INDUCED SUBGRAPHS AND TREE DECOMPOSITIONS XV. EVEN-HOLE-FREE GRAPHS WITH BOUNDED CLIQUE NUMBER HAVE LOGARITHMIC TREEWIDTH

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Abstract. We prove that for every integer $t \geq 1$ there exists an integer $c_t \geq 1$ such that every n-vertex even-hole-free graph with no clique of size t has treewidth at most $c_t \log n$. This resolves a conjecture of Sintiari and Trotignon, who also proved that the logarithmic bound is asymptotically best possible. It follows that several NP-hard problems such as Stable Set, Vertex Cover, Dominating Set and Coloring admit polynomial-time algorithms on this class of graphs. As a consequence, for every positive integer r, r-Coloring can be solved in polynomial time on even-hole-free graphs without any assumptions on clique size.

As part of the proof, we show that there is an integer d such that every even-hole-free graph has a balanced separator which is contained in the (closed) neighborhood of at most d vertices. This is of independent interest; for instance, it implies the existence of efficient approximation algorithms for certain NP-hard problems while restricted to the class of all even-hole-free graphs.

1. Introduction

All graphs in this paper are finite and simple. For a graph G, we denote by $V(G)$ the vertex set of G, and by $E(G)$ the edge set of G. For graphs H, G, we say that H is an *induced subgraph* of G if $V(H) \subseteq V(G)$, and $x, y \in V(H)$ are adjacent in H if and only if $xy \in E(G)$. A *hole* in a graph is an induced cycle with at least four vertices. The *length* of a hole is the

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number of vertices in it. A hole is *even* it it has even length, and *odd* otherwise. The class of even-hole-free graphs has attracted much attention in the past due to its somewhat tractable, yet very rich structure (see the survey [37]). In addition to stuctural results, much effort was put into designing efficient algorithms for even-hole-free graphs (to solve problems that are NP-hard in general). This is discussed in [37], while [1, 12, 15, 28] provide examples of more recent work. Nevertheless, many open questions remain. Among them is the complexity of several algorithmic problems: Stable Set, Vertex Cover, Dominating Set, k-Coloring and Coloring.

For a graph $G = (V(G), E(G))$, a *tree decomposition* (T, χ) of G consists of a tree T and a map $\chi : V(T) \to 2^{V(G)}$ with the following properties:

(i) For every vertex $v \in V(G)$, there exists $t \in V(T)$ such that $v \in \chi(t)$.
(ii) For every edge $v_1v_2 \in E(G)$, there exists $t \in V(T)$ such that $v_1, v_2 \in \chi(t)$.
(iii) For every vertex $v \in V(G)$, the subgraph of T induced by $\{t \in V(T) \mid v \in \chi(t)\}$ is connected.

For each $t \in V(T)$, we refer to $\chi(t)$ as a *bag of* (T, χ). The *width* of a tree decomposition (T, χ), denoted by width(T, χ), is $\max_{t \in V(T)} |\chi(t)| - 1$. The *treewidth* of G, denoted by tw(G), is the minimum width of a tree decomposition of G.

Treewidth, first introduced by Robertson and Seymour in the context of graph minors [33], is an extensively studied graph parameter, mostly due to the fact that graphs of bounded treewidth exhibit interesting structural [35] and algorithmic [11] properties. It is easy to see that large complete graphs and large complete bipartite graphs have large treewidth, but there are others (in particular a subdivided $k \times k$-wall, which is a planar graph with maximum degree three, and which we will not define here). A theorem of [35] characterizes precisely excluding which graphs as minors (and in fact as subgraphs) results in a class of bounded treewidth.

Bringing tree decompositions into the world of induced subgraphs is a relatively recent endeavor. It began in [36], where the authors observed yet more evidence of the structural complexity of the class of even-hole-free graphs: the fact that there exist even-hole-free graphs of arbitrarily large treewidth (even when large complete subgraphs are excluded). Closer examination of their constructions led the authors of [36] to make the following two conjectures (the *diamond* is the unique simple graph with four vertices and five edges):

Conjecture 1.1 (Sintiari and Trotignon [36])**.** *For every integer $t \geq 1$, there exists a constant c_t such that every even-hole-free graph G with no induced diamond and no clique of size t satisfies* $\text{tw}(G) \leq c_t$.

Conjecture 1.2 (Sintiari and Trotignon [36])**.** *For every integer $t \geq 1$, there exists a constant C_t such that every even-hole-free graph G with no clique of size t satisfies* $\text{tw}(G) \leq C_t \log |V(G)|$.

Conjecture 1.1 was resolved in [8]. Here we prove Conjecture 1.2, thus closing the first line of research on induced subgraphs and tree decompositions.

We remark that the construction of [36] shows that the logarithmic bound of Conjecture 1.2 is asymptotically best possible. Furthermore, our main result implies that many algorithmic problems that are NP-hard in general (among them that Stable Set, Vertex Cover, Dominating Set and Coloring) admit polynomial-time algorithms in the class of even-hole-free graphs with bounded clique number. As a consequence, for every positive integer r, r-Coloring can be solved in polynomial time on even-hole-free graphs without any assumptions on clique size.

Before we proceed, we introduce some notation and definitions. Let $G = (V(G), E(G))$ be a graph. For a set $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X. For $X \subseteq V(G)$, we denote by $G \setminus X$ the subgraph induced by $V(G) \setminus X$. In this paper, we use induced subgraphs and their vertex sets interchangeably.

For graphs G and H, we say that G *contains* H if some induced subgraph of G is isomorphic to H. For a family $\mathcal{H}$ of graphs, G *contains* $\mathcal{H}$ if G contains a member of $\mathcal{H}$, and we say that G is $\mathcal{H}$*-free* if G does not contain $\mathcal{H}$. A *clique* in a graph is a set of pairwise adjacent vertices, and a *stable set* is a set of vertices no two of which are adjacent. Let $v \in V(G)$. The *open neighborhood of* v, denoted by $N(v)$, is the set of all vertices in $V(G)$ adjacent to v. The *closed neighborhood of* v, denoted by $N[v]$, is $N(v) \cup \{v\}$. Let $X \subseteq V(G)$. The *open neighborhood of* X, denoted by $N(X)$, is the set of all vertices in $V(G) \setminus X$ with at least one neighbor in X. The *closed neighborhood of* X, denoted by $N[X]$, is $N(X) \cup X$. If H is an induced subgraph of G and $X \subseteq V(G)$, then $N_H(X) = N(X) \cap H$ and $N_H[X] = N_H(X) \cup (X \cap H)$. Let $Y \subseteq V(G)$ be disjoint from X. We say X is *complete* to Y if all possible edges with an end in X and an end in Y are present in G, and X is *anticomplete* to Y if there are no edges between X and Y.

Given a graph G, a *path in* G is an induced subgraph of G that is a path. If P is a path in G, we write $P = p_1\text{-}\cdots\text{-}p_k$ to mean that $V(P) = \{p_1, \ldots, p_k\}$, and p_i is adjacent to p_j if and only if $|i - j| = 1$. We call the vertices p_1 and p_k the *ends of* P, and say that P is a path *from* p_1 *to* p_k. The *interior of* P, denoted by P^*, is the set $V(P) \setminus \{p_1, p_k\}$. The *length* of a path P is the number of edges in P. We denote by C_k a cycle with k vertices.

Next, we describe a few types of graphs that we will need (see Figure 1).

A *theta* is a graph consisting of three internally vertex-disjoint paths $P_1 = a\text{-}\cdots\text{-}b$, $P_2 = a\text{-}\cdots\text{-}b$, and $P_3 = a\text{-}\cdots\text{-}b$, each of length at least 2, such that P_1^*, P_2^*, P_3^* are pairwise anticomplete. In this case we call a and b the *ends* of the theta.

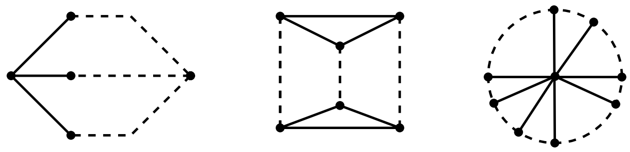

FIGURE 1. Theta, prism and an even wheel. Dashed lines represent paths of length at least one.

A *prism* is a graph consisting of three vertex-disjoint paths $P_1 = a_1\text{-}\cdots\text{-}b_1$, $P_2 = a_2\text{-}\cdots\text{-}b_2$, and $P_3 = a_3\text{-}\cdots\text{-}b_3$, each of length at least 1, such that $a_1a_2a_3$ and $b_1b_2b_3$ are triangles, and no edges exist between the paths except those of the two triangles.

A *pyramid* is a graph consisting of three paths $P_1 = a\text{-}\cdots\text{-}b_1$, $P_2 = a\text{-}\cdots\text{-}b_2$, and $P_3 = a\text{-}\cdots\text{-}b_3$ such that $P_1 \setminus \{a\}, P_2 \setminus \{a\}, P_3 \setminus \{a\}$ are pairwise disjoint, at least two of the three paths P_1, P_2, P_3 have length at least two, $b_1b_2b_3$ is triangle (called the *base* of the pyramid), and no edges exist between $P_1 \setminus \{a\}, P_2 \setminus \{a\}, P_3 \setminus \{a\}$ except those of the triangle $b_1b_2b_3$. The vertex a is called the *apex* of the pyramid.

A *wheel* (H, x) in G is a pair where H is a hole and x is a vertex with at least three neighbors in H. A wheel (H, x) is *even* if x has an even number of neighbors on H.

Let $\mathcal{C}$ be the class of (C_4, theta, prism, even wheel)-free graphs (sometimes called "C_4-free odd-signable" graphs). For every integer $t \geq 1$, let $\mathcal{C}_t$ be the class of all graphs in $\mathcal{C}$ with no clique of size t. It is easy to see that every even-hole-free graph is in $\mathcal{C}$.

The reader may be familiar with [3] where a special case of Conjecture 1.2 was proved; moreover, only one Lemma of [3] uses the fact that the set-up there is not the most general case. At the time, the authors of [3] thought that the full proof of Conjecture 1.2 would follow the general outline of [3], fixing the one missing lemma. That is not what happened. The proof in the present paper takes a different path: while it relies on insights and a general understanding of the class of even-hole-free graphs gained in [3], and uses several theorems proved there, a significant detour is needed.

The first part of the paper is not concerned with treewidth at all. Instead, we focus on the following question: given two non-adjacent vertices in a graph in $\mathcal{C}$ of bounded clique number, how many internally vertex-disjoint paths can there be between them? Given that if instead of "internally vertex-disjoint" we say "with pairwise anticomplete interiors", then the answer is "two", this is somewhat related to the recent work on the induced Menger theorem [21, 25, 31]. Let $k \geq 1$ be an integer and let $a, b \in V(G)$. We say that ab is a *k-banana* if a is non-adjacent to b, and there exist k pairwise internally-vertex-disjoint paths from a to b. Note that if ab is a k-banana in G, then ab is an l-banana in G for every $l \leq k$. We prove:

Theorem 1.3. *For every integer $t \geq 1$, there exists a constant c_t such that if $G \in \mathcal{C}_t$, then G contains no $c_t \log |V(G)|$-banana.*

The next step in the proof of Conjecture 1.2 is the following. Let $c \in [0, 1]$ and let w be a normal weight function on G, that is, $w : V(G) \to \mathbb{R}_{\geq 0}$ satisfies $\sum_{v \in V(G)} w(v) = 1$. A set $X \subseteq V(G)$ is a (w, c)*-balanced separator* if $w(D) \leq c$ for every component D of $G \setminus X$. The set X is a w*-balanced separator* if X is a $(w, \frac{1}{2})$-balanced separator. We show:

Theorem 1.4. *There is an integer d with the following property. Let $G \in \mathcal{C}$ and let w be a normal weight function on G. Then there exists $Y \subseteq V(G)$ such that*

- *$|Y| \leq d$, and*
- *$N[Y]$ is a w-balanced separator in G.*

We then use Theorem 1.3 and Theorem 1.4 to prove our main result:

Theorem 1.5. *For every integer $t \geq 1$, there exists a constant c_t such that every $G \in \mathcal{C}_t$ satisfies* $\text{tw}(G) \leq c_t \log |V(G)|$.

1.1. **Proof outline and organization.**

We only include a few definitions in this section; all the necessary definitions appear in later parts of the paper. Our first goal is to prove Theorem 1.3. Let $a, b \in V(G)$ be non-adjacent. Recall that a *separation* of a graph G is a triple (X, Y, Z) of pairwise disjoint subsets of G with $X \cup Y \cup Z = G$ such that X is anticomplete to Z. Similarly to [6], we use the fact that graphs in $\mathcal{C}_t$ admit a natural family of separations that correspond to special vertices of the graph called "hubs" and are discussed in Section 3. Unfortunately, the interactions between these separations may be complex, and, similarly to [5], we use degeneracy to partition the set of all hubs other than a, b (which yields a partition of all the natural separations) of an n-vertex graph G in $\mathcal{C}_t$ into collections $S_1, \ldots, S_p$, where each S_i is "non-crossing" (this property is captured in Lemma 4.9), $p \leq C(t) \log n$ (where $C(t)$ only depends on t and works for all $G \in \mathcal{C}_t$) and each vertex of S_i has at most d (where d depends on t) neighbors in $\bigcup_{j=i}^{p} S_j$. We prove a strengthening of Theorem 1.3, which asserts that the size of the largest ab-banana is bounded by a linear function of p.

We observe that a result of [3] implies that there exists a an integer k (that depends on t) such that if $G \in \mathcal{C}_t$ has no hubs other than a, b, then ab is not a k-banana in G. More precisely, the result of [3] states that if a and b are joined by k internally-vertex-disjoint paths $P_1, \ldots, P_k$, then for at least one $i \in \{1, \ldots, k\}$, the neighbor of a in P_i is a hub.

We now proceed as follows. Let $m = 2d + k$ (where d comes from the degeneracy partition and k from the previous paragraph); suppose that ab is a $(4m + 2)(m - 1)$-banana in G and let $\mathcal{P}$ be the set of all paths of G with ends a, b.

Let $S_1, \dots, S_p$ denote the partition of hubs as decribed above. We proceed by induction on p and describe a process below that finds an induced subgraph H of G in which:

- Vertices in S_1 are no longer hubs;
- If there are not many internally disjoint ab-paths in H, we can "lift" this to G.

We first consider a so-called m-lean tree decomposition (T, χ) of G (discussed in Section 2). By examining the intersection graphs of subtrees of a tree we deduce that there exist two (not necessarily distinct) vertices $t_1, t_2 \in V(T)$ such that for every ab path P, the interior of P meets $\chi(t_1) \cup \chi(t_2)$. We also argue that $a, b \in \chi(t_1) \cup \chi(t_2)$. A vertex u of S_1 is *bad* if u has large degree (at least $D = 2m(m-1)$) in the torso of $\chi(t_1)$, or u has large degree in the torso of $\chi(t_2)$, or u is adjacent to both a and b. We show that there are at most three bad vertices in S_1.

We would like to bound the size of the largest ab-banana in the union of the two torsos. Unfortunately, the torso of $\chi(t_i)$ may not be a graph in $\mathcal{C}_t$. Instead, we find an induced subgraph of G, which we call β, that consists of $\chi(t_1) \cup \chi(t_2)$ together with a collection of disjoint vertex sets $\mathrm{Conn}(t_i, t)$ for $t \in V(T) \setminus \{t_1, t_2\}$ and $i \in \{1, 2\}$ except vertices t on the t_1-t_2-path in T, where each $\mathrm{Conn}(t_i, t)$ "remembers" the component of $G \setminus (\chi(t_1) \cup \chi(t_2))$ that meets $\chi(t)$. Importantly, no vertex of $\beta \setminus (\chi(t_1) \cup \chi(t_2))$ is a hub of β, and all but at most three vertices of S_1 have bounded degree in the union of the torso of $\chi(t_1)$ and the torso of $\chi(t_2)$.

We next decompose β, simultaneously, by all the separations corresponding to the hubs in S_1 that are not bad, and delete the set of (at most three) bad vertices of S_1. We denote the resulting graph by $\beta^A(S_1)$ and call it the "central bag" for S_1. The parameter p is smaller for $\beta^A(S_1)$ than it is for G, and so we can use induction to obtain a bound on the largest size of an ab-banana in $\beta^A(S_1)$. Since our goal is to bound the size of an ab-banana in G by a linear function of p, we now need to show that the size of the largest ab-banana does not grow by more than an additive constant when we move from $\beta^A(S_1)$ to G.

We start with a smallest subset X of $\beta^A(S_1)$ that separates a from b in $\beta^A(S_1)$ (and whose size is bounded by Menger's theorem) and show how to transform in into a set Y separating a from b in β, while increasing the size of its intersection with $\chi(t_1) \cup \chi(t_2)$ by at most an additive constant, and ensuring the number of sets $\mathrm{Conn}(t_i, t)$ that Y meets is bounded by a constant. Then we repeat a similar procedure to obtain a set Z of vertices separating a from b in G, making sure that the increase in size is again bounded by an additive constant.

Let us now discuss how we obtain the bound on the growth of the separator. In the first step, to obtain Y, we add to X the neighbor sets of the vertices of $S_1 \cap X$. Since while constructing $\beta^A(S_1)$ we have deleted all the bad vertices of S_1, the number of vertices of $\chi(t_1) \cup \chi(t_2)$ added to X

is at most $2|S_1 \cap X|D$, and $Y \setminus X$ meets at most $2|S_1 \cap X|D$ of the sets $\text{Conn}(t_i, t)$. Note that the bound on the size of X that we have depends on p, which may be close to $\log|V(G)|$, so another argument is needed to obtain a constant bound on $|S_1 \cap X|$ and on the number of sets $\text{Conn}(t_i, t)$ that meet Y. This is a consequence of Theorem 6.3, because no vertex of S_1 is a hub in $\beta^A(S_1)$ (this is proved in Theorem 4.10), and no vertex of $\text{Conn}(t_i, t)$ is a hub in β. (The proof of Theorem 6.3 analyzes the structure of minimal separators in graphs in $\mathcal{C}$ using tools developed in Section 5.)

In the second growing step, we start with the set Y obtained in the previous paragraph. Then we add to Y the following subsets (here we describe the cases when $t_1 \neq t_2$; if $t_1 = t_2$ the argument is similar).

(1) $\chi(t_1) \cap \chi(t_1')$ where t_1' is the unique neighbor of t_1 in the path in T from t_1 to t_2.
(2) $\chi(t_2) \cap \chi(t_2')$ where t_2' is the unique neighbor of t_2 in the path in T from t_1 to t_2.
(3) $\chi(t_1) \cap \chi(t)$ for every $t \in N(t_1) \setminus \{t_1'\}$ such that $Y \cap \text{Conn}(t_1, t) \neq \emptyset$.
(4) $\chi(t_2) \cap \chi(t)$ for every $t \in N(t_2) \setminus \{t_2'\}$ such that $Y \cap \text{Conn}(t_2, t) \neq \emptyset$.
(5) The set of all bad vertices of S_1.

One of the properties of m-lean tree decompositions is that the size of each adhesion (intersection of "neighboring" bags) is less than m. The number of adhesions added to Y is again bounded since the number of distinct sets $\text{Conn}(t_i, t)$ that meet Y is bounded. This completes the proof of Theorem 1.3.

The next key ingredient in the proof of Theorem 1.5 is Theorem 1.4, asserting that there is an integer C such that for every graph $G \in \mathcal{C}$ and every normal weight function w on G, there is a w-balanced separator X in G such that X is contained in the union of the neighborhoods of at most C vertices of G. To prove that, we first prove a decomposition theorem, similar to the one giving us the natural separations associated with hubs, but this time the separations come from pyramids in G. We then use the two kinds of separations (the kind that come from hubs and the kind that come from pyramids) as follows.

For $X \subseteq V(G)$ we say that a set $Y \subseteq V(G)$ *dominates* X if $X \subseteq N[Y]$. By Lemma 8.5 (from [14]) there is a path $P = p_1\text{-}\cdots\text{-}p_k$ in G such that $w(D) \leq \frac{1}{2}$ for every component D of $G \setminus N[P]$. We now use a "sliding window" argument: we divide P into subpaths, each with the property that its neighborhood can be dominated by a small, but not too small, set of vertices. Using the decomposition theorems above, we find a bounded-size set $X(W)$ such that, except for our window W, the path P is disjoint from $N[X(W)]$, and $N[X(W)]$ separates the subpath of P before our current window from that after our current window. This means (unless $N[X]$ is a balanced separator) that the big component of $G \setminus N[X]$ does not contain either the subpath of P before our window, or the subpath after our window. By looking at the point in the path where this answer changes from “before” to ”after” (and showing that such a point exists), and by combining the separators for the two windows before and after this point as well as small dominating sets for the neighborhood of those two windows, we are able to find a w-balanced separator with bounded domination number. This completes the proof of Theorem 1.4. We remark that Theorem 1.4 applies to all graphs in $\mathcal{C}$ and does not assume a bound on the clique number.

The next step in the proof of Theorem 1.5 is to prove Theorem 9.2, asserting that for every integer L, if a graph G contains no L-banana, then for every normal weight function w on G, if G has a w-balanced separator $N[X]$, then G has an w-balanced separator Y and a clique $T \subseteq X$ such that $|Y \setminus T| \leq 3|X|L$. This step uses $3L$-lean tree decompositions of G and works for all graphs G.

Now Theorem 9.2 and Theorem 1.3 together imply that for every t, there exists an integer q, depending on t, such that for every $G \in \mathcal{C}_t$, G has a balanced separator of size at most $q \log |V(G)| + t$; that is Theorem 9.1. By Lemma 10.1 this immediately implies the desired bound on the treewidth of G.

The paper is organized as follows. In Section 2, we discuss lean tree decompositions and their properties, along with other classical results in graph theory. For some of them we prove variations tailored specifically to our needs. In Section 3 we summarize results guaranteeing the existence of separations associated with hubs. We also establish a stronger version of Theorem 1.3 for the case when the set of hubs in G is very restricted. In Section 4 we discuss the construction of the graphs β and $\beta^A(S_1)$, and how to use ab-separators there to obtains ab-separators in G. In Section 5 we analyze the structure of minimal separators in graphs of $\mathcal{C}$. In Section 6 we use the results of Section 5 to obtain a bound on the size of a stable set of non-hubs in an ab-separator of smallest size. Section 7 puts together the results of all the previous sections to prove Theorem 1.3. The goal of Section 8 is to prove Theorem 1.4. We start with Lemmas 8.2 and 8.3 to establish the existence of certain decompositions in graphs of $\mathcal{C}$, and then proceed with the sliding window argument. Section 9 is devoted to the proof of Theorem 9.1. The proof of Theorem 1.5 is completed in Section 10. Finally, Section 11 discusses algorithmic consequences of Theorem 1.4 and Theorem 1.5.

2. Special tree decompositions and connectivity

In this section we have collected several results from the literature that we need; for some of them we have also proved our own versions, tailored specifically to our needs. Along the way we also introduce some notation.

2.1. **Connectivity.** We start with a result on connectivity. For two vertices $u, v \in G$ and a set $X \subseteq G \setminus \{u, v\}$ we say that X *separates* u from v

if $P^* \cap X \neq \emptyset$ for every path P of G with ends u and v. The following is a well-known variant of a classical theorem due to Menger [30]:

Theorem 2.1 (Menger [30])**.** *Let $k \geq 1$ be an integer, let G be a graph and let $u, v \in G$ be distinct and non-adjacent. Then either there exists a set $M \subseteq G \setminus \{u, v\}$ with $|M| < k$ such that M separates u and v in G, or uv is a k-banana in G.*

2.2. **Lean tree decompositions.** We adopt notation from [3]: For a tree T and vertices $t, t' \in V(T)$, we denote by tTt' the unique path in T from t to t'. Let (T, χ) be a tree decomposition of a graph G. For every $uv \in E(T)$, the *adhesion at* uv, denoted by $\text{adh}(u, v)$, is the set $\chi(u) \cap \chi(v)$. For $u, v \in T$ such that $uv \notin E(T)$ (in particular, if $u = v$), we set $\text{adh}(u, v) = \emptyset$. We define $\text{adh}(T, \chi) = \max_{uv \in E(T)} |\text{adh}(u, v)|$. For every $x \in V(T)$, the *torso at* x, denoted by $\hat{\chi}(x)$, is the graph obtained from the bag $\chi(x)$ by, for each $y \in N_T(x)$, adding an edge between every two non-adjacent vertices $u, v \in \text{adh}(x, y)$.

In the proof of Theorem 1.3 and Theorem 1.5, we will use a special kind of a tree decomposition that we present next. Let $k > 0$ be an integer. A tree decomposition (T, χ) is called *k-lean* if the following hold:

- $\text{adh}(T, \chi) < k$; and
- for all $t, t' \in V(T)$ and sets $Z \subseteq \chi(t)$ and $Z' \subseteq \chi(t')$ with $|Z| = |Z'| \leq k$, either G contains $|Z|$ disjoint paths from Z to Z', or some edge ss' of tTt' satisfies $|\text{adh}(s, s')| < |Z|$.

For a tree T and an edge tt' of T, we denote by $T_{t \to t'}$ the component of $T \setminus t$ containing t'. Let $G_{t \to t'} = G[\bigcup_{v \in T_{t \to t'}} \chi(v)]$. A tree decomposition (T, χ) is *tight* if for every edge $tt' \in E(T)$ there is a component D of $G_{t \to t'} \setminus \chi(t)$ such that $\chi(t) \cap \chi(t') \subseteq N(D)$ (and therefore $\chi(t) \cap \chi(t') = N(D)$).

Next, we need a definition from [9]. Given a tree decomposition (T, χ) of an n-vertex graph G, its *fatness* is the vector $(a_n, \ldots, a_0)$ where a_i denotes the number of bags of T of size i. A tree decomposition (T, χ) of G is *k-atomic* if $\text{adh}(T, \chi) < k$ and the fatness of (T, χ) is lexicographically minimum among all tree decompositions of G with adhesion less than k.

It was observed in [13] that [9] contains a proof of the following:

Theorem 2.2 (Bellenbaum and Diestel [9], see Carmesin, Diestel, Hamann, Hundertmark [13], see also Weißauer [38])**.** *Every k-atomic tree decomposition is k-lean.*

We also need:

Theorem 2.3 (Weißauer [38])**.** *Every k-atomic tree decomposition is tight.*

Using ideas similar to those of Weißauer [38] and using Theorems 2.1 and 2.2, we prove:

Theorem 2.4. *Let G be a graph, let $k \geq 1$ and let (T, χ) be $3k$-atomic tree decomposition of G. Let $t_0 \in V(T)$ and let $u, v \in G$. Assume that*

$N[u]$ is not separated from $\chi(t_0) \setminus u$ by a set of size less than $3k$, and that $N[v]$ is not separated from $\chi(t_0) \setminus v$ by a set of size less than $3k$. Then u is not separated from v by a set of size less than k, and consequently uv is a k-banana.

Proof. By Theorems 2.2 and 2.3, we have that (T, χ) is tight and $3k$-lean. Suppose that u is separated from v by a set of size less than k. By Theorem 2.1, there exists a set $\mathcal{P}_u$ of pairwise vertex-disjoint (except u) paths, each with one end u and the other end in $\chi(t_0) \setminus u$, and such that $|\mathcal{P}_u| = 3k$. Let Z_u be the set of the ends of members of $\mathcal{P}_u$ in $\chi(t_0)$. Similarly, there exists a collection $\mathcal{P}_v$ of pairwise vertex-disjoint (except v) paths, each with one end v and the other end in $\chi(t_0) \setminus v$, and such that $|\mathcal{P}_v| = 3k$. Let Z_v be the set of the ends of members of $\mathcal{P}_v$ in $\chi(t_0)$. Since (T, χ) is $3k$-lean, there is a collection $\mathcal{Q}$ of pairwise vertex-disjoint paths from Z_u to Z_v. Let X be a set with $|X| < k$ separating u from v. Then, for every u-v path P in G we have $P^* \cap X \neq \emptyset$. But since $|Z_u| = |Z_v| = |\mathcal{Q}| = 3k$, there is a path $Q \in \mathcal{Q}$ with ends $z_u \in Z_u$ and $z_v \in Z_v$, a path $P_u \in \mathcal{P}_u$ from u to z_u, and a path $P_v \in \mathcal{P}_v$ from v to Z_v such that $X \cap (Q \cup (P_u \setminus u) \cup (P_v \setminus v)) = \emptyset$, contrary to the fact that u-P_u-z_u-Q-z_v-P_v-v is a path from u to v in G. This proves the first statement of the theorem. The second statement follows immediately by Theorem 2.1. ■

We finish this subsection with a theorem about tight tree decompositions in theta-free graphs that was proved in [3]. Note that by Theorem 2.3, the following result applies in particular to k-atomic tree decompositions for every k.

A *cutset* $C \subseteq V(G)$ of G is a (possibly empty) set of vertices such that $G \setminus C$ is disconnected. A *clique cutset* is a cutset that is a clique.

Theorem 2.5 (Abrishami, Alecu, Chudnovsky, Hajebi, Spirkl [3])**.** *Let G be a theta-free graph and assume that G does not admit a clique cutset. Let (T, χ) be a tight tree decomposition of G. Then for every edge t_1t_2 of T the graph $G_{t_1 \to t_2} \setminus \chi(t_1)$ is connected and $N(G_{t_1 \to t_2} \setminus \chi(t_1)) = \chi(t_1) \cap \chi(t_2)$. Moreover, if $t_0, t_1, t_2 \in V(T)$ and $t_1, t_2 \in N_T(t_0)$, then $\chi(t_0) \cap \chi(t_1) \neq \chi(t_0) \cap \chi(t_2)$.*

2.3. **Catching a banana.** In this subsection we discuss another important feature of tree decompositions that is needed in the proof of Theorem 1.3.

Theorem 2.6. *Let G be a theta-free graph and let $a, b \in V(G)$. Let (T, χ) be a tree decomposition of G. Then there exist $t_1, t_2 \in V(T)$ (not necessarily distinct) such that for every path P of G with ends a, b we have that $(\chi(t_1) \cup \chi(t_2)) \cap P^* \neq \emptyset$. Moreover, if D is a component of $G \setminus (\chi(t_1) \cup \chi(t_2))$, then $|N(D) \cap \{a, b\}| \leq 1$.*

Proof. Let $\mathcal{P}$ be the set of all paths of G with ends a, b. For every $P \in \mathcal{P}$ let $T(P) = \{t \in T \ : \ \chi(t) \cap P^* \neq \emptyset\}$. Since P^* is a connected subgraph of G, it follows that $T[T(P)]$ is a subtree of T.

Let H be a graph with vertex set $\mathcal{P}$ and such that $P_1P_2 \in E(H)$ if and only if $T(P_1) \cap T(P_2) \neq \emptyset$. Then, by the main result of [22], H is a chordal graph.

(1) *If $P_1, P_2 \in H$ are non-adjacent in H, then P_1^*, P_2^* are disjoint and anticomplete to each other in G.*

Suppose that there exists $v \in P_i^* \cap P_j^*$. Then for every $t \in T$ such that $v \in \chi(T)$, we have $t \in T(P_1) \cap T(P_2)$, contrary to the fact that P_1, P_2 are non-adjacent in H. Next suppose that there is an edge v_1v_2 of G such that $v_i \in P_i^*$. Let $t \in T$ be such that $v_1, v_2 \in \chi(t)$. Then $t \in T(P_1) \cap T(P_2)$, again contrary to the fact that P_1 is non-adjacent to P_2 in H. This proves (1).

(2) *H has no stable set of size three.*

Suppose P_1, P_2, P_3 is a stable set in H. By (1) the sets P_1^*, P_2^*, P_3^* are pairwise disjoint and anticomplete to each other in G. But now $P_1 \cup P_2 \cup P_3$ is a theta with ends a, b in G, a contradiction. This proves (2).

Since H is chordal, it follows from (2) and a result from [23] that there exist cliques K_1, K_2 of H such that $V(H) = K_1 \cup K_2$. Again by [23] we deduce that for each $i \in \{1, 2\}$, there exists $t_i \in K_i$ such that $t_i \in T(P)$ for every $P \in K_i$. Consequently, $(\chi(t_1) \cup \chi(t_2)) \cap P^* \neq \emptyset$ for every $P \in \mathcal{P}$, and the first statement of the theorem holds.

To prove the second statement, suppose $a, b \in N(D)$ for some component D of $G \setminus (\chi(t_1) \cup \chi(t_2))$. Then $D \cup \{a, b\}$ is connected, and so there is a path P from a to b with $P^* \subseteq D$. But now $P^* \cap (\chi(t_1) \cup \chi(t_2)) = \emptyset$, a contradiction. This completes the proof. ■

2.4. **Centers of tree decompositions.** We finish this section with a well-known theorem about tree decompositions. Recall that for a graph G, a function $w : V(G) \to [0, 1]$ is a *normal weight function* on G if $w(V(G)) = 1$, where we use the notation $w(X)$ to mean $\sum_{x \in X} w(x)$ for a set X of vertices.

Let G be a graph and let (T, χ) be a tree decomposition of G. Let $w : V(G) \to [0, 1]$ be a normal weight function on G. A vertex t_0 of T is a *center* of (T, χ) if for every $t' \in N_T(t_0)$ we have $w(G_{t_0 \to t'} \setminus \chi(t_0)) \leq \frac{1}{2}$.

The following is well-known; a proof can be found in [3], for example.

Theorem 2.7. *Let (T, χ) be a tree decomposition of a graph G. Then (T, χ) has a center.*

3. Wheels and star cutsets

Recall that a *wheel* in G is a pair (H, x) such that H is a hole and x is a vertex that has at least three neighbors in H. Wheels play an important role in the study of even-hole-free and odd-signable graphs. Graphs in this classes that contain no wheels are much easier to handle; on the other hand, the presence of a wheel forces the graph to admit a certain kind of decomposition.

A *sector* of (H, x) is a path P of H whose ends are distinct and adjacent to x, and such that x is anticomplete to P^*. A sector P is a *long sector* if P^* is non-empty. We now define several types of wheels that we will need.

A wheel (H, x) is a *universal wheel* if x is complete to H. A wheel (H, x) is a *twin wheel* if $N(x) \cap H$ induces a path of length two. A wheel (H, x) is a *short pyramid* if $|N(x) \cap H| = 3$ and x has exactly two adjacent neighbors in H. A wheel is *proper* if it is not a twin wheel or a short pyramid. We say that $x \in V(G)$ is a *hub* if there exists H such that (H, x) is a proper wheel in G. We denote by $\mathrm{Hub}(G)$ the set of all hubs of G.

We need the following result, which was observed in [6]:

Theorem 3.1 (Abrishami, Chudnovsky, Vušković [6]). *Let $G \in \mathcal{C}$ and let (H, v) be a proper wheel in G. Then there is no component D of $G \setminus N[v]$ such that $H \subseteq N[D]$.*

We will revisit this result in Section 8. A *star cutset* in a graph G is a cutset $S \subseteq V(G)$ such that either $S = \emptyset$ or for some $x \in S$, $S \subseteq N[x]$. A large portion of this paper is devoted to dealing with hubs and star cutsets arising from them in graphs in $\mathcal{C}$, but in the remainder of this section we focus on the case when $\mathrm{Hub}(G)$ is very restricted. To do that, we use a result from [3]:

Theorem 3.2 (Abrishami, Alecu, Chudnovsky, Hajebi, Spirkl [3]). *For every integer $t \geq 1$ there exists an integer $k = k(t)$ such that if $G \in \mathcal{C}_t$ and xy is a k-banana in G, then $N(x) \cap \mathrm{Hub}(G) \neq \emptyset$ and $N(y) \cap \mathrm{Hub}(G) \neq \emptyset$.*

We immediately deduce:

Theorem 3.3. *For every integer $t \geq 1$, there exists a constant $k = k(t)$ with the following property. Let $G \in \mathcal{C}_t$ and let $a, b \in V(G)$ be non-adjacent. Assume that $Hub(G) \subseteq \{a, b\}$. Then ab is not a k-banana in G. In particular, if $Hub(G) = \emptyset$, there is no k-banana in G.*

We will also use the following:

Theorem 3.4 (Abrishami, Alecu, Chudnovsky, Hajebi, Spirkl [3]). *For every integer $t \geq 1$, there exists an integer $\gamma = \gamma(t)$ such that every $G \in \mathcal{C}_t$ with $\mathrm{Hub}(G) = \emptyset$ satisfies $\mathrm{tw}(G) \leq \gamma - 1$.*

4. Stable sets of safe hubs

As we discussed in Section 1, in the course of the proof of Theorem 1.3, we will repeatedly decompose the graph by star cutsets arising from a stable set of appropriately chosen hubs (using Theorem 3.1). In this section, we prepare the tools for handling one such step: a stable set of safe hubs.

Throughout this section, we fix the following: Let $t, d \in \mathbb{N}$ and let $G \in \mathcal{C}_t$ be a graph with $|V(G)| = n$. Let $m = k + 2d$ where $k = k(t)$ is as in

Theorem 3.2. Let $a, b \in V(G)$ such that ab is a $(4m+2)(m-1)$-banana in G, and let $\mathcal{P}$ be the set of all paths in G with ends a, b. Let (T, χ) be an m-atomic tree decomposition of G. By Theorems 2.2 and 2.3, we have that (T, χ) is tight and m-lean. By Theorem 2.6, there exist $t_1, t_2 \in T$ such that $P^* \cap (\chi(t_1) \cup \chi(t_2)) \neq \emptyset$ for every $P \in \mathcal{P}$. We observe:

Lemma 4.1. *We have $a, b \in \chi(t_1) \cup \chi(t_2)$.*

Proof. Suppose that $a \notin \chi(t_1) \cup \chi(t_2)$. Let D be the component of $G \setminus (\chi(t_1) \cup \chi(t_2))$ such that $a \in D$. Then $N(D)$ is contained in the union of at most two adhesions of (T, χ), at most one for each of $\chi(t_1)$ and $\chi(t_2)$. Since (T, χ) is m-lean, it follows that $|N(D)| \leq 2(m-1)$. Since $P^* \cap (\chi(t_1) \cup \chi(t_2)) \neq \emptyset$ for every $P \in \mathcal{P}$, it follows that $P^* \cap N(D) \neq \emptyset$ for every $P \in \mathcal{P}$. But then $\mathcal{P}$ contains at most $|N(D)|$ pairwise internally vertex-disjoint paths, contrary to the fact that ab is a $(4m+2)(m-1)$-banana in G. ■

We say that a vertex v is *d-safe* if $|N(v) \cap \text{Hub}(G)| \leq d$. The goal of the next two lemmas is to classify d-safe vertices into "good ones" and "bad ones," and show that the bad ones are rare. Let $t_0 \in V(T)$. A vertex $v \in V(G)$ is *t_0-cooperative* if either $v \notin \chi(t_0)$, or $\deg_{\hat{\chi}(t_0)}(v) < 2m(m-1)$. It was shown in [3] that t_0-cooperative vertices have the following important property (note that [3] assumed that t_0 is a center of (T, χ), but this was not used in the proof of the following lemma):

Lemma 4.2 (Abrishami, Alecu, Chudnovsky, Hajebi, Spirkl [3])**.** *Let $t_0 \in V(T)$. If $u, v \in \chi(t_0)$ are d-safe and not t_0-cooperative, then u is adjacent to v.*

A vertex $v \in G$ is ab-cooperative if there exists a component D of $G \setminus N[v]$ such that $a, b \in N[D]$. We show:

Lemma 4.3. *If $v \in G$ is d-safe and not ab-cooperative, then v is adjacent to both a and b. In particular, the set of vertices that are not ab-cooperative is a clique.*

Proof. Suppose v is non-adjacent to a and v is not ab-cooperative. Since ab is a $(4m+2)(m-1)$-banana and $(4m+2)(m-1) \geq k+d$, we can choose $P_1, \ldots, P_{k+d} \in \mathcal{P}$ such that the sets $P_1^*, \ldots, P_{k+d}^*$ are pairwise vertex-disjoint. Since v is not ab-cooperative, it follows that v has a neighbors in each of $P_1^*, \ldots, P_{k+d}^*$, and so there exist pairwise vertex-disjoint paths $Q_1, \ldots Q_{k+d}$, each with ends a, v, and such that $Q_i^* \subseteq P_i^*$ for every $i \in \{1, \ldots, k+d\}$. Since v is d-safe, we may assume (by renumbering the paths $Q_1, \ldots, Q_{k+d}$ if necessary), that $N(v) \cap \text{Hub}(G) \cap Q_i = \emptyset$ for $i \in \{1, \ldots, k\}$. But now we get a contradiction applying Theorem 3.2 to the graph $H = Q_1 \cup \cdots \cup Q_k$ and the k-banana av in H. This proves the first statement of 4.3. Since G is C_4-free, the second statement follows. ■

In view of Theorem 2.6, it would be convenient if we could work with the graph $\hat{\chi}(t_1) \cup \hat{\chi}(t_2)$, but unfortunately this graph may not be in $\mathcal{C}_t$, or even

$\mathcal{C}$. In [3], a tool was designed to construct a safe alternative to torsos by adding a set of vertices for each adhesion, rather than turning the adhesion into a clique:

Theorem 4.4 (Abrishami, Alecu, Chudnovsky, Hajebi, Spirkl [3])**.** *Let $t_0 \in T$. Assume that G does not admit a clique cutset. For every $t \in N_T(t_0)$, there exists $\mathrm{Conn}(t_0,t) \subseteq G_{t_0 \to t}$ such that*

- *(1)* $\mathrm{adh}(t,t_0) = \chi(t) \cap \chi(t_0) \subseteq \mathrm{Conn}(t_0,t)$.
- *(2)* $\mathrm{Conn}(t_0,t) \setminus \chi(t_0)$ *is connected and* $N(\mathrm{Conn}(t_0,t) \setminus \chi(t_0)) = \chi(t_0) \cap \chi(t)$.
- *(3) No vertex of* $\mathrm{Conn}(t_0,t) \setminus \chi(t_0)$ *is a hub in the graph* $(G \setminus G_{t_0 \to t}) \cup \mathrm{Conn}(t_0,t)$.

Similarly to the construction in [3], we now define a graph that is a safe alternative to $\hat{\chi}(t_1) \cup \hat{\chi}(t_2)$. Let S' be a stable set of hubs of G with $S' \cap \{a,b\} = \emptyset$, and assume that every $s \in S'$ is d-safe. Let S_{bad} denote the set of all vertices in S' that are common neighbors of a and b, or are not t_1-cooperative, or are not t_2-cooperative. Since S' is stable, Lemma 4.2 implies that $|S_{bad}| \leq 3$. By Lemma 4.3, every vertex in $S' \setminus S_{bad}$ is ab-cooperative. If $t_1 \neq t_2$, for $i \in \{1,2\}$, let t_i' be the neighbor of t_i in the (unique) path in T from t_1 to t_2. If $t_1 = t_2$, set $t_1' = t_2' = t_1$. Let $S = S' \setminus S_{bad}$ and set

$$\beta(S') = \left(\chi(t_1) \cup \chi(t_2) \cup \left(\bigcup_{i \in \{1,2\}} \bigcup_{t \in N_T(t_i) \setminus \{t_i'\}} \mathrm{Conn}(t_i,t)\right)\right) \setminus S_{bad}.$$

Write $\beta = \beta(S')$. We fix S', S, S_{bad} and β throughout this section. It follows that for every $i \in \{1,2\}$ and for every $t \in N_T(t_i) \setminus \{t_i'\}$, we have that $\beta \subseteq (G \setminus G_{t_i \to t}) \cup \mathrm{Conn}(t_i,t)$. Let $X \subseteq \beta$. For $i \in \{1,2\}$, we define $\delta_i(X)$ to be the set of all vertices $t \in N_T(t_i) \setminus \{t_i'\}$ such that $X \cap (G_{t_i \to t} \setminus (\chi(t_1) \cup \chi(t_2))) \neq \emptyset$; let $\delta(X) = \delta_1(X) \cup \delta_2(X)$. Write

$$\Delta(X) = \bigcup_{t \in \delta_1(X)} \mathrm{adh}(t_1,t) \cup \bigcup_{t \in \delta_2(X)} \mathrm{adh}(t_2,t).$$

Next we summarize several key properties of β.

Lemma 4.5. *Suppose that G does not admit a clique cutset and let $s \in S \cap \mathrm{Hub}(\beta)$. Then the following hold.*

- *(1)* $s \in \chi(t_1) \cup \chi(t_2)$.
- *(2) For* $i \in \{1,2\}$, *we have* $|N_{\hat{\chi}(t_i)}(s)| < 2m(m-1)$.
- *(3)* $|N_{\chi(t_1) \cup \chi(t_2)}(s)| < 4m(m-1)$.
- *(4) There is a component* $B(s)$ *of* $\beta \setminus N[s]$ *such that* $a,b \in N[B(s)]$.

Proof. Let $s \in S \cap \mathrm{Hub}(\beta)$. It follows from Theorem 4.4(3) that $s \in \chi(t_1) \cup \chi(t_2)$. Since s is t_i-cooperative for $i \in \{1,2\}$, we have that $|N_{\hat{\chi}(t_i)}(s)| < 2m(m-1)$, and the second assertion of the theorem holds. Since $\chi(t_i)$ is a subgraph of $\hat{\chi}(t_i)$, the third assertion follows immediately from the second.

We now prove the fourth assertion. Suppose there is no component D of $\beta \setminus N[s]$ such that $a, b \in N[D]$. Write

$$\hat{\Delta}(s) = N_{\chi(t_1)\cup\chi(t_2)}[s] \cup \text{adh}(t_1, t_1') \cup \text{adh}(t_2, t_2') \cup \Delta(N_\beta[s]).$$

Since $\hat{\Delta}(s) \subseteq N_{\hat{\chi}(t_1)}[s] \cup N_{\hat{\chi}(t_2)}[s] \cup \text{adh}(t_1, t_1') \cup \text{adh}(t_2, t_2')$, and $\text{adh}(T, \chi) \leq m-1$, and by the second assertion of the theorem, we have that $|\hat{\Delta}(s)| < (4m+2)(m-1)$.

(3) *$\hat{\Delta}(s) \cap Q^* \neq \emptyset$ for every path in β with ends a, b.*

Suppose there is a path Q in β with ends a, b such that $\hat{\Delta}(s) \cap Q^* = \emptyset$. Since there is no component D of $\beta \setminus N[s]$ such that $a, b \in N[D]$, it follows that $N_\beta(s) \cap Q^* \neq \emptyset$. Consequently, there is an $i \in \{1, 2\}$ and $t \in N_T(t_i) \setminus \{t_i'\}$ such that $N_\beta(s) \cap Q^* \cap \text{Conn}(t_i, t) \neq \emptyset$. Since Q is a path from a to b in G, it follows that $Q^* \cap (\chi(t_1) \cup \chi(t_2)) \neq \emptyset$. Since Q^* is connected and $\text{adh}(t_i, t)$ separates $G_{t_i \to t} \setminus \chi(t_i)$ from $(\chi(t_1) \cup \chi(t_2)) \setminus \chi(t)$, we deduce that $Q^* \cap \text{adh}(t_i, t) \neq \emptyset$. But since s has a neighbor in $\text{Conn}(t_i, t)$, it follows that $\text{adh}(t_i, t) \subseteq \hat{\Delta}(s)$, a contradiction. This proves (3).

(4) *$P^* \cap \hat{\Delta}(s) \neq \emptyset$ for every path P of G with ends a, b.*

Let P be a path in G with ends a, b. We prove by induction on $|P \setminus \beta|$ that $\hat{\Delta}(s) \cap P^* \neq \emptyset$. If $P \subseteq \beta$, the claim follows from (3), and this is the base case.

Let $p \in P^* \setminus \beta$ and let $t_0 \in T$ such that $p \in \chi(t_0)$. Suppose first that t_0 belongs to the component T' of $T \setminus \{t_1, t_2\}$ such that $t_1' \in T'$. Then $t_2' \in T'$. Since P^* is connected and $P^* \cap (\chi(t_1) \cup \chi(t_2)) \neq \emptyset$, it follows that $P^* \cap (\text{adh}(t_1, t_1') \cup \text{adh}(t_2, t_2')) \neq \emptyset$, and so $P^* \cap \hat{\Delta}(s) \neq \emptyset$. Thus we may assume that $p \in G_{t_1 \to t} \setminus \chi(t_1)$ for some $t \in N_T(t_1) \setminus \{t_1'\}$. Since P^* is connected and $P^* \cap (\chi(t_1) \cup \chi(t_2)) \neq \emptyset$, it follows that $P^* \cap \text{adh}(t_1, t) \neq \emptyset$. Write $P = p_1 \text{-} \cdots \text{-} p_l$ where $p_1 = a$ and $p_l = b$. Let i be minimum and j maximum such that $p_i \in \text{adh}(t_1, t)$. Since $p \in G_{t_1 \to t} \setminus \chi(t_1)$, it follows that $i < j - 1$. By Theorem 4.4(2) there is a path R form p_i to p_j with $R^* \subseteq \text{Conn}(t_1, t)$. Then $P' = p_1 \text{-} \cdots \text{-} p_i \text{-} R \text{-} p_j \text{-} \ldots \text{-} p_l$ is a path from a to b in G with $|P' \setminus \beta| < |P \setminus \beta|$. Inductively, $\hat{\Delta}(s) \cap P'^* \neq \emptyset$. But $\hat{\Delta}(s) \subseteq \chi(t_1) \cup \chi(t_2)$ and $P'^* \cap (\chi(t_1) \cup \chi(t_2)) \subseteq P^* \cap (\chi(t_1) \cup \chi(t_2))$, and so $\hat{\Delta}(s) \cap P^* \neq \emptyset$. This proves (4).

Since ab is a $(4m+2)(m-1)$-banana in G, and since $\hat{\Delta}(s) < (4m+2)(m-1)$, (4) leads to a contradiction. This completes the proof.

■

Recall that a *separation* of β is a triple (X, Y, Z) of pairwise disjoint subsets of β with $X \cup Y \cup Z = \beta$ such that X is anticomplete to Z. We are now ready to move on to star cutsets. We will work in the graph β and take advantage of its special properties. At the end of the section we will explain how to connect our results back to the graph G that we are interested in.

As in other papers in the series, we associate a certain unique star separation to every vertex of $S \cap \mathrm{Hub}(\beta)$. The separation here is chosen differently from the way it was done in the past, but the behavior we observe is similar. The reason for the difference is that unlike in [2] or [6], our here goal is to disconnect two given vertices, rather than find a "balanced separator" in the graph (more on this in Section 10).

Let $v \in S \cap \mathrm{Hub}(\beta)$. By Theorem 4.5, there is a component D of $\beta \setminus N[v]$ such that $a, b \in N[D]$. Since $s \notin S_{bad}$, it follows that s is not complete to $\{a, b\}$; consequently $D \cap \{a, b\} \neq \emptyset$, and so D is unique. Let $B(v)$ be the unique component of $\beta \setminus N[v]$ such that $a, b \in N[B(v)]$ (this choice of $B(v)$ is different from what we have done in earlier papers). Let $C(v) = N(B(v)) \cup \{v\}$, and finally, let $A(v) = \beta \setminus (B(v) \cup C(v))$. Then $(A(v), C(v), B(v))$ is the *canonical star separation of β corresponding to v*. The next lemma is a key step in the proof of the fact that decomposing by canonical star separations makes the graph simpler. We show:

Lemma 4.6. *The vertex v is not a hub of $\beta \setminus A(v)$.*

Proof. Suppose that (H, v) is a proper wheel in $\beta \setminus A(v)$. Then $H \subseteq N[B(v)]$, contrary to Theorem 3.1. This proves Lemma 4.6. ■

We need just a little more set up. Let $\mathcal{O}$ be a linear order on $S \cap \mathrm{Hub}(\beta)$. Following [4], we say that two vertices of $S \cap \mathrm{Hub}(\beta)$ are *star twins* if $B(u) = B(v)$, $C(u) \setminus \{u\} = C(v) \setminus \{v\}$, and $A(u) \cup \{u\} = A(v) \cup \{v\}$. (Note that every two of these conditions imply the third.)

Let $\leq_A$ be a relation on $S \cap \mathrm{Hub}(\beta)$ defined as follows:

$$x \leq_A y \quad \text{if} \quad \begin{cases} x = y; \\ x \text{ and } y \text{ are star twins and } \mathcal{O}(x) < \mathcal{O}(y); \text{ or} \\ x \text{ and } y \text{ are not star twins and } y \in A(x). \end{cases}$$

Note that if $x \leq_A y$, then either $x = y$, or $y \in A(x)$.

The conclusion of the next lemma is the same as of Lemma 6.2 from [5], but the assumptions are different.

Lemma 4.7. *If $y \in A(x)$, then $A(y) \cup \{y\} \subseteq A(x) \cup \{x\}$.*

Proof. Since $C(y) \subseteq N[y]$ and y is anticomplete to $B(x)$, we have $B(x) \subseteq G \setminus N[y]$. Since $B(x)$ is connected, there is a component D of $G \setminus N[y]$ such that $B(x) \subseteq D$. Since $x \notin S_{bad}$, it follows that $\{a, b\} \cap B(x) \neq \emptyset$, and so $D = B(y)$. Let $v \in C(x) \setminus \{x\}$. Then v has a neighbor in $B(x)$ and thus in $B(y)$. If $v \in N[y]$, then $v \in C(y)$. If $v \notin N[y]$, then $v \in B(y)$. It follows that $C(x) \setminus \{x\} \subseteq C(y) \cup B(y)$. But now $A(y) \setminus \{x\} \subseteq A(x)$, as required. This proves Lemma 4.7. ■

The proofs of the next two lemmas are identical to the proofs of Lemmas 6.3 and Lemma 6.4 in [5] (using Lemma 4.7 instead of Lemma 6.2 of [5]), and we omit them.

Lemma 4.8. *$\leq_A$ is a partial order on $S \cup \mathrm{Hub}(\beta)$.*

In view of Lemma 4.8, let $\mathrm{Core}(S')$ be the set of all $\leq_A$-minimal elements of $S \cap \mathrm{Hub}(\beta)$.

Lemma 4.9. *Let $u, v \in \mathrm{Core}(S')$. Then $A(u) \cap C(v) = C(u) \cap A(v) = \emptyset$.*

We have finally reached our goal: we can define a subgraph of G that is simpler than G itself, but carries all the information we need. Define

$$\beta^A(S') = \bigcap_{v \in \mathrm{Core}(S')} (B(v) \cup C(v)).$$

The next theorem summarizes the important aspects of the behavior of $\beta^A(S')$.

Theorem 4.10. *The following hold:*

1. *For every $v \in \mathrm{Core}(S')$, we have $C(v) \subseteq \beta^A(S')$.*
2. *For every $v \in \mathrm{Core}(S')$, $|C(v) \cap (\chi(t_1) \cup \chi(t_2))| \leq 4m(m-1)$.*
3. *For every $v \in \mathrm{Core}(S')$, $|\Delta(C(v))| \leq 4m(m-1)$.*
4. *For every component D of $\beta \setminus \beta^A(S')$, there exists $v \in \mathrm{Core}(S')$ such that $D \subseteq A(v)$. Further, if D is a component of $\beta \setminus \beta^A(S')$ and $v \in \mathrm{Core}(S')$ such that $D \subseteq A(v)$, then $N_\beta(D) \subseteq C(v)$.*
5. *$S' \cap \mathrm{Hub}(\beta^A(S')) = \emptyset$.*

Proof. (1) is immediate from Lemma 4.9, and (2) follows from Lemma 4.5.

Next we prove (3). By Lemma 4.5(1), it follows that $v \in \chi(t_1) \cup \chi(t_2)$. Observe that if $t \in T \setminus \{t_1, t_2, t_1', t_2'\}$ and $(G_{t_i \to t} \setminus (\chi(t_1) \cup \chi(t_2)) \cap C(v) \neq \emptyset$ for some $i \in \{1, 2\}$, then $v \in \chi(t_1) \cap \chi(t)$ or $v \in \chi(t_2) \cap \chi(t)$. It follows that for $i \in \{1, 2\}$, $\mathrm{adh}(t_i, t) \subseteq \Delta(C(v))$ only if $v \in \mathrm{adh}(t_i, t)$. Consequently, $|\Delta(C(v))| \leq \deg_{\hat{\chi}(t_1)}(v) + \deg_{\hat{\chi}(t_2)}(v)$, and so (3) follows from Lemma 4.5(3).

Next we prove (4). Let D be a component of $\beta \setminus \beta^A(S')$. Since $\beta \setminus \beta^A(S') = \bigcup_{v \in \mathrm{Core}(S')} A(v)$, there exists $v \in \mathrm{Core}(S')$ such that $D \cap A(v) \neq \emptyset$. If $D \setminus A(v) \neq \emptyset$, then, since D is connected, it follows that $D \cap N(A(v)) \neq \emptyset$; but then $D \cap C(v) \neq \emptyset$, contrary to (1). Since $N_\beta(D) \subseteq \beta^A(S')$ and $N_\beta(D) \subseteq A(v) \cup C(v)$, it follows that $N_\beta(D) \subseteq C(v)$. This proves (4).

To prove (5), let $u \in S' \cap \mathrm{Hub}(\beta^A(S'))$. Since $\beta^A(S') \subseteq \beta$, we deduce that $u \notin S_{bad}$, and so $u \in S \cap \mathrm{Hub}(\beta)$. By Theorem 4.5(1), we have that $u \in \chi(t_1) \cup \chi(t_2)$. By Lemma 4.6, it follows that $\beta^A(S') \not\subseteq B(u) \cup C(u)$, and therefore $u \notin \mathrm{Core}(S')$. But then $u \in A(v)$ for some $v \in \mathrm{Core}(S')$, and so $u \notin \beta^A(S')$, a contradiction. This proves (5) and completes the proof of Theorem 4.10. ■

We now explain how $\beta^A(S')$ is used. In the course of the proof of Theorem 1.3, we will inductively obtain a small cutset separating a from b in $\beta^A(S')$.

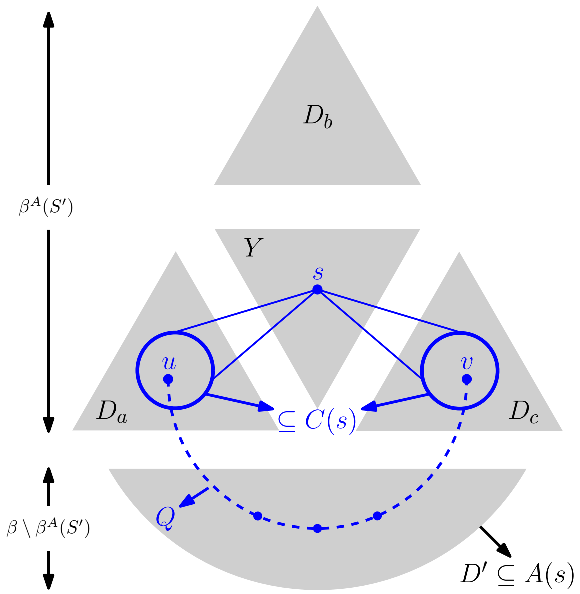

FIGURE 2. Proof of Theorem 4.11.

The next theorem allows us to transform this cutset into a cutset separating a from b in β.

Theorem 4.11. *Let (X, Y, Z) be a separation of $\beta^A(S')$ such that $a \in X$ and $b \in Z$. Let $Y' = (Y \cup \bigcup_{s \in Y \cap \text{Core}(S')} C(s)) \setminus \{a, b\}$. Then*

(1) Y' separates a from b in β.
(2) $|Y' \cap (\chi(t_1) \cup \chi(t_2))| \leq |Y| + 4m(m-1)|Y \cap \text{Core}(S')|$.
(3) $|\Delta(Y') \setminus \Delta(Y)| \leq 4m(m-1)|Y \cap \text{Core}(S')|$.

Proof. Suppose that Y' does not separate a from b in β. Let P be a path from a to b in $\beta \setminus Y'$. Let D_a, D_b be the components of $\beta^A(S') \setminus Y$ such that $a \in D_a$ and $b \in D_b$. Since the first vertex of P is in D_a and the last vertex of P is in D_b, it follows that there is a subpath Q of P such that Q has ends u, v and:

- The vertex u is in D_a;
- $Q^* \cap \beta^A(S') = \emptyset$;
- The vertex v is in $\beta^A(S') \setminus D_a$.

See Figure 2. Since $Y \subseteq Y'$, it follows that v is in a component D_c of $\beta^A(S') \setminus Y$ with $D_c \neq D_a$ (possibly $D_c = D_b$). This implies that $uv \notin E(G)$, and so $Q^* \neq \emptyset$.

By Theorem 4.10(4), there is an $s \in \text{Core}(S') \subseteq \beta^A(S')$ such that $Q^* \subseteq A(s)$ and $u, v \in C(s)$. Since D_a, D_c are distinct components of $\beta^A(S') \setminus Y$, and since $N_{\beta^A(S')}(s)$ meets both D_a and D_b, it follows that $s \in Y$. Therefore $C(s) \setminus \{a, b\} \subseteq Y'$, and so $u, v \in \{a, b\}$. Since $u \neq v$, it follows that $u = a$ and $b = v$, and so $a, b \in N(s)$, contrary to the fact that $s \notin S_{bad}$. This proves the first assertion of the theorem.

The second assertion follows from Theorem 4.10(2), and the third assertion follows from Theorem 4.10(3). ■

We finish this section with a theorem that allows us to transform a cutset separating a from b in β into a cutset separating a from b in G.

Theorem 4.12. *Assume that G does not admit a clique cutset. Let (X, Y, Z) be a separation of β such that $a \in X$ and $b \in Z$. Let*

$$Y' = (S_{bad} \cup \text{adh}(t_1, t_1') \cup \text{adh}(t_2, t_2') \cup (Y \cap (\chi(t_1) \cup \chi(t_2))) \cup \Delta(Y)) \setminus \{a, b\}.$$

Then Y' separates a from b in G.

Proof. Suppose not. Let D_a, D_b be the components of $\beta \setminus Y$ such that $a \in D_a$ and $b \in D_b$. Since Y' does not separate a from b in G, and $Y' \cap \{a, b\} = \emptyset$, there is a component D of $G \setminus Y'$ such that $a, b \in D$. Let P be a path from a to b with $P \subseteq D$.

As before, it follows that there is a subpath Q of P such that Q has ends u, v and:

- The vertex u is in D_a;
- $Q^* \cap \beta = \emptyset$;
- The vertex v is in $\beta \setminus D_a$.

We first consider the case that $v \in Y$ (and so $v \in P^*$). Then $v \in \text{Conn}(t_i, t) \setminus (\chi(t_1) \cup \chi(t_2))$ for some $i \in \{1, 2\}$ and $t \in N_T(t_i) \setminus \{t_i'\}$. It follows that $t \in \delta_i(Y)$, and so $\text{adh}(t_i, t) \setminus \{a, b\} \subseteq \Delta(Y) \setminus \{a, b\} \subseteq Y'$. It follows that $P^* \cap \text{adh}(t_i, t) = \emptyset$, and from Theorem 2.6, we know that $P^* \cap (\chi(t_1) \cup \chi(t_2)) \neq \emptyset$. Therefore, $P^* \cap (G_{t_i \to t} \setminus (\chi(t_1) \cup \chi(t_2))) = \emptyset$, contrary to the fact that $v \in \text{Conn}(t_i, t) \setminus (\chi(t_1) \cup \chi(t_2))$. This is a contradiction, and proves that $v \notin Y$.

Therefore, there is a component $D_c \neq D_a$ (possibly $D_b = D_c$) of $\beta \setminus Y$ such that $v \in D_c$. Since there are no edges between D_a and D_c, it follows that $Q^* \neq \emptyset$.

Consequently, there is a component $\tilde{T}$ of $T \setminus \{t_1, t_2\}$ such that $Q^* \subseteq (\bigcup_{t \in \tilde{T}} \chi(t)) \setminus (\chi(t_1) \cup \chi(t_2))$. Let $\tilde{D}$ be the component of $G \setminus (\chi(t_1) \cup \chi(t_2))$ such that $Q^* \subseteq \tilde{D}$. By Theorem 2.6, it follows that $|N(\tilde{D}) \cap \{a, b\}| \leq 1$.

Suppose first that $t_1' \in \tilde{T}$. Then $t_1' \neq t_2$, and $t_2' \neq t_1$, and $t_2' \in \tilde{T}$. Since $N_G(\tilde{D}) \subseteq \text{adh}(t_1, t_1') \cup \text{adh}(t_2, t_2')$, it follows that $N_{G \setminus Y'}(\tilde{D}) \subseteq \{a, b\}$, and so $|N_{G \setminus Y'}(\tilde{D})| \leq 1$. Since P is a path with ends $a, b \notin \tilde{D}$ and with $P \subseteq G \setminus Y'$, it follows that $P \cap \tilde{D} = \emptyset$, and so $Q \cap \tilde{D} = \emptyset$. Similarly, $t_2' \notin \tilde{T}$.

We deduce that exactly one of t_1, t_2 has a neighbor in $\tilde{T}$ (unless $t_1 = t_2$), and this neighbor is unique; denote it by t. By symmetry, we may assume that $tt_1 \in E(T)$, and so $\tilde{T} = T_{t_1 \to t}$. By Theorem 2.5, we deduce that $\tilde{D} = G_{t_1 \to t} \setminus \chi(t_1)$. Since $u \in D_a \cap N(Q^*) \subseteq \beta \cap N[\tilde{D}]$, it follows that $u \in \text{Conn}(t_1, t)$. Similarly, we have $v \in \text{Conn}(t_1, t)$. Since $\text{Conn}(t_1, t) \setminus \chi(t_1)$ is connected and $N(\text{Conn}(t_1, t) \setminus \chi(t_1)) = \chi(t_1) \cap \chi(t)$ by Theorem 4.4(2), it follows that $\text{Conn}(t_1, t)$ contains a path Q' with ends u, v and interior in $\text{Conn}(t_1, t) \setminus \chi(t_1)$. Since Q' is a path from D_a to D_c in β, it follows that $Y \cap Q'^* \neq \emptyset$, and so $Y \cap (\mathrm{Conn}(t_1,t) \setminus \chi(t_1)) \neq \emptyset$. Therefore, $Y \cap (G_{t_1 \to t} \setminus \chi(t_1)) \neq \emptyset$, which implies that $\mathrm{adh}(t_1,t) \setminus \{a,b\} \subseteq Y'$. It follows that $P^* \cap \mathrm{adh}(t_1,t) = \emptyset$, and from Theorem 2.6, we know that $P^* \cap (\chi(t_1) \cup \chi(t_2)) \neq \emptyset$. Therefore, $P^* \cap (G_{t_1 \to t} \setminus \chi(t_1)) = \emptyset$, contrary to the fact that $\emptyset \neq Q^* \subseteq G_{t_1 \to t} \setminus \chi(t_1)$. This is a contradiction, and concludes the proof. ■

5. Connectifiers

We start this section by describing minimal connected subgraphs containing the neighbors of a large number of vertices from a given stable set. We then use this result to deduce what a pair of two such subgraphs can look like (for the same stable set) assuming that they are anticomplete to each other and the graph in question is in $\mathcal{C}$. This generalizes results from [2] and [3].

What follows is mostly terminology from [3], but there are also some new notions. Let G be a graph, let $P = p_1\text{-}\cdots\text{-}p_n$ be a path in G and let $X = \{x_1,\dots,x_k\} \subseteq G \setminus P$. We say that (P,X) is an *alignment* if $N_P(x_1) = \{p_1\}$, $N_P(x_k) = \{p_n\}$, every vertex of X has a neighbor in P, and there exist $1 < j_2 < \cdots < j_{k-1} < j_k = n$ such that $N_P(x_i) \subseteq p_{j_i}\text{-}P\text{-}p_{j_{i+1}-1}$ for $i \in \{2,\dots,k-1\}$. We also say that $x_1,\dots,x_k$ is *the order on X given by the alignment* (P,X). An alignment (P,X) is *wide* if each of $x_2,\dots,x_{k-1}$ has at least two non-adjacent neighbors in P, *spiky* if each of $x_2,\dots,x_{k-1}$ has a unique neighbor in P and *triangular* if each of $x_2,\dots,x_{k-1}$ has exactly two neighbors in P and they are adjacent. An alignment is *consistent* if it is wide, spiky or triangular.

By a *caterpillar* we mean a tree C with maximum degree three such that there exists a path P of C such that all vertices of degree 3 in C belong to P. We call a minimal such path P the *spine* of C. By a *subdivided star* we mean a graph isomorphic to a subdivision of the complete bipartite graph $K_{1,\delta}$ for some $\delta \geq 3$. In other words, a subdivided star is a tree with exactly one vertex of degree at least three, which we call its *root*. For a graph H, a vertex v of H is said to be *simplicial* if $N_H(v)$ is a clique. We denote by $\mathcal{Z}(H)$ the set of all simplicial vertices of H. Note that for every tree T, $\mathcal{Z}(T)$ is the set of all leaves of T. An edge e of a tree T is said to be a *leaf-edge* of T if e is incident with a leaf of T. It follows that if H is the line graph of a tree T, then $\mathcal{Z}(H)$ is the set of all vertices in H corresponding to leaf-edges of T.

Let H be a graph that is either a path, or a caterpillar, or the line graph of a caterpillar, or a subdivided star with root r, or the line graph of a subdivided star with root r. We define an induced subgraph of H, denoted by $P(H)$, which we will use throughout the paper. If H is a path, we let $P(H) = H$. If H is a caterpillar, we let $P(H)$ be the spine of H. If H is the line graph of a caterpillar C, let $P(H)$ be the path in H consisting of the vertices of H that correspond to the edges of the spine of C. If H is

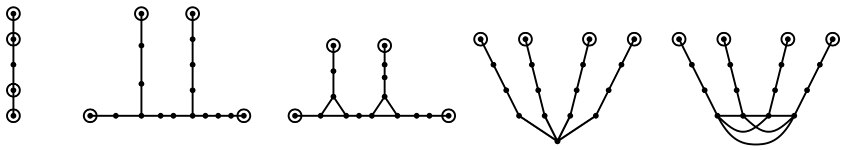

FIGURE 3. Outcomes of Theorem 5.2. Circled nodes are the vertices in $H \cap N(Y')$.

a subdivided star with root r, let $P(H) = \{r\}$. It H is the line graph of a subdivided star S with root r, let $P(H)$ be the clique of H consisting of the vertices of H that correspond to the edges of S incident with r. The *legs* of H are the components of $H \setminus P(H)$.

Next, let H be a caterpillar or the line graph of a caterpillar and let S be a set of vertices disjoint from H such that every vertex of S has a unique neighbor in H and $H \cap N(S) = \mathcal{Z}(H)$. Let X be the set of vertices of $H \setminus P(H)$ that have neighbors in $P(H)$. Then the neighbors of elements of X appear in $P(H)$ in order (there may be ties at the ends of $P(H)$, which we break arbitrarily); let $x_1, \dots, x_k$ be the corresponding order on X. Now, order the vertices of S as $s_1, \dots, s_k$ where s_i has a neighbor in the leg of H containing x_i for $i \in \{1, \dots, k\}$. We say that $s_1, \dots, s_k$ is *the order on S given by* (H, S).

Next, let H be an induced subgraph of G that is a caterpillar, or the line graph of a caterpillar, or a subdivided star or the line graph of a subdivided star and let $X \subseteq G \setminus H$ be such that every vertex of X has a unique neighbor in H and $H \cap N(X) = \mathcal{Z}(H)$. We say that (H, X) is a *consistent connectifier* and it is *spiky, triangular, stellar, or clique* respectively. If H is a single vertex and $X \subseteq N(H)$, we also call (H, X) a *stellar connectifer.* If H is a subdivided star, a singleton or the line graph of a subdivided star, we say that (H, X) is a *concentrated* connectifier.

The following was proved in [3]:

Theorem 5.1 (Abrishami, Alecu, Chudnovsky, Hajebi, Spirkl [3])**.** *For every integer $h \geq 1$, there exists an integer $\nu = \nu(h) \geq 1$ with the following property. Let G be a connected graph with no clique of cardinality h. Let $S \subseteq G$ such that $|S| \geq \nu$, $G \setminus S$ is connected and every vertex of S has a neighbor in $G \setminus S$. Then there is a set $\tilde{S} \subseteq S$ with $|\tilde{S}| = h$ and an induced subgraph H of $G \setminus S$ for which one of the following holds.*

- *H is a path and every vertex of $\tilde{S}$ has a neighbor in H; or*
- *H is a caterpillar, or the line graph of a caterpillar, or a subdivided star. Moreover, every vertex of $\tilde{S}$ has a unique neighbor in H and $H \cap N(\tilde{S}) = \mathcal{Z}(H)$.*

We now prove a version of Theorem 5.1 that does not assume a bound on the clique number. This result may be of independent use in the future (See Figure 3 for a depiction of the outcomes).

Theorem 5.2. *For every integer $h \geq 1$, there exists an integer $\mu = \mu(h) \geq 1$ with the following property. Let G be a connected graph. Let $Y \subseteq G$ such that $|Y| \geq \mu$, $G \setminus Y$ is connected and every vertex of Y has a neighbor in $G \setminus Y$. Then there is a set $Y' \subseteq Y$ with $|Y'| = h$ and an induced subgraph H of $G \setminus Y$ for which one of the following holds.*

- *H is a path and every vertex of Y' has a neighbor in H; or*
- *H is a caterpillar, or the line graph of a caterpillar, or a subdivided star or the line graph of a subdivided star. Moreover, every vertex of Y' has a unique neighbor in H and $H \cap N(Y') = \mathcal{Z}(H)$.*

Proof. Let $\mu(h) = \nu(h+1)$ where ν is as in Theorem 5.1. Let F be a minimal connected subset of $G \setminus Y$ such that every $y \in Y$ has a neighbor in F.

(5) *If F contains a clique of size h, then there exists $H \subseteq F$ and $Y' \subseteq Y$ with $|Y'| = h$ such that (H, Y') is a clique connectifier.*

Suppose that there is a clique K of size h in F. It follows from the minimality of F that for every $k \in K$ one of the following holds:

- There is $y(k) \in Y$ such that $y(k)$ is anticomplete to $F \setminus k$; in this case set $C(k, y(k)) = \emptyset$.
- $F \setminus \{k\}$ is not connected, and for every component C of $F \setminus k$ there is $y \in Y$ such that y is anticomplete to $F \setminus (C \cup \{k\})$. Since K is a clique, there is a component C of $F \setminus k$ such that $K \cap C = \emptyset$. Let $y(k)$ be a vertex of Y that is anticomplete to $F \setminus (C \cup \{k\})$; write $C(k, y(k)) = C$.

Let Y' be the set of all vertices $y(k)$ as above. For every $k \in K$, let H_k be a path from k to $y(k)$ with $H_k^* \subseteq C(k, y(k))$. Write $H = \bigcup_{k \in K} H_k$. We will show that $|Y'| = h$ and (H, Y') is a clique connectifier. It follows from the definition of H_k that every vertex of Y' has a neighbor in H.

Next we claim that if $k \neq k'$, then H_k is disjoint from and anticomplete to $H_{k'}$. Recall that $H_k \subseteq C(k, y(k)) \cup \{k\}$, and $y(k)$ is anticomplete to $F \setminus (C(k, y(k)) \cup \{k\})$. It follows that $H_k \cap K = \{k\}$, and so $k \notin H_{k'}$ and $k' \notin H_k$. Let D be the component of $F \setminus k$ such that $k' \in D$. By the definition of $C(k, y(k))$, we have that $D \neq C(k, y(k))$. Since $H_{k'}$ is connected and $k \notin H_{k'}$, it follows that $H_{k'} \subseteq D$. Consequently, H_k and $H_{k'}$ are disjoint and anticomplete to each other. Since $y(k)$ has no neighbor in D, we deduce that $y(k)$ is anticomplete to $H_{k'}$, and in particular $y(k) \neq y(k')$. Similarly, $y(k')$ is anticomplete to H_k. This proves the claim.

Now it follows from the claim that if $k \neq k'$, then $y(k) \neq y(k')$. Consequently, $|Y'| = |K| = h$, and (H, Y') is a clique connectifier. This proves (5).

By (5), we may assume that F is K_h-free. Since S is a stable set, it follows that $F \cup S$ is K_{h+1}-free. Now the result follows from Theorem 5.1 applied to $F \cup S$. ■

Now we move to two anticomplete connected subgraphs and prove the following:

Theorem 5.3. *For every integer $x \geq 1$, there exists an integer $\phi = \phi(x) \geq 1$ with the following property. Let $G \in \mathcal{C}$ and assume that $V(G) = D_1 \cup D_2 \cup Y$ where*

- *Y is a stable set with $|Y| = \phi$,*
- *D_1 and D_2 are components of $G \setminus Y$, and*
- *$N(D_1) = N(D_2) = Y$.*

Then there exist $X \subseteq Y$ with $|X| = x$, $H_1 \subseteq D_1$ and $H_2 \subseteq D_2$ (possibly with the roles of D_1 and D_2 reversed) such that either:

(1) Not both (H_1, X) and (H_2, X) are alignments, and
   - *(H_1, X) is a triangular connectifier or a clique connectifier or a triangular alignment; and*
   - *(H_2, X) is a stellar connectifier, or a spiky connectifier, or a spiky alignment or a wide alignment.*

   or

(2) Both (H_1, X) and (H_2, X) are alignments, and at least one of (H_1, X) and (H_2, X) is not a spiky alignment.

Moreover, if neither of $(H_1, X), (H_2, X)$ is a concentrated connectifier, then the orders given on X by (H_1, X) and by (H_2, X) are the same.

In this paper we do not need the full generality of Theorem 5.3; we are only interested in two special cases: when D_1 is a path and when the clique number of G is bounded. However, it is easier to prove the more general result first using the symmetry between D_1 and D_2, and then use it to handle the special cases. We also believe that Theorem 5.3 will be useful in the future.

We start by recalling a well known theorem of Erdős and Szekeres [20].

Theorem 5.4 (Erdős and Szekeres [20])**.** *Let $x_1, \ldots, x_{n^2+1}$ be a sequence of distinct reals. Then there exists either an increasing or a decreasing $(n+1)$-sub-sequence.*

We start with two lemmas.

Lemma 5.5. *Let $G \in \mathcal{C}$ and assume that $V(G) = H_1 \cup H_2 \cup X$ where X is a stable set with $|X| \geq 3$ and H_1 is anticomplete to H_2. Suppose that for $i \in \{1, 2\}$, the pair (H_i, X) is a consistent alignment, or a consistent connectifier. Assume also that if neither of $(H_1, X), (H_2, X)$ is concentrated, then the orders given on X by (H_1, X) and by (H_2, X) are the same. Then (possibly switching the roles of H_1 and H_2), we have that either:*

(1) Not both (H_1, X) and (H_2, X) are alignments, and
   - *(H_1, X) is a triangular connectifier or a clique connectifier or a triangular alignment; and*
   - *(H_2, X) is a stellar connectifier, or a spiky connectifier, or a spiky alignment or a wide alignment.*

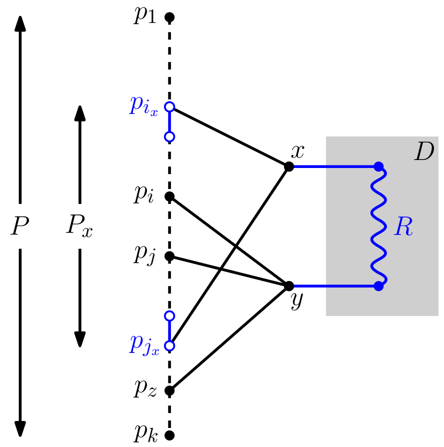

FIGURE 4. Proof of Lemma 5.6.

or

(2) Both (H_1, X) and (H_2, X) are alignments, and at least one of (H_1, X) and (H_2, X) is not a spiky alignment, and at least one of (H_1, X) and (H_2, X) is not a triangular alignment.

Proof. If at least one $(H_i, X) \subseteq \{(H_1, X), (H_2, X)\}$ is not a concentrated connectifier, we let $x_1, \dots, x_k$ be the order given on X by (H_i, X). If both of (H_i, X) are concentrated connectifiers, we let $x_1, \dots, x_k$ be an arbitrary order on X. Let H be the unique hole contained in $H_1 \cup H_2 \cup \{x_1, x_k\}$. For $j \in \{1, 2\}$ and $i \in \{1, \dots, k\}$, if H_j is a connectifier, let D_i^j be the leg of H_j containing a neighbor of x_i; and if H_j is an alignment let $D_i^j = \emptyset$.

Suppose first that (H_1, X) is a triangular alignment, a clique connectifier or a triangular connectifier. If (H_2, X) is a triangular alignment, a clique connectifier or a triangular connectifier, then for every $i \in \{2, \dots, k-1\}$, the graph $H \cup D_i^1 \cup D_i^2 \cup \{x_i\}$ is either a prism or an even wheel with center x_i, a contradiction. This proves that (H_2, X) is a stellar connectifier, or a spiky connectifier, or a spiky alignment, or a wide alignment, as required.

Thus we may assume that for $i \in \{1, 2\}$, the pair (H_i, X) is a stellar connectifier, or a spiky connectifier, or a spiky alignment, or a wide alignment. If (H_1, X) is a stellar connectifier or a spiky connectifier, then for every $x_i \in X \setminus \{x_1, x_k\}$, the graph $H \cup D_i^1 \cup D_i^2 \cup \{x_i\}$ contains a theta, a contradiction.

It follows that for $i \in \{1, 2\}$, the pair (H_i, X) is an alignment. We may assume that (H_1, X) and (H_2, X) are either both spiky alignments or both triangular alignments, for otherwise the second outcome of the theorem holds. But now for every $x_i \in X \setminus \{x_1, x_k\}$, the graph $H \cup \{x_i\}$ is either a theta or an even wheel, a contradiction. ■

Lemma 5.6. *Let G be a theta-free graph and let $P = p_1\text{-}\cdots\text{-}p_k$ be a path in G. Let D be a connected subset of G such that D is anticomplete to P. Let $x, y \in N(P) \cap N(D)$ such that xy is not an edge. Assume that* *each of x and y has at least two non-adjacent neighbors in P. Let i_x be minimum and j_x be maximum such that x is adjacent to p_{i_x} and p_{j_x} and write $P_x = p_{i_x}$-P-p_{j_x}. Suppose that y has a neighbor in P_x. Then y has a neighbor in $N_{P_x}[p_{i_x}] \cup N_{P_x}[p_{j_x}]$.*

Proof. Suppose that y is anticomplete to $N_P[p_{i_x}] \cup N_P[p_{j_x}]$. Let R be a path from x to y with $R^* \subseteq D$. (See Figure 4.) Let $i \in \{i_x, \dots, j_x\}$ be minimum and $j \in \{i_x, \dots, j_x\}$ be maximum such that y is adjacent to p_i, p_j. Then there there is a path P_i from x to y with interior in p_{i_x}-P-p_i, and a path P_j from x to y with interior in p_j-P-p_{j_x}. If $j > i+1$, then $P_i \cup P_j \cup R$ is a theta with ends x, y, a contradiction; therefore $j \leq i+1$. Since y has at least two non-adjacent neighbors in P, it follows that y has a neighbor p_z in $P \setminus P_x$. We may assume that $z > j_x$, and therefore there is a path Q from x to y with $Q^* \subseteq p_{j_x}$-P-p_z. Since y is anticomplete to $N_{P_x}[p_{i_x}] \cup N_{P_x}[p_{j_x}]$, we have that $j < j_x - 1$, and so P_i^* is anticomplete to Q^*. But now $P_i \cup Q \cup R$ is a theta with ends x, y, a contradiction. ■

We can now prove Theorem 5.3. The proof follows along the lines of [3], but the assumptions are different.

Proof. Let $z = 12^2 36^2 x^6$ and let $\phi(x) = \mu(\mu(z))$, where μ is as in Theorem 5.2. Applying Theorem 5.2 twice, we obtain a set $Z \subseteq Y$ with $|Z| = z$ and $H_i \subseteq D_i$ such that either

- H_i is a path and every vertex of Z has a neighbor in H_i; or
- (H_i, X) is a consistent connectifier

for every $i \in \{1, 2\}$.

(6) *Let $i \in \{1, 2\}$ and $y \in \mathbb{N}$. If H_i is a path and every vertex of Z has a neighbor in H_i, then either some vertex of H_i has y neighbors in Z, or there exists $Z' \subseteq Z$ with $|Z'| \geq \frac{|Z|}{12y}$ and a subpath H_i' of H_i such that (H_i', Z') is a consistent alignment.*

Let $H_i = h_1$-$\cdots$-h_k. We may assume that H_i is chosen minimal satisfying Theorem 5.2, and so there exist $z_1, z_k \in Z$ such that $N_{H_i}(z_j) = \{h_j\}$ for $j \in \{1, k\}$.

We may assume that $|N_Z(h)| < y$ for every $h \in H_i$. Let Z_1 be the set of vertices in Z with exactly one neighbor in H_i. Suppose that $|Z_1| \geq \frac{|Z|}{3}$. It follows that Z_1 contains a set Z' with $|Z'| \geq \frac{|Z_1|}{y} \geq \frac{|Z|}{3y}$ such that no two vertices in Z' have a common neighbor in H_i. We may assume that $z_1, z_k \in Z'$. Therefore, (H_i, Z') is a spiky alignment, as required. Thus we may assume that $|Z_1| < \frac{|Z|}{3}$.

Next, let Z_2 be the set of vertices in $z \in Z$ such that either $z \in \{z_1, z_k\}$ or z has exactly two neighbors in H_i, and they are adjacent. Suppose that $|Z_2| \geq \frac{|Z|}{3}$. By choosing Z' greedily, it follows that Z_2 contains a subset Z' with the following specifications:

- $z_1, z_k \in Z'$;

- $|Z'| \geq \frac{|Z_2|}{2y} \geq \frac{|Z|}{6y}$; and
- no two vertices in Z' have a common neighbor in H_i.

But then (H_i, Z') is a triangular alignment, as required. Thus we may assume that $|Z_2| < \frac{|Z|}{3}$.

Finally, let $Z_3 = \{z_1, z_k\} \cup (Z \setminus (Z_1 \cup Z_2))$. It follows that $|Z_3| \geq \frac{|Z|}{3}$. Let R be a path from z_1 to z_k with $R^* \subseteq H_{3-i}$, and let H be the hole z_1-H_i-z_k-R-z_1. Let $z \in Z_3 \setminus \{z_1, z_k\}$. Define $H_i(z)$ to be the minimal subpath of H_i containing $N_{H_i}(z)$. Let the ends of $H_i(z)$ be a_z and b_z. Next let $Bad(z) = N_{H_i(z)}[a_z, b_z]$. Since H_i is a path, it follows that $|Bad(z)| \leq 4$ for every z. Since $N_Z(h) < y$ for every $h \in H_i$, we can greedily choose $Z' \subseteq Z_3$ with $|Z'| \geq \frac{|Z_3|}{4y} \geq \frac{|Z|}{12y}$, where $z_1, z_k \in Z'$ and such that if $z, z' \in Z'$, then z' is anticomplete to $Bad(z)$. It follows from Lemma 5.6 that $H_i(z) \cap H_i(z') = \emptyset$ for every $z, z' \in Z'$, and so (H_i, Z') is a wide alignment. This proves (6).

(7) *There exist $Z' \subseteq Z$ with $|Z'| \geq x^2$, and $H_i' \subseteq H_i$ for $i = 1, 2$ such that (H_i', Z') is a consistent alignment or a consistent connectifier.*

If both (H_1, Z) and (H_2, Z) are consistent connectifiers, (7) holds. Thus we may assume that H_1 is a path and every vertex of Z has a neighbor in H_1. Suppose first that some $h \in H_1$ has at least $36x^2$ neighbors in Z. Let $H_1' = \{h\}$, and let $Z'' \subseteq Z \cap N(h)$ with $|Z''| = 36x^2$. If (H_2, Z) is a connectifier, we can clearly choose $H_2' \subseteq H_2$ such that (7) holds. So we may assume that H_2 is a path and every vertex of Z has a neighbor in H_2. Moreover, no vertex f of H_2 has three or more neighbors in Z', for otherwise $\{f, h\} \cup (N(f) \cap Z')$ contains a theta with ends h, f. Now by (6) applied with $y = 3$, there is a set $Z'' \subseteq Z'$ with $|Z''| \geq x^2$ such that (H_2, Z'') is a consistent alignment, and (7) holds. Similarly, we may assume that every $h \in H_2$ has fewer than $36x^2$ neighbors in Z.

Therefore, we may assume that every vertex $h \in H_1$ has strictly fewer than $36x^2$ neighbors in Z. Applying (6) with $y = 36x^2$, we conclude that there exists $Z_1 \subseteq Z$ with $|Z_1| \geq 36 \times 12x^4$ and a path $H_1' \subseteq H_1$ such that (H_1', Z_1) is a consistent alignment. If (H_2, Z) is a consistent connectifier, then (7) holds, so we may assume that H_2 is a path and every vertex of Z has a neighbor in H_2. Since every vertex $h \in H_2$ has fewer than $36x^2$ neighbors in Z, we apply (6) with $y = 36x^2$ again, and we conclude that there exists $Z' \subseteq Z_1$ with $|Z'| \geq x^2$ and a path $H_2' \subseteq H_2$ such that (H_2', Z') is a consistent alignment. This proves (7).

(8) *There exist $\hat{Z} \subseteq Z$ with $|\hat{Z}| \geq x$, and $\hat{H}_i \subseteq H_i$ for $i = 1, 2$ such that $(\hat{H}_i, \hat{Z})$ is a consistent alignment or a consistent connectifier. Moreover, if neither of $(\hat{H}_1, \hat{Z}), (\hat{H}_2, \hat{Z})$ is a concentrated connectifier, then the order given on $\hat{Z}$ by $(\hat{H}_1, \hat{Z})$ and $(\hat{H}_2, \hat{Z})$ is the same.*

Let Z', H_1', H_2' be as in (7). We may assume that neither of (H_1', Z') and (H_2', Z') is a concentrated connectifier. For $i \in \{1, 2\}$, let π_i be the order given on Z' by (H'_i, Z'). By Theorem 5.4 there exists $\hat{Z} \subseteq Z'$ such that (possibly reversing H'_i) the orders π_i restricted to $\hat{Z}$ are the same, as required. This proves (8).

Now Theorem 5.3 follows from Lemma 5.5. ■

We now refine Theorem 5.3 in the two special cases we need here. The first one is when D_1 is a path. It is useful to state this result in terms of the following definition.

Given a graph class $\mathcal{G}$, we say $\mathcal{G}$ is *amiable* if, for every integer $x \geq 2$, there exists an integer $\sigma = \sigma(x) \geq 1$ such that the following holds for every graph $G \in \mathcal{G}$. Assume that $V(G) = D_1 \cup D_2 \cup Y$ where

- Y is a stable set with $|Y| = \sigma$,
- D_1 and D_2 are components of $G \setminus Y$,
- $N(D_1) = N(D_2) = Y$,
- $D_1 = d_1\text{-}\cdots\text{-}d_k$ is a path, and
- for every $y \in Y$ there exists $i(y) \in \{1, \ldots, k\}$ such that $N(d_{i(y)}) \cap Y = \{y\}$.

Then there exist $X \subseteq Y$ with $|X| = x + 2$, $H_1 \subseteq D_1$ and $H_2 \subseteq D_2$ such that

(1) One of the following holds:
  - (H_1, X) is a triangular alignment and (H_2, X) is a stellar connectifier or a spiky connectifier;
  - (H_1, X) is a spiky alignment, and (H_2, X) is a triangular connectifier or a clique connectifier;
  - (H_1, X) is a wide alignment and (H_2, X) is a triangular connectifier or a clique connectifier; or
  - Both (H_1, X) and (H_2, X) are alignments, and at least one of (H_1, X) and (H_2, X) is not a spiky alignment, and at least one of (H_1, X) and (H_2, X) is not a triangular alignment.

(2) If (H_2, X) is not a concentrated connectifier, then the orders given on X by (H_1, X) and by (H_2, X) are the same.

(3) For every $x \in X$ except at most two, we have that $N_{D_1}(x) = N_{H_1}(x)$.

Theorem 5.7. *The class $\mathcal{C}$ is amiable.*

Proof. Let $x \geq 2$, and let $G \in \mathcal{C}, D_1 = d_1\text{-}\cdots\text{-}d_k, D_2, Y$ be as in the definition of an amiable class. Let $\sigma(x) = \phi(x+4)$ where ϕ is as in Theorem 5.3. We start with the following.

(9) *For every $d \in D_1$, we have that $|N_Y(d)| \leq 4$.*

Suppose there exists $i \in \{1, \ldots, k\}$ and $y_1, y_2, y_3, y_4, y_5 \in Y \cap N(d_i)$. We may assume that $i(y_1) < \cdots < i(y_5)$. By reversing D_1 if necessary, we may assume that $i(y_3) < i$, and so $i(y_2) < i - 1$. Let P be a path with ends y_1, y_2 and with interior in $d_{i(y_1)}\text{-}D_1\text{-}d_{i(y_2)}$. Let R be a path from y_1 to y_2

with interior in D_2. Now there is a theta in G with ends y_1, y_2 and paths P,R and y_1-d_i-y_2, a contradiction. This proves (9).

Now we apply Theorem 5.3 and obtain a set X' with $|X'| = x + 4$ satisfying one of the outcomes of Theorem 5.3. Let (H_1, X') and (H_2, X') be as in Theorem 5.3 where $H_1 \subseteq D_1$ (and so Theorem 5.3 may hold with the roles of H_1 and H_2 reversed). It follows from (9) that every vertex in D_1 has degree at most 6 in G and degree 2 in $G[D_1]$, and so (H_1, X') is an alignment. Let $z_1, \dots, z_{x+4}$ be the order on X' given by (H_1, X').

(10) *There exist at most two values of $i \in \{2, \dots, x + 3\}$ for which $N(z_i) \cap D_1 \neq N(z_i) \cap H_1$.*

Suppose there exist three such values of i. Since H_1 is a path, there exist $i, j \in \{1, \dots, k\}$ such that $H_1 = d_i$-D_1-d_j, and we may assume (by reversing D_1 if necessary) that for two values $p, q \in \{2, \dots, x + 3\}$, both z_p and z_q have neighbors in d_1-D_1-d_{i-1}. Let P be a path from z_p to z_q with $P^* \subseteq d_1$-D_1-d_{i-1}. Since $p, q > 1$, there is a path Q from z_p to z_q with $Q^* \subseteq H_1 \setminus d_i$. But now we get a theta with ends z_p, z_q and path P, Q, and a path from z_p to z_q with interior in D_2, a contradiction. This proves (10).

By (10), there exists $X \subseteq \{z_2, \dots, z_{x+3}\}$ with $|X| = x$ and such that $N_{D_1}(z) = N_{H_1}(z)$ for every $z \in Z$. We obtain that $(H_1, X \cup \{z_1, z_{x+4}\})$ and $(H_2, X \cup \{z_1, z_{x+4}\})$ satisfy the first statement of Theorem 5.7. The second statement of Theorem 5.7 follows immediately from Theorem 5.3, and the third statement holds by the choice of X. ■

The second special case is when the clique number of G is bounded, and $Y \cap \mathrm{Hub}(G) = \emptyset$; it is a result from [3].

Theorem 5.8 (Abrishami, Alecu, Chudnovsky, Hajebi, Spirkl [3]). *For every pair of integers $t, x \geq 1$, there exists an integer $\tau = \tau(t, x) \geq 1$ with the following property. Let $G \in \mathcal{C}_t$ and assume that $V(G) = D_1 \cup D_2 \cup Y$ where*

- *Y is a stable set with $|Y| = \tau$,*
- *D_1 and D_2 are components of $G \setminus Y$, and*
- *$N(D_1) = N(D_2) = Y$.*

Assume that $Y \cap \mathrm{Hub}(G) = \emptyset$. Then there exist $X \subseteq Y$ with $|X| = x$, $H_1 \subseteq D_1$ and $H_2 \subseteq D_2$ (possibly with the roles of D_1 and D_2 reversed) such that

- *(H_1, X) is a triangular connectifier or a triangular alignment;*
- *(H_2, X) is a stellar connectifier, or a spiky connectifier, or a spiky alignment or a wide alignment; and*
- *if (H_1, X) is a triangular alignment, then (H_2, X) is not a wide alignment.*

Moreover, if neither of $(H_1, X), (H_2, X)$ is a stellar connectifier, then the orders given on X by (H_1, X) and by (H_2, X) are the same.

6. Bounding the number of non-hubs in a minimal separator

The goal of this section is to prove Theorem 6.3. This is the second main ingredient of the proof of Theorem 1.3 (in addition to the machinery of Section 4). Theorem 6.3 is what allows us to iterate the construction of Section 4 for $\mathcal{O}(\log |V(G)|)$ rounds (rather than just constantly many), which is what we need in the proof. We need the following definition and result. Given a graph H, a $(\leq p)$-*subdivision* of H is a graph that arises from H by replacing each edge uv of H by a path from u to v of length at least 1 and at most p; the interiors of these paths are pairwise disjoint and anticomplete.

Theorem 6.1 (Lozin and Razgon [29])**.** *For all positive integers p and r, there exists a positive integer $m = m(p, r)$ such that every graph G containing a $(\leq p)$-subdivision of K_m as a subgraph contains either $K_{p,p}$ as a subgraph or a proper $(\leq p)$-subdivision of $K_{r,r}$ as an induced subgraph.*

Since graphs in $\mathcal{C}_t$ contain neither K_{t+1} nor $K_{2,2}$ as an induced subgraph, we conclude that graphs in $\mathcal{C}_t$ do not contain $K_{t+1,t+1}$ as a subgraph. Likewise, graphs in $\mathcal{C}_t$ do not contain subdivisions of $K_{2,3}$ (in other words, thetas) as an induced subgraph. Therefore, letting $m = m(t + 1, 3)$, we conclude:

Lemma 6.2. *Let $t \in \mathbb{N}$. Then there exists an $m = m(t) \in \mathbb{N}$ such that the following holds: Let G in $\mathcal{C}_t$. Then G does not contain a (≤ 2)-subdivision of K_m as a subgraph.*

Recall that the Ramsey number $R(t, s)$ is the minimum integer such that every graph on at least $R(t, s)$ vertices contains either a clique of size t or a stable set of size s.

Theorem 6.3. *For every integer $t \geq 1$ there exists $\Gamma = \Gamma(t)$ with the following property. Let $G \in \mathcal{C}_t$ and assume that $V(G) = D_1 \cup D_2 \cup Y$ where*

- *D_1 and D_2 are components of $G \setminus Y$.*
- *Y is a stable set.*
- *$N(D_1) = N(D_2) = Y$.*
- *$Y \cap \text{Hub}(G) = \emptyset$.*
- *There exist $a_1 \in D_1$ and $a_2 \in D_2$ such that a_1a_2 is a $|Y|$-banana in G.*

Then $|Y| \leq \Gamma$.

Proof. Let $k = k(t)$ be as in Theorem 3.2. We may assume that $k \geq 3$. Let $m = m(t)$ as in Lemma 6.2. Let $\lambda = R(m, k)$. Let $\tau = \tau(t, \lambda)$ be as in Theorem 5.8. Set $\Gamma = \tau$. Applying Theorem 5.8, we deduce that there exist $H_1 \subseteq D_1$, $H_2 \subseteq D_2$ and $X \subseteq Y$ with $|X| = \lambda$ such that:

- (H_1, X) is a triangular connectifier or a triangular alignment;
- (H_2, X) is a stellar connectifier, or a spiky connectifier, or a spiky alignment, or a wide alignment; and
- if (H_1, X) is a triangular alignment, then (H_2, X) is not a wide alignment.

(11) *Suppose (H_2, X) is a wide alignment, and let $x_1, \dots, x_\lambda$ be the order on X given by H_2. Then for every $i \in \{2, \dots, \lambda - 1\}$, x_i has exactly three neighbors in H_2, and two of them are adjacent.*

Let $i \in \{2, \dots, \lambda - 1\}$. Let R be a path from x_1 to x_λ with interior in H_1. Then $H = x_1\text{-}H_2\text{-}x_\lambda\text{-}R\text{-}x_1$ is a hole. Since (H_2, X) is a wide alignment, x_i has two non-adjacent neighbors in H. Since (H, x_i) is not a wheel in G, and x_i is non-adjacent to x_1 and x_λ, (11) follows.

Next we show:

(12) *Every vertex in D_1 has at most two neighbors in X.*

Suppose that there is a vertex $d \in D_1$ such that d has three distinct neighbors x_i, x_j, x_k in X; without loss of generality, we may assume that $i < j < k$. We consider two cases. First, if H_2 is a wide alignment, then G contains a wheel (C, x_j) where C is a cycle consisting of x_i, d, x_k and the path from x_k to x_i with interior in H_2. By (11), x_j has four neighbors in C: d, as well as three neighbors in H_2. But x_j is not a hub in G, a contradiction.

It follows that H_2 is not a wide alignment. Now, for $r, s \in \{i, j, k\}$, let P_{rs} be a path from x_r to x_s with interior in H_2. Then $\{d\} \cup P_{ij} \cup P_{jk} \cup P_{ik}$ is a theta in G (see Figure 5). This proves (12).

(13) *There exists $S \subseteq X$ with $|S| = k$, such that for every $s, s' \in S$ and for every path P from s to s' with interior in D_1 we have that $|E(P)| > 2$.*

Define a graph H with vertex set X such that $xx' \in E(H)$ if and only if there is a path of length two with ends x, x' and interior in D_1. Since $|X| = \lambda = R(m, k)$, there is $S \subseteq X$ with $|S| = k$ such that S is either a clique or a stable set in H.

If S is a stable set, then (13) holds, so we may assume that S is a clique of size m in H. Write $S = \{s_1, \dots, s_m\}$. For $i, j \in \{1, \dots, m\}$ with $i \neq j$, we let d_{ij} be a common neighbor of s_i and s_j in D_1 (which exists, since S is a clique in H). By (12), no vertex has three neighbors in S, and so the vertices d_{ij} are pairwise distinct. But now $S \cup \{d_{ij} : i, j \in \{1, \dots, m\}, i \neq j\}$ contains a (≤ 2)-subdivision of K_m as a subgraph, contrary to Lemma 6.2. This is proves (13).

From now on let $S \subseteq X$ be as in (13). Let G' be the graph obtained

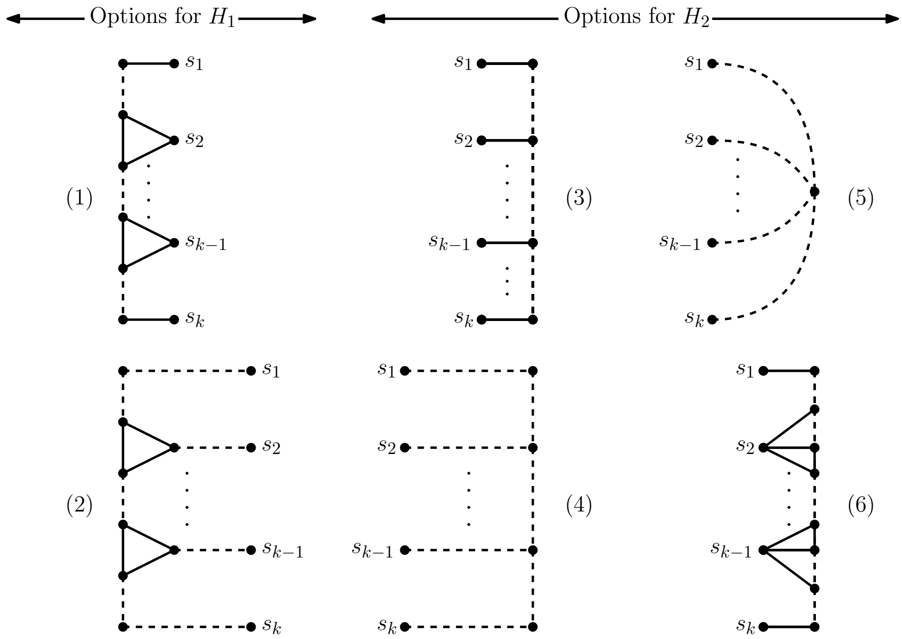

FIGURE 5. Options in the proof of Theorem 6.3 (except (1) and (6) cannot be the case together).

from $D_1 \cup S$ by adding a new vertex v with $N(v) = S$. We need the following two facts about G':

(14) $S \cap \mathrm{Hub}(G') = \emptyset$.

Suppose not, let $s \in S$ and let (H, s) be a wheel in G'. Since $S \cap \mathrm{Hub}(G) = \emptyset$, it follows that $v \in H$. Let $N_H(v) = \{s', s''\}$; then $H \setminus v$ is a path R from s' to s'' with $R^* \subseteq D_1$. Since (H, s) is a wheel, it follows that s has two non-adjacent neighbors in R. Consequently, there is a path P' from s to s' and a path P''^* form s to s'', both with interior in R, such that P'^* is disjoint from and anticomplete to P''^*.

Let T be a path from s' to s'' with interior in H_2. Then $H' = s'\text{-}R\text{-}s''\text{-}T\text{-}s'$ is a hole. Since $X \cap \mathrm{Hub}(G) = \emptyset$, it follows that (H', s) is not a wheel in G, and so s is anticomplete to T.

Suppose first that (H_2, X) is a wide alignment. Since s is anticomplete to T, it follows that (reversing the order on X if necessary, and exchanging the roles of s', s'' if necessary) s, s', s'' appear in this order in the order given by H_2 on X. Now we get a theta with ends s, s' and paths $s\text{-}P'\text{-}s'$, $s\text{-}P''\text{-}s''\text{-}H_2\text{-}s'$, and $s\text{-}H_2\text{-}s'$, a contradiction. This proves that (H_2, X) is not a wide alignment.

Since (H_2, X) is not a wide alignment, there is a vertex $a \in H_2$, and three paths Q, Q', Q'' all with interior in H_2, where Q is from a to s, Q' is from a to s', and Q'' is from a to s'', and the sets $Q \setminus a$, $Q' \setminus a$ and $Q'' \setminus a$ are pairwise disjoint and anticomplete to each other. Note that

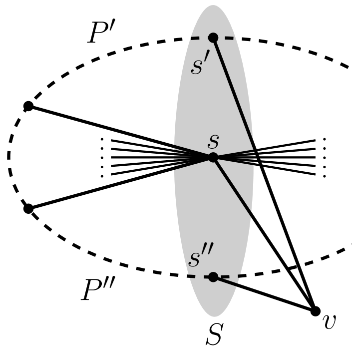

FIGURE 6. Proof of (14).

$T = s'$-Q'-a-Q''-s''. Since s is anticomplete to T, it follows that s is non-adjacent to a. But now we get a theta with ends a, s and path s-Q-a, s-P'-s'-Q'-a, and s-P''-s''-Q''-a, a contradiction. This proves (14).

(15) $G' \subseteq \mathcal{C}_t$.

Suppose not. Let $Z \subseteq G'$ be such that Z is a K_t, a C_4, a theta, a prism or an even wheel. Then $v \in Z$. Since $N_Z(v)$ is a stable set, it follows that Z is not a K_t. Suppose first that $|N_Z(v)| = 2$; then $N_Z(v) = \{s, s'\} \subseteq S$. Suppose that Z is a C_4. Then there is $d \in D_1$ such that $Z = v$-s-d-s'-v, contrary to the fact that S was chosen as in (13). It follows that Z is a theta, a prism or an even wheel. Let Q be a path from s to s' with interior in D_2. Now $Z' = (Z \setminus v) \cup Q$ is an induced subgraph of G. Moreover, if Z is a theta then Z' is a theta, and if Z is a prism then Z' is a prism. Since $G \in \mathcal{C}$, it follows that Z is an even wheel; denote it by (H, w). Then $v \in H$. By (14), $w \notin S$. It follows that $N_Z(w) = N_{Z'}(w')$, and so Z' is an even wheel, contrary to the fact that $G \in \mathcal{C}$.

This proves that $|N_Z(v)| > 2$. Since $N_Z(v)$ is a stable set, it follows that Z is not a prism. Suppose that Z is a theta. Then the ends of Z are v and $d \in D_1$; let the paths of Z be Z_1, Z_2, Z_3 where $N_{Z_i}(v) = s_i$. By renumbering Z_1, Z_2, Z_3, we may assume that s_1, s_2, s_3 appear in this order in the order on X given by H_1. Then for every $i, j \in \{1, \dots, 3\}$, there is a unique path F^1_{ij} from s_i to s_j with interior in H_1. Similarly, for all $i, j \in \{1, \dots, 3\}$ there is a unique path F^2_{ij} from s_i to s_j with interior in H_2. Observe that $F^1_{12} \cup F^1_{13} \cup F^1_{23} \cup F^2_{12} \cup F^2_{13} \cup F^2_{23}$ contains a unique pyramid Σ with the following specifications:

- The paths of Σ are P_1, P_2, P_3, where $s_i \in P_i$.
- Denote the apex of Σ by a, and let the base of Σ be $b_1b_2b_3$ where b_i is an end of P_i. Then either $a \in H_2$, or (H_2, X) is a wide alignment and $a = s_2$.
- $b_1, b_3 \in H_1$, and $b_2 \in H_1 \cup \{s_2\}$.

Let $\Sigma_2 = \Sigma \setminus D_1$. If d is non-adjacent to a, then $(Z \setminus v) \cup \Sigma_2$ is a theta with ends a, d and paths d-Z_i-s_i-P_i-a, contrary to the fact that $G \in \mathcal{C}_t$. This proves that d is adjacent to a. Consequently, $a = s_2$ and (H_2, X) is a wide alignment. But now $H = s_1\text{-}Z_1\text{-}d\text{-}Z_3\text{-}s_3\text{-}F^2_{13}\text{-}s_1$ is a hole, and, since s_2 is adjacent to d, (H, s_2) is a wheel, contrary to the fact that $S \cap \text{Hub}(G) = \emptyset$. This proves that Z is not a theta.

Consequently, Z is an even wheel (H, w). Since $|N_Z(v)| > 2$, it follows that either $v = w$, or v is adjacent to w. If v is adjacent to w, then $w \in S$, contrary to (14). It now follows that $w = v$, and so H is a hole in $D_1 \cup S$, where $|S \cap H| = l \geq 3$. Let $S \cap H = \{s_1, \dots, s_l\}$ where $s_1, \dots s_l$ appear in this order in the order given on S by (H_2, X). If (H_2, X) is a spiky alignment or a wide alignment, or a spiky connectifier, then we get a theta with ends s_1, s_2 two of whose paths are contained in H, and the third is the path from s_1 to s_2 with interior in H_2. It follows that (H_2, X) is a stellar connectifier. Therefore there exist paths $Q_1, \dots, Q_l$, all with a common end $a \in H_2$, such that Q_i is from a to s_i, $Q_i^* \subseteq H_2$, and the sets $Q_1 \setminus a, \dots, Q_k \setminus a$ are pairwise disjoint and anticomplete to each other. Since (H, a) is a not an even wheel in G, we deduce that a is not complete to $\{s_1, \dots, s_l\}$; let $s_i \in \{s_1, \dots, s_l\}$ be non-adjacent to a. Since $l \geq 3$, there exist distinct $p, q \in \{1, \dots, l\} \setminus \{i\}$ and paths R_q and R_p of H such that

- R_p is from s_i to s_p;
- R_q is from s_i to s_q;
- $R_p^* \cap R_q^* = \emptyset$; and
- $R_p^* \cap \{s_1, \dots, s_l\} = R_q^* \cap \{s_1 \dots, s_l\} = \emptyset$.

Now we get a theta with ends s_i, a and path $s_i\text{-}Q_i\text{-}a$, $s_i\text{-}R_p\text{-}s_p\text{-}Q_p\text{-}a$ and $s_i\text{-}R_q\text{-}s_q\text{-}Q_q\text{-}a$. This proves (15).

Observe that $a_1 v$ is a k-banana in G'. But by (15), $G' \in \mathcal{C}_t$, and by (14), $N_{G'}(v) \cap \text{Hub}(G') = \emptyset$, contrary to Theorem 3.2. ■

7. The proof of Theorem 1.3

We can now prove our first main result. The "big picture" of the proof is similar to [3] and [5], but the context here is different. For the remainder of the paper, all logarithms are taken in base 2. We start with the following theorem from [5]:

Theorem 7.1 (Abrishami, Chudnovsky, Hajebi, Spirkl [5])**.** *Let $t \in \mathbb{N}$, and let G be (theta, K_t)-free with $|V(G)| = n$. There exist an integer $d = d(t)$ and a partition $(S_1, \dots, S_k)$ of $V(G)$ with the following properties:*

(1) $k \leq \frac{d}{4} \log n$.
(2) S_i is a stable set for every $i \in \{1, \dots, k\}$.
(3) For every $i \in \{1, \dots, k\}$ and $v \in S_i$ we have $\deg_{G \setminus \bigcup_{j<i} S_j}(v) \leq d$.

Let $G \in \mathcal{C}_t$ be a graph and let $a, b \in V(G)$. A *hub-partition with respect to ab* of G is a partition $S_1, \dots, S_k$ of $\text{Hub}(G) \setminus \{a, b\}$ as in Theorem 7.1; we call k the *order* of the partition. We call the *hub-dimension* of (G, ab)

(denoting it by $\operatorname{hdim}(G, ab)$) the smallest k such that G has a hub-partition of order k with respect to ab.

For the remainder of this section, let us fix $t \in \mathbb{N}$. Let $d = d(t)$ be as in Theorem 7.1. Let $m = k + 2d$ where $k = k(t)$ is as in Theorem 3.2. Let $C_t = (4m + 2)(m - 1)$. Let $\Gamma = \max\{k, \Gamma(t)\}$ where $\Gamma(t)$ is as in Theorem 6.3.

Since in view of Theorem 7.1, we have $\operatorname{hdim}(G, ab) \leq \frac{d}{4} \log n$ for all $a, b \in V(G)$, Theorem 1.3 follows immediately from the next result:

Theorem 7.2. *Let $G \in \mathcal{C}_t$ and with $|V(G)| = n$ and let $a, b \in V(G)$. Then ab is not a $C_t + 8m^2\Gamma \operatorname{hdim}(G, ab)$-banana in G.*

Proof. Suppose that ab is a $C_t + 8m^2\Gamma \operatorname{hdim}(G, ab)$-banana in G. We will get a contradiction by induction on $\operatorname{hdim}(G, ab)$, and for fixed hdim by induction on n. Suppose that $\operatorname{hdim}(G, ab) = 0$. Then $\operatorname{Hub}(G) \subseteq \{a, b\}$ and by Theorem 3.3, we have that there is no k-banana in G. Now the statement holds since $k < C_t$. Thus we may assume $\operatorname{hdim}(G, ab) > 0$.

(16) *We may assume that G admits no clique cutset.*

Suppose C is a clique cutset in G. Since G is K_t-free, it follows that there is a component D of $G \setminus C$ such that ab is $C_t + 8m^2\Gamma \operatorname{hdim}(G, ab)$-banana in $G' = D \cup C$. But $|V(G')| < n$ and $\operatorname{hdim}(G', ab) \leq \operatorname{hdim}(G, ab)$, consequently we get contradiction (inductively on n). This proves (16).

Let $S_1, \ldots, S_q$ be a hub-partition of G with respect to ab and with $q = \operatorname{hdim}(G, ab)$. We now use notation and terminology from Section 4 (note that the definitions of k and m agree; we use d, a, b, as defined above). It follows from the definition of S_1 that every vertex in S_1 is d-safe. Let (T, χ) be an m-atomic tree decomposition of G. Let t_1, t_2 be as in Theorem 2.6. and let $\beta = \beta(S_1)$ and $\beta^A(S_1)$ be as in Section 4. Then by Lemma 4.1, we have $a, b \in \beta^A(S_1)$. By Theorem 4.10(5), we have that $S_1 \cap \operatorname{Hub}(\beta^A(S_1)) = \emptyset$ and $S_2 \cap \operatorname{Hub}(\beta^A(S_1)), \ldots, S_q \cap \operatorname{Hub}(\beta^A(S_1))$ is a hub-partition of $\beta^A(S_1)$ with respect to ab. It follows that $\operatorname{hdim}(\beta^A(S_1), ab) \leq q - 1$. Inductively (on $\operatorname{hdim}(\cdot, ab)$), we have that ab is not a $C_t + 8m^2\Gamma(q - 1)$-banana in $\beta^A(S_1)$.

Our first goal is to prove:

(17) *There is a set $Y' \subseteq \beta$ such that*
- *(1) Y' separates a from b in β;*
- *(2) $|Y' \cap (\chi(t_1) \cup \chi(t_2))| \leq C_t + 8m^2\Gamma(q - 1) + 4m^2\Gamma$; and*
- *(3) $|\Delta(Y')| \leq (4m + 1)(m - 1)\Gamma$.*

Since ab is not a $(C_t + 8m^2\Gamma(q - 1))$-banana in $\beta^A(S_1)$, Theorem 2.1 implies that there is a separation (X, Y, Z) of $\beta^A(S_1)$ such that $a \in X$ and $b \in Z$ and $|Y| \leq C_t + 8m^2\Gamma(q - 1)$. We may assume that (X, Y, Z) is chosen with $|Y|$ as small as possible. Let D_1 be the component of X such that $a \in D_1$, and and let D_2 be the component of Z such that $b \in D_2$. It follows from the minimality of $|Y|$ that $N(D_1) = N(D_2) = Y$ and ab is a $|Y|$-banana in $D_1 \cup D_2 \cup Y$.

We claim that $|Y \cap S_1| \leq \Gamma$. Let $G' = D_1 \cup D_2 \cup (Y \cap S_1)$. Again by the minimality of Y, it follows that ab is a $|Y \cap S_1|$-banana in G'. If ab is not a k-banana in G', then $|Y \cap S_1| < k$ and the claim follows immediately since $k \leq \Gamma$; thus we may assume that ab is a k-banana in G'. Now since S_1 is a stable set and since no vertex of S_1 is a hub in $\beta^A(S_1)$, applying Theorem 6.3 to G' we deduce that $|Y \cap S_1| \leq \Gamma$, as required. This proves the claim.

Next we claim that $|\delta_1(Y) \cup \delta_2(Y)| \leq \Gamma$, and $|\Delta(Y)| \leq (m-1)\Gamma$. Write $\beta_0 = \chi(t_1) \cup \chi(t_2)$. Observe that for distinct $t, t' \in \delta_1(Y) \cup \delta_2(Y)$, say with $t \in \delta_i(Y)$ and $t' \in \delta_{i'}(Y)$ for $i, i' \in \{1, 2\}$, the sets $G_{t_i \to t} \setminus \beta_0$ and $G_{t_{i'} \to t'} \setminus \beta_0$ are disjoint and anticomplete to each other. Let W be a subset of Y such that for all $i \in \{1, 2\}$ and $t \in \delta_i(Y)$, we have $|W \cap (G_{t_i \to t} \setminus \beta_0))| = 1$. Then W is stable set and $|W| = |\delta_1(Y) \cup \delta_2(Y)|$. Let $G' = D_1 \cup D_2 \cup W$. If follows from the minimality of Y that ab is a $|W|$-banana in G'. If ab is not a k-banana in G', then $|W| < k$ and the claim follows immediately since $k \leq \Gamma$; thus we may assume that ab is a k-banana in G'. By Theorem 4.4(3), $W \cap \text{Hub}(\beta) = \emptyset$. Now since W is a stable set, applying Theorem 6.3 to G' we deduce that $|W| \leq \Gamma$, as required. Since $\text{adh}(T, \chi) \leq m - 1$, it follows that $|\Delta(Y)| \leq (m-1)\Gamma$. This proves the claim.

Now let Y' be as in Theorem 4.11. Then Y' separates a from b in β. Moreover, by Theorem 4.11,

$$|Y' \cap (\chi(t_1) \cup \chi(t_2))| \leq |Y| + 4m(m-1)|Y \cap \text{Core}(S_1)|.$$

Since $|Y \cap \text{Core}(S_1)| \leq |Y \cap S_1| \leq \Gamma$, it follows that

$$|Y' \cap (\chi(t_1) \cup \chi(t_2))| \leq C_t + 8m^2\Gamma(q-1) + 4m^2\Gamma.$$

Finally, by Theorem 4.11, $|\Delta(Y') \setminus \Delta(Y)| \leq 4m(m-1)|Y \cap \text{Core}(S_1)|$. Since $|\Delta(Y)| \leq (m-1)\Gamma$ and $|Y \cap \text{Core}(S_1)| \leq \Gamma$, it follows that $|\Delta(Y')| \leq (4m+1)(m-1)\Gamma$. This proves (17).

Let Y' be as in (17), and let (X', Y', Z') be a separation of β such that $a \in X'$ and $b \in Z'$. Let Y'' be obtained from Y' as in Theorem 4.12. Then Y'' separates a from b in G. Recall that by (17), we have $|\Delta(Y')| \leq (4m+1)(m-1)\Gamma$. Now since $\text{adh}(T, \chi) < m$ and since $|S_{1bad}| \leq 3$, it follows from Theorem 4.12 that

$$\begin{aligned}|Y''| &\leq |Y' \cap (\chi(t_1) \cup \chi(t_2))| + (4m+1)(m-1)\Gamma + 2(m-1) + 3 \\ &\leq |Y' \cap (\chi(t_1) \cup \chi(t_2))| + 4m^2\Gamma.\end{aligned}$$

Since $|Y' \cap (\chi(t_1) \cup \chi(t_2))| \leq C_t + 8m^2\Gamma(q-1) + 4m^2\Gamma$ by (17), we deduce that $|Y''| \leq C_t + 8m^2\Gamma q$ as required. ■

8. Dominated balanced separators

Several more steps are needed to complete the proof of Theorem 1.5. We take the first one in this section. The goal of this section is to prove Theorem 1.4, which we restate:

Theorem 8.1. *There is an integer d with the following property. Let $G \in \mathcal{C}$ and let w be a normal weight function on G. Then there exists $Y \subseteq V(G)$ such that*

- *$|Y| \leq d$, and*
- *$N[Y]$ is a w-balanced separator in G.*

We start with two decomposition theorems that will become the engine for obtaining separators. The first is a more explicit version of Theorem 3.1; it shows that the presence of a wheel in the graph forces a decomposition that is an extension of what can be locally observed in the wheel (see [4] for detailed treatment of this concept).

Theorem 8.2 (Addario-Berry, Chudnovsky, Havet, Reed, Seymour [7], da Silva, Vušković [17])**.** *Let G be a C_4-free odd-signable graph that contains a proper wheel (H, x) that is not a universal wheel. Let x_1 and x_2 be the endpoints of a long sector Q of (H, x). Let W be the set of all vertices h in $H \cap N(x)$ such that the subpath of $H \setminus \{x_1\}$ from x_2 to h contains an even number of neighbors of x, and let $Z = H \setminus (Q \cup N(x))$. Let $N' = N(x) \setminus W$. Then, $N' \cup \{x\}$ is a cutset of G that separates Q^* from $W \cup Z$.*

The second is a similar kind of theorem, but we start with a pyramid rather than a wheel.

Theorem 8.3. *Let $G \in \mathcal{C}$. Let Σ be a pyramid with paths P_1, P_2, P_3, apex a, and base $b_1b_2b_3$ in G and let $i, j \in \{1, 2, 3\}$ be distinct. For $i \in \{1, \dots, 3\}$, let a_i be the neighbor of a in P_i. Let P be a path from a vertex of $P_i \setminus \{a, a_i, b_i\}$ to a vertex of $P_j \setminus \{a, a_j, b_j\}$. Then at least one one of a, b_1, b_2, b_3 has a neighbor in P^*.*

Proof. It follows from the defintion of P that there exist distinct $i, j \in \{1, 2, 3\}$ and a subpath $P' = p_1\text{-}\cdots p_k$ of P such that p_1 has a neighbor in $P_i \setminus \{a, a_i, b_i\}$, and p_k has a neighbor in $P_j \setminus \{a, a_j, b_j\}$. Let P' be chosen minimal with this property. It follows that $P' \subseteq P^*$, and so we may assume that $\{a, b_1, b_2, b_3\}$ is anticomplete to P'. We may also assume $i = 1$ and $j = 3$. See Figure 7. It follows that and $a_1 \neq b_1$, $a_3 \neq b_3$.

(18) *P' is anticomplete to $P_2 \setminus a_2$.*

Suppose not; then some vertex in P' has a neighbor in $P_2 \setminus \{a, a_2, b_2\}$. It follows from the minimality of P' that $k = 1$. Then p_1 has a neighbor in each of the paths $P_1 \setminus b_1, P_2 \setminus b_2$ and $P_3 \setminus b_3$. Now we get a theta with ends

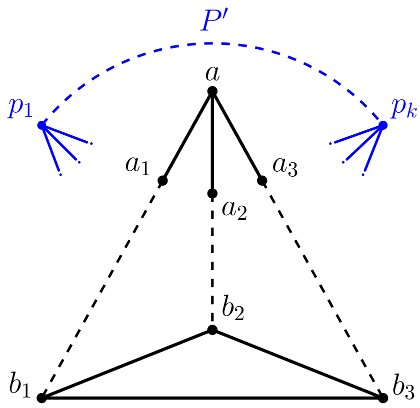

FIGURE 7. Proof of Theorem 8.3.

p_1, a whose paths are subpaths of P_1, P_2, P_3, a contradiction. This proves (18).

(19) *If $k > 1$ and a_1 is adjacent to p_k, then a_1 has a neighbor in $P' \setminus p_k$.*

Suppose that a_1 is adjacent to p_k and anticomplete to $P' \setminus p_k$. Since a_2-a-a_1-p_k-a_2 is not a C_4 in G, and by (18), it follows that p_k is anticomplete to P_2. If a_2 has a neighbor in $P' \setminus p_1$, then we get a theta with ends a_1, a_2, two of whose paths are contained in the hole b_1-P_1-a-P_2-b_2-b_1, and third one has interior on $P' \setminus p_1$, a contradiction. Consequently, a_2 is anticomplete to $P' \setminus p_1$. Next suppose that a_3 has a neighbor in P'. Since a_1-a-a_3-p_k-a_1 is not a C_4 in G, it follows that a_3 is not adjacent to p_k. Now we get a theta with ends a_3, p_k and path p_k-a_1-a-a_3, a path with interior in P' and a path with interior in P_3. This proves that a_3 is anticomplete to P'.

Let x be the neighbor of a_1 in $P_1 \setminus a$. Suppose that p_1 has a neighbor in $P_1 \setminus \{a, a_1, x\}$. Then there is a path T from p_1 to a with $T^* \subseteq (P_1 \cup P_2) \setminus \{a, a_1, x\}$ and we get a theta with ends p_k, a and paths p_k-a_1-a and p_k-P'-p_1-T-a, and a path with interior in P_3^*, a contradiction. It follows that p_1 is anticomplete to $P_1 \setminus \{a_1, a, x\}$ and therefore p_1 is adjacent to x. Since a_1 is anticomplete to $P' \setminus p_k$ we deduce that $N_{P_1}(p_1) = \{x\}$. Now we get a theta with ends x, p_k and paths x-a_1-p_k, x-P'-p_k and x-P_1-b_1-b_3-P_3-p_k, a contradiction. This proves (19).

(20) *Either $N_{P'}(a_1) \subseteq \{p_1\}$ or $N_{P'}(a_3) \subseteq \{p_k\}$.*

Suppose not. Then $k > 1$. If both a_1 and a_3 have a neighbor in P'^*, then there is a theta in G with ends a_1, a_3, and paths a_1-a-a_3, a_1-P_1-b_1-b_3-P_3-a_3 and a_1-P'-a_3, a contradiction. By symmetry we may assume that a_1 is anticomplete to P'^*; now by (19), it follows that a_1 is adjacent to both p_1 and p_k. Since a_1-p_1-a_3-a-a_1 is not a C_4, we deduce that a_3 is non-adjacent to p_1. Similarly a_3 is non-adjacent to p_k. It follows that a_3 has a neighbor in P'^*. Let S be a path from a_3 to p_k with interior in P'^*. Now we get a theta with ends a_3, p_k and path a_3-S-p_k, a_3-a-a_1-p_k and a path with interior in P_3, a contradiction. This proves (20).

(21) *If a_1 has a neighbor in $P' \setminus p_1$, then a_2 is anticomplete to P'.*

Suppose that a_1 has a neighbor in $P' \setminus p_1$ and a_2 has a neighbor in P'. Then $k > 1$. If both a_1 and a_2 have a neighbor in $P' \setminus p_1$, then there is a theta in G with ends a_1, a_2, and paths a_1-a-a_2, a_1-P_1-b_1-b_2-P_2-a_2 and a_1-P'-a_2, a contradiction. This proves that $N_{P'}(a_2) = \{p_1\}$.

Let S_1 be a path from a_1 to p_1 with $S_1^* \subseteq P'$. Since a_1-p_1-a_2-a-a_1 is not a C_4, it follows that a_1 is non-adjacent to p_1. But now we get a theta with ends a_1, p_1 and paths a_1-a-a_2-p_1, a_1-S_1-p_1 and a path from a_1 to p_1 with interior in P_1, a contradiction. This proves (21).

(22) *$N_{P'}(a_1) \subseteq \{p_1\}$ and $N_{P'}(a_3) \subseteq \{p_k\}$.*

Suppose a_1 has a neighbor in $P' \setminus p_1$. Then $k > 1$. By (20), a_3 is anticomplete to $P' \setminus p_k$, and by (21), a_2 is anticomplete to P'. Let H_1 be the hole b_1-P_1-p_1-P'-p_k-P_3-b_3-b_1, and let H_2 be the hole p_1-P_1-b_1-b_2-P_2-a-P_3-p_k-P'-p_1 (H_2 is a hole since p_k has a neighbor in $P_3 \setminus b_3$). Then $N_{H_2}(a_1) = N_{H_1}(a_1) \cup \{a\}$. Let p_0 be the neighbor of $p_1 \in H_1 \setminus P'$ and let Q be the path p_0-p_1-P'-p_k. Observe that $Q \subseteq H_2$, and $N_{H_1}(a_1) \subseteq Q$. Since neither of (H_1, a_1) and (H_2, a_1) is an even wheel, it follows that a_1 has exactly two neighbors in Q and they are adjacent. Let $s \in \{0, \dots, k-1\}$ be such that $N_Q(a_1) = \{p_s, p_{s+1}\}$. Since a_1 has a neighbor in $P_1 \setminus p_1$, it follows that $s > 0$. Now we get a prism with triangles $p_s a_1 p_{s+1}$ and $b_1 b_2 b_3$ and paths a_1-a-P_2-b_2, p_s-P'-p_1-p_0-P_1-b_1 and p_{s+1}-P'-p_k-P_3-b_3, a contradiction. This proves (22).

(23) *P_2 is anticomplete to P'.*

Suppose not. By (18), a_2 has a neighbor in P'. Then $a_2 \neq b_2$. Let H_1 be the hole p_1-P'-p_k-P_3-b_3-b_1-P_1-p_1 and let H_2 be the hole p_1-P'-p_k-P_3-a_3-a-a_1-P_1-p_1 (H_2 is a hole since $\{b_1, b_3\}$ is anticomplete to P'). Since $a_2 \neq b_2$, we have that $N_{H_2}(a_2) = N_{H_1}(a_2) \cup \{a\}$. Since since neither of (H_1, a_2), (H_2, a_2) is an even wheel, it follows that there exists $s \in \{1, \dots, k-1\}$ such that $N_{P'}(a_2) = \{p_s, p_{s+1}\}$. Now we get a prism with triangles $b_1 b_2 b_3$ and $p_s a_2 p_{s+1}$ and paths p_s-P'-p_1-P_1-b_1, a_2-P_2-b_2 and p_{s+1}-P'-p_k-P_3-b_3, a contradiction. This proves (23).

(24) *p_1 has at least two neighbors in P_1, and p_3 has at least two neighbors in P_3.*

By symmetry it is enough to prove the first assertion of (24). Suppose that p_1 has a unique neighbor x in P_1. It follows that $x \neq a_1$. By (23), P_2

is anticomplete to P'. Now we get a theta with ends x, a and paths x-P_1-a, x-P_1-b_1-b_2-P_2-a and x-p_1-P'-p_k-P_3-a, a contradiction. This proves (24).

(25) *p_1 has two non-adjacent neighbors in P_1, and p_3 has two non-adjacent two neighbors in P_3.*

By symmetry it is enough to prove the first statement. By (24), we may assume that p_1 has exactly two neighbors x, y in P_1, and x is adjacent to y. We may assume that P_1 traverses b_1, x, y, a_1 in this order. By (23), P_2 is anticomplete to P'. Let S be the path from p_k to b_3 with interior in P_3. It follows from the defintion of P' that a is anticomplete to S. Now we get a prism with triangles xyp_1 and $b_1b_2b_3$ and paths x-P_1-b_1, y-P_1-a-P_2-b_2 and p_1-P'-p_k-S-b_3, a contradiction. This proves (25).

By (25), there exist paths P_1' from p_1 to a and P_1'' from p_1 to b_1, both with interior in P_1^*, and such that $P_1' \setminus p_1$ is anticomplete to $P_1'' \setminus p_1$. Let P_3' be a path from p_k to a with interior in P_3^*. By (23), P_2 is anticomplete to P'. Now we get a theta with ends p_1, a and paths p_1-P_1'-a, p_1-P_1''-b_1-b_2-P_2-a and p_1-P'-p_k-P_3'-a, a contradiction. ■

We need a technical definition. Given an integer $m > 0$ and a graph class $\mathcal{G}$, we say $\mathcal{G}$ is *m-amicable* if $\mathcal{G}$ is amiable, and the following holds for every graph $G \in \mathcal{G}$. Let σ be as in the definition of an amiable class for $\mathcal{G}$ and let $V(G) = D_1 \cup D_2 \cup Y$ such that $D_1 = d_1$-$\cdots$-d_k, D_2 and Y satisfy the first five bullets from the definition of an amiable class with $|Y| = \sigma(7)$. Let $X \subseteq Y$, $H_1 \subseteq D_1$ and $H_2 \subseteq D_2$ be as in the definition of an amiable class with $|X| = 9$, and let $\{x_1, \dots, x_7\} \subseteq X$ such that:

- $x_1, \dots, x_7$ is the order given on $\{x_1, \dots, x_7\}$ by (H_1, X); and
- For every $x \in \{x_1, \dots, x_7\}$, we have $N_{D_1}(x) = N_{H_1}(x)$.

Let i be maximum such that x_1 is adjacent to d_i, and let j be minimum such that x_7 is adjacent to d_j. Then there exists a subset $Z \subseteq D_2 \cup \{d_{i+2}, \dots, d_{j-2}\} \cup \{x_4\}$ with $|Z| \leq m$ such that $N[Z]$ separates d_i from d_j. It follows that $N[Z]$ separates d_1-D_1-d_i from d_j-D_1-d_k.

We deduce:

Theorem 8.4. *The class $\mathcal{G}$ is 4-amicable.*

Proof. From Theorem 5.7, we know that $\mathcal{G}$ is amiable. With the notation as in the definition of an amicable class, we need to show that there exists a subset $Y \subseteq D_2 \cup \{d_{i+2}, \dots, d_{j-2}\} \cup \{x_4\}$ with $|Z| \leq 4$ such that $N[Z]$ separates d_i from d_j. Let R be the path from x_1 to x_7 with interior in H_2.

Let H be the hole x_1-H_1-x_7-R-x_1. Let $L = l_1$-$\cdots$-l_q be the path in $H_2 \cup \{x_4\}$ such that $l_1 = x_4$ and l_q has a neighbor in $P(H_2)$. Thus if (H_2, X) is an alignment then $L = x_4$, and if (H_2, X) is a connectifier then $L \setminus x_4$ is the leg of H_2 containing a neighbor of x_4. See Figure 8. (Note that $L \setminus x_4$ may be empty if (H_2, X) is a clique connectifier or a stellar connectifier.) We consider different possibilities for the behavior of (H_1, X) and (H_2, X).

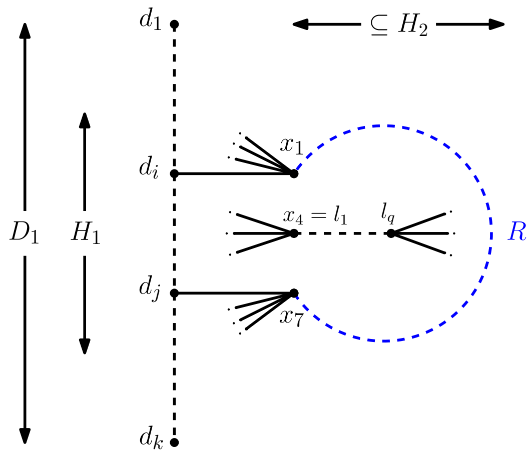

FIGURE 8. Proof of Theorem 8.4.

(26) *If* (H_1, X) *is a wide alignment, then the theorem holds.*

Suppose first that (H_2, X) is an alignment. Then (H, x_4) is a proper wheel, and we are done by Theorem 8.2 and setting $Z = \{x_4\}$. Thus we may assume that (H_2, X) is a connectifier. It now follows from Theorem 5.7 that (H_2, X) is a triangular connectifier or a clique connectifier. Let $N_H(l_q) = \{h, h'\}$. Then $H \cup L$ contains a pyramid Σ with apex x_4 and base $l_q hh'$. Since $\{d_i, d_j\}$ is anticomplete to $\{x_4, h, h', l_q\}$, it follows that d_i belongs to the interior of the path of Σ with ends x_4, h, and d_j belongs to the interior of the path of Σ with ends x_4, h' (switching the roles of h, h' if necessary), and that d_i, d_j are not adjacent to the apex x_4 of Σ. Since $\{d_i, d_j\} \cap N(x_4) = \emptyset$, Theorem 8.3 implies that $N[x_4, h, h', l_1]$ separates d_i from d_j. Since $h, h', l_q \in D_2$, we are done. This proves (26).

(27) *If* (H_2, X) *is a wide alignment, then the theorem holds.*

Since (H_1, X) is an alignment, we deduce that (H, x_4) is a proper wheel. Now by Theorem 8.2 and setting $Z = \{x_4\}$, we are done. This proves (27).

(28) *If* (H_1, X) *is a triangular alignment, then the theorem holds.*

Let $l \in \{i, \dots, j\}$ be such that $N(x_4) \cap D_1 = \{d_l, d_{l+1}\}$. Since x_2 has a neighbor in the path d_i-H_1-d_l, it follows that $l > i + 1$. Since x_6 has a neighbor in the path d_{l+1}-H_1-d_j, it follows that $l < j - 2$. By Theorem 5.7 and in view of (27), we may assume that (H_2, X) is a spiky alignment, a stellar connectifier or a spiky connectifier. In all cases, l_q has a unique neighbor in $P(H_2)$; denote it by s (where $P(H_2)$ is as defined in Section 5). Now $H \cup L$ is a pyramid Σ with apex s and base $x_4 d_l d_{l+1}$. Since $s \in D_2$ and since x_4 is non-adjacent to d_i, d_j, it follows that d_i belongs to the interior of the path of Σ with ends d_l, s, and d_j belongs to the interior of the path of Σ with ends d_{l+1}, s, and d_i, d_j are not adjacent to the apex s of Σ. Since $\{d_i, d_j\} \cap N(s) = \emptyset$, Theorem 8.3 implies that $N[x_4, s, d_l, d_{l+1}]$ separates d_i from d_j. Since $s \in D_2$ and $i+1 < l < j-2$, we are done. This proves (28).

By Theorem 5.7 and in view of (26) and (28), we deduce that (H_1, X) is a spiky alignment. Let d_l be the unique neighbor of x_4 in H_1. By Theorem 5.7 and in view of (27), it follows that (H_2, X) is either a triangular alignment or a triangular connectifier or a clique connectifier. Let h, h' be the neighbors of l_q in H. Now $H \cup L$ is a pyramid Σ with apex d_l and base $l_q hh'$. Since x_2 has a neighbor in the path d_i-H_1-d_l, it follows from the definition of a spiky alignment that $i < l-1$. Similarly, $j > l+1$. Since $h, h' \in D_2$, we deduce that d_i belongs to the interior of the path of Σ with ends d_l, h, and d_j belongs to the interior of the path of Σ with ends d_l, h' (possibly switching the roles of h, h'), and d_i, d_j are non-adjacent to the apex d_l of Σ since $i+1 < l < j-1$. Therefore, Theorem 8.3 implies that $N[d_l, h, h', l_q]$ separates d_i from d_j. Since $h, h' \in D_2$ and $i+1 < l < j-1$, we conclude that $\mathcal{C}$ is 4-amicable.

We now prove that $N[Z]$ separates d_1-D_1-d_i from d_j-D_1-d_k. Let D be the component of $G \setminus N[Z]$ such that $d_i \in D$, and let D' be the component of $G \setminus N[Z]$ such that $d_j \in D'$. Then $D \neq D'$. Since $Y \subseteq D_2 \cup \{d_{i+2}, \dots, d_{j-2}\} \cup \{x_4\}$, it follows that $N[Z]$ is disjoint from d_1-D_1-d_j and d_j-D_1-d_k. Therefore, d_1-D_1-$d_j \in D$ and d_j-D_1-$d_k \in D'$ as required. ■

We need the following lemma:

Lemma 8.5 (Chudnovsky, Pilipczuk, Pilipczuk, Thomassé [14], Lemma 5.3)**.** *Let G be a connected graph with a weight function w. Then there is an induced path $P = p_1$-$\dots$-p_k in G such that $N[P]$ is a w-balanced separator.*

For a graph G and a set $X \subseteq G$ we denote by $\gamma(X)$ the minimum size of a set Y such that $X \subseteq N[Y]$. We are now ready to prove Theorem 8.1. We prove the following strengthening, which along with Theorem 8.4 yields Theorem 8.1 at once. This will be used in a future paper.

Theorem 8.6. *For every integer $m > 0$ and every m-amicable graph class $\mathcal{G}$, there is an integer $d > 0$ with the following property. Let $G \in \mathcal{G}$ and let w be a normal weight function on G. Then there exists $Y \subseteq V(G)$ such that*

- *$|Y| \leq d$, and*
- *$N[Y]$ is a w-balanced separator in G.*

Proof. It suffices to consider the unique component of G with weight greater than 1/2; if there is no such component, we set $Y = \emptyset$. Therefore, we may assume that G is connected.

Since $\mathcal{G}$ is amicable, it is also amiable. Let $K = \sigma(7)$ where σ is as in the definition of an amiable class for $\mathcal{C}$. Let $d = 6K + 2m + 1$. We claim that d satisfies the conclusion of the theorem.

By Lemma 8.5, there is a path P in G such that $w(D) \leq \frac{1}{2}$ for every component D of $G \setminus N[P]$. Let $P = p_1\text{-}\cdots\text{-}p_k$ be such a path chosen with k as small as possible. It follows from the minimality of k that there is a component B of $G \setminus N[p_1\text{-}P\text{-}p_{k-1}]$ with $w(B) > \frac{1}{2}$. Let $T = N(P \setminus p_k) \cap N(B)$.

(29) *Let D be a connected subset of G such that $D \cap (T \cup N[p_k]) = \emptyset$. Then $w(D) \leq \frac{1}{2}$.*

Since $D \cap T = \emptyset$ and D is connected, it follows that either $D \subseteq B$ or $D \cap B = \emptyset$. If $D \cap B = \emptyset$, then $w(D) \leq 1 - w(B) < \frac{1}{2}$. Thus we may assume that $D \subseteq B$. Since $D \cap N[p_k] = \emptyset$, we deduce that D is contained in a component of $G \setminus N[P]$, and so $w(D) \leq \frac{1}{2}$. This proves (29).

Let r be minimum such that $\gamma[T \cap N(p_{r+1}, \dots, p_{k-1})] \leq 2K$.

(30) *We may assume that $r > 0$.*

Suppose $r = 0$. Then $\gamma(T) < 2K$. Let $Y' \subseteq V(G)$ be such that $T \subseteq N[Y']$ and with $|Y'| \leq 2K$. Let $Y = Y' \cup \{p_k\}$. Then every component of $G \setminus N[Y]$ is disjoint from $T \cup N[p_k]$ and (29) implies that Y satisfies the conclusion of the theorem. This proves (30).

Let $Y'_0 \subseteq V(G)$ be such that $T \cap N(p_r, \dots, p_{k-1}) \subseteq N[Y'_0]$ and with $|Y'_0| \leq 2K$. Let $Y_0 = Y'_0 \cup \{p_k\}$. For every $i \in \{1, \dots, r\}$, let $\text{end}(i)$ be the minimum $j > i$ such that $\gamma(T \cap N[\{p_i, \dots, p_{\text{end}(i)}\}]) = 2K$. Let Z_i be a set of size at most $2K$ such that $T \cap N[\{p_i, \dots, p_{\text{end}(i)}\}] \subseteq N[Z_i]$.

Our next goal is, for every $i \in \{1, \dots, r\}$, to define two sets Q_i and N_i. To that end, we fix $i \in \{1, \dots, r\}$. Let $T'_1 = T \cap N(\{p_i, \dots, p_{\text{end}(i)}\})$. Let $S'_1 \subseteq \{p_i, \dots, p_{\text{end}(i)}\}$ be such that $T'_1 \subseteq N(S'_1)$, and assume that S'_1 is chosen with $|S'_1|$ as small as possible. Order the vertices of S'_1 as $q_1, \dots, q_l$ in the order that they appear in P when P is traversed starting from p_1. It follows from the minimality of $|S'_1|$ that there is $n_1 \in T'_1$ such that $N_{S'_1}(n_1) = \{q_1\}$. Let $t(1) = 1$ and let $S_1 = \{q_{t(1)}\}$ and $T_1 = \{n_1\}$. So far we have defined S'_1, S_1, T'_1, T_1 and $t(1)$. Suppose that we have defined sequences of subsets $S'_1, \dots, S'_s$, $S_1, \dots, S_s$, $T'_1, \dots, T'_s$, and $T_1, \dots, T_s$ and a sequence of integers $t(1), \dots, t(s)$ such that $T'_s \subseteq N(S'_s)$ and with the following properties (for every $l \in \{1, \dots, s\}$):

- T_l is a stable set.
- If $t < t(l)$ and $q_t \in S'_l$, then q_t is anticomplete to T'_l.
- $t(j) < t(l)$ for every $j < l$.
- If $j < l$, then n_j is anticomplete to S'_l, and $N_{S'_l}(n_l) = q_{t(l)}$.
- For every $j \in \{1, \dots, l\}$, $N_{S_l}(n_j) = q_{t(j)}$.

See Figure 9. Now we either terminate the construction or construct the sets T'_{s+1}, T_{s+1}, S'_{s+1}, S_{s+1} and define $t(s+1)$, and verify that the properties above continue to hold. Let $T'_{s+1} = T'_s \setminus N[S_s \cup T_s]$. If $T'_{s+1} = \emptyset$, let $Q_i = S_s$

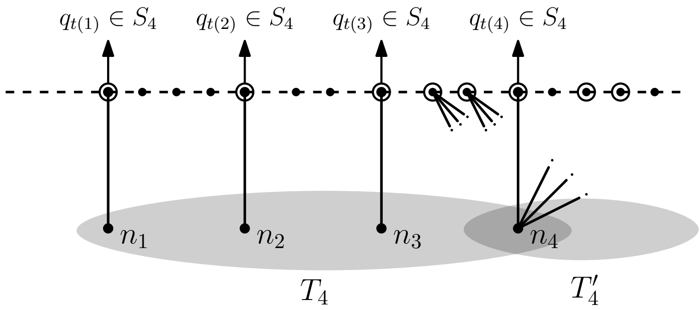

FIGURE 9. Proof of Theorem 8.6. The circled nodes depict the vertices in S'_4.

and $N_i = T_s$; the construction terminates here. Now suppose that $T'_{s+1} \neq \emptyset$. Let S'_{s+1} be a minimal subset of S'_s such that $T'_{s+1} \subseteq N(S'_{s+1})$. It follows from the second bullet and the minimality of $t(s)$ that if $q_t \in S'_s$ and $t < t(s)$, then q_t is anticomplete to T'_s. Since $S'_{s+1} \subseteq S'_s$ and $T'_{s+1} \subseteq T'_s$, we deduce that if $q_t \in S'_{s+1}$, then $t > t(s)$. Let t be minimum such that $q_t \in S'_{s+1}$ and set $t(s+1) = t$. Then $t(s+1) > t(s)$, and the third bullet continues to hold. By the minimality of t, the second bullet continues to hold. It follows from the minimality of S'_{s+1} that there exists $n_{s+1} \in T'_{s+1}$ such that $N_{S'_{s+1}}(n_{s+1}) = \{q_{t(s+1)}\}$. Set $S_{s+1} = S_s \cup \{q_{t(s+1)}\}$ and $T_{s+1} = T_s \cup \{n_{s+1}\}$. Since T_s is anticomplete to T'_{s+1}, the first bullet continues to hold. Since $N_{S'_{s+1}}(n_{s+1}) = q_{t(s+1)}$, and since $S'_{s+1} \subseteq S'_s \setminus \{q_{t(s)}\}$, it follows that the fourth bullet continues to hold, and consequently the fifth bullet continues to hold. Thus our construction maintains the required properties. When we finish, write $N_i = \{n_1, \dots, n_m\}$ and $Q_i = \{q_{t(1)}, \dots, q_{t(m)}\}$, and we have:

(31) *The following statements hold.*

- *N_i is a stable set.*
- *For every $j \in \{1, \dots, m\}$, $N_{Q_i}(n_j) = q_{t(j)}$.*
- *If $j < j'$, then $t(j) < t(j')$.*
- *$T \cap N(\{p_i, \dots, p_{\text{end}(i)}\}) \subseteq N[Q_i \cup N_i]$.*

To simplify notation, we renumber the elements of Q_i, replacing each index $t(j)$ by j. In the new notation we have $Q_i = \{q_1, \dots, q_m\}$, and for every $j \in \{1, \dots, m\}$, $N_Q(n_j) = \{q_j\}$. Observe that $q_1, \dots, q_m$ appear in this order in P when P is traversed from p_1 to p_k. Let $L_i = p_1\text{-}P\text{-}q_1$, $M_i = q_1\text{-}P\text{-}q_m$ and $R_i = q_{m+1}\text{-}P\text{-}p_k$.

(32) *For every i, there exists $r_i \in N_i$ and $Y_i \subseteq (G \setminus N[P]) \cup M_i \cup \{r_i\}$ with $|Y_i| \leq m$ such that $N[Y_i]$ separates L_i from R_i.*

By (31), N_i is a stable set. Since $T \cap N(\{p_i, \dots, p_{\text{end}(i)}\}) \subseteq N[Q_i \cup N_i]$, and $|Q_i| = |N_i|$, it follows from (31) that $|N_i| \geq K$. Let $N'_i \subseteq N_i$ with $|N'_i| = K$. Recall that $K = \sigma(7)$. Now (32) follows from the definition of an m-amicable class with $D_1 = P$, $D_2 = B$ and $Y = N'_i$. This proves (32).

We may assume that for every i there is a component D_i of $G \setminus N[Z_i \cup Y_i]$ with $w(D_i) > \frac{1}{2}$, for otherwise the conclusion of Theorem 8.1 holds setting $Y = Z_i \cup Y_i$. By (32), either D_i is anticomplete to L_i, or D_i is anticomplete to R_i. Let us say that Q_i is of *type L* if D_i is anticomplete to L_i, and that Q_i is of *type R* if D_i is anticomplete to R_i. By (32), every Q_i is of at least one type.

(33) *Q_1 is of type L.*

By (29), $D_1 \cap (T \cup N[p_k]) \neq \emptyset$. It follows that D_1 contains a vertex t such that $t \in N[p_k]$ or $t \in T \setminus N[Z_1]$. Since $T \cap N(\{p_1, \dots, p_{\text{end}(1)}\}) \subseteq N[Z_1]$, and since $L_1 \cup M_1$ is a subpath of p_1-P-$p_{\text{end}(1)}$, it follows that D_1 contains a vertex of $N[R_1]$. Consequently, Q_1 is not of type R, and so Q_1 is of type L. This proves (33).

(34) *We may assume that Q_r is of type R.*

Suppose that Q_r is of type L. Then D_r is anticomplete to L_r. We claim that the set $Y = Z_r \cup Y_r \cup Y_0$ satisfies the conclusion of the theorem. Let D be a component of $G \setminus N[Y]$, and suppose that $w(D) > \frac{1}{2}$. By (29) there exists $t \in D \cap (T \cup N[p_k])$. Since $Z_r \cup Y_r \subseteq Y$, we have that $D \subseteq D_r$. It follows that D is anticomplete to L_r, and so $t \notin N(L_r)$. Since $T \cap N(\{p_r, \dots, p_{\text{end}(r)}\}) \subseteq N[Z_r]$, it follows that $t \notin \{p_r, \dots, p_{\text{end}(r)}\}$. We deduce that $t \in N(\{p_{\text{end}(r)+1}, \dots, p_k\})$. But then $t \in N[Y_0]$, contrary to the fact that $Y_0 \subseteq Y$. This proves the claim, and (34) follows.

From now on we make the assumption of (34). By (34) we can choose j minimum such that Q_j is of type R. By (33), $j > 1$. Let

$$Y = Z_{j-1} \cup Y_{j-1} \cup Z_j \cup Y_j \cup Y_0.$$

Then $|Y| \leq 6k+2m+1 = d$. We claim that $w(D) \leq \frac{1}{2}$ for every component D of $G \setminus N[Y]$. Suppose not, and let D be a component of $G \setminus N[Y]$ with $w(D) > \frac{1}{2}$. Since $Z_{j-1} \cup Y_{j-1} \subseteq Y$, we have that $D \subseteq D_{j-1}$. Since Q_{j-1} is of type L, it follows that D is anticomplete to L_{j-1}. Since $Z_j \cup Y_j \subseteq Y$, we have that $D \subseteq D_j$. Since Q_j is of type R, it follows that D is anticomplete to R_j. Since $T \cap N(\{p_{j-1}, \dots, p_{\text{end}(j)}\}) \subseteq N[Z_{j-1} \cup Z_j]$ and since $p_k \in Y_0$, we deduce that $D \cap (T \cup N[p_k]) = \emptyset$. But then $w(D) < \frac{1}{2}$ by (29), a contradiction. ■

9. From balanced to small

The second step in the proof of Theorem 1.5 is to transform balanced separators with small domination number into balanced separators of small size. We show:

Theorem 9.1. *Let t be an integer, let d be as in Theorem 8.1 and let c_t be as in Theorem 1.3. Let $G \in \mathcal{C}_t$ and let w be a weight function on G. Then there exists $Y \subseteq V(G)$ such that*

- $|Y| \leq 3c_t d \log n + t$, *and*
- Y *is a w-balanced separator in G.*

The proof uses Theorems 1.3 and 8.1. In order to prove Theorem 9.1, we first prove a more general statement (below). We will then explain how to deduce Theorem 9.1 from it.

Theorem 9.2. *Let $L > 0$ be an integer, let G be a graph and let w be a weight function on G. Assume that there is no L-banana in G. There exists a clique K in G with the following property. Let $X \subseteq V(G) \setminus K$ be such that $w(D) \leq \frac{1}{2}$ for every component D of $G \setminus (K \cup N[X])$. Then there exists a balanced separator Y in G such that $|Y \setminus K| \leq 3L|X|$.*

Proof. Let (T, χ) be a $3L$-atomic tree decomposition of G. By Theorems 2.2 and 2.3, we have that (T, χ) is tight and $3L$-lean. By Theorem 2.7, there exists $t_0 \in T$ such that t_0 is a center for T. A vertex $v \in V(G)$ is *t_0-friendly* if v is not separated from $\chi(t_0) \setminus v$ by a set of size $< 3L$. We show that t_0-friendly vertices have the following important property.

(35) *If $u, v \in \chi(t_0)$ are not t_0-friendly, then u is adjacent to v.*

Suppose that u and v are not t_0-friendly and that u is non-adjacent to v. Since there is no L-banana in G, Theorem 2.1 implies that there exists a set X with $|X| < L$ such X separates u from v. But this contradicts Theorem 2.4. This proves (35).

Let K be the set of all the vertices of G that are not t_0-friendly. By (35), K is a clique. We show that K satisfies the conclusion of Theorem 9.2. Let $X \subseteq G \setminus K$ be such that $w(D) \leq \frac{1}{2}$ for every component D of $G \setminus (K \cup N[X])$.

For every $v \in G \setminus K$, define the set $\Delta(v)$ as follows. Let $X_v \subseteq V(G)$ be such that X_v separates v from $\chi(t_0) \setminus v$, chosen with $|X_v|$ minimum. Let $\Delta(v) = X_v \cup \{v\}$. Then $|\Delta(v)| \leq 3L$ and $N_{\chi(t_0)}(v) \subseteq \Delta(v)$. Let

$$Y = \bigcup_{v \in X} \Delta(v).$$

It follows that $|Y| \leq 3L|X|$.

To complete the proof it is enough to show that $Y \cup K$ is a w-balanced separator of G. Suppose not, and let D be a component of $G \setminus (Y \cup K)$ with $w(D) > \frac{1}{2}$.

(36) *If $D \cap (G_{t_0 \to t} \setminus \chi(t_0)) \neq \emptyset$ for some $t \in N_T(t_0)$, then X is anticomplete to D.*

Suppose that $v \in X$ has a neighbor d in D. Let (A, X_v, B) be a separation such that $\chi(t_0) \setminus X_v \subseteq B$ and $v \in A$. Since $d \notin X_v$ and d is adjacent to v, it follows that $d \in A$. But then, since D is a component of $G \setminus (K \cup Y)$ and $X_v \subseteq Y$, it follows that $D \subseteq A$. Consequently, $D \cap \chi(t_0) = \emptyset$. It follows that $D \subseteq G_{t_0 \to t} \setminus \chi(t_0)$, and so $w(D) \leq w(G_{t_0 \to t} \setminus \chi(t_0)) \leq \frac{1}{2}$, a contradiction. This proves (36).

(37) $D \cap N[X] = \emptyset$.

Suppose there is $v \in X$ such that v has a neighbor $d \in D$. By (36), $d \in \chi(t_0)$. But then $d \in \Delta(v) \subseteq Y$, a contradiction. This proves (37).

By (37), D is a component of $G \setminus (K \cup N[X])$, and therefore $w(D) \leq \frac{1}{2}$, a contradiction. ■

We now prove Theorem 9.1.

Proof. Let K be as in Theorem 9.2. Let c_t be as in Theorem 1.3 and let $L = c_t \log n$. Let d be as Theorem 8.1. Define $w' : V(G) \setminus K \to [0,1]$ as $w'(v) = \frac{w(v)}{1-w(K)}$. Then w' is a normal weight function on $G \setminus K$. By Theorem 8.1, there exists $X \subseteq V(G) \setminus K$ such that

- $|X| \leq d$, and
- $N[X]$ is a w'-balanced separator in $G \setminus K$.

It follows that $w(D) \leq \frac{1}{2}$ for every component D of $G \setminus (N[X] \cup K)$. Now Theorem 9.2 applied with $L = c_t \log_n$ implies that there exists a w-balanced separator Y in G such that $|Y \setminus K| \leq 3L|X| = 3c_t(\log n)|X| \leq 3c_t d \log n$. Since $G \in \mathcal{C}_t$, it follows that $|Y| \leq 3c_t d \log n + t$. ■

10. The proof of Theorem 1.5

We are now ready to complete the proof of Theorem 1.5. The following result was originally proved by Robertson and Seymour in [33], and tightened by Harvey and Wood in [24]. It was then restated and proved in the language of (w, c)-balanced separators in an earlier paper of this series [4].

Theorem 10.1 (Robertson, Seymour [33], see also [4, 24]). *Let G be a graph, let $c \in [\frac{1}{2}, 1)$, and let d be a positive integer. If G has a (w, c)-balanced separator of size at most d for every normal weight function $w : V(G) \to [0,1]$, then $\text{tw}(G) \leq \frac{1}{1-c}k$.*

We prove the following, which immediately implies Theorem 1.5.

Theorem 10.2. *Let t be an integer. Let let c_t be as in Theorem 1.3 and let d be as in Theorem 1.4. Then every $G \in \mathcal{C}_t$ satisfies $\text{tw}(G) \leq 6c_t d \log n + 2t$.*

Proof. By Theorem 9.1, for every normal weight function w of G there exists w-balanced separator of size at most $3c_t d \log n + t$. Now the result follows immediately from Lemma 10.1. ■

11. Algorithmic consequences

We now summarize the algorithmic consequences of our structural results, specifically of Theorems 1.4 and 1.5.

The consequences for graphs in $\mathcal{C}_t$ are the most immediate. In particular, using the factor 2 approximation algorithm of Korhonen [26] (or the simpler factor 4 approximation algorithm of Robertson and Seymour [34]) for treewidth, we can compute a tree decomposition of width at most $O(c_t \log n)$ in time $2^{O(c_t \log n)} n^{O(1)} = n^{O(c_t)}$. A number of well-studied graph problems, including Stable Set, Vertex Cover, Feedback Vertex Set Dominating Set and r-Coloring (for fixed r) admit algorithms with running time $2^{O(k)}n$ when a tree decomposition of G of width k is provided as input (See, for example, [16] chapters 7 and 11, as well as [10]). Since $2^{O(c_t \log n)} = n^{O(c_t)}$ this immediately leads to polynomial time algorithms for these problems in $\mathcal{C}_t$.

Theorem 11.1. *Let $t \geq 1$ be fixed and* P *be a problem which admits an algorithm running in time $\mathcal{O}(2^{\mathcal{O}(k)}|V(G)|)$ on graphs G with a given tree decomposition of width at most k. Then* P *is solvable in time $n^{\mathcal{O}(t)}$ in $\mathcal{C}_t$. In particular, Stable Set, Vertex Cover, Feedback Vertex Set, Dominating Set and r-Coloring (with fixed r) are all polynomial-time solvable in $\mathcal{C}_t$.*

This list of problems is far from exhaustive, it is worth mentioning the work of Pilipczuk [32] who provides a relatively easy-to-check sufficient condition (expressibility in Existential Counting Modal Logic) for a problem to admit such an algorithm.

Theorem 11.1 implies a polynomial time algorithm for r-Coloring (with fixed r) in $\mathcal{C}$ (without any assumptions on max clique size) because every graph that contains a clique of size $r+1$ can not be r-colored. Thus, to solve r-Coloring we can first check for the existence of an $(r+1)$-clique in time $n^{r+\mathcal{O}(1)}$. If an $(r+1)$-clique is present, report that no r-coloring exists, otherwise invoke the algorithm of Theorem 11.1 with $t = r+1$. This yields the following result.

Theorem 11.2. *For every positive integer r, r-Coloring is polynomial-time solvable in $\mathcal{C}$.*

Let us now discuss another important problem, and that is Coloring. It is well-known (and also follows immediately from Theorem 7.1), that for every t there exists a number $d(t)$ such that all graphs in $\mathcal{C}_t$ have chromatic number at most $d(t)$. Also, for each fixed r, by Theorem 11.1, r-Coloring is polynomial-time solvable in $\mathcal{C}_t$. Now by solving r-Coloring for every $r \in \{1, \dots, d(t)\}$, we also obtain a polynomial-time algorithm for Coloring in $\mathcal{C}_t$.

Theorem 11.1 quite directly leads to a polynomial-time approximation scheme (PTAS) for several problems on graphs in $\mathcal{C}$. We illustrate this using Vertex Cover as an example.

Theorem 11.3. *There exists an algorithm that takes as input a graph $G \in \mathcal{C}$ and $0 < \epsilon \leq 1$, runs in time $n^{O(c_{2/\epsilon})}$ and outputs a vertex cover of size at most $(1+\epsilon)\mathsf{vc}(G)$, where $\mathsf{vc}(G)$ is the size of the minimum vertex cover of G.*

Proof. Check in time $n^{O(1/\epsilon)}$ whether G contains as input a clique C of size at least $2/\epsilon$. If G does not have a clique of size at least $2/\epsilon$ then compute an optimal vertex cover of G in time $n^{O(c_{2/\epsilon})}$ using the algorithm of Theorem 11.1. If G contains such a clique C then run the algorithm recursively on $G - C$ to obtain a vertex cover S of $G - C$ of size at most $(1+\epsilon)\mathsf{vc}(G-C)$. Return $S \cup C$; clearly $S \cup C$ is a vertex cover of G. Furthermore, every vertex cover of G (and in particular a minimum one) contains at least $|C|-1$ vertices of C. It follows that $\mathsf{vc}(G-C) \leq \mathsf{vc}(G) - |C| + 1$ and that therefore

$$\begin{aligned}
|S \cup C| &= |S| + |C| \\
&\leq (1+\epsilon)\mathsf{vc}(G-C) + |C| \\
&\leq (1+\epsilon)(\mathsf{vc}(G) - |C| + 1) + |C| \\
&\leq (1+\epsilon)(\mathsf{vc}(G))
\end{aligned}$$

Here the last inequality follows because for $C \geq 2/\epsilon$ it holds that $(1+\epsilon)(|C|-1) \geq |C|$. ■

The PTAS of Theorem 11.3 generalizes without modification (except for the constant 2 in the $2/\epsilon$) to Feedback Vertex Set, and more generally, to deletion to any graph class which is closed under vertex deletion, excludes some clique, and admits an algorithm (for the deletion problem) with running time $2^{O(k)}n^{O(1)}$ on graphs of treewidth k. Despite the tight connection between Vertex Cover and Stable Set, Theorem 11.3 does not directly lead to a PTAS for Stable Set on graphs in $\mathcal{C}$. On the other hand, a QPTAS for Stable Set in $\mathcal{C}$ follows from Theorem 1.4 together with an argument of Chudnovsky, Pilipczuk, Pilipczuk and Thomassé [14], who gave a QPTAS for Stable Set in P_k-free graphs (they refer to the problem as Independent Set). For ease of reference, we repeat their argument here.

Theorem 11.4. *There exists an algorithm that takes as input a graph $G \in \mathcal{C}$ and $0 < \epsilon \leq 1$, runs in time $n^{O(\log^2 n/\epsilon)}$ and outputs a stable set S in G of size at least $(1-\epsilon)\alpha(G)$, where $\alpha(G)$ is the size of the maximum size stable set in G.*

Proof. The algorithm is recursive, taking as input the graph G and also an integer N. Initially, the algorithm is called with $N = |V(G)|$, in all subsequent recursive calls the value of N remains the same. If G is a single vertex the algorithm returns $V(G)$. If G is disconnected, the algorithm runs itself recursively on each of the connected components and returns the union of the stable sets returned for each of them.

Let d be the constant of Theorem 1.4. If G is a connected graph on at least two vertices the algorithm iterates over all subsets $S \subseteq V(G)$ of size at most $\frac{2d \log n \log N}{\epsilon}$ and all subsets Y of size at most d such that each connected component of $G - (N(S) \cup N[Y])$ has at most $|V(G)|/2$ vertices. For each such pair (S, Y) the algorithm calls itself recursively on $G - (N(S) \cup N[Y])$ and returns the maximum size stable set found over all such recursive calls.

Clearly, the set returned by the algorithm is a stable set. For each pair (S, Y) which results in a recursive call of the algorithm, each connected component of the graph $G - (N(S) \cup N[Y])$ that the algorithm is called on has at most $|V(G)|/2$ vertices. Therefore the running time of the algorithm is governed by the recurrence $T(n) \leq n^{\frac{2d \log n \log N}{\epsilon}+d} \cdot n \cdot T(n/2)$ which solves to $T(n) \leq n^{O(\frac{d \log^2 n \log N}{\epsilon})}$. Since the algorithm is initially called with $N = |V(G)|$ the running time of the algorithm is upper bounded by $|V(G)|^{O(\frac{d \log^3 |V(G)|}{\epsilon})}$.

It remains to argue that the size of the stable set output by the algorithm is at least $(1-\epsilon)\alpha(G)$. To that end, we show that the size of the independent set output by a recursive call on (G, N) is at least $\alpha(G)(1 - \epsilon\frac{\log |V(G)|}{\log N})$. Let I be a maximum size stable set of G. By a standard coupon-collector argument (see e.g. Lemma 4.1 in [14]) there exists a choice of $S \subseteq I$ of size at most $\frac{d \log N}{\epsilon} 2 \log n$ such that no vertex of $G - N(S)$ has at least $|I|\frac{\epsilon}{d \log N}$ neighbors in I. Let Y now be the set (of size at most d) given by Lemma 1.4 applied to $G - N[S]$. Since no vertex in Y has at least $\frac{\epsilon}{d \log N}$ neighbors in I it follows that $|I - (N(S) \cup N[Y])| \geq |I|(1 - \frac{\epsilon}{\log N})$. Furthermore, each connected component of $G - (N(S) \cup N[Y])$ has at most $|V(G)|/2$ vertices. By the inductive hypothesis the recursive call on $(G - (N(S) \cup N[Y]), N)$ outputs a stable set of size at least

$$|I| \left(1 - \frac{\epsilon}{\log N}\right) \left(1 - \frac{\epsilon(\log |V(G)| - 1)}{\log N}\right) \geq |I| \left(1 - \epsilon \frac{\log |V(G)|}{\log N}\right).$$

Since $\alpha(G) = |I|$ the recursive call on (G, N) outputs a stable set of size at least $\alpha(G)(1 - \epsilon\frac{\log |V(G)|}{\log N})$, as claimed. ■

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