EVEN PAIRS IN BERGE GRAPHS WITH NO BALANCED SKEW-PARTITIONS

TARA ABRISHAMI∗†, MARIA CHUDNOVSKY∗¶, AND YAQIAN TANG∗⨿

Abstract. Let $G$ be a Berge graph that has no odd prism and no antihole of length at least six as an induced subgraph. We show that every such graph $G$ with no balanced skew-partition is either complete or has an even pair.

1. Introduction

All graphs in this paper are finite and simple. Let $\chi(G)$ and $\omega(G)$ denote the chromatic number and the clique number of a graph $G$, respectively. A graph $G$ is perfect if every induced subgraph $H$ of $G$ satisfies $\chi(H) = \omega(H)$. The complement of a graph $G$, denoted by $\overline{G}$, has the same vertex set as $G$, and two distinct vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in $G$. A hole in a graph $G$ is an induced subgraph isomorphic to a cycle on at least five vertices, and an antihole is an induced subgraph whose complement is a hole in $\overline{G}$. The length of a hole (antihole) is equal to the number of its vertices. A graph is Berge if it contains no odd hole and no odd antihole as an induced subgraph. In the 1960s, Berge [1] conjectured that a graph is perfect if and only if it is Berge. The study of perfect graphs became a major area of research in structural graph theory after Berge’s conjecture. In 2002, Chudnovsky, Robertson, Seymour, and Thomas [4] proved the conjecture, which then became known as the Strong Perfect Graph Theorem (SPGT).

An even pair in a graph $G$ is a pair $\{u, v\}$ of nonadjacent vertices such that every induced path from $u$ to $v$ in $G$ has an even number of edges. Before the SPGT was proved, many results focused on properties of minimal imperfect graphs: imperfect graphs $G$ such that every proper induced subgraph of $G$ is perfect. In particular, Meyniel [14] proved that minimal imperfect graphs do not have an even pair. Also, the proof of the SPGT was simplified by Chudnovsky and Seymour in 2007 using even pairs [6].

A graph $G$ is complete if every pair of vertices in $G$ is adjacent. For a vertex $v \in V(G)$, we denote the set of vertices adjacent to $v$ by $N_G(v) = N(v)$. We say a graph $G'$ is obtained by contracting an even pair $\{u, v\}$ in $G$ if:

- $V(G') = (V(G) \setminus \{u, v\}) \cup \{w\}$;
- $G' \setminus \{w\} = G \setminus \{u, v\}$; and
- $N_{G'}(w) = N_G(u) \cup N_G(v)$

We denote the graph obtained by contracting the even pair $\{u, v\}$ by $G/\{u, v\}$. A sequence of contraction for a graph $G$ is a sequence of graphs $G_0, \ldots, G_k$ such that $G_0 = G$, $G_k$ has no even pair, and for all $0 \leq i \leq k - 1$, there exists an even pair $\{u, v\}$ in $G_i$ such that $G_{i+1} = G_i/\{u, v\}$. A graph is even-contractile if it has a sequence of contraction with $G_k$ being a complete graph. Fonlupt and Uhry [9] observed that if $G$ is Berge with an even pair $\{u, v\}$, then $G/\{u, v\}$ is also Berge and $\omega(G/\{u, v\}) = \omega(G)$. In particular, given a $\chi(G/\{u, v\})$-coloring of $G/\{u, v\}$, one can obtain a $\chi(G)$-coloring of $G$ by preserving the same colors for vertices in $G \setminus \{u, v\}$ and assigning

∗PRINCETON UNIVERSITY, PRINCETON, N.J., USA
† Supported by NSF-EPSRC Grant DMS-2120644.
‡ Supported by NSF-EPSRC Grant DMS-2120644 and by AFOSR grant FA9550-22-1-0083.
§ Supported by AFOSR grant FA9550-22-1-0083.
$u, v$ the color of the additional vertex $w$. Also, an even pair can be recognized in polynomial time using the algorithm for detecting an odd hole ([5] and [12]): Given a pair of nonadjacent vertices $\{u, v\}$, we add a new vertex $p$ to $G$ such that $N(p) = \{u, v\}$; if the graph $G \cup \{p\}$ contains no odd hole, then $\{u, v\}$ is an even pair. Therefore, we can find a sequence of contraction $G_0, \ldots, G_k$ for a Berge graph $G$ in polynomial time, and thus a $\chi(G)$-coloring of $G$ can be derived from a $\chi(G_k)$-coloring of $G_k$ by the procedure above in polynomial time. This algorithm works especially well when $G$ is even-contractile, as the $\chi(G_k)$-coloring of the complete graph $G_k$ is trivial. Therefore, a natural question is to identify which Berge graph is even-contractile.

To this end, Everett and Reed [8] proposed a conjecture for characterizing even-contractile Berge graphs. A prism in a graph $G$ is an induced subgraph that consists of two cliques $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ and three disjoint paths $P_1, P_2, P_3$ from $a_i$ to $b_i$ for each $i$, and with no other edge except for those in the two cliques and in the three paths. The paths $P_1, P_2, P_3$ are called the rungs of the prism. Under these conditions, a prism is odd (even) if all the three rungs have odd (even) number of edges. Note that every prism in a Berge graph is either even or odd. Everett and Reed conjectured the following:

**Conjecture 1.1** [8]. A Berge graph with no induced subgraph isomorphic to an antihole of length at least six or an odd prism is even-contractile.

This conjecture remains open, but several related theorems have been proved. Maffray and Trotignon [13] showed that a Berge graph that has no prism and no antihole of length at least six is even-contractile. Chudnovsky, Maffray, Seymour, and Spirkl [3] showed that if a Berge graph contains has no cycle on four vertices and no odd prism of a particular type, then it is either complete or has an even pair.

The main theorem of this paper is the following:

**Theorem 1.2.** Let $G$ be a Berge graph with no induced subgraph isomorphic to an antihole of length at least six or an odd prism. If $G$ does not admit a balanced skew-partition, then $G$ is either complete or has an even pair.

A balanced skew-partition is a type of decomposition that appears in the proof of the SPGT. In 2003, Chudnovsky [2] proved a structural decomposition theorem for trigraphs, which is a generalization of graphs with possible “undecided” edges called switchable pairs. In particular, the theorem implies that a Berge graph either belongs to some “basic” class, or has a balanced skew-partition, or a 2-join, or a 2-join in the complement. Our result is based on this decomposition theorem, and the notion of trigraph is very helpful to the proof.

The remainder of the paper is organized as follows. In Section 2 we introduce the definitions related to trigraphs and present relevant theorems that have been proved. We also define basic trigraphs and decompositions, namely balanced skew-partition, 2-join, and the complement of 2-join. In Section 3 we define a class $\mathcal{F}$ of Berge trigraphs and a subclass called favorable trigraphs that interact well with the 2-join decomposition. In particular, we will show that almost all trigraphs in $\mathcal{F}$ are favorable when forbidding antihole of length six and balanced skew-partition. In Section 4 we show that basic trigraphs have even pairs, and favorable basic trigraphs have even pairs in certain desirable location. In Section 5 we apply the technique of block of decompositions introduced in [7] to handle 2-join and its complement. This technique allows us to decompose any trigraph in $\mathcal{F}$ with no balanced skew-partition into basic trigraphs while keeping track of even pairs. Finally, we prove a generalization of our main theorem 1.2 for trigraphs.

2. Trigraphs

In this paper, we mainly adopt the notation regarding trigraphs from the work by Chudnovsky, Trotignon, Trunck, and Vušković [7]. For the sake of clarity, we restate relevant definitions and introduce new definitions that will appear in the paper.
For a set $X$, we denote by $\binom{X}{2}$ the set of all subsets of $X$ of size 2. For brevity, an element \{u, v\} of $\binom{X}{2}$ is also denoted by $uv$, or equivalently, $vu$. A trigraph $T$ consists of a finite vertex set $V(T)$, called the vertex set of $T$, and a map $\theta : \binom{V(T)}{2} \to \{-1, 0, 1\}$, called the adjacency function of $T$. Two distinct vertices of $T$ are strongly adjacent if $\theta(uv) = 1$, strongly antiadjacent if $\theta(uv) = -1$, and semiadjacent if $\theta(uv) = 0$. We say that $u$ and $v$ are adjacent if $\theta(uv) \in \{0, 1\}$ and antiadjacent if $\theta(uv) \in \{0, -1\}$. If $u$ and $v$ are adjacent (antiadjacent), we also say that $u$ is adjacent (antiadjacent) to $v$, or $u$ is a neighbor (antineighbor) of $v$. Similarly, if $u$ and $v$ are strongly adjacent (strongly antiadjacent), we say $u$ is a strong neighbor (strong antineighbor) of $v$. For $v \in V(T)$, let $N(v)$ denote the set of all vertices in $V(T) \setminus \{v\}$ that are adjacent to $v$, and let $\overline{N}[v]$ denote $N(v) \cup \{v\}$. An edge (antiedge) is a pair of adjacent (antiadjacent) vertices. A switchable pair is a pair of semiadjacent vertices, and a strong edge (antiedge) is a pair of strongly adjacent (strongly antiadjacent) vertices. An edge $uv$ (antiedge, strong edge, strong antiedge, switchable pair) is between two sets $A \subseteq V(T)$ and $B \subseteq V(T)$ if $u \in A$ and $v \in B$, or if $u \in B$ and $v \in A$.

Let $T$ be a trigraph. The complement of $T$, denoted by $\overline{T}$, is a trigraph with $V(T) = V(T)$ and the adjacency function $\overline{\theta} = -\theta$. Let $A \subseteq V(T)$ and $b \in V(T) \setminus A$. We say that $b$ is strongly complete (strongly anticomplete) to $A$ if $b$ is strongly adjacent (strongly antiadjacent) to every vertex of $A$; $b$ is complete (anticomplete) to $A$ if $b$ is adjacent (antiadjacent) to every vertex of $A$. For two disjoint subsets $A \subseteq V(T)$ and $B \subseteq V(T)$, $B$ is strongly complete (strongly anticomplete, complete, anticomplete) to $A$ if every vertex of $B$ is strongly complete (strongly anticomplete, complete, anticomplete) to $A$.

A clique of $T$ is a set of pairwise adjacent vertices of $T$, and a strong clique is a set of pairwise strongly adjacent vertices of $T$. A trigraph $T$ is complete if $V(T)$ is a clique. A stable set of $T$ is a set of pairwise antiedge-adjacent vertices of $T$. For $X \subseteq V(T)$, the trigraph induced by $T$ on $X$, denoted by $T|X$, has vertex set $X$ and adjacency function $\theta|X$, the restriction of $\theta$ to $\binom{X}{2}$. We denote by $T \setminus X$ the trigraph $T|(V(T) \setminus X)$. Isomorphism between trigraphs is defined in the natural way. For two trigraphs $T$ and $H$, $H$ is an induced subtrigraph of $T$ (or $T$ contains $H$ as an induced subtrigraph) if $H$ is isomorphic to $T|X$ for some $X \subseteq V(T)$. Since this paper mainly considers the induced subtrigraph containment relation, we say that $T$ contains $H$ if $T$ contains $H$ as an induced subtrigraph.

Let $\eta(T)$ denote the set of all strong edges of $T$, $\nu(T)$ the set of all strong antiedges of $T$, $\sigma(T)$ the set of all switchable pairs of $T$. If $\sigma(T)$ is empty, $T$ is a graph. A semirealization of $T$ is a trigraph $T'$ with vertex set $V(T)$ that satisfies $\eta(T) \subseteq \eta(T')$ and $\nu(T) \subseteq \nu(T')$. A realization of $T$ is any graph that is semirealization of $T$. For $S \subseteq \sigma(T)$, we denote by $G^T_S$ the realization of $T$ with edge set $\eta(T) \cup S$. The realization $G^T_{\sigma(T)}$ is called the full realization of $T$.

Let $T$ be a trigraph. For $X \subseteq V(T)$, we say that $X$ and $T|X$ are connected (anticonnected) if the graph $G^T_{\sigma(T), X}(G^T_{\emptyset, X})$ is connected. A connected component (or simply component) of $X$ is maximal connected subset of $X$, and an anticonnected component (or simply anticomponent) of $X$ is a maximal anticonnected subset of $X$.

A path $P$ of $T$ is a sequence of distinct vertices $p_1, \ldots, p_k$ such that either $k = 1$, or for $i, j \in \{1, \ldots, k\}$, $p_i$ is adjacent to $p_j$ if $|i - j| = 1$ and $p_i$ is antiadjacent to $p_j$ if $|i - j| > 1$. We say that $P$ is a path from $p_1$ to $p_k$, and the endpoints of $P$ are $p_1$ and $p_k$. Under these conditions, let $V(P) = \{p_1, \ldots, p_k\}$, the interior of $P$, denoted by $P^*$, is the induced subtrigraph of $P$ with $V(P^*) = V(P) \setminus \{p_1, p_k\}$, and the length of $P$ is $k - 1$. We say $P$ is even (odd) if it has even (odd) length. Two paths $P_1$ and $P_2$ are disjoint if $V(P_1) \cap V(P_2) = \emptyset$, and they are internally disjoint if $V(P_1^*) \cap V(P_2^*) = \emptyset$; $P_1$ is a subpath of $P_2$ if $P_1$ is a connected induced subtrigraph of $P_2$. Sometimes we denote $P$ by $p_1, \ldots, p_k$. Notice that, as a graph is also a trigraph, our definition of a path of a graph here is equivalent to a chordless path of a graph in some literature.
A cycle in a trigraph \( T \) is an induced subtrigraph \( H \) of \( T \) with vertices \( h_1, \ldots, h_k \) such that \( k \geq 3 \), and for \( i, j \in \{1, \ldots, k\} \), \( h_i \) is adjacent to \( h_j \) if \( |i - j| = 1 \) or \( |i - j| = k - 1 \); a hole is a cycle that further satisfies that \( h_i \) is antiaacent to \( h_j \) if \( 1 < |i - j| < k - 1 \). The length of a hole (cycle) is the number of vertices in it. Sometimes we denote \( H \) by \( h_1 \cdots h_k h_1 \). An antipath (anti-hole) in \( T \) is an induced subtrigraph of \( T \) whose complement is a path (hole) in \( T \).

A prism in a trigraph \( T \) is an induced subtrigraph \( H \) such that the full realization of \( H \) is a prism. A trigraph \( T \) is Berge if it contains no odd hole and no odd antihole. By this definition, \( T \) is Berge if and only if \( T \) is Berge. Also, \( T \) is Berge if and only if every realization (semirealization) of \( T \) is Berge. An even pair in \( T \) is a strongly nonadjacent pair \( uv \) such that every path from \( u \) to \( v \) in \( T \) is even.

### 2.1. Basic Trigraphs.

Here, we define the classes of basic trigraphs. A trigraph \( T \) is bipartite if its vertex set can be partitioned into two strongly stable sets, called a bipartition. A trigraph \( T \) is a line trigraph if its full realization is the line graph of a bipartite graph and every clique of size at least 3 in \( T \) is a strong clique. A trigraph is a doubled graph if it has a good partition. A good partition is a partition \((X, Y)\) of \( V(T) \) satisfying the following:

- Every component of \( T \mid X \) has at most two vertices, and every anticomponent of \( T \mid Y \) has at most two vertices.
- No switchable pair of \( T \) is between \( X \) and \( Y \).
- For every component \( C_x \) of \( T \mid X \) and every anticomponent \( C_y \) of \( T \mid Y \), every vertex \( v \) of \( C_x \cup C_y \) is incident with at most one strong edge and at most one strong antiedge between \( C_x \) and \( C_y \).

A trigraph is basic if it is either a bipartite trigraph, the complement of a bipartite trigraph, a line trigraph, the complement of a line trigraph, or a doubled trigraph. The following is Theorem 2.3 from [7]:

**Theorem 2.1 (7).** Basic trigraphs are Berge, and are closed under taking induced subtrigraphs, semirealizations, realizations, and complementation.

### 2.2. Decompositions.

We now describe the decompositions for trigraphs. First, a 2-join in a trigraph \( T \) is a partition \((X_1, X_2)\) of \( V(T) \) such that there exist disjoint sets \( A_1, B_1, C_1, A_2, B_2, C_2 \subseteq V(T) \) satisfying:

- \( X_1 = A_1 \cup B_1 \cup C_1 \) and \( X_2 = A_2 \cup B_2 \cup C_2 \);
- \( A_1, A_2, B_1 \) and \( B_2 \) are non-empty;
- no switchable pair is between \( X_1 \) and \( X_2 \);
- every vertex of \( A_1 \) is strongly adjacent to every vertex of \( A_2 \), and every vertex of \( B_1 \) is strongly adjacent to every vertex of \( B_2 \);
- there are no other strong edges between \( X_1 \) and \( X_2 \);
- for \( i = 1, 2 \), \(|X_i| \geq 3\); and
- for \( i = 1, 2 \), if \(|A_i| = |B_i| = 1\), then the full realization of \( T \mid X_i \) is not a path of length two containing the members of \( A_i \) and \( B_i \).

Under these conditions, we say that \((A_1, B_1, C_1, A_2, B_2, C_2)\) is a split of \((X_1, X_2)\). A 2-join is proper if for \( i = 1, 2 \), every component of \( T \mid X_i \) meets both \( A_i \) and \( B_i \). A complement 2-join of a trigraph \( T \) is a 2-join of \( T \). We need the following fact about 2-joins (Theorem 2.4 of [7]):

**Theorem 2.2 (7).** Let \( T \) be a Berge trigraph and \((A_1, B_1, C_1, A_2, B_2, C_2)\) a split of a proper 2-join of \( T \). Then all paths with one end in \( A_i \), one end in \( B_i \) and interior in \( C_i \), for \( i = 1, 2 \), have lengths of the same parity.

Next, a partition \((A, B)\) of \( V(T) \) is a skew-partition if \( A \) is not connected and \( B \) is not anticonnected. A skew-partition \((A, B)\) is balanced if there is no odd path of length greater than one with ends in \( B \) and interior in \( A \), and there is no odd antipath of length greater than one with ends
in $A$ and interior in $B$. Given a balanced skew-partition $(A, B)$, the 4-tuple $(A_1, A_2, B_1, B_2)$ is a split of $(A, B)$ if $A_1, A_2, B_1,$ and $B_2$ are disjoint non-empty sets, $A_1 \cup A_2 = A$, $B_1 \cup B_2 = B$, $A_1$ is strongly anticomplete to $A_2$, and $B_1$ is strongly complete to $B_2$. Note that there exists at least one split for every balanced skew-partition.

When $(A, B)$ is a skew-partition of a trigraph $T$, we say that $B$ is a star cutset of $T$ if at least one anticomponent of $B$ has size one. The following is Theorem 5.9 from [2].

**Theorem 2.3** ([2]). If a Berge trigraph admits a star cutset, then it admits a balanced skew-partition.

We will often use the following corollary:

**Theorem 2.4** ([2]). If $T$ is a Berge trigraph with no balanced skew-partition, then $T$ does not admit a star cutset.

3. Decomposing Trigraphs

3.1. Decomposing Trigraphs from $F$. In order to handle 2-join partitions and their complements in Section 5 we define a class of trigraphs that will be useful.

Let $T$ be a trigraph. Denote by $\Sigma(T)$ the graph with vertex set $V(T)$ and edge set $\sigma(T)$ (the switchable pairs of $T$). The connected components of $\Sigma(T)$ are called the switchable components of $T$. Let $F$ be the class of Berge trigraphs $T$ such that the following hold:

1. $T$ has at most one switchable component, and the switchable component $D$ of $T$ has at most two edges.
2. If $D$ contains exactly one edge $xy$, then $N(x) \cap N(y) = \emptyset$ in the trigraph $T$. In this case, we say it is a small switchable component.
3. Next, assume that $D$ has two edges. Let $v \in V(T)$ be the vertex of degree two in $\Sigma(T)$, denote its neighbors by $x$ and $y$. Then $v$ is strongly anticomplete to $V(T) \setminus \{v, x, y\}$ in $T$, $x$ is strongly antiaadjacent to $y$ in $T$, and $N(x) \cap N(y) = \{v\}$ in $T$. In this case, we say that the switchable component is light.

Our class $F$ of trigraph is a subclass of the class of the same name studied in [7], and we make use of several of their results.

**Theorem 3.1** ([7]). Let $T$ be a trigraph from $F$ with no balanced skew-partition, and let $(A_1, B_1, C_1, A_2, B_2, C_2)$ be a split of a 2-join $(X_1, X_2)$ in $T$. Then the following hold:

1. $(X_1, X_2)$ is a proper 2-join;
2. if $C_i = \emptyset$, then $|A_i| \geq 2$ and $|B_i| \geq 2$, $i = 1, 2$;
3. $|X_i| \geq 4$, $i = 1, 2$.

**Theorem 3.2** ([7]). Every trigraph in $F$ is either basic, or admits a proper 2-join, or admits a proper 2-join in the complement.

3.2. Favorable Trigraphs. Let $T$ be a trigraph in $F$. We say a pair $uv$ of vertices of $T$ is disjoint from its switchable component if $D$ is the switchable component of $T$ and $V(D) \cap \{u, v\}$ is empty. In particular, if the switchable component $D$ of $T$ is empty, every pair of vertices is disjoint from its switchable component. A trigraph $T \in F$ is favorable if it satisfies the following conditions:

1. $|V(T)| \geq 5$;
2. $T$ has at least one pair of strongly nonadjacent vertices $uv$ disjoint from $D$; and
3. if $D$ is small and $V(D) = \{x, y\}$, then at least one of $T \setminus (D \cup N(x))$ or $T \setminus (D \cup N(y))$ is not a clique.

A trigraph is unfavorable if it is not favorable. By this definition, if $T$ is complete, then $T$ is unfavorable; if $T$ is a graph with at least five vertices and is not complete, then $T$ if favorable as it has empty switchable component. Notice that condition (2) and (3) of being a favorable
trigraph are also necessary conditions for trigraphs to have even pairs disjoint from the switchable component.

Next, we will show that, with a few exceptions, a trigraph $T$ in $F$ with no balanced skew-partition and no antihole is favorable. Further, we prove in section 4 that a basic favorable trigraph has an even pair disjoint from its switchable component. Both results are essential for handling 2-joins in section 5.

**Theorem 3.3.** Let $T$ be a trigraph in $F$ with no balanced skew-partition and no antihole of length six. If $T$ is unfavorable, then either $T$ is complete or $|V(T)| < 5$.

**Proof.** We may assume that $|V(T)| \geq 5$. Let $D$ be the switchable component of $T$, and let $T' = T \setminus V(D)$ be the induced subtrigraph of $T$. If $D$ is small, we denote the pair by $x$ and $y$; if $D$ is light, we denote the vertex of degree two in $\Sigma(T)$ by $v$ and its neighbors by $x$ and $y$. Therefore, we can partition $V(T')$ into four sets: $T_1 = T' \setminus (N(x) \cup N(y))$, $T_2 = T'|N(x)$, $T_3 = T'|N(y)$.

First, suppose that $D$ is a light switchable component. Since $D$ is unfavorable, it follows that $V(T) \setminus V(D)$ is a clique. If both $T_2$ and $T_3$ are nonempty, then $x-v-y-a-b-x$ with $a \in T_2$ and $b \in T_3$ is a hole of length five, contradicting that $T$ is Berge, so we may assume up to symmetry that $T_3 = \emptyset$. Since $|V(T)| \geq 5$, it follows that $T_1 \cup T_2$ contains two distinct vertices $s$ and $t$. Since $T$ is connected, we may assume $t \in T_2$. Now, $V(T) \setminus \{v, y, s\}$ is a star cutset, contradicting 2.4.

Therefore, $D$ is a small switchable component. Now, if $T_1 \neq \emptyset$, $T_2$ is strongly complete to $T_3$ since $T$ contains no hole of length five. In this case, $T_2 \cup T_3$ is a star cutset, contradicting 2.4. Thus, $T_1 = \emptyset$. By the definition of unfavorable, since $T$ is not complete, it follows that both $T_2$ and $T_3$ are cliques. As $T$ has at least five vertices, at least one of $T_2$ or $T_3$ has more than two vertices. Without loss of generality, suppose $T_2$ contains two distinct vertices. Let $s$ be the vertex in $T_2$ such that $|N(s) \cap T_3|$ is the maximum, and let $t$ be a vertex in $T_2$ distinct from $s$. By 2.4, we may assume $N(s) \cup \{s\} \setminus \{t\}$ is not a star cutset. It follows that there exists a vertex $p \in T_3 \setminus N(s)$ adjacent to $t$. By maximality of $|N(s) \cap T_3|$, there exists $q \in N(s) \cap T_3$ such that $q$ is not connected to $t$. Now, $T\{x, y, s, t, p, q\}$ is an antiholes of length six, a contradiction. This completes the proof.

4. Even Pairs in Basic Trigraphs

The goal of this section is to prove the following theorem by analyzing each class of basic trigraph:

**Theorem 4.1.** Let $T$ be a basic trigraph in $F$ with no odd prism and no antiholes. Then the following statements hold:

1. $T$ is either complete or has an even pair.
2. If $T$ is favorable, then $T$ has an even pair disjoint from its switchable component.

4.1. Bipartite Trigraph. Let $T$ be a bipartite trigraph with bipartition $(A, B)$, where $A$ and $B$ are strongly stable sets. We have the following observation.

**Theorem 4.2.** Let $T$ be a bipartite trigraph in $F$. Then the following statements hold:

1. $T$ is either complete or has an even pair.
2. If $T$ is favorable, then $T$ has an even pair disjoint from its switchable component.

**Proof.** By the definition of bipartite trigraph, it holds that $T$ is complete or has an even pair, so the first statement holds. For the second statement, suppose that $T$ is favorable and has a nonempty switchable component $D$. If either $A' = A \setminus V(D)$ or $B' = B \setminus V(D)$ contains at least two vertices, then any two vertices $a_1, a_2 \in A'$ (or $b_1, b_2 \in B'$) form an even pair disjoint from the switchable component, so we may assume that $|A'| = |B'| = 1$. Since $T$ is favorable, it follows that $|V(T)| \geq 5$. Thus $|V(D)| \geq 3$, and so $T$ has a light switchable component, and $|V(T)| = 5$. Assume up to symmetry that $A = \{v, a\}$ and $B = \{x, y, b\}$, where $v$ is the vertex of degree two in $D$, and $x$ and $y$ are neighbors of $v$ in $D$. Since $T$ is favorable, it follows that $ab$ is a strong antiedge and $b$ is
paths along the cycle. Suppose the contrary that $H$ isomorphic to a $K_2$; thus, following the notation for a path, we call $H$ a bipartite graph, denoted by $L(H)$. Let $H$ be a bipartite graph such that its line graph, denoted by $L$, is the full realization of $T$. Let $(A, B)$ be a bipartition of $H$. A pair $(a_1b_1, a_2b_2)$ of disjoint edges in $H$ with $a_1, a_2 \in A$ and $b_1, b_2 \in B$ is a good pair of $H$ if both of the followings are satisfied:

- Every path $P_1$ with endpoints $a_1$ and $a_2$ satisfies $V(P_1) \cap \{b_1, b_2\} \neq \emptyset$; and
- every path $P_2$ with endpoints $b_1$ and $b_2$ satisfies $V(P_2) \cap \{a_1, a_2\} \neq \emptyset$.

We prove that a good pair in $H$ corresponds to an even pair in $T$. This is analogous to a result by Hougardy in [11].

**Proposition 4.3.** Let $H$ be a bipartite graph, let $(a_1b_1, a_2b_2)$ be a good pair of $H$, and let $u$ and $v$ be the vertices in $L$ that represent $a_1b_1$ and $a_2b_2$, respectively. Let $P$ be a trigraph such that $L$ is the full realization of $T$. Then, $uv$ is an even pair in $T$.

**Proof.** First, note that $uv$ is a chord path in $T$, as $a_1b_1$ and $a_2b_2$ are disjoint in $H$. Suppose that there is an odd path $P$ from $u$ to $v$ in $T$. Then, $P$ corresponds to an inclusion-wise minimal path $Q$ in $H$ with one end in $\{a_1, b_1\}$ and one end in $\{a_2, b_2\}$ of even length. Therefore, up to symmetry, we may assume that $Q$ has endpoints $a_1$ and $a_2$. As $Q$ is minimal and even, $V(Q) \cap \{b_1, b_2\} = \emptyset$. However, this contradicts that $(a_1b_1, a_2b_2)$ is a good pair.

Next, we show that forbidding odd prisms guarantees even pairs in line trigraphs. Let $H$ be a bipartite graph. (Note that the following theorems consider all subgraphs of $H$, which are not necessarily induced subgraphs.) A path $Q$ of $H$ is a chord path if its endpoints are contained in the vertex set of a cycle $C$ in $H$ and $V(Q^*) \cap V(C) = \emptyset$. An even theta is a graph composed of three internally disjoint even paths with the same endpoints. A path along the cycle $C$ is an induced subgraph of $C$ that is a path in $H[C]$. A graph is series-parallel if and only if it has no subgraph isomorphic to a $K_4$-minor.

**Proposition 4.4.** Let $T$ be a line trigraph with no odd prism, and let $H$ be a bipartite graph such that $L$ is the full realization of $T$. Then, $H$ has no subgraph isomorphic to an even theta, and $H$ is series-parallel.

**Proof.** Since the line graph of an even theta is an odd prism, it follows that $H$ contains no even theta as a subgraph. As $H$ is bipartite, all cycles in $H$ have even length. It follows that a chord path $P$ of a cycle $C$ in $H$ must has odd length, and the endpoints of $P$ in $C$ divide $C$ into two odd paths along the cycle. Suppose the contrary that $H$ is not series-parallel and thus has a subgraph isomorphic to a $K_4$-minor. Since $K_4$ has maximum degree three, it follows that $H$ has a subgraph $J$ isomorphic to a $K_4$-subdivision. Let $a, b, c, d$ be the vertices of degree three of $J$, and let $P_1, P_2, P_3, P_4, P_5, P_6$ denote the paths with endpoints $(a, b), (b, c), (c, d), (d, a), (b, d),$ and $(a, c)$, respectively, in $J$. Notice that each $P_i$ is a chord path, so they are all odd. Now, $P_1 \cup P_4 \cup P_6$ is an odd cycle, contradicting that $H$ is bipartite.

Finally, we prove the main result of this subsection.

**Theorem 4.5.** Let $T$ be a line trigraph in $F$ with no odd prism. The following statements hold:

1. $T$ is either complete or has an even pair.
2. If $T$ is favorable, then $T$ has an even pair disjoint from its switchable component.

**Proof.** Let $H$ be a bipartite graph such that $L$ is the full realization of $T$. We may assume that $H$ is connected and that $T$ is not complete. If $T$ has a nonempty switchable component $D$, let $J$ be the subgraph of $H$ such that $T[V(L(J))] = D$. In particular, $J$ is a path $p_1 \cdots p_k$ of length either two or three. Thus, following the notation for a path, we call $p_1$ and $p_k$ the endpoints of $J$.
and denote \( V(J) \setminus \{p_1, p_k\} \) by \( V(J^*) \). Also, note that any vertex \( v \in V(J^*) \) has degree at most two in \( H \): Otherwise, the line graph induced by the edges adjacent to \( v \) is a clique \( K \) of size at least three, and \( T \) contains a switchable pair, which contradicts the definition of a line trigraph.

By [4.3] to prove the first statement, it suffices to find a good pair in \( H \). Also, to prove the second statement, it suffices to find a good pair in \( H \setminus V(J^*) \). Thus, in the following discussion, the proof is completed when the corresponding good pair is found.

**Case 1:** \( H \) is a tree. Since \( T \) is not complete, it follows that \( H \) is not a star, so \( H \) has a path \( a_1-b_1-a_2-b_2 \) of length three. Now, \( (a_1b_2, a_2b_2) \) is a good pair. This proves the first statement for this case.

Next, suppose that \( T \) is favorable and has a nonempty switchable component \( D \). Let \( x \) and \( y \) be the endpoints of \( J \), and let \( H_x \) and \( H_y \) be the components of \( H \setminus V(J^*) \) containing \( x \) and \( y \) correspondingly. It suffices to show that \( H_x \cup H_y \) contains a good pair. If either \( H_x \) or \( H_y \) contains a path \( a_i-b_i-a_j-b_j \) of length three, then \( (a_i, a_j, b_i, b_j) \) is a good pair. Thus, we may assume both \( H_x \) and \( H_y \) are either empty or isomorphic to a star. If \( D \) is small, then \( T \) contradicts the third condition of being favorable. So we may assume \( D \) is light. By the second condition of being favorable, both \( H_x \) and \( H_y \) are nonempty. In particular, \( H_x \) contains an edge \( xx' \), and \( H_y \) contains an edge \( yy' \).

Now, \( (xx', yy') \) is a good pair. This completes the proof of the second statement for this case.

**Case 2:** \( H \) has a cycle of length at least six. Let \( C = a_1-b_1-a_2-b_2-a_3 \) where \( k \geq 6 \) is a cycle (not necessarily induced) of maximum length in \( H \). If \( C \) has no chord path, then every pair of disjoint edges \( (a_i, b_i, a_j, b_j) \) is a good pair. In particular, as \( |E(J)| \leq 3 \), there is a good pair in \( C \setminus V(J^*) \). Thus, we may assume that \( C \) has a chord path \( P \). By [4.4] \( P \) is odd and has ends \( a_i \) and \( b_j \) for \( 1 \leq i \leq j \leq k \). Let \( Q_1 \) and \( Q_2 \) be the two disjoint paths along the cycle \( C \) with endpoints \( a_i \) and \( b_j \). We may assume by symmetry that \( E(J) \cap E(Q_1) = \emptyset \) as any \( v \in V(J^*) \) has degree two. Now, to prove both statements for this case, it suffices to show that \( Q_1 \) contains a good pair.

Let \( S_1 \) be a minimal subpath of \( Q_1 \) such that the endpoints of \( S_1 \) are joined by a chord path of \( C \), and let this chord path be \( P' \). Thus, \( S_1 \) has odd length. If \( S_1 \) has length one, then \( P' \cup (C \setminus S_1) \) is a chord, a contradiction. So \( S_1 = a_1-b_1-a_2-b_2 \) has length at least three. Further, if there is a chord path \( P'' \) of \( C \) with exactly one endpoint in \( V(S_1^*) \), then \( C \cup P' \cup P'' \) forms a \( K_4 \) minor, contradicting [4.4]. Therefore, there is no path in \( H \setminus \{a_s, a_t\} \) with endpoints \( b_s \) and \( b_t \), and there is no path in \( H \setminus \{b_s, b_t\} \) with endpoints \( a_s \) and \( a_t \). So \( (a_1b_1, a_2b_2) \) is a good pair of \( H \) contained in \( Q_1 \). This completes the proof.

**Case 3:** All the cycles in \( H \) have length four. Let \( C = a_1-b_1-a_2-b_2-a_3 \) be a cycle of length four in \( H \). By [4.4] there is no chord path of \( C \) with endpoints \( a_1 \) and \( a_2 \) (or \( b_1 \) and \( b_2 \)). Also, if there is a chord path with endpoints \( a_i \) and \( b_j \) with \( i, j \in \{1, 2\} \), then \( G \) contains a cycle of length greater than four, a contradiction. Thus, there is no path in \( H \setminus \{a_1, a_2\} \) with endpoints \( b_1 \) and \( b_2 \), and there is no path in \( H \setminus \{b_1, b_2\} \) with endpoints \( a_1 \) and \( a_2 \). So \( (a_1b_2, a_2b_2) \) is a good pair of \( H \). In particular, this proves that in this case every cycle \( C \) in \( H \) contains a good pair, and thus the first statement follows.

Now, suppose \( T \) is favorable with nonempty switchable component \( D \). We may assume that \( E(J) \cap E(C) \neq \emptyset \), and \( H \setminus V(J^*) \) contains no cycle. As any vertex \( v \in V(J^*) \) has degree two, we have \( E(J) \subseteq E(C) \) and \( V(J) \subseteq V(C) \). Thus, we may assume that \( H \setminus V(C) \) is a tree. If \( D = \{x, y, v\} \) (where \( v \) has degree two in the switchable component) is a light switchable component, then \( N(x) \cap N(y) \neq \emptyset \) as the endpoints of \( J \) are adjacent, contrary to the fact that \( T \in \mathcal{F} \). Therefore, we may assume that \( D \) is small and \( J = a_1-b_1-a_2 \). As \( T \) is favorable, \( T \setminus V(D) \) is not a clique, which means that there is an edge \( a_ib_i \) in \( H \setminus V(J^*) \) such that \( b_i \neq b_2 \). Also, since \( b_1 \in V(J^*) \) has degree two in \( H \), we have \( b_1 \neq b_2 \). If \( a_1 = a_2 \), then \( (a_1b_1, a_2b_2) \) is a good pair in \( H \setminus V(J^*) \). Thus, by symmetry, we may assume that \( \{a_1, b_1\} \cap V(C) = \emptyset \), and every edge between \( C \) and \( H \setminus V(C) \) has \( b_2 \) as a vertex. In this case, as \( C \) has no chord path and \( H \setminus V(C) \) is a tree, \( \{a_1b_1, a_2b_2\} \) is a good pair in \( H \setminus V(J^*) \). This completes the proof.
4.3. Complement of a Bipartite Trigraph and Complement of a Line Trigraph. A diamond in a trigraph $T$ is an induced subtrigraph $H$ such that the full realization of $H$ is $K_4$ minus an edge. A claw in a trigraph $T$ is an induced subtrigraph $H$ such that the full realization of $H$ is the complete bipartite graph $K_{1,3}$. We will need the following characterization of line trigraph, which is a generalization of the main theorem of [10].

Proposition 4.6. Let $T$ be a line trigraph. Then, $T$ has no induced subtrigraph isomorphic to a diamond or a claw.

Proof. Suppose that $T$ contains a diamond or a claw, then the full realization of $T$ contains a diamond or a claw as an induced subgraph. By definition, the full realization of $T$ is a line graph. This contradicts to the main theorem of [10], which states that a line graph of a bipartite graph is (claw,diamond)-free.

Basic trigraphs which are the complement of a bipartite trigraph and the complement of a line trigraph share the following key property.

Proposition 4.7. Let $T$ be the complement of a bipartite trigraph or the complement of a line trigraph. Then, a path $P$ of odd length in $T$ has length at most three.

Proof. First, if $T$ is the complement of a bipartite trigraph, then for all $X \subseteq V(T)$ with $|X| \geq 3$, there exists an edge with both ends in $X$. Therefore, the path of maximal length in $T$ has length three, so the result follows. Now, we may suppose that $T$ is the complement of a line trigraph, and $P$ is a path of $T$ of length at least five. In this case, $T[V(P)]$ contains a diamond, which contradicts 4.6. This completes the proof.

Proposition 4.8. Let $T$ be a trigraph in $\mathcal{F}$ such that $T$ is either the complement of a bipartite trigraph or the complement of a line trigraph. If $T$ is favorable, then either $T$ is a graph, or $T$ has an even pair disjoint from the switchable component.

Proof. Recall that by the definitions, every clique of size at least three is a strong clique in line trigraphs and bipartite trigraphs. If $T$ has a light switchable component $D = \{x,y,v\}$, then, $T[D]$ is a clique of size three with two switchable pairs, contradicting that $T$ is the complement of a bipartite trigraph or the complement of a line trigraph. Thus, we may assume that $T$ has a small switchable component $D = \{x,y\}$. Suppose that $T$ is the complement of a bipartite trigraph with bipartition $(A,B)$. By definition, $T[A]$ and $T[B]$ are strong cliques, so we may assume $x \in A$ and $y \in B$ up to symmetry. In this case, $T \backslash (D \cup N(x)) \subseteq B$ and $T \backslash (D \cup N(y)) \subseteq A$ are both cliques, contradicting that $T$ is favorable.

Now, we may assume that $T$ is the complement of a line trigraph with a small switchable component. If there is a vertex $v$ contained in $T \backslash (N(x) \cup N(y))$, then $T[x,y,t]$ is a clique of size three but not a strong clique, contradicting the definition of line trigraph. So $T \backslash (N(x) \cup N(y)) = \emptyset$. If there are two vertices $s,t \in N(x)$ (or $s,t \in N(y)$) such that $st$ is an edge in $T$, then $T[x,s,t,y]$ is a claw, contradicting 4.6. Therefore, $T[N(x)]$ and $T[N(y)]$ are stable sets. Since $T$ is favorable, we may assume up to symmetry that $|N(x)| \geq 2$. Let $s$ and $t$ be two vertices in $N(x)$, and we claim that $\{s,t\}$ is an even pair: Suppose not, then there is an odd path $s-v_1-v_2-t$ of length three by 4.7. As $T[N(x)]$ is a clique, $\{v_1,v_2\} \subseteq T \backslash (N(x) \cup \{x,y\}) = N(y)$. So $v_1v_2$ is an edge in $T[N(y)]$, contradicting that $T[N(y)]$ is a stable set. This completes the proof.

The proof of following proposition is inspired by the main idea of [13].

Theorem 4.9. Let $T$ be a trigraph in $\mathcal{F}$ with no antihole such that $T$ is either the complement of a bipartite trigraph or the complement of a line trigraph. The following statements hold:
1. $T$ is either complete or has an even pair.
2. If $T$ is favorable, then $T$ has an even pair disjoint from its switchable component.
Proof. By 4.8 we may assume that $T$ is a graph, and it suffices to prove the first statement. Let $T$ be the vertex-minimal counterexample. We may assume that $T$ is not complete. Let $M$ be a maximal anticonnected set in $T$ such that there are at least two nonadjacent vertices in $V(T) \setminus M$ that are complete to $M$. Notice that $M$ is nonempty: since $T$ is not complete, it holds that $T$ contains at least one path of length at least two. Let $C(M)$ be the set of all vertices that are complete to $M$. By 2.1, each class of basic graphs is closed under taking induced subgraphs. Thus, since $T$ is minimal, it follows that $C(M)$ has an even pair $\{a, b\}$ as $C(M)$ is not complete by our construction.

Suppose the contrary that $\{a, b\}$ is not an even pair in $T$. Thus, by 4.7, there is a path $P = a-c-d-b$ of length three in $T$. Since $\{a, b\}$ is complete to $M$, it follows that $V(P) \cap M = \emptyset$. First, suppose $\{c, d\} \subseteq V(T) \setminus (M \cup C(M))$. Since both $c$ and $d$ are not in $C(M)$, it follows that $c$ and $d$ each has at least one strong antineighbor in $M$. So there exists an antipath $Q$ with ends $c$ and $d$ and $Q^* \in M$. Then, $c-Q-d-a-b-c$ is an antihole of length at least five in $T$, a contradiction. Thus, we may assume up to symmetry that $c \in C(M)$. Since $\{a, b\}$ is an even pair in $C(M)$, it follows that $P \not\subseteq C(M)$, and so $d \in V(T) \setminus (M \cup C(M))$. Now, $M \cup \{d\}$ is also an anticonnected set in $T$ such that there are at least two vertices in $V(T) \setminus (M \cup \{d\})$ complete to $M \cup \{d\}$, contradicting that $M$ is maximal. Therefore, $\{a, b\}$ is an even pair in $T$. This completes the proof. 

4.4. Doubled Graph. We first state a proposition regarding even pairs in doubled graphs.

Proposition 4.10. Let $T$ be a doubled graph with good partition $(X,Y)$. Then the following two statements hold:

1. Let $C_y = \{y\}$ be an anticomponent of size one in $Y$. If $y$ has an antineighbor $x \in X$, then $xy$ is an even pair. In particular, if $X$ has an edge $x_1x_2$, then $T$ has an even pair.
2. If $T|Y$ has a strong antiedge $y_1y_2$ and $T|X$ has no edge, then $y_1y_2$ is an even pair.

Proof.

1. In this case, $y$ is complete to $N(x)$, so all paths between $x$ and $y$ have length 2. If $X$ contains an edge $x_1x_2$, then either $x_1$ or $x_2$ is an antineighbor of $y$, so one of $x_1y$ or $x_2y$ is an even pair.

2. In this case, $N(y_1) \cap X$ and $N(y_2) \cap X$ are two disjoint stable sets that partition $X$. So there is no path from $y_1$ and $y_2$ whose interior is contained in $X$. Since $\{y_1, y_2\}$ is complete to other vertices in $Y$, all paths from $y_1$ to $y_2$ have length 2.

Theorem 4.11. Let $T$ be a doubled graph in $F$ with no antihole of length six. The following statements hold:

1. $T$ is either complete, or has an even pair.
2. If $T$ is favorable, then $T$ has an even pair disjoint from its switchable component.

Proof. We may assume that $T$ is connected and not complete. Let $(X,Y)$ be a good partition of $T$. Note that by the definition of doubled graph, every switchable component of $T$ is either an edge of $T|X$ or an edge of $T|Y$. If $T$ has a switchable component $D$, it must be small: otherwise, $T|D$ is both a component and an anticomponent of size 3, contradicting that $T$ is a doubled graph.

Case 1: $T|X$ has a component $C_1 = \{x_1, x_2\}$ of size two. If $T|Y$ is empty, then $T = C_1$ and $T$ is a clique, so we may assume that $T|Y$ is nonempty. If $T|Y$ has two distinct anticomponents $C_2, C_3$ of size two, $C_1 \cup C_2 \cup C_3$ is an antihole of length six, a contradiction. Therefore, $T|Y$ has at most one anticomponent of size two.

First, suppose $T|Y$ has an anticomponent $C_4 = \{v\}$ of size one. By symmetry, we may assume that $v$ is strongly adjacent to $x_1$ and strongly antiaadjacent to $x_2$. By (1) of 4.10 it follows that $\{v, x_2\}$ is an even pair of $T$, and this proves the first statement for this subcase. Next, assume that $T$ has a small switchable component $D$. If $\{x_1, x_2\}$ is not the switchable component of $T$,
then \( \{v, x_2\} \) is an even pair disjoint from its switchable component, so assume that \( \{x_1, x_2\} \) is the switchable component. Since \( T \) is favorable, there exists a vertex \( x_3 \) in \( T \mid X \setminus \{x_1, x_2\} \) such that \( x_3 \) has a non-neighbor \( y_1 \in T \mid Y \). If \( \{y_1\} \) is an anticomponent of size one in \( T \mid Y \), \( \{x_3, y_1\} \) is an even pair disjoint from the switchable component by (4.10). So we may assume \( y_1 \) is in an anticomponent \( C_5 = \{y_1, y_2\} \) of size two in \( T \mid Y \). Since \( T \mid Y \) has less than two anticomponent of size two, we may assume that \( X \setminus \{x_1, x_2\} \) is complete to \( Y \setminus \{y_1, y_2\} \). Thus, \( X \setminus \{x_1, x_2\} \) is a nonempty stable set. Now, \( \{x_3, y_1\} \) is an even pair disjoint from the switchable component \( \{x_1, x_2\} \): \( y_1 \) is complete to \( N(x) \setminus \{y_2\} \), so a path from \( x_3 \) to \( y_1 \) either has length two, or is exactly \( x_3-y_2-x_2-x_1-y_1 \), which has length four. This proves the second statement for this subcase.

Therefore, we may assume that \( T \mid Y \) contains an antiedge \( y_1y_2 \) and \( Y = \{y_1, y_2\} \). By symmetry, we may assume that \( x_i \) is strongly adjacent to \( y_i \) for \( i = 1, 2 \). Notice that all paths from \( y_1 \) to \( y_2 \) have length three. Every path \( P \) from \( y_1 \) to \( y_2 \) goes through either \( x_1 \) or \( x_2 \). If \( x_1 \in P \), then \( P = y_1-x_1-x_2 \) has length two; if \( y_2 \in P \), then \( P = y_1-\cdots-y_2-x_2 \) has length four. Therefore, \( y_1x_2 \) is an even pair, and \( x_1y_2 \) is also an even pair by symmetry. Thus, this proves the first statement for this subcase. Next, assume that \( T \) is favorable. We may also suppose that either \( \{x_1, x_2\} \) \( \{y_1, y_2\} \) is the switchable component. As \( |V(T)| \geq 5 \), there exists a vertex \( x_3 \in X \setminus \{x_1, x_2\} \). By symmetry, we may assume \( x_3 \) is strongly adjacent to \( y_1 \) and strongly antiadjacent to \( y_2 \). Suppose \( x_3 \) has a neighbor \( x_4 \in X \setminus \{x_1, x_2\} \). Then, \( x_3y_2 \) and \( x_4y_1 \) are even pairs by the same argument above. Also, \( \{x_1, x_2\} \) and \( \{x_2, x_4\} \) are even pairs. So at least one of them is disjoint from the switchable component, which is either \( \{x_1, x_2\} \) or \( \{y_1, y_2\} \). Thus, we may suppose \( x_3 \) has no neighbor in \( X \). Then, both \( \{x_3, y_2\} \) and \( \{x_1, x_3\} \) are even pairs: \( x_3 \) is complete to \( N(x_3) = \{y_1\} \), and every path from \( x_3 \) to \( y_2 \) that contains \( y_1 \) has length four. So at least one even pair is disjoint from the switchable component. This completes the proof of both statements for Case 1.

**Case 2: \( T \mid X \) has no edge.** It follows that the switchable component of \( T \) is contained in \( T \mid Y \). By (2) of (4.10) we may assume that \( T \mid Y \) is a clique. Suppose that there is no switchable pair in \( T \). Then \( Y \) is a strong clique, and thus \( X \) is complete to \( Y \). Since \( T \) is not complete, there exist nonadjacent vertices \( x_1, x_2 \in X \). Now, \( \{x_1, x_2\} \) is an even pair of \( T \). Next, assume that there exists a switchable pair \( y_1y_2 \) in \( T \). As \( T \in F \), \( N(y_1) \cap N(y_2) = \emptyset \), and thus \( Y = \{y_1, y_2\} \). Since \( T \) is not complete, there exists a vertex \( x_1 \in X \cap N(y_1) \). Now, \( \{x_1, y_2\} \) is an even pair. This proves the first statement for Case 2.

Suppose \( T \) is favorable with switchable pair \( y_1y_2 \). Then, there exists \( x_3, x_4 \in X \) such that either \( \{x_3, x_4\} \subseteq N(y_1) \) or \( \{x_3, x_4\} \subseteq N(y_2) \). In either cases, \( \{x_3, x_4\} \) is an even pair because \( N(x_3) = N(x_4) \). This completes the proof.

4.5. **Proof of Theorem 4.1** By 4.1, 4.5, 4.9 and 4.11 we have checked that the statements hold for each class of basic trigraphs. So 4.1 follows.

5. **Even Pairs in Non-Basic Trigraphs**

5.1. **Block of Decomposition.** To handle 2-join partitions and their complements, we need the following definitions and a theorem regarding trigraphs with no balanced skew-partition from [7].

A set \( X \subseteq V(T) \) is a fragment of a trigraph \( T \) if \( (X, V(T) \setminus X) \) is a proper 2-join of \( T \). A proper 2-join is even or odd according to the parity of the paths described in (2.2). The block of decomposition \( T_X \) with respect to a fragment \( X \) is defined as follows. Let \( X_1 = X \), \( X_2 = V(T) \setminus X \), and \( (A_1, B_1, C_1, A_2, B_2, C_2) \) be a split of the proper 2-join \( (X_1, X_2) \).

- **Case 1: \((X_1, X_2)\) is odd.** We build the block of decomposition \( T_X = T_{X_1} \) as follows: starting with \( T \mid X_1 \), we add two vertices \( a \) and \( b \), called marker vertices, such that \( a \) is strongly complete to \( A_1 \), \( b \) is strongly complete to \( B_1 \), \( ab \) is a switchable pair, and there are no other edges between \( \{a, b\} \) and \( X_1 \). Note that \( \{a, b\} \) is a small switchable component of \( T_X \), and we call it the marker component of \( T_X \).
• **Case 2:** \((X_1, X_2)\) is even. We build the block of decomposition \(T_X = T_{X_1}\) as follows: starting with \(T\mid X_1\), we add three vertices \(a, b, c\), called marker vertices, such that \(a\) is strongly complete to \(A_1\), \(b\) is strongly complete to \(B_1\), \(ac\) and \(bc\) are switchable pairs, and there are no other edges between \(\{a, b, c\}\) and \(X_1\). Note that \(\{a, b, c\}\) is a light switchable component of \(T_X\), and we call it the marker component of \(T_X\).

In both cases, we say that \(a\) and \(b\) are the ends of the marker component. Again, as our class \(\mathcal{F}\) is a subclass of the class of the same name studied in [7], we make use of the following result.

**Theorem 5.1 ([7]).** If \(X\) is a fragment of a trigraph \(T\) in \(\mathcal{F}\) with no balanced skew-partition, then the block of decomposition \(T_X\) is Berge and has no balanced skew-partition.

**Theorem 5.2.** Let \(X\) be a fragment of a trigraph \(T\) in \(\mathcal{F}\) with no balanced skew-partition, no odd prism, and no antihole. Then the block of decomposition \(T_X\) is Berge, and \(T_X\) has no balanced skew-partition, no odd prism, and no antihole of length at least five.

**Proof.** By [5,1] it suffices to show \(T_X\) has no odd prism and no antihole of length at least six. Let \(M\) denote the marker component of \(T_X\). Suppose \(T_X\) has an odd prism \(Q\) and assume that \(Q\) is chosen among odd prisms of \(T_X\) so that \(|V(Q) \cap V(M)|\) is minimum. If \(Q \cap M = \emptyset\), then \(Q\) is an odd prism in \(T\), a contradiction. Therefore, \(M\) may assume up to symmetry between \(a\) and \(b\) that \(a \in V(Q) \cap V(M)\). Suppose that \(N(a) \cap V(M) \subseteq V(Q)\). Let \(y \in A_2\) and let \(Q' = (Q \setminus \{a\}) \cup \{y\}\). If \(V(Q') \subseteq V(T)\), then \(Q'\) is an odd prism of \(T\), a contradiction. Therefore, \(M = \{a, b, c\}\), \(a\) is not adjacent to \(b\), and \(b \in V(Q')\). Let \(z \in B_2\) and let \(Q'' = (Q' \setminus \{b\}) \cup \{z\}\). Now, \(Q''\) is an odd prism of \(T\), a contradiction. Therefore, \(V(Q) \cap V(M) = V(M)\). Let \((A_1, C_1, B_1, A_2, C_2, B_2)\) be a split of \((X, V(T) \setminus X)\). Since \((X, V(T) \setminus X)\) is a proper 2-join, it follows that there is a path \(P\) of \(T\) with ends in \(A_2\) and \(B_2\) and interior in \(C_2\) such that \(P\) has the same parity as \(M\). Now, \((Q \setminus M) \cup P\) is an odd prism of \(T\), a contradiction. Therefore, \(T_X\) does not contain an odd prism.

Next, suppose that \(T_X\) contains an antihole, and let \(H = v_1 \cdots v_k v_1\) be the shortest antihole in \(T_X\). Since \(T_X\) is Berge and an antihole of length six is an odd prism, we may assume that \(k \geq 7\). When \(T_X\) has the marker component \(\{a, b, c\}\), it follows that \(c \notin V(H)\) because \(c\) is strongly anticomplete to \(T_X\). If \(|V(H) \cap V(M)| = 1\), we may assume by symmetry that \(a \in V(H) \cap V(M)\).

Now, we may replace \(a\) by a vertex \(a' \in A_2\), and \(T[V(H)] \setminus \{a\} \cup \{a'\}\) is an antihole of the same length in \(T\), a contradiction. Therefore, we may assume that \(V(H) \cap V(M) = \{a, b\}\), and let \(a = v_i\) and \(b = v_j\) where \(i < j\). First, suppose \(ab\) is an antiedge of the antihole such that \(i - j = 1\) or \(i - j = k - 1\). Because \(\{a, b\}\) is strongly anticomplete to \(C_1\), it follows that no vertex of the antihole is contained in \(C_1\). Also, at most one vertex of \(H\) is in \(A_1\) and at most one vertex of \(H\) is in \(B_1\), since \(a\) is strongly anticomplete to \(B_1\) and \(b\) is strongly anticomplete to \(A_1\). Therefore, \(k \leq 4\), a contradiction. Hence, we may suppose that \(1 < i - j < k - 1\). However, as \(k \geq 7\), at least one of \(v_i v_{i+1} \cdots v_j v_i\) or \(v_j v_{j+1} \cdots v_i v_j\) is an antihole and has length less than \(k\), which is a contradiction that \(H\) is the shortest antihole in \(T_X\). Therefore, \(T_X\) does not contain an antihole. This completes the proof.

**Theorem 5.3.** Let \(X\) be a fragment of a trigraph \(T\) in \(\mathcal{F}\). If the block of decomposition \(T_X\) has an even pair \(uv\) disjoint from its marker component, then \(uv\) is also an even pair in \(T\).

**Proof.** Let \((A_1, B_1, C_1, A_2, B_2, C_2)\) be a split of a proper two join \((X_1, X_2)\) with \(X = X_1\). Suppose that there is a path \(P\) from \(u\) to \(v\) in \(T\) of odd length. If \(P \subseteq T\mid X_1\), then \(P\) is a path of \(T_X\), a contradiction, so \(P \cap (T\mid X_2)\) is not empty. First, suppose \(P \cap (T\mid X_2)\) is a path \(Q\) of \(X_2\) from \(A_2\) to \(B_2\). Let \(a\) and \(b\) be the ends of the marker component of \(T_X\). By construction of the marker component, it follows that the path from \(a\) to \(b\) in the marker component of \(T_X\) has the same parity as \(Q\). Let \(Q'\) be the path from \(a\) to \(b\) in the marker component of \(T_X\). Now, \(P' = (P \setminus Q) \cup Q'\) induces a path of \(T_X\) of the same parity as \(P\), a contradiction.

Therefore, if \(Q = P \cap (T\mid X_2)\) is a path, we may assume up to symmetry that the endpoints of \(P \cap (T\mid X_2)\) are contained in \(A_2\). However, as \(A_1\) is strongly complete to \(A_2\), any vertex \(a\) in \(A_2\)
induces a cycle with $Q$, contradicting that $P$ is a path with endpoints in $X_1$. Thus, no edge of $P$ is contained in $T[X_2]$, and either $P \cap (T[X_2]) = \{a_2, b_2\}$, or $P \cap (T[X_2]) = \{a_2\}$, or $P \cap (T[X_2]) = \{b_2\}$, where $a_2 \in A_2$ and $b_2 \in B_2$. Let $P' \subseteq T_X$ be the path obtained by replacing $a_2$ with the marker vertex $a$ if $P \cap A_2 \neq \emptyset$, and replacing $b_2$ with the marker vertex $b$ if $P \cap B_2 \neq \emptyset$. Now, $P'$ is a path of $T_X$ of the same parity as $P$, a contradiction. 

5.2. Handling 2-joins and their complements. First, we show that it remains to consider trigraphs that admit proper 2-joins.

**Theorem 5.4.** Let $T$ be a trigraph in $F$ with no balanced skew-partition and no antihole. Then, either $T$ is basic, or $T$ admits a proper 2-join.

**Proof.** By Lemma 3.2 we may assume that $\overline{T}$ admits a proper 2-join $(X_1, X_2)$ with split $(A_1, C_1, B_1, A_2, C_2, B_2)$. By the definition of balanced skew-partition, a trigraph admits a balanced skew-partition if and only if its complement admits a balanced skew-partition. Thus, by Lemma 2.3, $\overline{T}$ has no star cutset. Suppose that $C_1 \neq \emptyset$, and assume up to symmetry between $A_1$ and $B_1$ that there is a vertex $c \in C_1$ adjacent to a vertex $a \in A_1$. Let $Q$ be a path in $\overline{T}[X_2]$ from a vertex in $A_2$ to a vertex in $B_2$. We claim that $N[a] \setminus \{c\}$ is a star cutset separating the component containing $c$ in $\overline{T} \setminus (N[a] \setminus \{c\})$ from the rest of the trigraph: If not, there exists a path $P \in \overline{T}[X_1 \setminus (N[a] \setminus \{c\})]$ connecting $c$ and a vertex $b \in B_1$; however, $T[(a-c-P^*-b-Q-a)]$ is an antihole of length at least five, a contradiction. Therefore, $C_1 = \emptyset$, and $C_2 = \emptyset$ by symmetry. Now, $(A_1, C_1, B_1, B_2, C_2, A_2)$ is a split of a proper 2-join of $T$. This completes the proof.

**Theorem 5.5.** Let $T$ be a trigraph in $F$ with no balanced skew partition, no odd prism, and no antihole. Then, $T$ is either complete or has an even pair.

**Proof.** Let $T$ be a vertex-minimal counterexample to the claim. If $T$ is basic, the theorem follows from 4.1. Thus, we may assume $T$ admits a proper 2-join. Let $(A_1, B_1, C_1, A_2, B_2, C_2)$ be a split of a proper 2-join of $T$ with $X_1 = A_1 \cup B_1 \cup C_1$ and $X_2 = A_2 \cup B_2 \cup C_2$. If $T$ has a switchable component $D$, we may assume up to symmetry that $V(D) \subseteq X_2$, as no switchable pair meets both $X_1$ and $X_2$. By our construction, $T_{X_1}$ has at most one light or small switchable component. By 5.2, $T_{X_1}$ is Berge and thus $T_{X_1} \in \mathcal{F}$. It also follows from 5.2 that $T_{X_1}$ admits no balanced skew-partition, no antihole, and no odd prism.

Next, we show that $T_X$ is favorable. By 3.1, $|X_i| \geq 4$ for $i = 1, 2$, and the marker component of $T_{X_1}$ has either two or three vertices (depending on the parity of the 2-join $(X_1, X_2)$). In either cases, we have $|T| > |T_{X_1}| \geq 6$. By 3.3, it remains to show $T_{X_1}$ is not complete: if $C_1$ is not empty, then a vertex $y \in C_1$ is not adjacent to any vertex of the marker component of $T_{X_1}$; if $C_1$ is empty, then by 3.1, $|A_1| \geq 2$ and $T_{X_1}[A_1]$ is strongly anticompact to $b$.

Therefore, since $T$ is the vertex-minimal counterexample, $T_{X_1}$ contains an even pair $uv$ disjoint its marker component. However, by 5.3, $uv$ is an even pair in $T$, a contradiction.

5.3. Proof of the main theorem. Now, we are ready to prove 1.2. We restate it here for the sake of clarity.

**Theorem 5.6 [1.2].** If $G$ is a Berge graph with no odd prism and no antihole, and $G$ does not admit a balanced skew-partition, then $G$ is either complete or has an even pair.

**Proof.** First, $G \in \mathcal{F}$ as $G$ is Berge and has no switchable component. So the result follows from 5.5.
References


