INDUCED EQUATORS IN FLAG SPHERES

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Abstract. We propose a combinatorial approach to the following strengthening of Gal's conjecture: $\gamma(\Delta) \geq \gamma(E)$ coefficientwise, where Δ is a flag homology sphere and $E \subseteq \Delta$ an induced homology sphere of codimension 1. We provide partial evidence in favor of this approach, and prove a nontrivial nonlinear inequality that follows from the above conjecture, for boundary complexes of flag d-polytopes: $h_1(\Delta)h_i(\Delta) \geq (d-i+1)h_{i-1}(\Delta) + (i+1)h_{i+1}(\Delta)$ for all $0 \leq i \leq d$.

1. Introduction

The γ-vector, reviewed in the next section, encodes the face numbers of simplicial complexes which are homology spheres. A simplicial complex is *flag* if it is the clique complex of its 1-skeleton. For example, barycentric subdivisions of the boundary complex of polytopes are flag homology spheres. Gal [6] conjectured the following tight analog of the Generalized Lower Bound Thoerem inequalities [11, 15, 2] in the flag case.

Conjecture 1.1 (Gal [6]). *If Δ is the boundary complex of a flag polytope, or more generally a flag homology sphere, then $\gamma(\Delta) \geq 0$, coefficientwise.*

This conjecture includes the Charney–Davis conjecture [5] as a special case.

As a vertex link in a flag homology sphere is again a flag homology sphere, the following conjecture immediately implies Gal's conjecture.

Conjecture 1.2 (Link Conjecture). *If v is a vertex in a flag homology sphere Δ, then its link satisfies $\gamma(\mathrm{lk}_v(\Delta)) \leq \gamma(\Delta)$, coefficientwise.*

An *equator* in a flag homology sphere is any *induced* subcomplex which is a flag homology sphere of codimension 1, see e.g. [9]. Each vertex link is an example of an equator. We prove in Proposition 3.1 that the following formal generalization of Conjecture 1.2 is in fact equivalent to it.

Conjecture 1.3 (Equator Conjecture). *If E is an equator in a flag homology sphere Δ, then $\gamma(E) \leq \gamma(\Delta)$, coefficientwise.*

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Athanasiadis [4] showed that Gal's conjecture follows from his conjecture that $\gamma(\Delta') \leq \gamma(\Delta)$ when Δ is a certain subdivision of Δ', called vertex-induced homology subdivision. However, the Link Conjecture, which amounts to dimension reduction, does not follow from Athanasiadis' conjecture, as Δ may not be such a subdivision of the suspension of $\mathrm{lk}_v(\Delta)$. To see this, let Δ be the boundary of the icosahedron. Every vertex of Δ has degree five, and in particular there are three pairwise-adjacent vertices of degree five. However, in the suspension $\Delta' = \Sigma_{a,b}\,\mathrm{lk}_v(\Delta)$ there are only two vertices of degree five, namely a and b. Since the inverse image of every vertex w of Δ' in Δ has at least the same degree as w, it follows that two adjacent vertices of Δ have a (or b) as their image, which is impossible.

Let $\mathcal{R}$ be the subfamily of *minimal* flag homology spheres, i.e. those that do not admit edge contractions that keep them flag, excluding the octahedral ones; equivalently, those where each edge belongs to an induced 4-cycle, excluding the octahedral sphere in each dimension. Clearly Gal's conjecture holds for the octahedral spheres. Then, it is known and easy to show that Gal's conjecture reduces to proving it for all $\Delta \in \mathcal{R}$ (see Lemma 2.2). In [10, Conj.6.1] it is conjectured that $\gamma_2(\Delta) > 0$ for all $\Delta \in \mathcal{R}$. In Proposition 3.2 we show that the Equator Conjecture holds if it holds for all $\Delta \in \mathcal{R}$. Unconditionally, we verify the validity of the Link conjecture for the following family: Let $\mathcal{S}$ be the family of boundary complexes of flag polytopes obtained from a cross-polytope by successive edge subdivisions.

Proposition 1.4. *Conjecture 1.2 holds for all $\Delta \in \mathcal{S}$.*

Replacing Conjecture 1.2 with 1.3 in above proposition is left open. We remark that Aisbett [3] and Volodin [18] proved that for any $\Delta \in \mathcal{S}$, $\gamma(\Delta)$ is the f-vector of some flag complex, supporting a conjecture of Nevo and Petersen [13].

We show in Proposition 3.5 that Conjecture 1.3 follows from the following structural conjecture.

Problem 1.5 (Structure)**.** For all flag homology spheres Δ, one of the following three alternatives must hold:

(0) Δ is a suspension, or

(i) there exists an edge in Δ which belongs to no induced 4-cycle, or

(ii) for every vertex $v \in \Delta$ there exists an equator E in Δ which is not a vertex link and which does not contain v.

Observe that (0) or (i) must hold if some vertex v in Δ is nonadjacent to *at most two* vertices: never to zero as Δ is not a cone, if to exactly one then Δ is a suspension (over v and the unique nonneighbor of it), and if to exactly two then the two nonneighbors of v form an edge which is in no induced 4-cycle by [9, Lem.3.4]. We prove the structural conjecture above holds when the dimension of Δ is at most two in Theorem 3.7, using this observation.

Let Δ_0 denote the vertex set of Δ. Note that the Link Conjecture implies the average assertion $\sum_{v\in\Delta_0}\gamma(\mathrm{lk}_v(\Delta)) \leq f_0(\Delta)\gamma(\Delta)$, which implies the h-polynomial inequality

$$(1) \qquad (1+t)\sum_{v\in\Delta_0} h_{\mathrm{lk}_v(\Delta)}(t) \leq f_0(\Delta)h_\Delta(t).$$

Recall that McMullen's proof of the Upper Bound Theorem for polytopes used the inequality

$$\sum_{v\in\Delta_0} h_{\mathrm{lk}_v(\Delta)}(t) \leq f_0(\Delta)h_\Delta(t).$$

The inequality (1) gives stronger upper bounds for flag homology spheres, however they are not tight. See [13, 1, 20] for the statement and progress on the Upper Bound Conjecture for flag homology spheres. Here we prove (1) in the polytope case.

Theorem 1.6. *The inequality (1) holds for all flag polytopes; it is tight only for the cross-polytopes.*

The proof combines a simple shelling argument with the following result, which may be of independent interest. A *half-integral perfect matching* in a graph G is a function $f : E(G) \to \{0, 1, \frac{1}{2}\}$ such that for every vertex v of G, $\Sigma_{e \text{ incident with } v} f(e) = 1$. Given a graph H, a graph G is *the complement of* H if G has the same vertex set as H, and two vertices are adjacent in G if and only if they are non-adjacent in H. We prove the following.

Theorem 1.7. *Let G be the complement of the 1-skeleton of a flag homology sphere. Then G has a half-integral perfect matching; equivalently, the vertex set can be partitioned into a matching and odd cycles in G.*

Combining Theorem 1.6 with McMullen's formula (see e.g.[17, Prop.2.3])

$$\sum_{v\in\Delta_0} h_{i-1}(\mathrm{lk}_v(\Delta)) = ih_i(\Delta) + (d-i+1)h_{i-1}(\Delta)$$

gives the following inequality on the h-vector, which seems new.

Corollary 1.8. *For Δ the boundary complex of a flag d-polytope, its h-vector satisfies*

$$h_1h_i \geq (d-i+1)h_{i-1} + (i+1)h_{i+1} \tag{2}$$

for all i.

For comparison, for the cyclic d-polytope and $i < d/2$, $h_1h_i < ih_{i+1}$. In Section 5 we show that (2) holds for boundary complexes of balanced d-polytopes as well, namely when the 1-skeleton is vertex d-colorable.

Outline. In Section 2 we set notation, recall the γ-vector and its relation to vertex splits and other basic constructions. In Section 3 we prove results towards the Equator Conjecture. In Section 4 we prove Theorems 1.6 and 1.7. Section 5 applies ideas from Section 4 to the balanced case.

2. Preliminaries

For the basics on face enumeration needed here we refer to e.g. Stanley's book [16] or the recent surveys by Klee-Novik [8] and Zheng [21]; for basics on polytopes refer to e.g. the textbooks by Grünbaum [7] and Ziegler [22].

2.1. **Simplicial complexes.** A *simplicial complex* Δ is a finite collection of subsets of $[n] := \{1, \ldots, n\}$, called *faces*, closed under containment. A face of cardinality $k+1$ has *dimension* k, and is called a k-face; the dimension of Δ is $\dim \Delta := -1+\max\{|\sigma| : \sigma \in \Delta\}$. Faces of dimension 0 (resp. 1) are called *vertices* (resp. *edges*); together they form the 1-skeleton, or *graph* of Δ. Then Δ is *flag* if its faces are exactly the cliques over its graph.

Given a face $\sigma \in \Delta$, the (closed) *star*, *antistar*, and *link* of σ in Δ are the following subcomplexes of Δ:

$$\begin{aligned}
\mathrm{st}_\sigma \Delta &:= \{\tau \in \Delta : \sigma \cup \tau \in \Delta\}, \\
\mathrm{ast}_\sigma \Delta &:= \{\tau \in \Delta : \sigma \not\subseteq \tau\}, \\
\mathrm{lk}_\sigma \Delta &:= \{\tau \in \Delta : \sigma \cup \tau \in \Delta, \quad \sigma \cap \tau = \emptyset\}.
\end{aligned}$$

Then for any vertex $v \in \Delta$, $\Delta = \mathrm{st}_v \Delta \cup_{\mathrm{lk}_v \Delta} \mathrm{ast}_v \Delta$. (We will keep abusing notation writing v for the singleton $\{v\}$.)

Call Δ a *homology sphere* (over a field $\mathbb{F}$) if for all faces $\sigma \in \Delta$ the reduced homology groups with coefficients in $\mathbb{F}$ satisfy

$$\widetilde{H}_i(\mathrm{lk}_\sigma \Delta, \mathbb{F}) = \begin{cases} 0 & \text{if } i < \dim \Delta - |\sigma|, \\ \mathbb{F} & \text{if } i = \dim \Delta - |\sigma|. \end{cases}$$

Call Δ *pure* if all its maximal faces (w.r.t. inclusion) have the same dimension. A pure $(d-1)$-dimensional simplicial complex B is a *homology ball* (over $\mathbb{F}$) if (i) for all faces $\sigma \in \Delta$ the link $\mathrm{lk}_\sigma \Delta$ is either a $(d-1-|\sigma|)$-dimensional homology sphere over $\mathbb{F}$ or is homologically $\mathbb{F}$-acyclic, and (ii) the faces of B with acyclic link form a $(d-2)$-dimensional homology sphere over $\mathbb{F}$.

Recall a subcomplex X of Δ is *induced* if $X = \Delta[W] := \{\sigma \in \Delta : \ \sigma \subseteq W\}$ for some subset W of the vertex set of Δ. Given a homology sphere Δ, an induced codimension 1 homology sphere $E \subseteq \Delta$ is called an *equator* of Δ. By the Jordan–Alexander theorem, Δ is decomposed into two homology balls intersecting in E, denoted $\Delta = B_1 \cup_E B_2$.

The *join* of two simplicial complexes Δ_i, $i = 1, 2$, on disjoint vertex sets is

$$\Delta_1 * \Delta_2 := \{\sigma_1 \cup \sigma_2 : \quad \sigma_i \in \Delta_i, \ i = 1, 2\}.$$

Important instances are the case of a *cone*, where $\Delta_2 = \{\emptyset, \{v\}\}$ and we simply write $\Delta_1 * v$ for the join, and the case of *suspension*, where $\Delta_2 = \{\emptyset, \{v\}, \{u\}\}$ and we simply write $\Sigma_{u,v} \Delta_1$ for the join. The join of the two-point complex with itself d times is the *octahedral* $(d-1)$-sphere; it can be realized as the boundary of the d-cross-polytope. It is the unique minimizer of the number of vertices (and i-faces, for all i) among all flag homology $(d-1)$-spheres, e.g. [6], [12].

The *contraction* of Δ by an edge $e = uv \in \Delta$ is the complex

$$\Delta' := \mathrm{ast}_v \Delta \cup u * \mathrm{ast}_u \mathrm{lk}_v \Delta,$$

obtained by replacing v by u in faces containing v in Δ. Then Δ is obtained from Δ' by a *vertex split* at u. We recall the following known facts, see e.g. [9, Lemma 2.1].

Lemma 2.1. *Let Δ be a $(d-1)$-dimensional flag homology sphere, $\sigma \in \Delta$, and $e \in \Delta$ an edge. Then:*

i) The link $\mathrm{lk}_\sigma \Delta$ is a flag induced homology sphere, hence an equator when $\sigma = \{v\}$.

ii) The contraction of Δ by e is a flag homology sphere if and only if e is not contained in an induced 4-cycle in the graph of Δ.

A particularly simple case of vertex split is that of stellar subdivision of Δ' at an edge $e = uv$, by introducing a new vertex v_e. This operation preserves being flag. Then the inverse operation is contracting the edge uv_e (or vv_e, they both give back the original complex). In the case when $\Delta' = \partial P$ is the boundary complex of a simplicial polytope P, subdividing e can be realized by placing v_e *beyond* e, thus the resulted Δ is again the boundary complex of a simplicial polytope.

2.2. $f-, h-, \gamma$**-vectors.** For a $(d-1)$-dimensional simplicial complex Δ let $f_i(\Delta) := |\{\sigma \in \Delta : \ |\sigma| = i+1\}|$ denote the number of i-faces of Δ, and $f(\Delta) := (f_{i-1}(\Delta))_{i=0}^d$ denote its f-vector; equivalently, let $f_\Delta(t) := \sum_{i=0}^d f_{i-1}(\Delta)t^i$ denote its f-polynomial.

Define the h-polynomial and h-vector of Δ by the equality

$$x^d \sum_{i=0}^d h_i(\Delta)\left(\frac{1}{x}\right)^i = (x-1)^d \sum_{i=0}^d f_{i-1}(\Delta)\left(\frac{1}{x-1}\right)^i.$$

When Δ is a flag homology sphere, the Dehn-Sommerville relations assert that $h_i(\Delta) = h_{d-i}(\Delta)$ for all $0 \le i \le d$. Being a palindrome, one can express $h_\Delta(t)$ as

$$h_\Delta(t) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i t^i (1+t)^{d-2i}.$$

Then the γ_is define the γ-vector and γ-polynomial of Δ, namely $\gamma_\Delta(t) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i t^i$. (We will switch between the f, h, γ-vectors and polynomials freely as convenient, where coefficientwise $\ge$ or $=$ between vectors of different length means by interpreting them as polynomials.)

We collect the following easy facts on the behavior of γ-polynomials under basic constructions.

Lemma 2.2 ([6])**.** *Let Δ be a flag homology sphere and $e \in \Delta$ an edge. Then,*

i) the suspension satisfies $\gamma(\Sigma_{a,b}\Delta) = \gamma(\Delta)$;

ii) if Δ' is the contraction of Δ by e, then $\gamma_\Delta(t) = \gamma_{\Delta'}(t) + t\gamma_{\mathrm{lk}_e \Delta}(t)$.

3. Towards the Equator conjecture

First, we reduce the Equator conjecture to the Link conjecture.

Proposition 3.1. *Fix d and n. Then the assertion of Conjecture 1.2 holds for all homology d-spheres with at most n vertices if and only if the assertion of Conjecture 1.3 holds for all homology d-spheres with at most n vertices.*

Proof. Let Δ be a flag homology sphere. As every vertex link is an induced subcomplex, and hence an equator, the assertion of Conjecture 1.3 for Δ clearly implies the assertion of Conjecture 1.2 for Δ.

For the converse implication, let E be an equator of Δ and not a vertex link. Thus, it decomposes Δ as the union of two homology balls B_1 and B_2 with common boundary E, such that in each B_i there are at least *two* interior vertices.

Consider the flag homology spheres $\Delta_i = B_i \cup (E * v_i)$ where the cone vertex v_i of E is not in B_i, for $i = 1, 2$. Then the f-polynomials satisfy

$$f_\Delta(t) = f_{\Delta_1}(t) + f_{\Delta_2}(t) - f_{\Sigma_{v_1,v_2}E}(t).$$

Translating into γ-polynomials, and using the fact that suspension does not change the γ-polynomial, gives

$$\gamma_\Delta(t) = \gamma_{\Delta_1}(t) + \gamma_{\Delta_2}(t) - \gamma_E(t). \tag{3}$$

Now, E is a vertex link in Δ_i, and Δ_i has fewer vertices than Δ and the same dimension as Δ, so by Conjecture 1.2 $\gamma_E(t) \leq \gamma_{\Delta_i}(t)$. Combining with (3) gives $\gamma_\Delta(t) \geq 2\gamma_E(t) - \gamma_E(t) = \gamma_E(t)$ as claimed. □

Next, we reduce the Equator conjecture for all flag homology spheres to the subfamily of minimal ones.

Proposition 3.2. *Let $e = uv$ be an edge in a flag homology sphere Δ and in no induced 4-cycle in Δ. If Conjecture 1.3 holds for all flag homology spheres Δ' such that $\dim \Delta' \leq \dim \Delta$ and $f_0(\Delta') < f_0(\Delta)$ (and all equators in Δ'), then it holds for Δ (and all equators E in Δ).*

Proof. Let E be an equator in Δ. There are exactly 3 cases: either (i) e is disjoint from E, or (ii) e is contained in E, or (iii) e intersects E in a single vertex, say u.

As $e = uv$ is in no induced C_4, its contraction results in a smaller flag homology sphere Δ'. In case (i) we get by induction $\gamma(E) \leq \gamma(\Delta')$, so we are done as $\gamma(\Delta') \leq \gamma(\Delta)$ by Lemma 2.2 ii). In case (ii) the contraction of e in E results in an equator E' of Δ'. Note that $\mathrm{lk}_e E$ is an equator of $\mathrm{lk}_e \Delta$, so by induction and Lemma 2.2 ii) we get

$$\gamma_\Delta(t) = \gamma_{\Delta'}(t) + t\gamma_{\mathrm{lk}_e \Delta}(t) \geq \gamma_{E'}(t) + t\gamma_{\mathrm{lk}_e E}(t) = \gamma_E(t).$$

In case (iii), as E decomposes Δ into the union of two homology balls, $\Delta = B_1 \cup_E B_2$, the complexes $\Delta_i = B_i \cup (E * v_i)$ and Δ satisfy equation (3). If $f_0(\Delta) > f_0(\Delta_i)$ for $i = 1, 2$ then by induction $\gamma(E) \leq \gamma(\Delta_i)$ for $i = 1, 2$ and combined with (3) we are done.

Else, w.l.o.g. e is contained in B_1.

Case $f_0(\Delta) = f_0(\Delta_1)$: Then, Δ and Δ_1 are isomorphic. Indeed, as B_2 has exactly one interior vertex, denote it by w, and as E is induced in Δ, for each maximal face F in E, the facet of B_2 containing it is $F \cup \{w\}$; this describes all the facets of B_2. In other words, $E = \mathrm{lk}_w(\Delta)$, but then also $E = \mathrm{lk}_w(\Delta')$.

By induction and Lemma 2.2 ii) conclude $\gamma(E) \leq \gamma(\Delta') \leq \gamma(\Delta)$.

Case $f_0(\Delta) = f_0(\Delta_2)$: then, as argued in the previous case, Δ and Δ_2 are isomorphic, and $E = \mathrm{lk}_v \Delta$.

As uv is in no induced C_4, the boundary of the homology ball $B = \mathrm{st}_v\,\Delta \cup \mathrm{st}_u\,\Delta$ is an induced subcomplex of Δ, denote it by E''; so E'' is an equator of Δ. Consider the flag homology sphere $\Delta'' = B \cup E'' * w$ with w not a vertex of B. Applying (3) to Δ, Δ'' and the third sphere Δ''' obtained by coning the boundary of the complementary ball to B, we get by induction $\gamma(E'') \leq \gamma(\Delta''')$; hence $\gamma(\Delta'') \leq \gamma(\Delta)$.

If $\Delta \neq \Delta''$ then by induction $\gamma(E) \leq \gamma(\Delta'')$ and we are done.

Else, $\Delta = \Delta''$. Note that in this case Δ is the union of the stars of u, v and w; we have

$$f_\Delta(t) = (1+t)(f_{\mathrm{lk}_v\,\Delta}(t) + f_{\mathrm{lk}_u\,\Delta}(t)) - (1+t)^2 f_{\mathrm{lk}_e\,\Delta}(t) + t f_{\mathrm{lk}_w\,\Delta}(t).$$

Translating into h-polynomials we get

$$h_\Delta(t) = h_{\mathrm{lk}_v\,\Delta}(t) + h_{\mathrm{lk}_u\,\Delta}(t) - h_{\mathrm{lk}_e\,\Delta}(t) + t h_{\mathrm{lk}_w\,\Delta}(t).$$

Further, contracting e in Δ gives the suspension over $\mathrm{lk}_w\,\Delta$ which by Lemma 2.2 ii) gives

$$h_\Delta(t) = t h_{\mathrm{lk}_e\,\Delta}(t) + (1+t) h_{\mathrm{lk}_w\,\Delta}(t).$$

Equating the RHSs of the last two equations gives in γ-terms

$$\gamma(\mathrm{lk}_w\,\Delta) + \gamma(\mathrm{lk}_e\,\Delta) = \gamma(\mathrm{lk}_v\,\Delta) + \gamma(\mathrm{lk}_u\,\Delta).$$

By Lemma 2.2 ii) and induction $\gamma(\mathrm{lk}_w\,\Delta) \leq \gamma(\Delta)$, and by induction $\gamma(\mathrm{lk}_e\,\Delta) \leq \gamma(\mathrm{lk}_u\,\Delta)$, thus

$$\gamma(\Delta) \geq \gamma(\mathrm{lk}_w\,\Delta) = \gamma(\mathrm{lk}_v\,\Delta) + (\gamma(\mathrm{lk}_u\,\Delta) - \gamma(\mathrm{lk}_e\,\Delta)) \geq \gamma(\mathrm{lk}_v\,\Delta) = \gamma(E),$$

completing the proof. □

Recall the family $\mathcal{R}$ from the Introduction, of minimal flag homology spheres. Proposition 3.2 immediately implies the following.

Corollary 3.3. *If Conjecture 1.3 holds for all $\Delta \in \mathcal{R}$ then it holds in general.*

Next we discuss Proposition 1.4. Recall the family $\mathcal{S}$ from the Introduction, of flag simplicial spheres obtained from an octahedral sphere by successive edge subdivisions. We need the following straightforward observation:

Lemma 3.4. *The family $\mathcal{S}$ is closed under (i) suspension and (ii) links.*

Proof. For (i), note that if Δ is obtained from a homology sphere Δ' by stellar subdivision at the edge $e \in \Delta'$, then the suspension $\Sigma_{a,b}\Delta$ is obtained from $\Sigma_{a,b}\Delta'$ by stellar subdivision at the same edge e.

For (ii), the assertion clearly holds for octahedral spheres. We argue by induction. Keeping the notation of the proof of part (i), we distinguish cases according to the vertex v whose link is being considered, for $v \in \Delta \in \mathcal{S}$: cases are (1.)$v \in e$, (2.)$v = v_e$ is the new vertex, (3.)$v \in \mathrm{lk}_e\,\Delta'$, and (4.)otherwise. See e.g. [3, Sec.3] for details. Specifically, in case (1.) $\mathrm{lk}_v\,\Delta \cong \mathrm{lk}_v\,\Delta'$ and we are done by induction on number of vertices, in case (2.) $\mathrm{lk}_{v_e}\,\Delta \cong \Sigma_{a,b}\,\mathrm{lk}_e\,\Delta'$ so we are done using part (i) and induction on dimension, in case (3.) $\mathrm{lk}_v\,\Delta$ is obtained from $\mathrm{lk}_v\,\Delta'$ by a stellar subdivision at the edge e so we are done by induction on dimension, and in case (4.) $\mathrm{lk}_v\,\Delta = \mathrm{lk}_v\,\Delta'$ and there is nothing new to prove. □

Proof of Proposition 1.4. We argue by induction on dimension and number of vertices (equivalently, on the number of edge subdivisions). Let $v \in \Delta \in \mathcal{S}$, and Δ obtained from Δ' by a stellar subdivision at edge e. Consider the 4 cases in the proof of Lemma 3.4, whose assertion we also use.

In case (1.),

$$\gamma(\mathrm{lk}_v \Delta) = \gamma(\mathrm{lk}_v \Delta') \leq \gamma(\Delta') \leq \gamma(\Delta),$$

where first inequality is by induction and second one is by Lemma 2.2 ii), where nonnegativity of $\gamma(\mathrm{lk}_e(\Delta'))$ is known by Lemma 3.4 and induction.

In case (2.),

$$\gamma(\mathrm{lk}_v \Delta) = \gamma(\mathrm{lk}_e \Delta') \leq \gamma(\Delta') \leq \gamma(\Delta),$$

where for the first inequality we applied induction twice, as for $e = uw$, $\mathrm{lk}_e \Delta' = \mathrm{lk}_u(\mathrm{lk}_v \Delta')$.

In case (3.),

$$\gamma(\mathrm{lk}_v \Delta) = \gamma(\mathrm{lk}_v \Delta') + t\gamma(\mathrm{lk}_e(\mathrm{lk}_v \Delta')) \leq \gamma(\Delta') + t\gamma(\mathrm{lk}_e \Delta') = \gamma(\Delta),$$

where we used that for a partition of a face $\sigma = \sigma_1 \cup \sigma_2$, links operators satisfy $\mathrm{lk}_\sigma \Delta' = \mathrm{lk}_{\sigma_2}(\mathrm{lk}_{\sigma_1} \Delta') = \mathrm{lk}_{\sigma_1}(\mathrm{lk}_{\sigma_2} \Delta')$.

In case (4.) we are immediately done by induction. □

Next we consider the relevance of the Structure Problem 1.5.

Proposition 3.5. *Problem 1.5 implies Conjecture 1.3.*

Proof. Let Δ be a flag homology sphere. By Proposition 3.1, it is enough to show that for every vertex w of Δ we have $\gamma(\mathrm{lk}_w(\Delta)) \leq \gamma(\Delta)$. We may assume that the assertion of Conjecture 1.3 holds for all flag homology spheres Δ' such that $f_0(\Delta') < f_0(\Delta)$ and $\dim(\Delta') \leq \dim(\Delta)$, and that one of the outcomes listed in Problem 1.5 holds for Δ. Thus either

(0) Δ is a suspension, or

(i) there exists an edge in Δ which belongs to no induced 4-cycle, or

(ii) for every vertex $v \in \Delta$ there exists an equator E in Δ which is not a vertex link and which does not contain v.

Let w be a vertex of Δ. Assume first that $\Delta = \Sigma_{a,b}\Delta'$. If $w \in \{a, b\}$, then the result follows immediately from the first statement of Lemma 2.2. Thus $w \in \Delta'$, and $\mathrm{lk}_w(\Delta) = \Sigma_{a,b}\, \mathrm{lk}_w(\Delta')$. Inductively we have that $\gamma(\mathrm{lk}_w(\Delta')) \leq \gamma(\Delta')$. But now, again by the first statement of Lemma 2.2, we deduce:

$$\gamma(lk_w(\Delta)) = \gamma(\mathrm{lk}_w(\Delta')) \leq \gamma(\Delta') = \gamma(\Delta)$$

and thus the assertion of Conjecture 1.3 holds for Δ.

If there exists an edge in Δ which belongs to no induced 4-cycle, then the assertion of Conjecture 1.3 for Δ follows immediately from Proposition 3.2.

Thus we may assume that outcome (ii) above holds. Then there exists an equator E in Δ which is not a vertex link and which does not contain w. Then E decomposes Δ as the union of two homology balls B_1 and B_2 with common boundary E, such that in each B_i there are at least *two* interior vertices. We may assume that w is an interior vertex of B_1, and therefore $\mathrm{lk}_w(\Delta)$ is contained in B_1. Consider the flag homology spheres

$\Delta_i = B_i \cup (E * v_i)$ where the cone vertex v_i of E is not in B_i, for $i = 1, 2$. Since E is not a vertex link, we deduce that for $i = 1, 2$ $f_0(\Delta_i) < f_0(\Delta)$. Consequently, $\gamma(\mathrm{lk}_w(\Delta_1)) \leq \gamma(\Delta_1)$ and $\gamma(E) \leq \gamma(\Delta_2)$. Note that $\mathrm{lk}_w(\Delta) = \mathrm{lk}_w(\Delta_1)$. We now use (3) to deduce the following (coefficientwise):

$$\gamma(\Delta) = \gamma(\Delta_1) + \gamma(\Delta_2) - \gamma(E) \geq \gamma(\Delta_1) \geq \gamma(\mathrm{lk}_w(\Delta_1)) = \gamma(\mathrm{lk}_w(\Delta)),$$

as required. □

The structure conjectured in Problem 1.5 clearly holds for spheres of dimension ≤ 1. Further, it holds in dimension 2 due to the following lemma.

Lemma 3.6. *If Δ is a flag (homology) 2-sphere, different from the octahedron's boundary, then there exists an edge $e \in \Delta$ such that e is not contained in any induced 4-cycle.*

This statement is a flag analog of Whiteley [19, Lemma 6]; we omit its simple proof. Thus, for flag 2-spheres one of the alternatives (0) and (i) in Problem 1.5 holds. The point in Theorem 3.7 below is to show how alternative (ii) in Problem 1.5 can be found, when a strong condition that implies (0) or (i) fails to hold.

Theorem 3.7. *For every vertex v in a flag (homology) 2-sphere Δ, either*

(i) some vertex of Δ is non-adjacent to at most two vertices such that v is one of them, or

(ii) there exists an equator E in Δ which is not a vertex link and does not contain v.

Proof. Let v be a vertex of Δ. Since Δ is a flag homology 2-sphere, $\mathrm{lk}_v(\Delta)$ is an induced cycle. Let the vertices of $\mathrm{lk}_v(\Delta)$ be $u_1, \ldots, u_t$, where $u_i u_{i+1}$ is an edge of Δ for every $i \in \{1, \ldots, t-1\}$, and u_1 is adjacent to u_t. Then there are no other adjacent pairs among $\{u_1, \ldots, u_t\}$. If no vertex of $\mathrm{lk}_{u_t}(\Delta) \setminus \mathrm{st}_v(\Delta)$ has a neighbor in $\mathrm{lk}_v(\Delta) \setminus \mathrm{st}_{u_t}(\Delta)$, then the edge vu_t is in no induced C_4, and so $E = (\mathrm{lk}_v(\Delta) \cup \mathrm{lk}_{u_t}(\Delta)) \setminus \{v, u_t\}$ is an induced cycle, and therefore an equator. If there exists w such that $E = \mathrm{lk}_w(\Delta)$, then outcome (i) holds, and otherwise outcome (ii) holds. Thus by symmetry we may assume that for every $i \in \{1, \ldots, t\}$ u_i has a neighbor $w_i \notin \mathrm{st}_v(\Delta)$ and such that w_i has a neighbor in $\mathrm{lk}_v(\Delta) \setminus \mathrm{st}_{u_i}(\Delta)$.

For a vertex w, a *w-interval* is a circular interval $[u_j, \ldots, u_k]$ of $\mathrm{lk}_v(\Delta)$ such that u_j is non-adjacent to u_k, w is adjacent to u_j, u_k, and w has no other neighbor in this interval.

Now, no w_i-interval exists iff $\mathrm{lk}_{w_i}(\Delta) = \mathrm{lk}_v(\Delta)$ in which case outcome (i) holds. Thus, we may assume there exists a w_i-interval I_{w_i} for all $1 \leq i \leq t$. Let C_{w_i} be the induced graph in Δ on the vertex set $I_{w_i} \cup \{w_i\}$. Then C_{w_i} is an induced cycle not containing v. If C_{w_i} is not a vertex link then outcome (ii) holds; thus assume $C_{w_i} = \mathrm{lk}_{s_i}(\Delta)$ for some vertex s_i, for all $1 \leq i \leq t$.

This implies a specific structure on Δ, which means both outcomes (i) and (ii) hold, as follows. By renaming we may assume that, for some i, $I_{w_i} = [u_1, \ldots, u_k]$ is a w_i-interval. Then $\mathrm{lk}_{s_i}(\Delta) \cap \mathrm{lk}_v(\Delta) = I_{w_i}$, thus the unique s_i-interval is $[u_k, \ldots, u_1]$. Regard now s_i as w_2, then $s_2 = w_i$, and Δ has exactly two vertices outside $\mathrm{st}_v(\Delta)$, and outcome (i) holds. Also, outcome (ii) holds, as $(w_2, u_k, \ldots, u_1)$ is an equator. □

4. Half-integral matchings

Here we prove Theorems 1.6 and 1.7. For background on shelling see e.g. [22].

Let u, v be two vertices of a simplicial d-polytope P ($d \geq 2$) such that uv is not an edge of P. Consider the line through u and v, and perturb it to obtain an oriented line l that crosses each facet hyperplane in a different point; orient l so that the line shelling it defines shells the facets containing v first and the facets containing u last. Then the restriction faces of the facets containing v are the same as in a corresponding shelling of $\mathrm{lk}_v\,\partial P$. The restriction faces of the facets containing u also contain u, and removing u from them gives the restriction faces of a corresponding shelling of $\mathrm{lk}_u\,\partial P$. Thus, by the expression for $h_{\partial P}(t)$ in terms of the restriction faces of the shelling above, one has the following.

Lemma 4.1. *For all nonedges uv as above,*

$$h_{\mathrm{lk}_v\,\partial P}(t) + th_{\mathrm{lk}_u\,\partial P}(t) \leq h_{\partial P}(t).$$

Further, equality holds iff all facets of P contain either u or v, namely ∂P is a suspension over the vertices u and v.

Proof of Theorem 1.6. By Theorem 1.7, proved below, the vertex set of P admits a partition into a matching and odd cycles in the complement of the 1-skeleton of P. Orient the edges in the odd cycles cyclically and consider each edge of the matching as a cyclically oriented 2-cycle.

Summing the inequality of Lemma 4.1 over all oriented edges given above, gives (1).

For the equality case, again by Lemma 4.1, it happens iff Δ is a suspension over *each* nonedge. In particular the nonedges give a perfect matching, so Δ has the same graph as the d-cross-polytope, and by flagness we are done. □

Before we prove Theorem 1.7 we need the following lemma.

Lemma 4.2. *Let Δ be a flag homology sphere and let F be a facet of Δ. Then there exists a facet F' of Δ that is disjoint from F.*

Proof. The proof is by induction on the dimension of Δ. Let $v \in F$. Then $F_1 = F \setminus \{v\}$ is a facet of $\mathrm{lk}_v(\Delta)$. Inductively, there exists a facet F_2 in $\mathrm{lk}_v(\Delta)$ such that F_2 is disjoint from F_1. Since Δ is a flag homology sphere, each F_i is contained in two facets of Δ, and therefore there exists a vertex $w \neq v$ of Δ such that $F_2 \cup \{w\}$ is a facet of Δ. But now F and $F' = F_2 \cup \{w\}$ are two disjoint facets of Δ as required. □

We will also use Theorem 2.2.4 of [14]:

Lemma 4.3. *A graph G has a half-integral perfect matching if and only if for every $X \subseteq V(G)$, $G \setminus X$ has at most $|X|$ isolated vertices.*

Proof of Theorem 1.7. Let Δ be a flag homology sphere, and let G be the complement of the 1-skeleton of Δ. First we prove the G has a half-integral perfect matching. We need to show that G satisfies the assumption of Lemma 4.3. Let $X \subseteq V(G)$ and let Y be the set of isolated vertices of $G \setminus X$. Then Y is a clique in the 1-skeleton of Δ, and every vertex of Y is adjacent in Δ to every vertex of $\Delta \setminus (X \cup Y)$. It follows that Y is contained in a facet F of Δ. By Lemma 4.2 there is a facet F' of Δ disjoint from F. Since every vertex of Y is adjacent in Δ to every vertex of $F' \setminus X$, we deduce that $|F' \setminus X| + |Y| \leq |F'|$. Consequently, $|F' \cap X| \geq |Y|$, and so $|Y| \leq |X|$ as required.

The second assertion of Theorem 1.7 now follows immdiatly by Proposition 2.2.2 of [14]. □

5. Balanced polytopes

In fact, (1) holds also for (completely) balanced simplicial polytopes, for a very similar reason as in the flag case, as we show in this section.

Recall that a simplicial complex Δ is *balanced* if its 1-skeleton is $(\dim(\Delta)+1)$-colorable.

Observation 5.1. Let Δ be the boundary complex of a balanced d-polytope, and v a vertex in Δ. Then there exists another vertex $v \neq u \in \Delta$ such that uv is not an edge in Δ.

Just take u of same color as v; it exists else Δ would be a cone over v, a contradiction.

Using line shellings, starting with all facets containing v and ending with all facets containing its non-neighbor u as above, (1) follows from showing that the graph G complementary to the graph of Δ admits a half-integral perfect matching; equivalently, by showing that for any subset X of the vertex set Δ_0 of Δ, there are at most $|X|$ isolated vertices in the induced graph $G[\Delta_0 \setminus X]$.

Indeed, let $Y = \{y_1, \ldots, y_t\}$ be a maximal set of isolated vertices in $G[\Delta_0 \setminus X]$. Then all vertices in $\Delta_0 \setminus X$ are in the intersection of the closed stars $\mathrm{st}_{y_i}(\Delta)$. In particular, the induced graph on Y in Δ is complete so they all have distinct colors. By Observation 5.1 there exist distinct $x_1, \ldots, x_t$ with x_i of same color as y_i, $x_i \neq y_i$, and so $\{x_1, \ldots, x_t\} \subseteq X$, showing $|Y| \leq |X|$.

Thus, Corollary 1.8 holds also when replacing *flag* by *balanced*.

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