideration is a minor of G. When the substructure under consideration is an induced subgraph of G, the situation is more complicated and is still the subject of much research. To understand this better, several questions and conjectures (together with a survey) are proposed by Hajebi in [8].

Conjecture 1.3 (Hajebi, see [8, Conjecture 1.14])**.** *For every $t \geq 1$, every graph of large enough treewidth has an induced subgraph of treewidth t which is either complete or K_4-free.*

Our construction (or more precisely a variant of it) disproves this conjecture, even in a weaker form, where K_4 is replaced in the statement by K_c for any constant $c \geq 4$, see Theorem 5.3 below. Note that when $c \in \{1, 2\}$ the statement is trivially false, and for $c = 3$ it is already disproved in [11] with the construction of K_4-free graphs of high treewidth with no even holes. For every integer c, a graph is *c-degenerate* if each of its induced subgraphs contains a vertex of degree at most c. A variant of Conjecture 1.3 is proposed in the same paper.

Conjecture 1.4 (Hajebi, see [8, Conjecture 1.15])**.** *For every integer $t \geq 1$, every graph of large enough treewidth has an induced subgraph of treewidth t which is either complete, complete bipartite, or 2-degenerate.*

Our construction also disproves Conjecture 1.4, see Theorem 5.3 below (even with "2-degenerate" replaced by higher degeneracy).

Unanswered questions

We would like to point out several questions that our construction does not answer. It is easily seen to produce graphs whose treewidth is logarithmic into the number of vertices. This implies (see for instance [1, Theorem 9.2]) that the Max Weight Independent Set problem is polynomial time solvable for our construction. Hence, we believe the following question might still have an affirmative answer, and that logarithmic treewidth might be a key ingredient of an algorithm.

Question 1.5 (Dallard, Milanič and Štorgel, see [4])**.** *Is the Max Weight Independent Set problem solvable in polynomial time in every* (tw, ω)*-bounded graph class?*

In our counter-example to Conjecture 1.2, we need that the class is (tw, ω)-bounded by a super-linear function. Hence, our construction does not seem to help answering the following question.

Question 1.6. *Does every hereditary class of graph $\mathcal{C}$ such that for some constant c, every graph $G \in \mathcal{C}$ satifisfies* $\mathrm{tw}(G) \leq c\,\omega(G)$*, has bounded tree-independence number?*

We leave as an open question the existence of an even-hole-free variant of our construction. If it exists, it might disprove the following conjecture.

Conjecture 1.7 (Hajebi, see [8])**.** *For every $t \geq 1$, every even-hole-free graph of large enough treewidth has an induced subgraph of treewidth t which is either complete or K_4-free.*

In our counter-example to Conjecture 1.3, we need a graph G such that $\omega(G) = c + 1$ to guaranty that K_c-free graphs in the class have bounded treewidth. So, we do not know the answer to the following question (quite suprisingly, Theorem 5.1 does not seem to help).

Question 1.8. *Does there exist a function $f : \mathbb{N} \setminus \{0\} \times \mathbb{N} \setminus \{0\} \to \mathbb{N}$ such that for all integers $k, t \geq 1$, all graphs G such that* $\mathrm{tw}(G) \geq f(k,t)$ *and $\omega(G) = k$ contain an induced subgraph H such that* $\mathrm{tw}(H) \geq t$ *and $\omega(H) = k - 1$.*

It might be of interest to study what induced subgraphs are contained in our construction. The original layered wheels from [11] suggest that maybe the so-called 3-paths-configurations (see the definition in [11]) are not contained in our construction, or in some variant of our construction.

It is easy to check that our construction contains as an induced subgraph an already known construction of Corneil and Rotics, designed to find graphs of large cliquewidth (not worth defining here) and small treewidth, see [2]. We therefore wonder whether our construction improves bounds about differents notions of widths of graphs.

Tools to bound the treewidth

To prove our results, we need to bound the treewidth and the tree-independence number of graphs produced by our construction. For that we rely on several known concepts and theorems.

To bound the treewidth and the tree-independence number from below, we rely on *clique minors*. For a graph G, two disjoint sets $Y \subseteq V(G)$ and $Z \subseteq V(G)$ are *adjacent* if some edge of G has an end in Y and an end in Z. A *clique minor* of a graph G is a family of disjoint, connected, and pairwise adjacent subsets $L_1, \dots, L_t$ of $V(G)$. The following is a well-known consequence of the Helly property of subtrees of trees.

Lemma 1.9. *If $\mathcal{T} = (T, (X_s)_{s \in V(T)})$ is a tree decomposition of some graph G and $(L_1, \dots, L_t)$ is a clique minor of G, then there exists $s \in V(T)$ such that X_s contains at least one vertex of each L_i, $i \in \{1, \dots, t\}$. In particular, $\mathrm{tw}(G) \geq t - 1$ and the independent width of $\mathcal{T}$ is at least $\alpha(G[X_s])$.*

To bound the treewidth from above, we rely on *balanced separations*. A *separation* of a graph G is a pair (A, B) of subsets of $V(G)$ such that $A \cup B = V(G)$ and no edge of G has one end in $A \setminus B$ and the other in $B \setminus A$. The *order* of the separation is $|A \cap B|$. It is *balanced* if $|A \setminus B| \leq 2n/3$ and $|B \setminus A| \leq 2n/3$ where $n = |V(G)|$. The *separation number* $\mathrm{sn}(G)$ of G is the smallest integer s such that every subgraph (or equivalently induced subgraph) of G has a balanced separation of order s.

Theorem 1.10 (Dvorák and Norin, see [7])**.** *The treewidth of any graph G is at most $15\,\mathrm{sn}(G)$.*

We also need the following classical results. A graph is *chordal* if it contains no hole. A classical characterization due to Rose [10] tells that a graph G is chordal graphs if and only if all induced subgraphs H of G contain a simplicial vertex (in H), where a vertex is *simplicial* if its neighborhood is a clique (for our purpose, we only need this characterization of chordal graphs). The following is usually refered to as the perfection of chordal graphs.

Theorem 1.11 (Dirac, see [6])**.** *If G is a chordal graph, then $\chi(G) = \omega(G)$.*

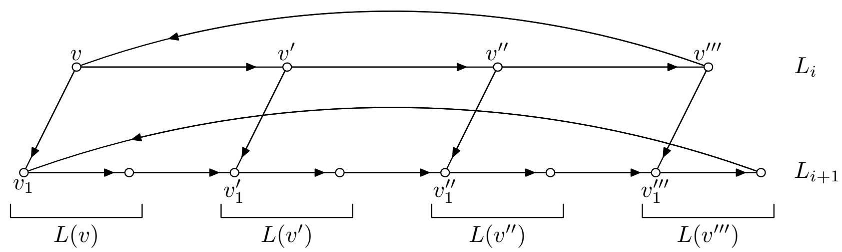

Figure 1: Rules R2–R4

Outline of the paper

We define what we call the (f, ℓ)-layered wheels in Section 2. We study some of their structural properties in Section 3. The (f, ℓ)-layered wheels are infinite graphs and their finite induced subgraphs are studied in Section 4. We provide the answers to questions and counter-examples to conjectures in Section 5.

2 Definition of layered wheels

A function $f : \mathbb{N} \setminus \{0\} \to \mathbb{N} \setminus \{0\}$ is *slow* if $f(1) = 1$, $f(2) = 2$, $f(3) = 3$ and for every $i \in \mathbb{N} \setminus \{0\}$, $f(i) \leq f(i+1) \leq f(i)+1$. Observe that a slow function is non-decreasing. So, every slow function is either ultimately constant or tends to $+\infty$.

For every slow function f and every integer $\ell \geq 4$, we define an oriented graph G called the (f, ℓ)-*layered wheel*. The vertex-set of G is countably infinite. The set of arcs of G is denoted by $A(G)$. Note that all the theorems in this paper are about the underlying graph of G. The orientations of the arcs are only used to facilitate the description of several subsets of $V(G)$. Before defining G precisely, we list five rules giving some properties and terminology that help stating the formal description.

R1 $V(G)$ is partitioned into sets L_i, $i \in \mathbb{N} \setminus \{0\}$, called the *layers* of G.

R2 Each layer L_i, $i \in \mathbb{N} \setminus \{0\}$, induces a directed cycle of length at least ℓ.

R3 If u and v are adjacent vertices in different layers, say $v \in L_i$, $u \in L_j$ and $i < j$, then the arc linking them is oriented from v to u.

R4 For every $i \in \mathbb{N} \setminus \{0\}$ and every vertex $v \in L_i$, there exists an integer n_v and a directed path $L(v) = v_1 \dots v_{n_v}$ such that $V(L(v)) \subseteq L_{i+1}$ and $L(v)$ contains all the neighbors of v in L_{i+1}. The vertex v_1 is adjacent to v. Moreover the paths $L(v)$, $v \in L_i$, are vertex-disjoint, $L_{i+1} = \cup_{v \in L_i} V(L(v))$ and if $vv' \in A(G)$, then $v_{n_v} v'_1 \in A(G)$.

It follows that a vertex $u \in L_{i+1}$ has at most one neighbor in L_i. If such a neighbor v exists, we say that v is the *parent* of u and u is a *child* of v. Observe that every vertex in G has at least one child and at most one parent.

Informally, the directed paths $L(v)$, $v \in L_i$, vertex-wise partition L_{i+1}, and parents in L_i and their children in L_{i+1} appear in the same cyclic order, see Fig. 1.

R5 If $v \in L_i$, then v has at most $f(i)-1$ neighbors in $\bigcup_{1 \leq j < i} L_j$. Moreover, these neighbors induce a (possibly empty) clique that contains at most one vertex in each layer. We denote this clique by $N^{\uparrow}(v)$ and we set $N^{\uparrow}[v] = \{v\} \cup N^{\uparrow}(v)$.

Now we define the (f, ℓ)-layered wheel G. The first layer L_1 is a directed cycle of length ℓ. We suppose inductively that for some integer $i \geq 1$, the graph induced by the i first layers (so $G[L_1 \cup \dots \cup L_i]$) is defined, and we explain how to add the next layer L_{i+1}. We assume that rules R1, …, R5 hold for layers L_1, …, L_i. Since f is slow, $f(i) \in \{f(i+1)-1, f(i+1)\}$. Hence, by rule R5, for every $v \in L_i$, $|N^{\uparrow}(v)| \leq f(i+1)-1$.

To fully describe $G[L_1 \cup \dots \cup L_{i+1}]$, it is sufficient to define for each $v \in L_i$ the integer n_v, and for each vertex $v_j \in L(v)$, the set $N^{\uparrow}(v_j)$; then L_{i+1} can be described as the vertex-set of the directed cycle formed by the consecutive directed paths $L(v)$, so the neighborhood of every vertex from L_{i+1} is fully described. So, let v be a vertex in L_i. There are two cases (see Fig. 2 and Fig. 3):

R6 If $|N^{\uparrow}(v)| < f(i+1)-1$, then we set $n_v = \ell - 2$, so $L(v) = v_1 \dots v_{\ell-2}$. Moreover, we set:

- $N^{\uparrow}(v_1) = N^{\uparrow}[v]$
- $N^{\uparrow}(v_2) = \dots = N^{\uparrow}(v_{\ell-2}) = \emptyset$.

R7 If $|N^{\uparrow}(v)| = f(i+1)-1$, then by rule R5, we have $N^{\uparrow}(v) = \{w_1, \dots, w_{f(i+1)-1}\}$, where for all $j \in \{1, \dots, f(i+1)-1\}$, $w_j \in L_{i_j}$, and $1 \leq i_1 < i_2 < \dots < i_{f(i+1)-1} < i$. We set $n_v = (f(i+1)-1)(\ell-2)$, so $L(v) = v_1 \dots v_{(f(i+1)-1)(\ell-2)}$. Moreover, we set:

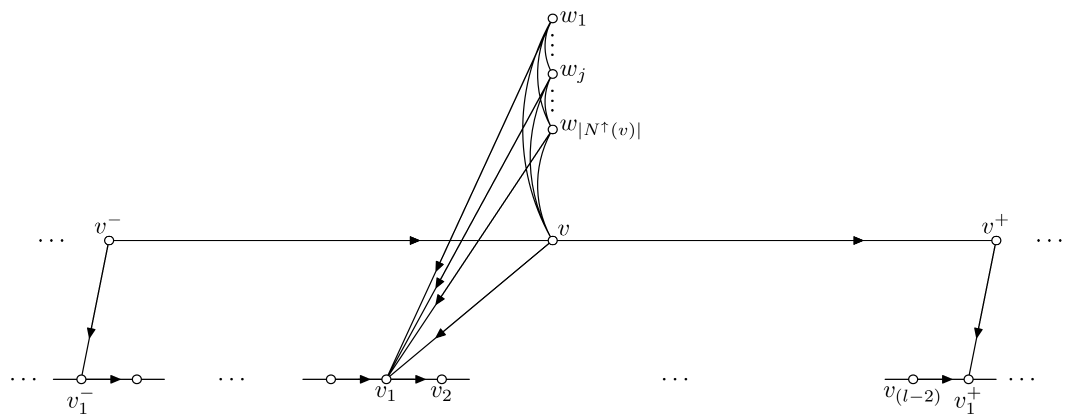

Figure 2: Rule R6

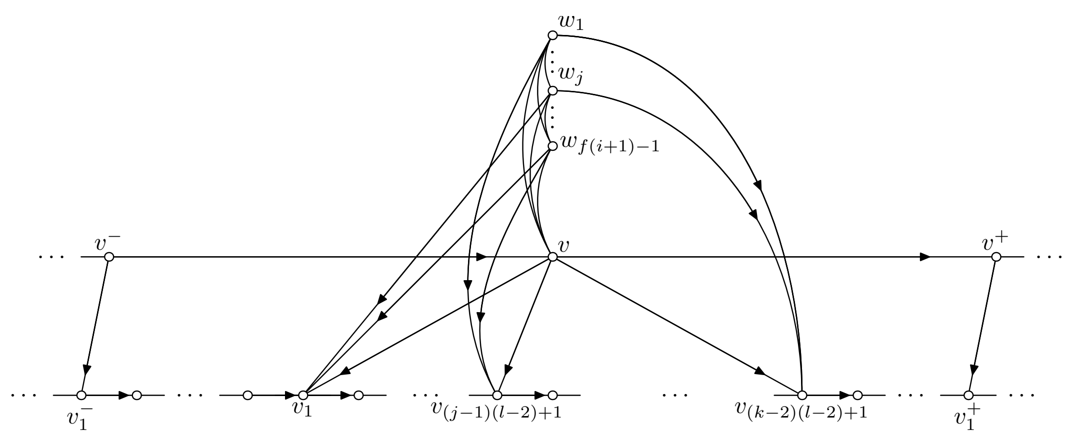

Figure 3: Rule R7

- For all $j \in \{1, \dots, f(i+1)-1\}$,

$$N^{\uparrow}(v_{(j-1)(\ell-2)+1}) = N^{\uparrow}[v] \setminus \{w_j\}.$$

- For all $j \in \{1, \dots, f(i+1)-1\}$ and $j' \in \{2, \dots, \ell-2\}$,

$$N^{\uparrow}(v_{(j-1)(\ell-2)+j'}) = \emptyset.$$

The description of $G[L_1 \cup \cdots \cup L_{i+1}]$ is now completed (see Fig. 4 for an example). Note that the new layer L_{i+1} satisfies rules R1–R5, so the inductive process can go on, and G is inductively defined. We now give several informal statements that will hopefully help the reader.

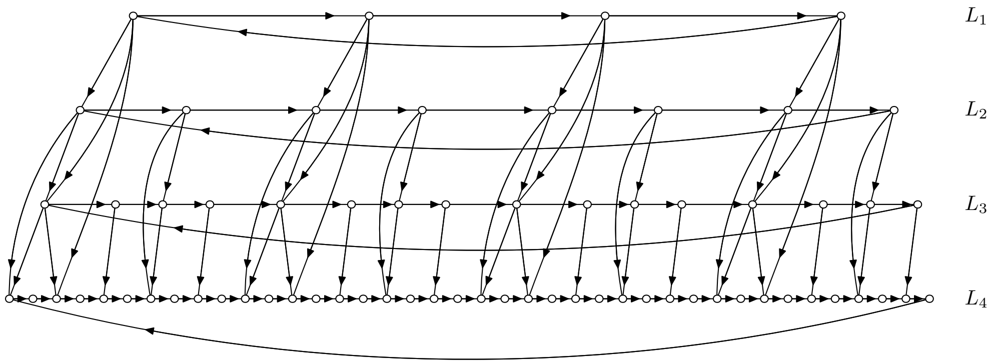

Figure 4: Layers L_1 to L_4 of the $(f, 4)$-layered wheel when $f(4) = 3$

The (f, ℓ)-layered wheel is infinite, but we are only interested in its finite induced subgraphs, that form a hereditary class of graphs. Recall that the orientations of the arcs are here only to help describing some sets of vertices later in the proofs. The statements of the theorems are only about the underlying undirected graph of G.

The layers of G should be thought of as the sets of a clique minor. More precisely, it will be shown in Lemma 3.2 that for every pair of layers L_i, L_j there exists an edge with one end in L_i and the other in L_j. It follows that for all integers $t \geq 1$, the layers $L_1, \dots, L_t$ form a clique minor of $G[L_1 \cup \cdots \cup L_t]$, that has therefore treewidth at least $t-1$ by Lemma 1.9.

The integer ℓ should be thought of as the length of a smallest hole in G. It controls the number of new vertices with no parent that are introduced in each new layer. The larger is ℓ, the more of them are needed to prevent creating short holes.

For every integer $i \geq 1$, the integer $f(i+1)$ should be thought of as the size of the cliques that are allowed to be introduced when the layer L_{i+1} is built. When a vertex $v \in L_i$ is such that $|N^{\uparrow}(v)| < f(i+1) - 1$, its children (all of which are in L_{i+1}) can be made complete to $N^{\uparrow}[v]$ whithout creating a clique of size larger than $f(i+1)$, and this explains the rule R6. When $|N^{\uparrow}(v)| = f(i+1) - 1$, the children of v cannot be made complete to $N^{\uparrow}[v]$ whithout creating a clique of size larger than $f(i+1)$. This explains why in rule R7, in the neighborhood of a child of v, we have to exclude one vertex from $N^{\uparrow}[v]$. Still, we want to give a chance of later augmentation to as many cliques as possible, and this is why we create $f(i+1) - 1$ children of v that cover all possible ways of extending a clique of size $f(i+1) - 1$ that contains v.

The function f should be thought of as the speed at which bigger and bigger cliques are introduced in the construction. We keep the flexibility of tuning f for different purposes. In particular it is convenient that f can be eventually constant. Then the layered wheel will have bounded clique size, and yet unbounded treewidth, see Theorem 5.3. On the other hand, if f tends to $+\infty$, then the layered wheel provides a class of graphs whose treewidth is bounded by a function of ω that is closely related to f, see Theorem 5.1 and Theorem 5.2.

3 Structure of layered wheels

Throughout this section, f is a slow function, $\ell \geq 4$ is an integer and G is the (f, ℓ)-layered wheel.

Lemma 3.1. *All holes of G have length at least ℓ.*

Proof. Let H be a hole of G and i be the maximum integer such that H contains vertices of L_i. By rule R2, if $V(H) = L_i$, then H has length at least ℓ, so we may assume that $V(H) \neq L_i$, implying that some vertex of L_i is not in H since L_i induces a hole. Let $u_1 \dots u_j$ be a subpath of L_i that is included in H and maximal with respect to this property. We have $j > 1$ since $N^{\uparrow}(u_1)$ is a clique by rule R5. Hence, by rule R6 or R7, $j \geq \ell - 1$. Hence, H has length at least ℓ. □

The following shows that the layers of G form a clique minor of G.

Lemma 3.2. *For all integers $i \geq 1$ and $i' > i$, every vertex $u \in L_i$ has at least one neighbor in $L_{i'}$.*

Proof. We prove the property by induction on i'. If $i' = i + 1$, then the conclusion follows directly from rule R4.

Assume that the property holds for some fixed $i' \geq i + 1$. Let us prove it for $i' + 1$. Note that $i' + 1 \geq 3$. So since f is slow, $f(i' + 1) \geq 3$. Let w be a vertex in L_i. By the induction hypothesis, w has a neighbor $v \in L_{i'}$. It is enough to check that the path $L(v)$ defined in R6 or R7 contains a neighbor of w. We use the notation of rules R6 and R7.

If $|N^{\uparrow}(v)| < f(i' + 1) - 1$, then rule R6 applies. So, v_1 is adjacent to w since $N^{\uparrow}(v_1) = N^{\uparrow}[v]$. If $|N^{\uparrow}(v)| = f(i' + 1) - 1$, then rule R7 applies. Since $f(i' + 1) - 1 \geq 2$, there exists $j \in \{1, \dots, f(i' + 1) - 1\}$ such that $w_j \neq w$. So, the vertex $v_{(j-1)(\ell-2)+1}$ from $L(v)$ is adjacent to w since $N^{\uparrow}(v_{(j-1)(\ell-2)+1}) = N^{\uparrow}[v] \setminus \{w_j\}$. □

Lemma 3.3. *For all integers $i \geq 2$, $\omega(G[L_1 \cup \dots \cup L_i]) = f(i)$.*

Proof. Let us prove by induction on i that for all $i \geq 1$, there exists a clique K of $G[L_1 \cup \dots \cup L_i]$ on $f(i)$ vertices such that $|K \cap L_i| = 1$. This is clearly true for $i = 1$. We suppose inductively that such K exists for some fixed $i \geq 1$ and call v the unique vertex of $K \cap L_i$. Since f is slow, $f(i+1) = f(i)+1$ or $f(i + 1) = f(i)$. In the former case, $|N^{\uparrow}(v)| = f(i) - 1 = f(i + 1) - 2$, so rule R6 applies and v has a unique child u satisfying $N^{\uparrow}(u) = N^{\uparrow}[v]$. So, $K \cup \{u\}$ is a clique on $f(i) + 1 = f(i + 1)$ vertices that contains u. In the latter case, $|N^{\uparrow}(v)| = f(i) - 1 = f(i+1) - 1$, so rule R7 applies, and for any child of u of v, $N^{\uparrow}(u) = N^{\uparrow}[v] \setminus \{w\}$ for some $w \in N^{\uparrow}(v)$. Hence, we again find a clique satisfying the conclusion.

We proved that $\omega(G[L_1 \cup \dots \cup L_i]) \geq f(i)$. Let us prove the converse inequality. For $i = 2$, it trivially holds. So, suppose $i \geq 3$. Let K be a maximum clique of $G[L_1 \cup \dots \cup L_i]$ and j be the maximum integer such that $K \cap L_j \neq \emptyset$. Note that $|K| \geq 3$. By rules R6 and R7, no adjacent vertices of L_j have a common neighbor in $L_{j'}$ if $j \geq j'$, so $|K \cap L_j| = \{u\}$ for some $u \in L_j$; consequently $K \subseteq N^{\uparrow}[u]$. It follows by rule R5 that $|K| \leq f(i)$. □

An infinite directed path P of G is a *vertical path starting in layer i* if $i \in \mathbb{N}^*$, $P = p_i p_{i+1} p_{i+2} \dots$ and for all $j \geq i$, $p_j \in L_j$. Observe that P may not be induced.

Lemma 3.4. *Let $P = p_i p_{i+1} p_{i+2} \dots$ and $Q = q_i q_{i+1} q_{i+2} \dots$ be two vertical paths starting in the same layer. If $p_i \neq q_i$, then $V(P) \cap V(Q) = \emptyset$.*

Proof. Otherwise, the common vertex of P and Q in L_j such that j is minimal has two parents, a contradiction to rule R4. □

Lemma 3.5. *If $p_i p_{i+1} p_{i+2} \dots$ is a vertical path, then for all $j \geq i$,*

$$N^{\uparrow}[p_j] \subseteq V(P) \cup N^{\uparrow}(p_i).$$

Proof. Let us prove the lemma by induction on j. If $j = i$ it trivially holds, so suppose $j > i$. By rules R6 or R7, $N^{\uparrow}(p_j) \subseteq N^{\uparrow}[p_{j-1}]$. So, by the induction hypothesis, $N^{\uparrow}[p_j] \subseteq V(P) \cup N^{\uparrow}(p_i)$. □

Let p and q be two vertices of L_i. We denote by $p\overrightarrow{L_i}q$ the vertex-set of the unique directed path of $G[L_i]$ from p to q. Note that if $p = q$, then $p\overrightarrow{L_i}q = \{p\}$ and if $qp \in A(G)$, then $p\overrightarrow{L_i}q = L_i$. We set $p\overleftarrow{L_i}q = \{p, q\} \cup (L_i \setminus p\overrightarrow{L_i}q)$. Observe that $G[p\overrightarrow{L_i}q]$ and $G[p\overleftarrow{L_i}q]$ edge-wise partition $G[L_i]$, so $(p\overrightarrow{L_i}q, p\overleftarrow{L_i}q)$ is a separation of $G[L_i]$ of order 2 (or 1 if $p = q$) since $p\overrightarrow{L_i}q \cap p\overleftarrow{L_i}q = \{p, q\}$.

Let P and Q be two vertical paths starting in the same layer L_i. We define

$$A(P, Q) = \bigcup_{u \in p_i\overrightarrow{L_i}q_i} N^{\uparrow}[u] \cup \bigcup_{j>i} p_j\overrightarrow{L_j}q_j$$

and

$$B(P, Q) = \bigcup_{1 \leq j \leq i} L_j \cup \bigcup_{j>i} p_j\overleftarrow{L_j}q_j.$$

Lemma 3.6. *If P and Q are two vertical paths starting in the same layer L_i, then:*

$$A(P, Q) \cup B(P, Q) = V(G)$$

and

$$A(P, Q) \cap B(P, Q) = V(P) \cup V(Q) \cup \bigcup_{u \in V(p_i\overrightarrow{L_i}q_i)} N^{\uparrow}[u].$$

Proof. By its definition, $B(P, Q)$ contains all layers $L_1, \dots, L_i$. Vertices of some layer L_j, $j > i$, are all either in $p_j\overrightarrow{L_j}q_j$ or $p_j\overleftarrow{L_j}q_j$ since these two sets form a separation of L_j. So, they are either in $A(P, Q)$ or in $B(P, Q)$. This proves the first equality.

The only vertices of $A(P, Q)$ that are in layers $L_1, \dots, L_i$ are those from $\bigcup_{u \in p_i\overrightarrow{L_i}q_i} N^{\uparrow}[u]$, and it turns out that they are all in $B(P, Q)$. In the next layers (so the L_j's, $j > i$), the only vertices that are both in $A(P, Q)$ and $B(P, Q)$ are the vertices of P and Q since for all $j > i$, $p_j\overrightarrow{L_j}q_j \cap p_j\overleftarrow{L_j}q_j = \{p_j, q_j\}$. This proves the second equality. □

A vertex $w \in L_j$ is an *ancestor* of a vertex $u \in L_i$ if $wu \in A(G)$ and $j < i$. A vertex $w \in L_j$ is a *descendant* of a vertex $u \in L_i$ if $uw \in A(G)$ and $j > i$.

Lemma 3.7. *If P and Q are two vertical paths starting in the same layer L_i and $u \in A(P,Q) \setminus B(P,Q)$, then all the descendants of u are in $A(P,Q) \setminus B(P,Q)$ and all ancestors of u are in $A(P,Q)$.*

Proof. Let us first prove the claim about the descendants of u. Suppose that the claim does not hold. Then there exists $uw \in A(G)$ such that $u \in (A(P,Q) \setminus B(P,Q)) \cap L_j$, $w \in B(P,Q) \cap L_{j'}$ and $j' > j$. We choose such a pair u, w subject to the minimality or $j' - j$. Since $uw \in A(G)$, by rule R6 or R7, w has a parent v such that $u \in N[v]$. We have $v \neq u$ because otherwise, w is a child of u and by rule R4, the children of u are in the interior of $p_{j+1}\overrightarrow{L_{j+1}}q_{j+1}$, so in $A(P,Q) \setminus B(P,Q)$. If $v \in V(P)$, then by Lemma 3.5, $u \in V(P) \cup N^{\uparrow}(p_i)$, a contradiction to $u \in A(P,Q) \setminus B(P,Q)$. So $v \notin V(P)$, and symmetrically $v \notin V(Q)$. Hence, v and w (if $v \in p_{j'-1}\overrightarrow{L_{j'-1}}q_{j'-1}$) or u, v (if $v \in p_{j'-1}\overleftarrow{L_{j'-1}}q_{j'-1}$) contradicts the minimality of $j' - j$.

Let us now prove the claim about the ancestors of u. Suppose that the claim does not hold. Then there exists $wu \in A(G)$ such that $u \in (A(P,Q) \setminus B(P,Q)) \cap L_j$, $w \in (B(P,Q) \setminus A(P,Q)) \cap L_{j'}$ and $j' < j$. We choose such a pair u, w subject to the minimality or $j - j'$. Since $wu \in A(G)$, by rule R6 or R7, u has a parent v such that $w \in N[v]$. If $v \in V(P)$, then by Lemma 3.5, $w \in V(P) \cup N^{\uparrow}(p_i)$, a contradiction to $w \in B(P,Q) \setminus A(P,Q)$. So $v \notin V(P)$, and symmetrically $v \notin V(Q)$. If $v \in p_i\overrightarrow{L_i}q_i$, then $w \in N^{\uparrow}[v]$, a contradiction to $w \in B(P,Q) \setminus A(P,Q)$. So, by rule R4, $j \geq i + 2$ and $v \in p_{j-1}\overrightarrow{L_{j-1}}q_{j-1} \setminus \{p_{j-1}, q_{j-1}\}$. It follows that $v \in (A(P,Q) \setminus B(P,Q)) \cap L_{j-1}$. So, w and v contradict the minimality of $j - j'$ unless $j - j' = 1$, in which case w and u contradict rule R4. □

Lemma 3.8. *If P and Q are two vertical paths starting in the same layer L_i, then $S = (A(P,Q), B(P,Q))$ is a separation of G.*

Proof. Suppose that S is not a separation. Since by Lemma 3.7 $V(G) = A(P,Q) \cup B(P,Q)$, there exists in G an edge uv such that $u \in A(P,Q) \setminus B(P,Q)$ and $v \in B(P,Q) \setminus A(P,Q)$. If u and v are in the same layer L_j, then $j > i$ because $B(P,Q)$ contains all layers $L_1, \ldots, L_i$; so we have $u \in p_j\overrightarrow{L_j}q_j \setminus \{p_j, q_j\}$ and $v \in p_j\overleftarrow{L_j}q_j \setminus \{p_j, q_j\}$, a contradiction since u and v are adjacent. Otherwise, v is a descendant or an ancestor of u, so by Lemma 3.7, $v \in A(P,Q)$, a contradiction again. □

4 Finite induced subgraphs of layered wheels

Throughout this section, f is a slow function, $\ell \geq 4$ is an integer and G is the (f, ℓ)-layered wheel. Moreover, we consider a finite set $X \subseteq V(G)$ and an integer $k \geq 1$ such that $\omega(G[X]) \leq k$ and we study $G[X]$. We set $n = |X|$.

Lemma 4.1. *If X contains at most one vertex in each layer of G, then $G[X]$ is a chordal graph.*

Proof. Consider the maximum integer i such that $L_i \cap X \neq \emptyset$. The unique vertex v of $L_i \cap X$ has all it neighbors are in layers L_j such that $j < i$ (it has no neighbor in L_i by assumption, and no neighbors in L_j, $j > i$, by the maximality of i). So $N(v) \cap X = N^{\uparrow}(v) \cap X$ and v is simplicial by rule R5. This proof can be applied to any induced subgraph of $G[X]$, so $G[X]$ is chordal. $\square$

The arc $vu \in A(G)$ is *augmenting with respect to X* if u is a child of v and

$$N^{\uparrow}(u) \cap X = N^{\uparrow}[v] \cap X.$$

In what follows, we will omit to write "with respect to X" since X is fixed for the entire section.

Lemma 4.2. *If $v \in L_i$ and $f(i+1) \geq k+2$, then there exists at least one child u of v such that vu is augmenting.*

Proof. Assume first that $|N^{\uparrow}(v)| < f(i+1) - 1$. Then by rule R6, the edge vu is augmenting, where u is the only child of v. Indeed, we have $N^{\uparrow}(u) = N^{\uparrow}[v]$, so $N^{\uparrow}(u) \cap X = N^{\uparrow}[v] \cap X$ trivially holds. Thus we may assume that $|N^{\uparrow}(v)| \geq f(i+1) - 1$. Now:

$$\begin{aligned} f(i) - 1 \quad &\geq |N^{\uparrow}(v)| \text{ by rule R5} \\ &\geq f(i+1) - 1 \text{ by assumption} \\ &\geq f(i) - 1 \text{ because } f \text{ is slow} \end{aligned}$$

So $|N^{\uparrow}(v)| = f(i+1) - 1 \geq k+1$. Since $\omega(G[X]) \leq k$, there exists $w \in N^{\uparrow}(v) \setminus X$. By rule R7, v has a child u such that $N^{\uparrow}(u) = N^{\uparrow}[v] \setminus \{w\}$. Since $w \notin X$, we have $N^{\uparrow}(u) \cap X = N^{\uparrow}[v] \cap X$, so the edge vu is augmenting. $\square$

For every vertex v in G, we denote by $a(v)$ the *augmenting child of* v that is a vertex defined as follows: if v has a child u such that vu is augmenting, then we choose such a child u, and set $a(v) = u$. Otherwise, we choose any child u of v and set $a(v) = u$. There might be many ways to choose $a(v)$, but we choose one of them and keep it for the rest of the proof. For every vertex v, there exists a vertical path induced by $\{v, a(v), a(a(v)), \dots\}$ that we call the *augmenting path out of* v.

When f is a slow function, we set $F(k) = \sup\{i \in \mathbb{N} \setminus \{0\} | f(i) \le k\}$. The function F should be thought of as the maximum number of layers where f is at most k. Observe that in the case where f is eventually constant, say $f(i) = c$ for all sufficiently large i, we have $F(k) = +\infty$ for all $k \ge c$.

Lemma 4.3. *If $v \in V(G)$ and P is the augmenting path out of v, then $(V(P) \cap (\bigcup_{i \ge F(k+1)} L_i)) \cap X$ is a clique. In particular,*

$$|V(P) \cap X| \le F(k+1) + k - 1.$$

Proof. If $F(k+1) = +\infty$, then the conclusion trivially holds (in particular $\{i \in \mathbb{N} \setminus \{0\} | i \ge F(k+1)\} = \emptyset$). Otherwise, $V(P) \cap (\bigcup_{i \ge F(k+1)} L_i)$ induces an infinite vertical path $p_1 p_2 \dots$, and each p_j for $j \ge 1$ is in a layer L_i such that $f(i+1) \ge k+2$. So, by Lemma 4.2, there exists an augmenting arc $p_j u$ for all $j \in \mathbb{N} \setminus \{0\}$, so by the definition of augmenting paths, the arc $p_j p_{j+1}$ is augmenting.

Let us now prove that $\{p_1, p_2, \dots\} \cap X$ induces a clique. We prove by induction on j a stronger fact: $\{p_1, \dots, p_j\} \cap X \subseteq N^{\uparrow}[p_j] \cap X$ (which induces a clique by rule R5). This is clear for $j = 1$, and assuming it is proved for a fixed j, it follows for $j+1$ from:

$$\begin{aligned}
\{p_1, \dots, p_{j+1}\} \cap X \quad &= (\{p_1, \dots, p_j\} \cap X) \cup (\{p_{j+1}\} \cap X) \\
&\subseteq (N^{\uparrow}[p_j] \cap X) \cup (\{p_{j+1}\} \cap X) \text{ (induction hypothesis)} \\
&= (N^{\uparrow}(p_{j+1}) \cap X) \cup (\{p_{j+1}\} \cap X) \ (p_j p_{j+1} \text{ augmenting}) \\
&= N^{\uparrow}[p_{j+1}] \cap X
\end{aligned}$$

Hence, $\{p_1, p_2, \dots\} \cap X$ is a clique and therefore contains at most k vertices. Together with the vertices potentially in layers from 1 to $F(k+1)-1$, we obtain that $V(P) \cap X$ contain at most $F(k+1)+k-1$ vertices. $\square$

A separation (A, B) of G is *fair* if there exists a pair of vertical paths $P = p_i p_{i+1} \dots$, $Q = q_i q_{i+1} \dots$ such that:

- p_i and q_i are in the same layer L_i,
- $|p_i \overrightarrow{L_i} q_i| \le \ell - 1$,
- The paths $P \setminus p_i = p_{i+1} p_{i+2} \dots$ and $Q \setminus q_i = q_{i+1} q_{i+2} \dots$ are augmenting paths,

- $A = A(P, Q)$ and $B = B(P, Q)$ and
- $|A \cap X| \geq n/3$ (recall that $n = |X|$).

Note that the notion of fair separation is defined for G (and not only for $G[X]$), but it depends on X. Observe that P and Q are possibly not augmenting (but removing their first vertex yields an augmenting path).

Lemma 4.4. *There exists a fair separation in G.*

Proof. Consider two distinct and non-adjacent vertices p and q in the first layer L_1. Note that $|p\overrightarrow{L_1}q| \leq \ell - 1$ and $|q\overrightarrow{L_1}p| \leq \ell - 1$ because L_1 induces a cycle of length ℓ by the definition of layered wheels. Let P and Q be the two augmenting paths starting at p and q respectively. By Lemma 3.8, $(A(P, Q), B(P, Q))$ and $(A(Q, P), B(Q, P))$ are separations of G. To prove that one of them is fair, only the condition on $|A \cap X|$ remains to be checked. Since P and Q are vertex-disjoint by Lemma 3.4, we have $A(P, Q) \cup A(Q, P) = V(G)$. So, either $|A(P, Q) \cap X| \geq n/3$ or $|A(Q, P) \cap X| \geq n/3$, so the condition holds for at least one separation. □

Observe that in the following lemma, $F(k+1) = +\infty$ is possible. In this case, the statement becomes trivial, but is stays true : it says that some subset of X (which is finite) has size at most $+\infty$.

Lemma 4.5. *There exists a balanced separation of $G[X]$ of order at most*

$$2F(k + 1) + (\ell + 1)k - 2.$$

Proof. If $n \leq 5$, then the conclusion trivially holds because $\ell + 1 \geq 5$ and $F(k + 1) \geq 2$, so (X, X) is a separation satisfying the conclusion. We therefore assume from here on that $n \geq 6$.

Since a fair separation exists by Lemma 4.4, consider a fair separation (A, B) of G with notation as in the definition, such that i is maximal and, among all separations with i maximal, such that $|p_{i+1}\overrightarrow{L_{i+1}}q_{i+1}|$ is minimal. The order of $(A \cap X, B \cap X)$ is at most $2F(k + 1) + (\ell + 1)k - 2$ because by Lemma 3.6,

$$A(P, Q) \cap B(P, Q) = V(P) \cup V(Q) \cup \bigcup_{u \in p_i\overrightarrow{L_i}q_i} N^{\uparrow}[u],$$

by Lemma 4.3, $|V(P \setminus p_i) \cap X| \leq F(k + 1) + k - 1$, a similar inequality holds for Q, and by rule R5, for all $u \in p_i\overrightarrow{L_i}q_i$, $|N^{\uparrow}[u]| \leq k$.

It remains to prove that $(A\cap X, B\cap X)$ is a balanced separation of $G[X]$. Suppose not. By the definition of fair separations, $|A \cap X| \geq n/3$. Hence, $|(B \cap X) \setminus (A \cap X)| \leq 2n/3$. So, the reason why $(A \cap X, B \cap X)$ is not balanced is that $|(A \cap X) \setminus (B \cap X)| > 2n/3$.

Suppose that no internal vertex of $p_{i+1}\overrightarrow{L_{i+1}}q_{i+1}$ has a parent. Then by rule R4, either $p_i = q_i$ or $p_iq_i \in A(G)$. Moreover, by rules R6 and R7, $p_{i+1}\overrightarrow{L_{i+1}}q_{i+1}$ induces a path on $\ell - 1$ vertices. We then set $P' = P \setminus p_i$ and $Q' = Q \setminus q_i$, $A' = A(P', Q')$ and $B' = B(P', Q')$. It is a routine matter to check that (A', B') is a fair separation, except for the condition on $|A' \cap X|$. But at most two vertices of A' are not in A, because by rule R6 and R7, at most one vertex is in $N^{\uparrow}[p_i] \setminus N^{\uparrow}(p_{i+1})$ and at most one vertex is in $N^{\uparrow}[q_i] \setminus N^{\uparrow}(q_{i+1})$. Hence, $|A' \cap X| \geq |A \cap X| - 2 > 2n/3 - 2 = n/3 + (n - 6)/3 \geq n/3$ since $n \geq 6$. Consequently, the condition on the size of A' is satisfied and (A', B') contradicts the optimality of (A, B) since $i + 1 > i$.

We may therefore assume that some vertex u in the interior of $p_{i+1}\overrightarrow{L_{i+1}}q_{i+1}$ has a parent v. By rule R4, we have $v \in p_i\overrightarrow{L_i}q_i$. Let R' be the augmenting path out of u and $R = vuR'$. Set $A' = A(P, R)$, $A'' = A(R, Q)$, $B' = B(P, R)$ and $B'' = B(R, Q)$. It is a routine matter to check that (A', B') and (A'', B'') are fair separations, except for the condition on the size of A' or A''. We have $A = A' \cup A''$. Hence, since $|A \cap X| \geq 2n/3$, either $|A' \cap X| \geq n/3$ or $|A'' \cap X| \geq n/3$. So, one of (A', B') or (A'', B'') is fair and contradicts the minimality of $|p_{i+1}\overrightarrow{L_{i+1}}q_{i+1}|$. □

The following is the main result about the treewidth of layered wheels. Observe that if $F(\omega(H)+1) = +\infty$, then the first conclusion trivially holds.

Lemma 4.6. *For all integers $\ell \geq 4$ and all slow functions f, the (f, ℓ)-layered wheel G satisfies:*

- *For every finite induced subgraph H of G:*

$$\mathrm{tw}(H) \leq 15\left(F(\omega(H) + 1) + (\ell + 1)\omega(H) - 2\right).$$

- *For all integers $k \geq 2$ such that $F(k - 1)$ is finite and all integers $t \leq F(k)$, there exists a finite induced subgraph H of G satisfying:*

$$\omega(H) = k \text{ and } \mathrm{tw}(H) \geq t - 1.$$

Proof. Let us prove the first statement. Let H' be an induced subgraph of H. Set $k = \omega(H')$. By Lemma 4.5, H' has a balanced separation of order at most $2F(k+1)+(\ell+1)k-2 \leq 2F(\omega(H)+1)+(\ell+1)\omega(H)-2$. Hence by Theorem 1.10,

$$\operatorname{tw}(H) \leq 15\,(F(\omega(H)+1)+(\ell+1)\omega(H)-2)\,.$$

To prove the second statement, set $t' = \max(F(k-1)+1, t)$. Consider the graph H induced by layers $L_1, \dots, L_{t'}$ of G. Since $F(k-1)+1 \leq t' \leq F(k)$ (because $t \leq F(k)$), $f(t') = k$. So by Lemma 3.3, $\omega(H) = k$. By Lemma 3.2, $L_1, \dots, L_{t'}$ forms a clique minor of H. So, since $t' \geq t$, by Lemma 1.9, $\operatorname{tw}(H) \geq t-1$. □

Observe that when $F(k)$ is finite and $F(k+1)$ is infinite, Lemma 4.6 does not tell us weather the treewidth of induced subgraphs of G with clique number exactly k is bounded or not. This is why we do not know the answer to Question 1.8.

5 Applications of layered wheels

Recall that when f is a slow function, we set

$$F(k) = \sup\{i \in \mathbb{N} \setminus \{0\} | f(i) \leq k\}.$$

Call F the *cumulative function of f*. Recall that informally, $f(i)$ tells us what size of clique is obtained when adding the layer L_i. This number is 1 at the start, then 2, then 3, and then it grows by at most 1 at each new layer. Informally, $F(k)$ is the number of layers where the clique number is at most k. Since f is slow, we have

$$\begin{aligned} &F : \mathbb{N} \setminus \{0\} \to \mathbb{N} \setminus \{0\} \cup \{+\infty\}, \\ &F(1) = 1, F(2) = 2 \text{ and} \\ &F(k+1) \geq F(k)+1 \text{ for all } k \in \mathbb{N} \setminus \{0\}. \end{aligned} \qquad (\star)$$

It is clear that the (f, ℓ)-layered wheel could be defined by giving F instead of f. This one-to-one correspondance between f and F could be formalized by the fact that for all $i \in \mathbb{N} \setminus \{0\}$, we have

$$f(i) = \min\{k \in \mathbb{N} \setminus \{0\} | F(k) \geq i\}.$$

but we do not need this. We will just use freely the fact that any slow function f can be defined by describing its corresponding cumulative function F, provided that F satisfies the property $(\star)$.

The following theorem answers Question 1.1.

Theorem 5.1. *For every function $g : \mathbb{N} \setminus \{0\} \to \mathbb{N} \setminus \{0\}$ and every integer $\ell \geq 4$, there exists a (tw, ω)-bounded class of graphs $\mathcal{C}$ such that every hole in $\mathcal{C}$ has length at least ℓ and for all integers $k \geq 2$, there exists a graph $H \in \mathcal{C}$ satisfying*

$$\omega(H) = k \text{ and } \mathrm{tw}(H) \geq g(k).$$

Proof. Consider a triangle-free graph J whose treewidth is at least $g(2)$ (J can be a wall, or a (theta, triangle)-free layered wheel as defined in [11]). Set $F(1) = 1$, $F(2) = 2$ and for all integers $k \geq 3$,

$$F(k) = \max\{F(k-1) + 1, g(k) + 1\}.$$

Consider the slow function f whose cumulative function is F. It exists since $F(1) = 1$, $F(2) = 2$, and $F(k+1) \geq F(k) + 1$ for all $k \in \mathbb{N} \setminus \{0\}$. Also, for all integers $k \geq 3$, $F(k) \geq g(k) + 1$.

Consider the class $\mathcal{C}$ of all finite induced subgraphs of either J or the (f, ℓ)-layered wheel G. By Lemma 4.6, $\mathcal{C}$ is (tw, ω)-bounded. For all integers $k \geq 3$, the second conclusion of Lemma 4.6 for $t = F(k)$ yields a graph $H \in \mathcal{C}$ such that $\omega(H) = k$ and $\mathrm{tw}(H) \geq F(k) - 1 \geq g(k)$. For $k = 2$, a graph $H \in \mathcal{C}$ such that $\omega(H) = k$ and $\mathrm{tw}(H) \geq g(k)$ also exists since $J \in \mathcal{C}$. □

The following theorem disproves Conjecture 1.2.

Theorem 5.2. *Let $\ell \geq 4$ be an integer and $F : \mathbb{N} \setminus \{0\} \to \mathbb{N} \setminus \{0\}$ be any super-linear function such that $F(1) = 1$, $F(2) = 2$, and $F(k+1) \geq F(k)+1$ for all $k \in \mathbb{N} \setminus \{0\}$. Then there exists a hereditary class of graphs $\mathcal{C}$ such that every hole in $\mathcal{C}$ has length at least ℓ, $\mathcal{C}$ contains graphs of arbitrarily large tree-independence number and every $H \in \mathcal{C}$ satisfies*

$$\mathrm{tw}(H) \leq 15\left(F(\omega(H) + 1) + (\ell + 1)\omega(H) - 2\right).$$

Proof. Consider the slow function f whose cumulative function is F. Let G be the (f, ℓ)-layered wheel and $\mathcal{C}$ be the class of finite induced subgraphs of G. By Lemma 3.1, the holes in $\mathcal{C}$ all have length at least ℓ. By Lemma 4.6, the required bound on the treewidth holds.

It remains to prove that $\mathcal{C}$ contains graphs of arbitrarily large tree-independence number. So let $c \geq 1$ be an integer. Since F is super-linear, let $k \in \mathbb{N} \setminus \{0\}$ be such that $F(k) \geq ck$. Consider the graph H induced by the layers $L_1, \ldots, L_{F(k)}$ of G. By Lemma 3.3, $\omega(H) = k$ and by Lemma 3.2, the layers $L_1, \ldots, L_{F(k)}$ form a clique minor of H.

Consider any tree-decomposition $\mathcal{T} = (T, (X_s)_{s \in V(T)})$ of H. By Lemma 1.9, there exists a vertex $s \in V(T)$ such that X_s contains at least one vertex of each L_i, $i \in \{1, \ldots, F(k)\}$. Consider a subset Y of X_s that contains exactly one vertex in each layer L_i, $i \in \{1, \ldots, F(k)\}$. We have $|Y| = F[k]$ and $\omega(H[Y]) \leq k$. By Lemma 4.1, $H[Y]$ is chordal. Hence by Theorem 1.11,

$$\alpha(H[X_s]) \geq \alpha(H[Y]) \geq \frac{|Y|}{\chi(H[Y])} = \frac{F[k]}{\omega(H[Y])} \geq \frac{ck}{k} = c.$$

Hence, for all tree-decompositions of H, some bag contains a stable of size at least c. It follows that tree-$\alpha(H) \geq c$. Since this can be performed for any integer c, $\mathcal{C}$ contains graph of arbitrarily large tree-independence number. $\square$

By allowing f to be eventually constant (or equivalently by allowing infinite values of F), we obtain the following, that disproves both Conjecture 1.3 and Conjecture 1.4.

Theorem 5.3. *For all integers $\ell \geq 5$, $c \geq 2$ and $t \geq 1$, there exists a graph G of treewidth t such that $\omega(G) = c+1$, every hole of G has length at least ℓ (in particular, G contains no complete bipartite graph of treewidth at least 2) and every K_c-free induced subgraph of G (in particular every $(c-2)$-degenerate induced subgraph of G) has treewidth at most $15(c + (\ell+1)(c-1) - 2)$.*

Proof. Let f be the function defined by $f(i) = \min\{i, c+1\}$. So, f is slow and the cumulative function F of f satisfies $F(c) = c$ and $F(c+1) = +\infty$. Let G be the graph induced by the layers $L_1, \ldots, L_{t+1}$ of the (f, ℓ)-layered wheel. By Lemma 3.3, $\omega(G) \leq c+1$. Since the layers $L_1, \ldots, L_{t+1}$ form a clique minor of G by Lemma 3.2, G has treewidth at least t by Lemma 1.9.

Let H be a K_c-free induced subgraph of G. So $\omega(H) \leq c-1$. By Lemma 4.6 and since $F(c) = c$, $\mathrm{tw}(H) \leq 15(c + (\ell+1)(c-1) - 2)$. So G satisfies the conclusion. $\square$

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