INDUCED SUBGRAPHS AND TREE DECOMPOSITIONS X. TOWARDS LOGARITHMIC TREEWIDTH FOR EVEN-HOLE-FREE GRAPHS

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ABSTRACT. A generalized t-pyramid is a graph obtained from a certain kind of tree (a subdivided star or a subdivided cubic caterpillar) and the line graph of a subdivided cubic caterpillar by identifying simplicial vertices. We prove that for every integer t there exists a constant $c(t)$ such that every n-vertex even-hole-free graph with no clique of size t and no induced subgraph isomorphic to a generalized t-pyramid has treewidth at most $c(t)\log n$. This settles a special case of a conjecture of Sintiari and Trotignon; this bound is also best possible for the class. It follows that several NP-hard problems such as STABLE SET, VERTEX COVER, DOMINATING SET and COLORING admit polynomial-time algorithms on this class of graphs. Results from this paper are also used in later papers of the series, in particular to solve the full version of the Sintiari-Trotignon conjecture.

1. INTRODUCTION

All graphs in this paper are finite and simple. Let $G = (V(G), E(G))$ be a graph. For a set $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X. For $X \subseteq V(G)$, $G \setminus X$ denotes the subgraph induced by $V(G) \setminus X$. In this paper, we use induced subgraphs and their vertex sets interchangeably.

For graphs G and H, we say that G *contains* H if some induced subgraph of G is isomorphic to H. For a family $\mathcal{H}$ of graphs, G *contains* $\mathcal{H}$ if G contains a member of $\mathcal{H}$. Finally, G is *$\mathcal{H}$-free* if G does not contain $\mathcal{H}$.

Let $v \in V(G)$. The *open neighborhood of* v, denoted by $N(v)$, is the set of all vertices in $V(G)$ adjacent to v. The *closed neighborhood of* v, denoted by $N[v]$, is $N(v) \cup \{v\}$. Let $X \subseteq V(G)$. The *open neighborhood of* X, denoted by $N(X)$, is the set of all vertices in $V(G) \setminus X$ with at least one neighbor in X. The *closed neighborhood of* X, denoted by $N[X]$, is $N(X) \cup X$. If H is an induced subgraph of G and $X \subseteq V(G)$, then $N_H(X) = N(X) \cap H$ and $N_H[X] = N_H(X) \cup (X \cap H)$. Let $Y \subseteq V(G)$ be disjoint from X. We say X is *complete* to Y if all possible edges with an end in X and an end in Y are present in G, and X is *anticomplete* to Y if there are no edges between X and Y.

For a graph $G = (V(G), E(G))$, a *tree decomposition* (T, χ) of G consists of a tree T and a map $\chi : V(T) \to 2^{V(G)}$ with the following properties:

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(i) For every $v \in V(G)$, there exists $t \in V(T)$ such that $v \in \chi(t)$.
(ii) For every $v_1v_2 \in E(G)$, there exists $t \in V(T)$ such that $v_1, v_2 \in \chi(t)$.
(iii) For every $v \in V(G)$, the subgraph of T induced by $\{t \in V(T) \mid v \in \chi(t)\}$ is connected.

For each $t \in V(T)$, we refer to $\chi(t)$ as a *bag of* (T, χ). The *width* of a tree decomposition (T, χ), denoted by width(T, χ), is $\max_{t \in V(T)} |\chi(t)| - 1$. The *treewidth* of G, denoted by tw(G), is the minimum width of a tree decomposition of G. The term "treewidth" and the study of the structure of graphs with large treewidth were introduced by Robertson and Seymour [17] as part of the Graph Minors series.

A *hole* in a graph is an induced cycle with at least four vertices. The *length* of a hole is the number of vertices in it. A hole is *even* it it has even length, and *odd* otherwise. The class of even-hole-free graphs has been studied extensively (see the survey [20]), but many open questions remain. Among them are several algorithmic problems: Stable Set, Vertex Cover, Dominating Set, k-Coloring and Coloring. The structural complexity of this class of graphs is further evidenced by the fact that there exist even-hole-free graphs of arbitrarily large tree-width [19] (even when the clique number is bounded). Closer examination of the constructions of [19] led the authors of [19] to make the following two conjectures (the *diamond* is the unique simple graph with four vertices and five edges):

Conjecture 1.1 (Sintiari and Trotignon [19])**.** *For every integer t, there exists a constant c_t such that every even-hole-free graph G with no diamond and no clique of size t satisfies* $\text{tw}(G) \leq C_t$.

Conjecture 1.2 (Sintiari and Trotignon [19])**.** *For every integer t, there exists a constant C_t such that every even-hole-free graph G with no clique of size t satisfies* $\text{tw}(G) \leq C_t \log |V(G)|$.

(In fact, [19] only states the two conjectures above for $t = 4$.) Conjecture 1.1 was recently proved in [6]. Here we prove a special case of Conjecture 1.2. We remark that the full version of the conjecture is proved, by a different set of authors, in a forthcoming paper in the series [11]. However, the contributions of the present work are of independent interest, as we will explain later. A *generalized t-pyramid* is a graph obtained from a certain kind of tree (a subdivided star or a subdivided cubic caterpillar) and the line graph of a subdivided cubic caterpillar by identifying their simplicial vertices (we give a precise definition later). We prove:

Theorem 1.3. *For every integer t, there exists a constant C_t such that every even-hole-free graph G with no clique of size t and no generalized t-pyramid satisfies* $\text{tw}(G) \leq c_t \log |V(G)|$.

We remark that the construction of [19] shows that the logarithmic bound of Theorem 1.3 is best possible for this class. Furthermore, before Theorem 1.3 was proved, the complexity of Stable Set, Vertex Cover, Dominating Set, k-Coloring and Coloring when restricted to this class of graphs was not known.

Given a graph G, a *path in G* is an induced subgraph of G that is a path. If P is a path in G, we write $P = p_1\text{-}\dots\text{-}p_k$ to mean that $V(P) = \{p_1, \dots, p_k\}$, and p_i is adjacent to p_j if and only if $|i - j| = 1$. We call the vertices p_1 and p_k the *ends of P*, and say that P is a path *from p_1 to p_k*. The *interior of P*, denoted by P^*, is the set $V(P) \setminus \{p_1, p_k\}$. The *length* of a path P is the number of edges in P. We denote by C_k a cycle with k vertices.

Next we describe a few types of graphs that we will need (see Figures 1 and 2). A *theta* is a graph consisting of three internally vertex-disjoint paths $P_1 = a\text{-}\dots\text{-}b$, $P_2 = a\text{-}\dots\text{-}b$, and $P_3 = a\text{-}\dots\text{-}b$, each of length at least 2, such that P_1^*, P_2^*, P_3^* are pairwise anticomplete. In this case we call a and b the *ends* of the theta.

A *prism* is a graph consisting of three vertex-disjoint paths $P_1 = a_1\text{-}\dots\text{-}b_1$, $P_2 = a_2\text{-}\dots\text{-}b_2$, and $P_3 = a_3\text{-}\dots\text{-}b_3$, each of length at least 1, such that $a_1a_2a_3$ and $b_1b_2b_3$ are triangles, and no edges exist between the paths except those of the two triangles.

Given an integer k, a *generalized k-pyramid* is a graph whose vertex set is the disjoint union of $k + 2$ paths $P, Q, R_1, \dots, R_k$, such that the following hold (here P is the bottom path in the graphs in Figure 2 and Q is the top path):

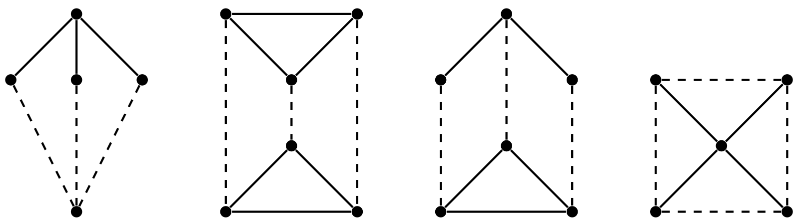

FIGURE 1. Theta, prism, pyramid and an even wheel. Dashed lines represent paths of length at least one.

- $P \cup Q$ is a hole.
- For every $i \in \{1, \ldots, k\}$, the path R_i has ends a_i and b_i.
- For every $i \in \{1, \ldots, k\}$, a_i has exactly two neighbors x_i, y_i in P. Moreover, $x_i, y_i \in P^*$ and x_i is adjacent to y_i.
- For every $i \in \{1, \ldots, k\}$, b_i has exactly one neighbor z_i in Q.
- P traverses $x_1, y_1, x_2, y_2, \ldots, x_k, y_k$ in this order.
- Q traverses $z_1, z_2, \ldots, z_k$ in this order (but note that some of these vertices may coincide).
- For every $i \in \{1, \ldots, k-1\}$, we have that $y_i \neq x_{i+1}$ (and so $x_1, y_1, x_2, y_2, \ldots, x_k, y_k$ are all distinct).

A generalized k-pyramid where $|Q^*| = 1$ is sometimes referred to as a *k-pyramid*, and a 1-pyramid is usually called a *pyramid*.

A *wheel* (H, x) consists of a hole H and a vertex x such that x has at least three neighbors in H. A wheel (H, x) is *even* if x has an even number of neighbors on H.

Our main result is a slight strengthening of Theorem 1.3. Let $\mathcal{C}$ be the class of (C_4, theta, prism, even wheel)-free graphs (these are sometimes called "C_4-free odd-signable graphs"). For every integer $t \geq 1$, let $\mathcal{C}_t$ be the class of all graphs in $\mathcal{C}$ with no clique of size t, and let $\mathcal{C}_{tt}$ be the class of all graphs in $\mathcal{C}_t$ that are also generalized t-pyramid-free. It is easy to see that every even-hole-free graph is in $\mathcal{C}$. We prove:

Theorem 1.4. *For every integer t, there exists a constant c_t such that every $G \in \mathcal{C}_{tt}$ satisfies* $\mathrm{tw}(G) \leq C_t \log |V(G)|$.

The general idea of the proof of 1.4 is similar to the proof of the main result of [2]. However, there were several technical steps in [2], dealing with so called "balanced vertices", that we managed to circumvent here, using a much more elegant approach of k-lean tree decompositions. In addition to making the proof less technical, using this tool also allowed us to only exclude generalized t-pyramids instead of pyramids (note that every generalized t-pyramid contains a pyramid).

1.1. **Contributions beyond the main result.** In this paper we introduce new techniques that are nicer than the techniques used earlier in the series. These include the use of k-lean tree decompositions, and the construction of Theorem 4.2 that is a way to emulate the well-known concept of "torsos" in the induced subgraph world.

Several results of the present paper are of independent interest, beyond their applications to Theorem 1.4. First and foremost, in Section 5 we prove several results describing the structure of minimal connected subgraphs containing the neighbors of a (large subset of) a given set of vertices. These structural results are used repeatedly in forthcoming papers in the series. Secondly, the full proof of [11] uses results of Section 3 of the current paper, as well as methods developed in Sections 4 and 7. Finally, this paper pushes the results of Section 6 to their limit. The bound obtained here does not seem to hold in more general settings, and methods of completely different nature were needed to make further progress.

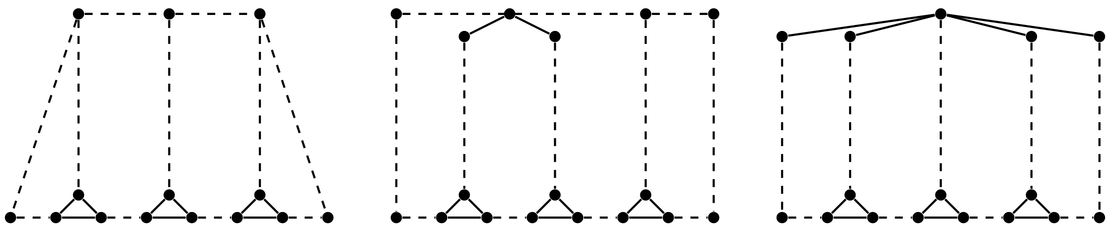

FIGURE 2. Examples of generalized 3-pyramids. Dashed lines represent paths of length at least one.

1.2. **Proof outline and organization.** Let us now discuss the main ideas of the proof of Theorem 1.4. We will give precise definitions of the concepts used below later in the paper; our goal here is to sketch a road map of where we are going. Obtaining a tree decomposition is usually closely related to producing a collection of "non-crossing separations," which roughly means that the separations "cooperate" with each other, and the pieces that are obtained when the graph is simultaneously decomposed by all the separations in the collection "line up" to form a tree structure.

We remark that most of our arguments work for graphs in the larger class $\mathcal{C}_t$; we will specify the point when we need to make the assumption that generalized t-pyramids are excluded. In the case of graphs in $\mathcal{C}_t$ (as well as some other graph classes addressed in other papers of this series), there is a natural family of separations to turn to; they correspond to special vertices of the graph called "hubs," and are discussed in Section 3. Unfortunately, these natural separations are very far from being non-crossing, and therefore we cannot use them in traditional ways to get tree decompositions. Similarly to [2], we use degeneracy to partition the set of all hubs (which yields a partition of all the natural separations) of an n-vertex graph G in $\mathcal{C}_t$ into collections $S_1, \dots, S_p$, where each S_i is "non-crossing" (this property is captured in Lemma 4.6), $p \leq C(t) \log n$ (where $C(t)$ only depends on t and works for all $G \in \mathcal{C}_t$) and each vertex of S_i has at most d (where d depends on t) neighbors in $\bigcup_{j=i}^{p} S_j$. Our main result is that the treewidth of G is bounded by a linear function of $p + \log n$.

First we will show that graphs in $\mathcal{C}_t$ that do not have hubs have treewidth that is bounded as a function of t; thus we may assume that $p > 0$. In fact, we will prove that there exists a constant k, depending on t, such that if two non-adjacent vertices u, v of a graph in $\mathcal{C}_t$ are joined by k vertex-disjoint paths $P_1, \dots, P_k$, then for at least one $i \in \{1, \dots, k\}$, the neighbor of v in P_i is a hub.

We now proceed as follows. Let $m = 2d + k$. We first consider a so-called m-lean tree decomposition (T, χ) of G (discussed in Section 2). By standard arguments on tree decompositions, we deduce that some bag $\chi(t_0)$ of (T, χ) is in some sense central to the tree decomposition, and we focus on it. We can then show that all but at most one vertex of S_1 has bounded degree in the torso of $\chi(t_0)$. We would like to use this fact in order to construct a tree decomposition of the torso of $\chi(t_0)$ and then use it to obtain a tree decomposition of G. Unfortunately, the torso of $\chi(t_0)$ is not a graph in $\mathcal{C}_t$. Instead, we find an induced subgraph of G, which we call β, that consists of $\chi(t_0)$ together with a collection of disjoint vertex sets $Conn(t)$ for $t \in N_T(t_0)$, where each $Conn(t)$ "remembers" the component of $G \setminus \chi(t_0)$ that meets $\chi(t)$. Moreover, no vertex of $\beta \setminus \chi(t_0)$ is a hub of β, and all but one vertex of S_1 have bounded degree in β.

Next, we decompose β, simultaneously, by all the separations corresponding to the hubs in S_1 whose degree in β is bounded, and delete the unique vertex of S_1 of high degree (if one exists). We denote the resulting graph by $\beta^A(S_1)$ and call it the "central bag" for S_1. The parameter p is smaller for $\beta^A(S_1)$ than it is for G, and so we can use induction to obtain a bound on the treewidth of $\beta^A(S_1)$. We then start with a special optimal tree decomposition of $\beta^A(S_1)$, where each bag is a "potential maximal clique" (see Section 2). Also inductively (this time on the number of vertices) we have tree decompositions for each component of $\beta \setminus \beta^A(S_1)$.

Now we use the special nature of our "natural separations" and properties of potential maximal cliques to combine the tree decompositions above into a tree decomposition of β, where the size of the bag only grows by an additive constant.

Then we repeat a similar procedure to combine the tree decomposition of β that we just obtained with tree decompositions of the components of $G \setminus \chi(t_0)$ to obtain a tree decomposition of G, where again the size of the bag only grows by an additive constant.

Let us now discuss how we obtain the bound on the growth of a bag. In the first step of the growing process, we add to each existing bag B the neighbor sets of the vertices of $S_1 \cap B$. The number of vertices of S_1 in each bag is bounded by Theorem 6.3, because no vertex of S_1 is a hub in $\beta^A(S_1)$ (this is proved in Theorem 4.7).

In the second growing step, we first turn the tree decomposition of β that we just constructed into a tree decomposition of the same width with the additional property that every bag is a potential maximal clique of G. Next, for each bag B of this tree decomposition, and for every $t \in N_T(t_0)$ such that $B \cap Conn(t) \neq \emptyset$, we add to B the adhesion $\chi(t_0) \cap \chi(t)$. One of the properties of m-lean tree decompositions is that the size of each adhesion is bounded. The number of adhesions added to a given bag is again bounded by Theorem 6.3 since distinct sets $Conn(t)$ are pairwise disjoint and anticomplete to each other, and no vertex of $Conn(t)$ is a hub.

Theorem 6.3 is the only result in the paper that uses the stronger assumption that generalized t-pyramids are excluded.

The paper is organized as follows. In Section 2, we discuss several types of tree decompositions that we use in this paper. In Section 3, we summarize results guaranteeing the existence of useful separations. In Section 4, we discuss the construction of the graphs β and $\beta^A(S_1)$, and how to use their tree decompositions to obtain a tree decomposition of G. In Section 5, we analyze the structure of minimal separators in graphs of $\mathcal{C}_t$. In Section 6.3, we use the results of Section 5 to obtain a bound on the size of a stable set of non-hubs in a potential maximal clique of a graph in $\mathcal{C}_{tt}$. Section 7 puts together the results of all the previous sections to prove Theorem 1.4. Finally, Section 8 discusses algorithmic consequences of Theorem 1.4.

2. Special tree decompositions and connectivity

In this section we discuss several known results related to connectivity, and a describe a few special kinds of tree decompositions. Let G be a graph. Let X, Y, Z be subsets of $V(G)$. By a *path from Y to Z* we mean a path from some $y \in Y$ to some $z \in Z$. We say that X *separates Y from Z* (in G) if every path P with an end in Y and an end in Z satisfies $P \cap X \neq \emptyset$. In this case we also say that *Y is separated from Z by X*. We start by recalling a classical result of Menger:

Theorem 2.1 (Menger [16])**.** *Let $k \geq 1$ be an integer, let G be a graph and let $X, Y \subseteq V(G)$ with $|X| = |Y| = k$. Then either there exists $M \subseteq V(G)$ with $|M| < k$ such that M separates X from Y, or or there are k pairwise vertex-disjoint paths in G from X to Y.*

Theorem 2.1 immediately implies:

Theorem 2.2 (Menger [16])**.** *Let $k \geq 1$ be an integer, let G be a graph and let $u, v \in G$ be distinct and non-adjacent. Then either there exists a set $M \subseteq G \setminus \{u, v\}$ with $|M| < k$ such that M separates u and v in G, or there are k pairwise internally vertex-disjoint paths in G from u to v.*

For a tree T and vertices $t, t' \in V(T)$, we denote by tTt' the unique path of T from t to t'. Let (T, χ) be a tree decomposition of a graph G. For every $x \in V(T)$, the *torso at x*, denoted by $\hat{\chi}(x)$, is the graph obtained from the bag $\chi(x)$ by, for each $y \in N_T(x)$, adding an edge between every two non-adjacent vertices $u, v \in \chi(x) \cap \chi(y)$. For every $uv \in E(T)$, the *adhesion at* uv, denoted by $adh(uv)$, is the set $\chi(u) \cap \chi(v)$. We define $adh(T, \chi) = \max_{uv \in E(T)} |adh(uv)|$.

In the proof of Theorem 1.4, we will use several special kinds of tree decompositions that we explain now.

2.1. **Potential Maximal Cliques.** For a graph G and a set $F \subseteq \binom{V(G)}{2} \setminus E(G)$, we denote by $G + F$ the graph obtained from G by making the pairs in F adjacent, that is, $E(G + F) = E(G) \cup F$. A set $F \subseteq \binom{V(G)}{2} \setminus E(G)$ is a *chordal fill-in* of G if $G + F$ is chordal; in this case, $G+F$ is a chordal completion of G. A chordal fill-in (and the corresponding chordal completion) is *minimal* if it is inclusion-wise minimal.

Let $X \subseteq V(G)$. The set X is a *minimal separator* if there exist $u, v \in V(G)$ such that u and v are in different connected components of $G \setminus X$, and u and v are in the same connected component of $G \setminus Y$ for every $Y \subsetneq X$. A component D of $G \setminus X$ is a *full component* for X if $N(D) = X$. It is well-known that a set $X \subseteq V(G)$ is a minimal separator if and only if there are at least two distinct full components for X.

A *potential maximal clique* (PMC) of a graph G is a set $\Omega \subseteq V(G)$ such that Ω is a maximal clique of some minimal chordal completion $G+F$ of G. The following result characterizes PMCs:

Theorem 2.3 (Bouchitté and Todinca [9])**.** *A set $\Omega \subseteq V(G)$ is a PMC of G if and only if:*

(1) for all distinct $x, y \in \Omega$ with $xy \notin E(G)$, there exists a component D of $G \setminus \Omega$ such that $x, y \in N(D)$; and
(2) for every component D of $G \setminus \Omega$ it holds that $N(D) \neq \Omega$ (that is, there are no full components for Ω).

We also need the following result relating PMCs and minimal separators:

Theorem 2.4 (Bouchitté and Todinca [9])**.** *Let $\Omega \subseteq V(G)$ be a PMC of G. Then, for every component D of $G \setminus \Omega$, the set $N(D)$ is a minimal separator of G.*

We remind the reader of the following well-known property of tree decompositions:

Theorem 2.5 (see Diestel [13])**.** *Let G be a graph, let K be a clique of G, and let (T, χ) be a tree decomposition of G. Then, there is $v \in V(T)$ such that $K \subseteq \chi(v)$.*

Let us say that a tree decomposition (T, χ) of a graph G is *structured* if $\chi(v)$ is a PMC of G for every $v \in V(T)$. We need the following well-known result which allows us to turn a tree decomposition into a structured tree decomposition; we include the proof for completeness.

Theorem 2.6. *Let G be a graph and let (T, χ) be a tree decomposition of G. There exists a structured tree decomposition (T', χ') of G such that for every $v' \in T'$ there exists $v \in T$ with $\chi'(v') \subseteq \chi(v)$.*

Proof. Let (T, χ) be a tree decomposition of G. It is easy to check that the graph G' obtained from G by adding all edges xy such that $x, y \in \chi(v)$ for some $v \in V(T)$ is chordal and that (T, χ) is a tree decomposition of G'. It follows that there exists a minimal chordal fill-in F of G such that $F \subseteq E(G') \setminus E(G)$; let $G'' = G + F$. In particular, every clique of G'' is a subset of a clique of G'. Since by Theorem 2.5 every clique of G' is contained in a bag $\chi(v)$ for some $v \in V(T)$, it follows that every clique of G'' is contained in a bag $\chi(v)$ for some $v \in T$.

Next, since G'' is chordal, there is a tree decomposition (T'', χ'') of G'' such that $\chi''(v)$ is a clique of G'' (and therefore a PMC of G) for every $v \in V(T'')$. Lastly, since G is a subgraph of G'', it follows that (T'', χ'') is a tree decomposition of G. This proves Theorem 2.6. ■

2.2. **Lean tree decompositions.** Let $k > 0$ be an integer. A tree decomposition (T, χ) is called *k-lean* if the following hold:

- $adh(T, \chi) < k$; and

- for all $t, t' \in V(T)$ and sets $Z \subseteq \chi(t)$ and $Z' \subseteq \chi(t')$ with $|Z| = |Z'| \leq k$, either G contains $|Z|$ disjoint paths from Z to Z', or some edge ss' of tTt' satisfies $|adh(ss')| < |Z|$.

For a tree T and an edge tt' of T, we denote by $T_{t \to t'}$ the component of $T \setminus t$ containing t'. Let $G_{t \to t'} = G[\bigcup_{v \in T_{t \to t'}} \chi(t)]$. A tree decomposition (T, χ) is *tight* if for every edge $tt' \in E(T)$ there is a component D of $G_{t \to t'} \setminus \chi(t)$ such that $\chi(t) \cap \chi(t') \subseteq N(D)$ (and therefore $\chi(t) \cap \chi(t') = N(D)$).

The following definition first appeared in [7]. Given a tree decomposition (T, χ) of an n-vertex graph G, its *fatness* is the vector $(a_n, \dots, a_0)$ where a_i denotes the number of bags of T of size i. A tree decomposition (T, χ) of G is k*-atomic* if $adh(T, \chi) < k$ and the fatness of (T, χ) is lexicographically minimum among all tree decompositions of G with adhesion less than k.

The following is immediate from the definition:

Theorem 2.7. *For every $k > 1$, every graph admits a k-atomic tree decomposition.*

It was observed in [10] that [7] contains a proof of the following:

Theorem 2.8 (Bellenbaum and Diestel [7], see Carmesin, Diestel, Hamann, Hundertmark [10], see also Weißauer [21])**.** *Every k-atomic tree decomposition is k-lean.*

The same proof also gives the following, which is in fact part of the definition of k-leanness in [10]:

Theorem 2.9 (Bellenbaum and Diestel [7], see Carmesin, Diestel, Hamann, Hundertmark [10], see also Weißauer [21])**.** *Let (T, χ) be a k-atomic tree decomposition of G and let tt' be an edge of T. Then both $\chi(t) \setminus \chi(t') \neq \emptyset$ and $\chi(t') \setminus \chi(t) \neq \emptyset$.*

We also need the following:

Theorem 2.10 (Weißauer [21])**.** *Every k-atomic tree decomposition is tight.*

We also have the following, which can be easily deduced by combining Lemmas 7 and 9 of Weißauer [21] and using Theorem 2.8.

Theorem 2.11. *Let G be a graph, let $k \geq 3$ and let (T, χ) be k-atomic tree decomposition of G. Let $t \in V(T)$. If $u, v \in \chi(t)$ have degree at least $(2k-2)(k-2)$ in $\hat{\chi}(t)$, then u and v are not separated in G by a set $X \subseteq V(G) \setminus \{u, v\}$ of size less than k.*

Using Theorem 2.2, we deduce:

Theorem 2.12. *Let G be a graph, let $k \geq 3$ and let (T, χ) be k-atomic tree decomposition of G. Let $t \in V(T)$. If $u, v \in \chi(t)$ have degree at least $(2k-2)(k-2)$ in $\hat{\chi}(t)$, then there are k pairwise internally vertex-disjoint paths in G from u to v.*

We finish this subsection with a theorem about tight tree decompositions in theta-free graphs. Note that by Theorem 2.10, the following result applies in particular to k-atomic tree decompositions for every k.

A *cutset* $C \subseteq V(G)$ of G is a (possibly empty) set of vertices such that $G \setminus C$ is disconnected. A *clique cutset* is a cutset that is a clique.

Theorem 2.13. *Let G be a theta-free graph and assume that G does not admit a clique cutset. Let (T, χ) be a tight tree decomposition of G. Then for every edge t_1t_2 of T the graph $G_{t_1 \to t_2} \setminus \chi(t_1)$ is connected and $N(G_{t_1 \to t_2} \setminus \chi(t_1)) = \chi(t_1) \cap \chi(t_2)$. Moreover, if $t_0, t_1, t_2 \in V(T)$ and $t_1, t_2 \in N_T(t_0)$, then $\chi(t_0) \cap \chi(t_1) \neq \chi(t_0) \cap \chi(t_2)$.*

Proof. Suppose that $G_{t_1 \to t_2} \setminus \chi(t_1)$ is not connected. Then there exists a component D_1 of $G_{t_1 \to t_2} \setminus \chi(t_1)$ such that $N(D_1) = \chi(t_1) \cap \chi(t_2)$. Let D_0 be a component of $G_{t_1 \to t_2} \setminus \chi(t_1)$ different from D_1. Since (T, χ) is tight, there exists a component D_2 of $G_{t_2 \to t_1} \setminus \chi(t_2)$ such that $N(D_2) = \chi(t_1) \cap \chi(t_2)$. Since $N(D_0)$ is not a clique cutset in G, there exist non-adjacent

$x, y \in N(D_0) \subseteq \chi(t_1) \cap \chi(t_2)$. But now we get a theta with ends x, y and paths with interiors in D_0, D_1 and D_2, respectively, a contradiction. This proves that $G_{t_1 \to t_2} \setminus \chi(t_1)$ is connected.

To prove the second assertion, let $t_1, t_2 \in N_T(t_0)$ and assume that $\chi(t_0) \cap \chi(t_1) = \chi(t_0) \cap \chi(t_2)$. Then, there exist $v_0 \in \chi(t_0) \setminus \chi(t_2)$ and $v_2 \in \chi(t_2) \setminus \chi(t_0)$. Since $\chi(t_0) \cap \chi(t_1) = \chi(t_0) \cap \chi(t_2)$ and $\chi(t_1) \cap \chi(t_2) \subseteq \chi(t_0)$, it follows that $v_0, v_2 \notin \chi(t_1)$. But then v_0 and v_2 belong to different components of $G_{t_1 \to t_0} \setminus \chi(t_1)$, contrary to the claim of the previous paragraph. This proves Theorem 2.13. ■

2.3. **Centers of tree decomposition.** We discuss another important feature of tree decompositions that we need. Let G be an n-vertex graph and let (T, χ) be a tree decomposition of G. A vertex t_0 of T is a *center* of (T, χ) if for every $t' \in N_T(t_0)$ we have $|G_{t_0 \to t'} \setminus \chi(t_0)| \leq \frac{n}{2}$. The following lemma is a analogous to the standard proof that every tree has a centroid.

Theorem 2.14. *Let (T, χ) be a tree decomposition of a graph G. Then (T, χ) has a center.*

Proof. Write $|V(G)| = n$. Let D be the directed graph obtained from T as follows. Let tt' be an edge of T; direct tt' from t to t' if $|G_{t \to t'} \setminus \chi(t)| > \frac{n}{2}$. As $G_{t \to t'} \setminus \chi(t)$ is disjoint from $G_{t' \to t} \setminus \chi(t')$ (because $G_{t \to t'} \cap G_{t' \to t} = \chi(t) \cap \chi(t')$), this does not prescribe conflicting orientations for edges of T. For each edge whose direction is not determined by this, direct it arbitrarily.

It follows that D has a sink t_0. Let $t' \in N_T(t_0)$. Then, because the edge $t_0 t'$ is not directed from t_0 to t', it follows that $|G_{t_0 \to t'} \setminus \chi(t_0)| \leq \frac{n}{2}$. This proves Theorem 2.14. ■

2.4. **Small separators.** We conclude this section by proving one more straightforward and well-known lemma about tree decompositions.

Lemma 2.15. *Let G be a graph. Let $X \subseteq V(G)$, and let $D_1, \ldots, D_s$ be the components of $G \setminus X$. Then* $\mathrm{tw}(G) \leq |X| + \max_{i \in \{1,\ldots,s\}} \mathrm{tw}(D_i)$.

Proof. For every i, we let (T_i, χ_i) be a tree decomposition of D_i of width $\mathrm{tw}(D_i)$. Let T be a tree obtained from the union of $T_1, \ldots, T_s$ by adding a new vertex t and making t adjacent to exactly one vertex of each T_i. We now construct a tree decomposition (T, χ) of G. Let $\chi(t) = X$, and for every $t' \in T \cap T_i$ let $\chi(t') = \chi_i(t) \cup X$. It is easy to check that (T, χ) is a tree decomposition of G, and $\mathrm{width}(T, \chi) \leq \max_{i \in \{1,\ldots,s\}} \mathrm{tw}(D_i) + |X|$. ■

3. Star cutsets, wheels and blocks

A *star cutset* in a graph G is a cutset $S \subseteq V(G)$ such that either $S = \emptyset$ or for some $x \in S$, $S \subseteq N[x]$.

Recall that a *wheel* (H, x) of G consists of a hole H and a vertex x that has at least three neighbors in H (and therefore $x \notin H$). A *sector* of (H, x) is a path P of H whose ends are distinct and adjacent to x, and such that x is anticomplete to P^*. A sector P is a *long sector* if P^* is non-empty. We now define several types of wheels that we will need.

A wheel (H, x) is a *universal wheel* if x is complete to H. A wheel (H, x) is a *twin wheel* if $N(x) \cap H$ induces a path of length two. A wheel (H, x) is a *short pyramid* if $|N(x) \cap H| = 3$ and x has exactly two adjacent neighbors in H. A wheel is *proper* if it is not a twin wheel or a short pyramid. We say that $x \in V(G)$ is a *wheel center* or a *hub* if there exists H such that (H, x) is a proper wheel in G. We denote by $\mathrm{Hub}(G)$ the set of all hubs of G.

We need the following result, which was observed in [1]:

Theorem 3.1 (Abrishami, Chudnovsky, Vušković [1])**.** *Let $G \in \mathcal{C}$ and let (H, v) be a proper wheel in G. Then there is no component D of $G \setminus N[v]$ such that $H \subseteq N[D]$.*

The majority of this paper is devoted to dealing with hubs and star cutsets arising from them in graphs in $\mathcal{C}$, but in the remainder of this section we focus on the case when $\mathrm{Hub}(G) = \emptyset$. To do that, we combine several earlier results from this series.

Let k be a positive integer and let G be a graph. A *k-block* in G is a set B of at least k vertices in G such that for every $\{x, y\} \subseteq B$, there exists a collection $\mathcal{P}_{\{x,y\}}$ of at least k distinct and pairwise internally vertex-disjoint paths in G from x to y. A slight strengthening of the following was proved in [5]:

Theorem 3.2 (Abrishami, Alecu, Chudnovsky, Hajebi, Spirkl [5])**.** *For all integers $k, t \geq 1$, there exists an integer $\beta = \beta(k, t)$ such that if $G \in \mathcal{C}_t$ and G has no k-block, then* $\operatorname{tw}(G) \leq \beta(k, t)$.

We also need the following result from [5]:

Theorem 3.3. *For all integers $\nu, t \geq 1$, there exists an integer $\psi = \psi(t, \nu) \geq 1$ with the following property. Let $G \in \mathcal{C}_t$, let $a, b \in G$ be distinct and non-adjacent and let $\{P_i : i \in [\psi]\}$ be a collection of ψ pairwise internally disjoint paths in G from a to b. For each $i \in [\nu]$, let a_i be the neighbor of a in P_i (so $a_i \neq b$). Then there exists $I \subseteq [\phi]$ with $|I| = \nu$ for which the following holds.*

- *$\{a_i : i \in I\} \cup \{b\}$ is a stable set in G.*
- *For all $i, j \in I$ with $i < j$, a_i has a neighbor in $P_j^* \setminus \{a_j\}$.*

Additionally, we use a result of [3]:

Theorem 3.4 (Abrishami, Alecu, Chudnovsky, Hajebi, Spirkl [3])**.** *Let G be an even wheel-free graph, let H be a hole of G, and let $v_1, v_2 \in V(G)$ be adjacent vertices each with at least two non-adjacent neighbors in H. Then, v_1 and v_2 have a common neighbor in H.*

We also remind the reader the following well-known version of Ramsey's theorem [18]:

Theorem 3.5. *For all positive integers a, b, c , there is a positive integer $M = M(a, b, c)$ such that if G is a graph with no K_a and no induced subgraph isomorphic to $K_{b,b}$, and G contains a collection $\mathcal{M}$ of M pairwise disjoint subsets of vertices, each of size at most a, then some two members of $\mathcal{M}$ are anticomplete to each other.*

Next we show:

Theorem 3.6. *For every integer $t \geq 1$ there exists an integer $k = k(t)$ such that if $G \in \mathcal{C}_t$ and $x, y \in V(G)$ are non-adjacent and $N(x) \cap \operatorname{Hub}(G) = \emptyset$, then there do not exist k pairwise internally vertex-disjoint paths in G from x to y.*

Proof. Let $M = M(2, t, 2)$ be as in Theorem 3.5 and let $\nu = \psi(\psi(M, t), t)$ be as in Theorem 3.3. Let $k = \psi(6\nu + 3, t)$ be as in Theorem 3.3, and suppose that there are k vertex disjoint paths from x to y in G. Let I be as in the conclusion of Theorem 3.3 applied with $a = x$ and $b = y$; renumbering the paths if necessary we may assume that $I = \{1, \ldots, 6\nu + 3\}$. Then each of a_1, a_2, a_3 has a neighbor in each of the paths $P_4, \ldots, P_{6\nu+3}$. For $i \geq 4$, let Q_i be a minimal subpath of P_i such that all of a_1, a_2, a_3 have neighbors in Q_i; let l_i and r_i be the ends of Q_i. By the minimality of Q_i, for every i there exist distinct $s, r \in \{a_1, a_2, a_3\}$ such that l_i is the unique neighbor of s in Q_i, and r_i is the unique neighbor of r in Q_i. By permuting a_1, a_2 and a_3 if necessary, we may assume that there exists $J \subseteq \{4, \ldots, 6\nu + 3\}$ with $|J| = \nu$ such that for every $i \in J$, l_i is the unique neighbor of a_1 in Q_i, and r_i is the unique neighbor of a_3 in Q_i. Renumbering $a_4, \ldots a_{6\nu} + 3$ we write $J = \{4, \ldots, \nu + 3\}$. For every $i \in J$ let H_i be the hole a_1-l_i-Q_i-r_i-a_3-x-a_1. Since (H_i, a_2) is not a proper wheel (recall that $N(x) \cap \operatorname{Hub}(G) = \emptyset$), and since $H_i \cup a_2$ is not a theta, it follows that a_2 has exactly two neighbors x_i, y_i in Q_i and they are adjacent. We may assume that Q_i traverses l_i, x_i, y_i, r_i in this order.

We now apply Theorem 3.3 again to the paths a_1-l_i-Q_i-x_i-a_2 (with $i \in \{4, \ldots, \nu+3\}$) from a_1 to a_2 with $a = a_2$ and $b = a_1$. Let K be the set of indices as in the conclusion of the theorem, so $|K| = \psi(M, t)$. Now apply Theorem 3.3 for the third time, to the paths a_3-r_i-Q_i-y_i-a_2 (with $i \in K$) from a_2 to a_3 with $a = a_2$ and $b = a_3$. Let L be the set of indices as in the conclusion of the theorem. Since $G \in \mathcal{C}_t$, applying Theorem 3.5 to the collections $\mathcal{M} = \{\{x_i, y_i\} : i \in L\}$ we deduce that there exist two values $i < j \in L$ such that

- $\{x_i, y_i\}$ is anticomplete to $\{x_j, y_j\}$, and
- x_i has a neighbor in l_j-Q_j-x_j, and
- y_i has a neighbor in y_j-Q_j-r_j.

We claim that x_i has exactly two neighbors in Q_j and they are adjacent. Suppose first that x_i has a unique neighbor u in Q_j. Then $u \in l_j$-Q_j-x_j, and the hole l_j-Q_j-x_j-a_2-x-a_1-l_j together with x_i is a theta with ends a_2, u, a contradiction. Next suppose that x_i has at least two non-adjacent neighbors in Q_j. Then by Theorem 3.4 applied to H_j, a_2 and x_i, it follows that a_2 and x_i have a common neighbor in H_j, and therefore x_i is adjacent to one of x_j, y_j, a contradiction. We deduce that x_i has exactly two neighbors p_j, q_j in Q_j and they are adjacent. Since x_i is non-adjacent to x_j and has a neighbor in l_j-Q_j-x_j, it follows that $p_j, q_j \in l_j$-Q_j-x_j. Similarly y_i has exactly two neighbors s_j, t_j in Q_j, s_j is adjacent to t_j, and $s_j, t_j \in y_j$-Q_j-r_j. We may assume that Q_j traverses $l_j, p_j, q_j, x_j, y_j, s_j, t_j, r_j$ in this order (possibly $l_j = p_j$ or $t_j = r_j$). But now we get a prism with triangles $x_i p_j q_j$ and $y_i t_j s_j$ and paths x_i-y_i, q_j-Q_j-s_j and p_j-Q_j-l_j-a_1-x-a_3-r_j-Q_j-t_j, a contradiction. This proves Theorem 3.6. ■

From Theorems 3.2 and 3.6 we deduce:

Theorem 3.7. *For every integer t, there exists an integer $\gamma = \gamma(t)$ such that every $G \in \mathcal{C}_t$ with $\mathrm{Hub}(G) = \emptyset$ satisfies $\mathrm{tw}(G) \leq \gamma$.*

4. Stable sets of safe hubs

Let $c \in [0, 1]$. A set $X \subseteq V(G)$ is a *c-balanced separator* if $|D| \leq c|V(G)|$ for every component D of $G \setminus X$. The set X is a *balanced separator* if X is a $\frac{1}{2}$-balanced separator.

Let t, d be integers. We make the following assumptions throughout this section. Let $G \in \mathcal{C}_t$ with $|V(G)| = n$. Let $m = k + 2d$ where $k = k(t)$ is as in Theorem 3.6, and assume that G does not have a balanced separator of size less than m. Let (T, χ) be an m-atomic tree decomposition of G. We say that a vertex v is *d-safe* if $|N(v) \cap \mathrm{Hub}(G)| \leq d$.

By Theorems 2.8 and 2.10, we have that (T, χ) is tight and m-lean. By Theorem 2.14, there exists $t_0 \in T$ such that t_0 is a center for T. A vertex $v \in V(G)$ is *t_0-cooperative* if either $v \notin \chi(t_0)$, or $\deg_{\hat{\chi}(t_0)}(v) < 2m(m-1)$. We show that t_0-cooperative vertices have the following important property.

Lemma 4.1. *If $u, v \in \chi(t_0)$ are d-safe and not t_0-cooperative, then u is adjacent to v.*

Proof. Since u, v are not t_0-cooperative, it follows that both u, v have degree at least $2m(m-1)$ in $\hat{\chi}(t_0)$. Since (T, χ) is an m-atomic tree decomposition, Theorem 2.12 implies that there are m pairwise internally disjoint paths in G from u to v. Let $Y = (N(u) \cup N(v)) \cap \mathrm{Hub}(G)$ and let $G' = G \setminus Y$. Then $|Y| \leq 2d$, and so in G' there are k pairwise internally vertex disjoint paths from u to v. But $N_{G'}(u) \cap \mathrm{Hub}(G') = N_{G'}(v) \cap \mathrm{Hub}(G') = \emptyset$, contrary to Theorem 3.6. This proves Lemma 4.1. ■

In the theory of tree decompositions, working with the torso of a bag is a natural way to focus on the bag while still keeping track of its relation to the rest of the graph. Unfortunately, taking torsos is not an operation closed under induced subgraphs, and so this method does not work when general hereditary classes are considered. The goal of the next theorem is to design a tool that will allow us to construct a safe alternative to torsos.

Theorem 4.2. *Assume that G does not admit a clique cutset. For every $t \in N_T(t_0)$, there exists $Conn(t) \subseteq G_{t_0 \to t}$ such that*

(1) We have $\chi(t) \cap \chi(t_0) \subseteq Conn(t)$.
(2) $Conn(t) \setminus \chi(t_0)$ is connected and $N(Conn(t) \setminus \chi(t_0)) = \chi(t_0) \cap \chi(t)$.

(3) No vertex of $Conn(t) \setminus \chi(t_0)$ is a hub in the graph $(G \setminus G_{t_0 \to t}) \cup Conn(t)$.

Proof. Write $\chi(t_0) \cap \chi(t) = M$. Let K be a minimal connected subgraph of $G_{t_0 \to t} \setminus \chi(t_0)$ such that $\chi(t_0) \cap \chi(t) \subseteq N(K)$ (such K exists by Theorem 2.13). Let $M = \{m_1, \dots, m_s\}$. Since G does not admit a clique cutset, we have that $s \geq 2$.

(1) *No vertex of K is a hub of $(G \setminus G_{t_0 \to t}) \cup (K \cup M)$.*

Let $v \in K$ and let (H, v) be a proper wheel in $G' = (G \setminus G_{t_0 \to t}) \cup (K \cup M)$. Then (H, v) is a proper wheel in G. Since $v \in K$, it follows that $N_G(v) \subseteq G_{t_0 \to t}$. By Theorem 2.13, $G \setminus G_{t_0 \to t}$ is connected, and since (T, χ) is tight, every vertex of M has a neighbor in $G \setminus G_{t_0 \to t}$. It follows that $G \setminus G_{t_0 \to t}$ is contained in a component D of $G \setminus N[v]$, and $M \subseteq N_G[D]$. Theorem 3.1 implies that $H \not\subseteq N_G[D]$. Since $H \subseteq G' \setminus \{v\} = (G \setminus G_{t_0 \to t}) \cup M \cup (K \setminus \{v\}) \subseteq N_G[D] \cup (K \setminus \{v\})$, it follows that H contains a vertex in $(K \setminus \{v\}) \setminus N_G[D]$. Let $K' = \{v\} \cup (K \cap N_G[D])$. Then $K' \subsetneq K$, and so from the minimality of K, it follows that either K' is not connected, or $M \not\subseteq N(K')$. Suppose first that the latter happens. Then there is an $m_i \in M \setminus N(K')$. It follows that $m_i \notin N(v)$ as $v \in K'$, and therefore $m_i \in D$. Let k be a neighbor of m_i in K. Then $k \in N[D]$, and so $k \in K'$, a contradiction; this shows that $M \subseteq N(K')$. It follows that K' is not connected. Since K is connected, it follows that for every $x \in K$, there is a path P_x from x to v with $P_x \subseteq K$. Moreover, since D is a component of $G \setminus N[v]$, if $x \in N[D]$, then $P_x \subseteq (N[D] \cap K) \cup \{v\}$. Therefore, for every $x \in K'$, we have $P_x \subseteq K'$; so K' is connected. This is a contradiction and proves (1).

Setting $Conn(t) = K \cup M$, Theorem 4.2 follows. ■

We now construct a graph that is a safe alternative for $\hat{\chi}(t_0)$. Let S' be a stable set of hubs of G, and assume that every $s \in S'$ is d-safe. Let S_{bad} denote the set of all vertices in S' that are not t_0-cooperative; by Lemma 4.1, we have that $|S_{bad}| \leq 1$. Let $S = S' \setminus S_{bad}$ and set

$$\beta(S') = \left(\chi(t_0) \cup \bigcup_{t \in N_T(t_0)} Conn(t) \right) \setminus S_{bad}.$$

Write $\beta = \beta(S')$. It follows that for every $t \in N_T(t_0)$, we have that $\beta \subseteq (G \setminus G_{t_0 \to t}) \cup Conn(t)$.

From now on, assume that β does not admit a balanced separator of size at most $2m(m-1) + \gamma(t) + 1$ where $\gamma(t)$ is as in Theorem 3.7. Let us say that a vertex v is *unbalanced* if there is a component D of $\beta \setminus N[v]$ such that $|D| > \frac{|\beta|}{2}$.

Lemma 4.3. *Suppose that G does not admit a clique cutset and let $s \in S \cap \chi(t_0)$. Then the following hold.*

(1) $|N_{\hat{\chi}(t_0)}(s)| < 2m(m-1)$.
(2) $|N_{\chi(t_0)}(s)| < 2m(m-1)$.
(3) The vertex s is unbalanced (in β).

Proof. Let $s \in S$. Since s is t_0-cooperative, we have that $|N_{\hat{\chi}(t_0)}(s)| < 2m(m-1)$, and the first assertion of the theorem holds. Since $\chi(t_0)$ is a subgraph of $\hat{\chi}(t_0)$, the second assertion follows immediately from the first.

We now prove the third assertion. Let $\delta(s)$ be the set of all vertices $t \in N_T(t_0)$ such that $s \in \chi(t_0) \cap \chi(t)$. Write $\Delta(s) = N_{\chi(t_0)}(s) \cup \bigcup_{t \in \delta(s)} (\chi(t) \cap \chi(t_0))$. Since $\Delta(s) \subseteq N_{\hat{\chi}(t_0)}[s]$, by the first assertion of the theorem, we have that $|\Delta(s)| \leq 2m(m-1)$. Since β does not admit a $2m(m-1)$-balanced separator, it follows that there is a component D' of $\beta \setminus \Delta(s)$ with $|D'| > \frac{|\beta|}{2}$.

(2) *We have that $D' \cap Conn(t) = \emptyset$ for every $t \in \delta(s)$.*

Suppose not, and let $t \in \delta(s)$ be such that $D' \cap Conn(t) \neq \emptyset$. Since D' is a component of $\beta \setminus \Delta(s)$, it follows that $D' = Conn(t) \setminus \chi(t_0)$. By Theorem 4.2(3), we have that $\text{Hub}(D') = \emptyset$. Now by Theorem 3.7, we conclude that D' has a balanced separator X of size $\gamma(t)+1$. But now $X \cup \Delta(s)$ is a balanced separator of β of size $2m(m-1)+\gamma(t)+1$, a contradiction. This proves (2).

Since $N_\beta(s) \setminus \Delta(s) \subseteq \bigcup_{t\in\delta(s)} Conn(t)$, statement (2) implies that $D' \cap N_\beta(s) = \emptyset$. Let D be a component of $\beta \setminus N_\beta[s]$ such that $D' \subseteq D$; then $|D| > \frac{|\beta|}{2}$, as required. This proves that s is unbalanced and completes the proof of Lemma 4.3. ■

As we mentioned in Section 1, handling balanced vertices was somewhat technical in [2]; it required constructing an artificial auxiliary graph. Using leanness provides a much more natural framework to deal with this obstacle.

A *separation* of a graph G is a triple (Y, X, Z) of pairwise disjoint subsets of $V(G)$ with $X \cup Y \cup Z = V(G)$ such that Y is anticomplete to Z. As in earlier papers in this series, we associate a certain unique star separation to every vertex of $S \cap \chi(t_0)$. For $v \in S \cap \chi(t_0)$, let $B(v)$ be the unique component of $\beta \setminus N[v]$ with $|B(v)| > \frac{|\beta|}{2}$, let $C(v) = N(B(v)) \cup \{v\}$, and finally, let $A(v) = \beta \setminus (B(v) \cup C(v))$. Then $(A(v), C(v), B(v))$ is the *canonical star separation of β corresponding to v*. We show:

Lemma 4.4. *The vertex v is not a hub of $\beta \setminus A(v)$.*

Proof. Suppose that (H, v) is a proper wheel in $\beta \setminus A(v)$. Then $H \subseteq N[B(v)]$, contrary to Theorem 3.1. This proves Lemma 4.4. ■

Let $\mathcal{O}$ be a linear order on $S \cap \chi(t_0)$. Following [2], we say that two unbalanced vertices of $S \cap \chi(t_0)$ are *star twins* if $B(u) = B(v)$, $C(u) \setminus \{u\} = C(v) \setminus \{v\}$, and $A(u) \cup \{u\} = A(v) \cup \{v\}$. (Note that every two of these conditions imply the third.)

Let $\leq_A$ be a relation on $S \cap \chi(t_0)$ defined as follows:

$$x \leq_A y \quad \text{if} \quad \begin{cases} x = y, \text{ or} \\ x \text{ and } y \text{ are star twins and } \mathcal{O}(x) < \mathcal{O}(y), \text{ or} \\ x \text{ and } y \text{ are not star twins and } y \in A(x). \end{cases}$$

Note that if $x \leq_A y$, then either $x = y$, or $y \in A(x)$. We need the following result from [2]:

Lemma 4.5 (Abrishami, Chudnovsky, Hajebi, Spirkl [2])**.** *The relation $\leq_A$ is a partial order on $S \cap \chi(t_0)$.*

In view of Lemma 4.5, let $\text{Core}(S')$ be the set of all $\leq_A$-minimal elements of $S \cap \chi(t_0)$. Define

$$\beta^A(S') = \bigcap_{v \in \text{Core}(S')} (B(v) \cup C(v)).$$

The following was also proved in [2]:

Lemma 4.6 (Abrishami, Chudnovsky, Hajebi, Spirkl [2])**.** *Let $u, v \in \text{Core}(S')$. Then $A(u) \cap C(v) = C(u) \cap A(v) = \emptyset$.*

Next, we need an analogue of a result of [2], summarizing the behavior of $\beta^A(S')$.

Theorem 4.7. *The following hold:*

(1) For every $v \in \mathrm{Core}(S')$, we have $C(v) \subseteq \beta^A(S')$.
(2) $|C(v) \cap \chi(t_0)| \leq 2m(m-1)$ for every $v \in \mathrm{Core}(S')$.
(3) For every component D of $\beta \setminus \beta^A(S')$, there exists $v \in \mathrm{Core}(S')$ such that $D \subseteq A(v)$. Further, if D is a component of $\beta \setminus \beta^A(S')$ and $v \in \mathrm{Core}(S')$ such that $D \subseteq A(v)$, then $N_\beta(D) \subseteq C(v)$.
(4) $S' \cap \mathrm{Hub}(\beta^A(S')) = \emptyset$.

Proof. (1) is immediate from Lemma 4.6, and (2) follows from Lemma 4.3.

Next we prove (3). Let D be a component of $\beta \setminus \beta^A(S')$. Since $\beta \setminus \beta^A(S') = \bigcup_{v \in \mathrm{Core}(S')} A(v)$, there exists $v \in \mathrm{Core}(S')$ such that $D \cap A(v) \neq \emptyset$. If $D \setminus A(v) \neq \emptyset$, then, since D is connected, it follows that $D \cap N(A(v)) \neq \emptyset$; but then $D \cap C(v) \neq \emptyset$, contrary to (1). Since $N_\beta(D) \subseteq \beta^A(S')$ and $N(D) \subseteq A(v) \cup C(v)$, it follows that $N_\beta(D) \subseteq C(v)$. This proves (3).

To prove (4), let $u \in S' \cap \mathrm{Hub}(\beta^A(S'))$. By Theorem 4.2(3), we have that $u \in \chi(t_0)$. Since $\beta^A(S') \subseteq \beta$, we deduce that $u \notin S_{bad}$. By Lemma 4.4, it follows that $\beta^A(S') \not\subseteq B(u) \cup C(u)$, and therefore $u \notin \mathrm{Core}(S')$. But then $u \in A(v)$ for some $v \in \mathrm{Core}(S')$, and so $u \notin \beta^A(S')$, a contradiction. This proves (4) and completes the proof of Theorem 4.7. ∎

In the course of the proof of Theorem 1.4, we will inductively obtain a bound on $\mathrm{tw}(\beta^A(S))$. Next we explain how to transform a tree decomposition of $\beta^A(S')$ into a tree decomposition of β.

Let $D_1, \ldots, D_s$ be the components of $\beta \setminus \beta^A(S')$. For $i \in \{1, \ldots, s\}$, let $r(D_i)$ be the $\mathcal{O}$-minimal vertex v of $\mathrm{Core}(S')$ such that $D_i \subseteq A(v)$ (such v exists by Theorem 4.7(3)). Write $\beta_0 = \chi(t_0)$.

Let (T_0, χ_0) be a tree decomposition of $\beta^A(S')$, and for $i \in \{1, \ldots, s\}$ let (T_i, χ_i) be a tree decomposition of D_i. Let T_β be the tree obtained from the union of $T_0, T_1, \ldots, T_s$ as follows. Let $i \in \{1, \ldots, s\}$. Choose $t \in T_0$ such that $r(D_i) \in \chi_0(t)$ and add an edge from t to an arbitrarily chosen vertex of T_i. For $u \in T_\beta$, let $\chi_\beta(u)$ be defined as follows.

- If $u \in V(T_0)$, let
$$\chi_\beta(u) = \chi_0(u) \cup \bigcup_{v \in \mathrm{Core}(S') \cap \chi_0(u)} C(v).$$
- If $u \in V(T_i)$ for $i \in \{1, \ldots, s\}$, let
$$\chi_\beta(u) = \chi_i(u) \cup C(r(D_i)).$$

Theorem 4.8. *With the notation as above, (T_β, χ_β) is a tree decomposition of β. Moreover,*

- *for every $t \in T_0$,*
$$|\chi_\beta(t) \cap \beta_0| \leq |\chi_0(t)| + 2m(m-1)(|\chi_0(t) \cap \mathrm{Core}(S')|),$$
and
- *for every $i \in \{1, \ldots, s\}$ and $t \in T_i$,*
$$|\chi_\beta(t) \cap \beta_0| \leq |\chi_i(t)| + 2m(m-1).$$

Proof. The fact that (T_β, χ_β) is a tree decomposition of β follows from Theorem 6.6 of [2]. Recall that by Lemma 4.3, $|(C(v) \setminus v) \cap \beta_0| < 2m(m-1)$ for every $s \in \mathrm{Core}(S')$. Now the theorem follows directly from the definition of (T_β, χ_β). ∎

We finish this section with a theorem that allows us to transform a tree decomposition of β into a tree decomposition of G.

Write $N_T(t_0) = \{t_1, \ldots, t_r\}$, and let $D'_i = G_{t_0 \to t_i} \setminus \chi(t_0)$ for $i \in \{1, \ldots, r\}$.

Let (T'_0, χ'_0) be a tree decomposition of β, and for $i \in \{1, \ldots, r\}$ let (T'_i, χ'_i) be a tree decomposition of D'_i. Let U be the tree obtained from the union of $T'_0, T'_1, \ldots, T'_r$ as follows. Let $i \in \{1, \ldots, r\}$. Choose $t \in T'_0$ such that $\chi'_0(t) \cap (Conn(t_i) \setminus \chi(t_0)) \neq \emptyset$ and add an edge from t to an arbitrarily chosen vertex of T'_i.

For $u \in U$, let $\psi(u)$ be defined as follows.

- If $u \in V(T'_0)$, let
$$\psi(u) = S_{bad} \cup (\chi'_0(u) \cap \beta_0) \cup \bigcup_{t_i \text{ s.t. } \chi'_0(u)\cap(Conn(t_i)\setminus\chi(t_0))\neq\emptyset} (\beta_0 \cap \chi(t_i)).$$
- If $u \in V(T'_i)$ for $i \in \{1, \dots, r\}$, let
$$\psi(u) = S_{bad} \cup \chi'_i(u) \cup (\beta_0 \cap \chi(t_i)).$$

Theorem 4.9. *With the notation as above, (U, ψ) is a tree decomposition of G.*

Proof. Since U is obtained by adding a single edge from T'_0 to each of the trees $T'_1, \dots, T'_r$, it follows that U is a tree. Clearly, every vertex of G is in $\psi(v)$ for some $v \in V(U)$. Next we check that for every edge xy of G, there exists $v \in V(U)$ such that $x, y \in \psi(v)$. This is clear if $x, y \in \chi(t_0)$ or if $x, y \in D'_i$ for some i; thus we may assume that $x \in \chi(t_0)$ and $y \in D'_1$, say. Since (T, χ) is a tree decomposition of G, it follows that $x \in \chi(t_0) \cap \chi(t_1)$. Let $v \in V(T'_1)$ such that $y \in \chi'_1(v) \subseteq \psi(v)$; then $x, y \in \psi(v)$ as required.

Let $x \in V(G)$. For $i \in \{0, 1, \dots, r\}$, define $F_i(x) = \{v \in V(T'_i)$ such that $x \in \psi(v)\}$. Let $F(x) = \bigcup_{i=0}^r F_i(x)$. We need to show that for every $x \in V(G)$, the induced subgraph $U[F(x)]$ is connected. This is true if $x \in S_{bad}$, since then $F(x) = V(U)$. Therefore, we now assume that $x \notin S_{bad}$. If $x \notin \beta_0$, then $F_0(x) = \emptyset$, and there exists a unique i such that $x \in D'_i$; therefore $F(x) = F_i(x)$, and $U[F(x)]$ is connected because $T'_i[F_i(x)]$ is connected. Thus we may assume that $x \in \beta_0$. Let $I \subseteq \{1, \dots, r\}$ be the set of all i such that $x \in \chi(t_0) \cap \chi(t_i)$. It follows that $F_i(x) = \emptyset$ for all $i \in \{1, \dots, r\} \setminus I$, and $F_i(x) = V(T'_i)$ for all $i \in I$.

First we show that for every $i \in I$, there is an edge between $T'_i[F_i(x)] = T'_i$ and $T'_0[F_0(x)]$. Let $i \in I$; and let $v \in T'_0$ be such that v is adjacent to a vertex of T'_i. Then $(Conn(t_i)\setminus\chi(t_0))\cap\chi'_0(v) \neq \emptyset$; consequently $\chi(t_0) \cap \chi(t_i) \subseteq \psi(v)$, and therefore, $x \in \psi(v)$. Thus there is an edge between $T'_i[F_i(x)]$ and $T'_0[F_0(x)]$, as required.

Now to show that $U[F(x)]$ is connected, it is enough to prove that $T'_0[F_0(x)]$ is connected. Write $\chi_0'^{-1}(x) := \{v \in T'_0 : x \in \chi'_0(v)\}$ and observe that
$$F_0(x) = \chi_0'^{-1}(x) \cup \bigcup_{t \in I} \chi_0'^{-1}(Conn(t) \setminus \chi(t_0)).$$
Since by Theorem 4.2(2), we have that $\{x\} \cup (Conn(t) \setminus \chi(t_0))$ is connected, it follows from basic properties of a tree decomposition that $T'_0[\chi_0'^{-1}(\{x\} \cup (Conn(t) \setminus \chi(t_0)))]$ is connected. Since $\chi_0'^{-1}(\{x\}) \neq \emptyset$, it follows that the union $T'_0\left[\bigcup_{t\in I} \chi_0'^{-1}(\{x\} \cup (Conn(t) \setminus \chi(t_0)))\right]$ is connected. We deduce that $T'_0[F_0(x)]$ is connected, as required. This proves Theorem 4.9. ■

5. Connectifiers revisited

We start by recalling a well known theorem of Erdős and Szekeres [14].

Theorem 5.1 (Erdős and Szekeres [14])**.** *Let $x_1, \dots, x_{n^2+1}$ be a sequence of distinct reals. Then there exists either an increasing or a decreasing $(n+1)$-sub-sequence.*

We will also need the following easy lemma, whose proof we include for completeness.

Lemma 5.2. *Let k be an integer and let D be a directed graph in which every vertex has at most k outneighbors. Let D^- be the underlying undirected graph of D. Then there exists $X \subseteq V(D)$ with $|X| \geq \frac{|V(D)|}{2k+1}$ such that X is a stable set of D^-.*

Proof. Every induced subgraph H of D^- has at most $k|V(H)|$ edges, and therefore contains a vertex of degree at most $2k$. This shows that D^- is $2k$-degenerate, and therefore $(2k+1)$-colorable. So D^- has a stable set of size at least $|V(D)|/(2k+1)$. This proves Lemma 5.2. ■

A vertex v in a graph G is said to be a *branch vertex* if v has degree more than two. By a *caterpillar* we mean a tree C with maximum degree three such that there exists a path P of C where all branch vertices of C belong to P. We call a maximal such path P the *spine* of P. (Our definition of a caterpillar is non-standard for two reasons: a caterpillar is often allowed to be of arbitrary maximum degree, and a spine often contains all vertices of degree more than one.) By a *subdivided star* we mean a graph isomorphic to a subdivision of the complete bipartite graph $K_{1,\delta}$ for some $\delta \geq 3$. In other words, a subdivided star is a tree with exactly one branch vertex, which we call its *root*. For a graph H, a vertex v of H is said to be *simplicial* if $N_H(v)$ is a clique. We denote by $\mathcal{Z}(H)$ the set of all simplicial vertices of H. Note that for every tree T, $\mathcal{Z}(T)$ is the set of all leaves of T. An edge e of a tree T is said to be a *leaf-edge* of T if e is incident with a leaf of T. It follows that if H is the line graph of a tree T, then $\mathcal{Z}(H)$ is the set of all vertices in H corresponding to the leaf-edges of T. The following is proved in [4] based on (and refining) a result from [12].

Theorem 5.3 (Abrishami, Alecu, Chudnovsky, Hajebi, Spirkl [4])**.** *For every integer $h \geq 1$, there exists an integer $\mu = \mu(h) \geq 1$ with the following property. Let G be a connected graph with no clique of cardinality h and let $S \subseteq G$ such that $|S| \geq \mu$. Then either some path in G contains h vertices of S, or there is an induced subgraph H of G with $|H \cap S| = h$ for which one of the following holds.*

- *H is either a caterpillar or the line graph of a caterpillar with $H \cap S = \mathcal{Z}(H)$.*
- *H is a subdivided star with root r such that $\mathcal{Z}(H) \subseteq H \cap S \subseteq \mathcal{Z}(H) \cup \{r\}$.*

Let H be a graph that is either a path, or a caterpillar, or the line graph of a caterpillar, or a subdivided star with root r. We define an induced subgraph of H, denoted by $P(H)$, which we will use throughout the paper. If H is a path let $P(H) = H$. If H is a caterpillar, let $P(H)$ be the spine of H. If H is the line graph of a caterpillar C, let $P(H)$ be the path of H consisting of the vertices of H that correspond to the edges of the spine of C. If H is a subdivided claw with root r, let $P(H) = \{r\}$. The *legs* of H are the components of $H \setminus P$. Our first goal is to prove the following variant of Theorem 5.3:

Theorem 5.4. *For every integer $h' \geq 1$, there exists an integer $\nu = \nu(h') \geq 1$ with the following property. Let G be a connected graph with no clique of cardinality h'. Let $S' \subseteq G$ such that $|S'| \geq \nu$, $G \setminus S'$ is connected and every vertex of S' has a neighbor in $G \setminus S'$. Then there is a set $\tilde{S} \subseteq S'$ with $|\tilde{S}| = h'$ and an induced subgraph H' of $G \setminus S'$ for which one of the following holds.*

- *H' is a path and every vertex of $\tilde{S}$ has a neighbor in H'; or*
- *H' is a caterpillar, or the line graph of a caterpillar, or a subdivided star. Moreover, every vertex of $\tilde{S}$ has a unique neighbor in H' and $H' \cap N(\tilde{S}) = \mathcal{Z}(H')$.*

Proof. Write $G' := G \setminus S'$, let $h = 1 + h' + 2h'^2$, and let $\nu = h'\mu(h)$, where μ is as in Theorem 5.3. Assume that the first bullet point above does not hold, that is:

(3) *For every path Q of G', $|N(Q) \cap S'| \leq h' - 1$.*

Now choose, for every $s \in S'$, a neighbor $n(s)$ of s in G', and let $S := \{n(s) : s \in S'\}$. By (3), $|n^{-1}(v)| \leq h' - 1$ for all $v \in G'$ (and in particular for all $v \in S$), and so $|S| \geq |S'|/h' = \mu(h)$. Let $S'' \subseteq S'$ be minimal such that $S = \{n(s) : s \in S''\}$. It follows that n is a bijection between S'' and S.

We now apply Theorem 5.3 to G', S and h. By (3), the path outcome of the theorem does not happen, so there is an induced subgraph H of G' with $|H \cap S| = h$, for which one of the following holds:

- H is either a caterpillar or the line graph of a caterpillar with $H \cap S = \mathcal{Z}(H)$.
- H is a subdivided star with root r such that $\mathcal{Z}(H) \subseteq H \cap S \subseteq \mathcal{Z}(H) \cup \{r\}$.

Let $P := P(H)$, and let $A := S'' \cap n^{-1}(\mathcal{Z}(H) \setminus P)$; in other words, A consists of the vertices of S'' whose selected neighbors are simplicial vertices of H other than the endpoint(s) of P. In particular, $|A| \geq h - 2 = 2h'^2 + h' - 1$. Moreover, for each $v \in A$, $n(v)$ belongs to a unique leg of H, which yields a bijective correspondence between A and the set of legs of H (henceforth, if $n(v) \in D$ for a vertex $v \in A$ and a leg D of H, we will say that v *corresponds* to D, and vice-versa).

Let $A'' \subseteq A$ be the set of vertices of A anticomplete to P. We note that, by (3), $|A''| \geq |A| - (h' - 1) \geq 2h'^2$. Let H'' be obtained from H by removing all legs corresponding to vertices in $A \setminus A''$. In particular, we note that H'' is still a caterpillar, line graph of a caterpillar, or star, according to what H was, and that the vertices of A'' correspond bijectively to the legs of H''.

Define now a directed graph F as follows. $V(F)$ is the set of legs of H'', and we have an arc from $D_1 \in F$ to $D_2 \in F$ if the vertex s_2 corresponding to D_2 has a neighbor in D_1. By (3), every vertex of F has at most $h' - 1$ outneighbours, and so by Lemma 5.2, the underlying undirected graph of F contains a stable set T of size $\frac{|V(F)|}{2(h'-1)+1} \geq \frac{2h'^2}{2h'-1} \geq h'$. Let $\tilde{S}$ be the set corresponding to the legs in T, and let H' be obtained from H'' by deleting the legs not in T. It is routine to check that H' and $\tilde{S}$ satisfy the second outcome of Theorem 5.4. ■

Next we introduce more terminology. Let G be a graph, let $P = p_1\text{-}\dots\text{-}p_n$ be a path of G and let $X = \{x_1, \dots, x_k\} \subseteq G \setminus P$. We say that (P, X) is an *alignment* if $N_P(x_1) = \{p_1\}$, $N_P(x_k) = \{p_n\}$, every vertex of X has a neighbor in P, and there exist $1 < j_2 < \dots < j_{k-1} < j_k = n$ such that $N_P(x_i) \subseteq p_{j_i}\text{-}P\text{-}p_{j_{i+1}-1}$ for $i \in \{2, \dots, k-1\}$. We also say that $x_1, \dots, x_k$ is *the order on X given by the alignment (P, X)*. An alignment (P, X) is *wide* if each of $x_2, \dots, x_{k-1}$ has two non-adjacent neighbors in P, *spiky* if each of $x_2, \dots, x_{k-1}$ has a unique neighbor in P and *triangular* if each of $x_2, \dots, x_{k-1}$ has exactly two neighbors in P and they are adjacent. An alignment is *consistent* if it is wide, spiky or triangular. Next, let H be a caterpillar or the line graph of a caterpillar and let S be a set of vertices disjoint from H such that every vertex of S has a unique neighbor in H and $H \cap N(S) = \mathcal{Z}(S)$. Let X be the set of vertices of $H \setminus P(H)$ that have neighbors in $P(H)$. Then the neighbors of elements of X appear in $P(H)$ in order (in fact, (X, Q) is an alignment for some subpath Q of $P(H)$); let $x_1, \dots, x_k$ be the corresponding order on X. Now, order the vertices of S as $s_0, s_1, \dots, s_k, s_{k+1}$ where s_i has a neighbor in the leg of H containing x_i for $i \in \{1, \dots, k\}$, and s_0, s_{k+1} are the ends of $P(H)$ in this order. We say that $s_0, s_1, \dots, s_k, s_{k+1}$ is *the order on S given by (H, S)*.

Next, let H be an induced subgraph of G that is a caterpillar, or the line graph of a caterpillar, or a subdivided star and let $X \subseteq G \setminus H$ be such that every vertex of X has a unique neighbor in H and $H \cap N(X) = \mathcal{Z}(H)$. We say that (H, X) is a *consistent connectifier* and it is *spiky, triangular, or stellar* respectively.

Our next goal is, starting with a graph $G \in \mathcal{C}_t$ and a stable set $X \subseteq V(G)$, to produce certain consistent connectifiers. We start with a lemma.

Lemma 5.5. *Let $G \in \mathcal{C}$ and assume that $V(G) = H_1 \cup H_2 \cup X$ where X is a stable set with $|X| \geq 3$ and $X \cap \mathrm{Hub}(G) = \emptyset$. Suppose that for $i \in \{1, 2\}$, the pair (H_i, X) is a consistent alignment, or a consistent connectifier. Assume also that if neither of $(H_1, X), (H_2, X)$ is stellar, then the orders given on X by (H_1, X) and by (H_2, X) are the same. Then (possibly switching the roles of H_1 and H_2), we have that:*

- *(H_1, X) is a triangular alignment or a triangular connectifier; and*
- *(H_2, X) is a spiky connectifier, a stellar connectifier, a spiky alignment or a wide alignment.*

Moreover, if (H_1, X) is a triangular alignment, then (H_2, X) is not a wide alignment.

Proof. If at least one $(H_i, X) \subseteq \{(H_1, X), (H_2, X)\}$ is not a stellar connectifier, we let $x_1, \dots, x_k$ be the order given on X by (H_i, X). If both of (H_i, X) are stellar connectifiers, we let $x_1, \dots, x_k$

be an arbitrary order on X. Let H be the unique hole contained in $H_1 \cup H_2 \cup \{x_1, x_k\}$. For $j \in \{1,2\}$ and $i \in \{1,\dots,k\}$, if H_j is a connectifier, let D_i^j be the leg of H_j containing a neighbor of x_i; and if H_j is an alignment let $D_i^j = \emptyset$.

Suppose first that (H_1, X) is a triangular alignment or a triangular connectifier. If (H_2, X) is a triangular alignment or a triangular connectifier, then for every $i \in \{2,\dots,k-1\}$, the graph $H \cup D_i^1 \cup D_i^2 \cup \{x_i\}$ is either a prism or an even wheel with center x_i, a contradiction. This proves that (H_2, X) is a stellar connectifier, or a spiky connectifier, or a spiky alignment, or a wide alignment. We may assume that (H_1, X) is a triangular alignment and (H_2, X) is a wide alignment, for otherwise the theorem holds. But now for every $x \in X \setminus \{x_1, x_k\}$, (H, x) is a proper wheel, a contradiction.

Thus we may assume that for $i \in \{1,2\}$, the pair (H_i, X) is a a stellar connectifier, or a spiky connectifier, or a spiky alignment, or a wide alignment. Now for every $x_i \in X \setminus \{x_1, x_k\}$, the graph $H \cup D_i^1 \cup D_i^2 \cup \{x_i\}$ is either a theta or a proper wheel with center x, a contradiction. This proves Lemma 5.5. ■

We now prove the main result of this section, which is a refinement of Theorem 5.4 for graphs in $\mathcal{C}$.

Theorem 5.6. *For every integer $x \geq 1$, there exists an integer $\tau = \tau(x) \geq 1$ with the following property. Let $G \in \mathcal{C}_x$ and assume that $V(G) = D_1 \cup D_2 \cup Y$ where*

- *Y is a stable set with $|Y| = \tau$,*
- *D_1 and D_2 are components of $G \setminus Y$, and*
- *$N(D_1) = N(D_2) = Y$.*

Assume that $Y \cap \mathrm{Hub}(G) = \emptyset$. Then there exist $X \subseteq Y$ with $|X| = x$, $H_1 \subseteq D_1$ and $H_2 \subseteq D_2$ (possibly with the roles of D_1 and D_2 reversed) such that

- *(H_1, X) is a triangular connectifier or a triangular alignment;*
- *(H_2, X) is a stellar connectifier, or a spiky connectifier, or a spiky alignment or a wide alignment; and*
- *if (H_1, X) is a triangular alignment, then (H_2, X) is not a wide alignment.*

Moreover, if neither of $(H_1, X), (H_2, X)$ is a stellar connectifier, then the orders given on X by (H_1, X) and by (H_2, X) are the same.

Proof. Let $z = (27)^2 \cdot 36x^4$ and let $\tau(x) = \nu(\nu(z))$, where ν is as in Theorem 5.4. Applying Theorem 5.4 twice, we obtain a set $Z \subseteq Y$ with $|Z| = z$ and $H_i \subseteq D_i$ such that either

- H_i is a path and every vertex of Z has a neighbor in H_i; or
- (H_i, X) is a consistent connectifier

for every $i \in \{1,2\}$.

(4) *Let $i \in \{1,2\}$ and $y \in \mathbb{N}$. If H_i is a path and every vertex of Z has a neighbor in H_i, then either some vertex of H_i has y neighbors in Z, or there exists $Z' \subseteq Z$ with $|Z'| \geq \frac{|Z|}{27y}$ and a subpath H_i' of H_i such that (H_i, Z') is a consistent alignment.*

Let $H_i = h_1\text{-}\dots\text{-}h_k$. We may assume that H_i is chosen minimal satisfying Theorem 5.4, and so there exist $z_1, z_k \in Z$ such that $N_{H_i}(z_j) = \{h_j\}$ for $j \in \{1,k\}$.

We may assume that $|N_Z(h)| < y$ for every $h \in H_i$. Let Z_1 be the set of vertices in Z with exactly one neighbor in H_i. Then, if $|Z_1| \geq |Z|/3$, it follows that Z_1 contains a set Z' with $|Z'| \geq |Z_1|/y \geq |Z|/(3y)$ such that no two vertices in Z' have a common neighbor in H_i. We may assume that $z_1, z_k \in Z'$. Therefore, (H_i, Z') is a spiky alignment.

Next, let Z_2 be the set of vertices in $z \in Z$ such that either $z \in \{z_1, z_k\}$ or has exactly two neighbors in H_i, and they are adjacent. Now, if $|Z_2| \geq |Z|/3$, by choosing Z' greedily, it follows that Z_2 contains a subset Z' with the following specifications:

- $z_1, z_k \in Z'$;
- $|Z'| \geq |Z_2|/(2y) \geq |Z|/(6y)$; and
- no two vertices in Z' have a common neighbor in H_i.

But then (H_i, Z') is a triangular alignment, as desired.

Let $Z_3 = \{z_1, z_k\} \cup (Z \setminus (Z_1 \cup Z_2))$. From the previous two paragraphs, we may assume that $|Z_3| \geq |Z|/3$. Let R be a path from z_1 to z_k with $R^* \subseteq H_{3-i}$, and let H be the hole z_1-H_i-z_k-R-z_1. If some $z \in Z \setminus \{z_1, z_k\}$ has at least four neighbors in H_i, then (H, z) is a proper wheel in G, a contradiction. This proves that $|N_{H_i}(z)| \leq 3$ for every $z \in Z$.

Let $z \in Z$. Define $Bad(z) = N_{H_i}[N_{H_i}(z)]$. Since $|N_{H_i}(z)| \leq 3$ and H_i is a path, it follows that $|Bad(z)| \leq 9$. Since $N_Z(h) < y$ for every $h \in H_i$, we can greedily choose $Z' \subseteq Z$ with $|Z'| \geq \frac{|Z_3|}{9y} \geq \frac{|Z|}{27y}$, $z_1, z_k \in Z'$ and such that if $z, z' \in Z'$, then z' is anticomplete to $Bad(z)$.

We claim that (H_i, Z') is an alignment. Suppose not; then there exist $i < j < k$ such that $h_i, h_k \in N(z)$ and $h_j \in N(z')$ for $z, z' \in Z'$ with $z \neq z'$. We may assume that i, j, k are chosen with $|k - i|$ minimum. It follows that z has no neighbor in $\{h_{i+1}, \ldots, h_{k-1}\}$. We consider three cases:

- If z' has a neighbor h_l with $l > k$, then we define $P_1 = z$-h_i-P-h_j-z' and $P_2 = z$-h_k-P-h_l-z'.
- If z' has a neighbor h_l with $l < i$, then we define $P_1 = z$-h_i-P-h_l-z' and $P_2 = z$-h_k-P-h_j-z'.
- Otherwise, all neighbors of z' are in $\{h_{i+1}, \ldots, h_{k-1}\}$. Let h_j and h_l be the first and last neighbor of z' in h_{i+1}-$\ldots$-h_{k-1}, respectively. Then, since $z' \in Z_3$, it follows that $|l - j| > 1$. We define $P_1 = z$-h_i-P-h_j-z' and $P_2 = z$-h_k-P-h_l-z'.

Now we get a theta with ends z, z' and paths P_1, P_2 and a third path with interior in H_{3-i}, a contradiction. This proves that (H_i, Z') is an alignment. Since $Z' \subseteq Z_3$, it follows that (H_i, Z') is a wide alignment. This proves (4).

(5) *There is a subset $\hat{Z} \subseteq Z$ with $|\hat{Z}| \geq x$, and a path $\hat{H}_i \subseteq H_i$ for $i = 1, 2$ such that $(\hat{H}_i, \hat{Z})$ is a consistent alignment or a connectifier. Moreover, if neither of $(\hat{H}_1, \hat{Z}), (\hat{H}_2, \hat{Z})$ is a stellar connectifier, then the order given on $\hat{Z}$ by $(\hat{H}_1, \hat{Z})$ and $(\hat{H}_2, \hat{Z})$ is the same.*

Suppose first that some $h \in H_1$ has at least $6x$ neighbors in Z. Let $\hat{H}_1 = \{h\}$, and let $\hat{Z} \subseteq Z \cap N(h)$ with $|\hat{Z}| = 6x$. If (H_2, Z) is a connectifier, then setting $\hat{H}_2 = H_2$, we have that (5) holds. So we may assume that H_2 is a path and every vertex of Z has a neighbor in H_2. Let $\hat{H}_2$ be a minimal subpath of H_2 such that every $z \in \hat{Z}$ has a neighbor in $\hat{H}_2$. Then $H_2 = h_1$-$\ldots$-h_k and there exist $z_1, z_k \in \hat{Z}$ such that $N_{\hat{H}_2}(z_i) = \{h_i\}$ for $i \in \{1, k\}$. Since for every $z \in \hat{Z} \setminus \{z_1, z_k\}$, the graph $H_2 \cup \{z_1, z_k, z, h\}$ is not a theta and not a proper wheel with center z, it follows that every $z \in \hat{Z} \setminus \{z_1, z_k\}$ has exactly two neighbors in H_2 and they are adjacent. Moreover, no vertex x of H_2 has three or more neighbors in $\hat{Z}$, for otherwise $\{x, h\} \cup (N(x) \cap \hat{Z})$ contains a $K_{2,3}$. Now, by choosing $Z' \subseteq \hat{Z}$ greedily, we find a set Z' with the following specifications:

- $z_1, z_k \in Z'$;
- $|Z'| \geq \hat{Z}/6$; and
- no two vertices in Z' have a common neighbor in H_2.

But then (H_1, Z') is a triangular alignment, and (5) holds. Therefore, each vertex $h \in H_1$ has at most $6x$ neighbors in Z; and by symmetry, it follows that each vertex $h \in H_2$ has at most $6x$ neighbors in Z.

In view of this, applying (4) with $y = 6x$ (possibly twice), we conclude that there exists $Z' \subseteq Z$ with $|Z'| \geq x^2$ and subgraphs $H_i' \subseteq H_i$ such that (H_i', Z') is a consistent alignment or a consistent connectifier. If none of them is a stellar connectifier, for $i \in \{1, 2\}$, let π_i be the order given on Z' by (H_i, Z'). By Theorem 5.1 there exists $\hat{Z} \subseteq Z'$ such that (possibly reversing H_i) the orders π_i restricted to $\hat{Z}$ are the same, as required. This proves (5).

Now Theorem 5.6 follows from Lemma 5.5. ■

6. Bounding the number of non-hubs

In this section, we show that a bag of a structured tree decomposition of a graph $G \in \mathcal{C}_{tt}$ contains a small number of vertices of $G \setminus \mathrm{Hub}(G)$. This is the only place in the paper where the assumption that $G \in \mathcal{C}_{tt}$ (rather than $G \in \mathcal{C}_t$) is used.

Theorem 6.1. *Let $G \in \mathcal{C}$ and let (T, χ) be a structured tree decomposition of G. Let $v \in T$ and $Y \subseteq \chi(v)$ be a stable set. Then there exist components $D_1, \ldots, D_k$ of $G \setminus \chi(v)$ such that $Y \subseteq \bigcup_{i=1}^{k} N(D_i)$ and $k \leq 4$.*

Proof. Let D_1 be a component of $G \setminus \chi(v)$ such that $N(D_1) \cap Y$ is maximal. We may assume there exists $x_1 \in Y \setminus N(D_1)$. Since (T, χ) is structured and since x_1 is not complete to $\chi(v) \setminus \{x_1\}$, by Theorem 2.3 there exists a component D_2 of $G \setminus \chi(v)$ such that $x_1 \in N(D_2)$; choose D_2 with $N(D_2) \cap Y \cap N(D_1)$ is maximal. By the maximality of $N(D_1) \cap Y$, there exists $x_2 \in (Y \cap N(D_1)) \setminus N(D_2)$. Since (T, χ) is structured, Theorem 2.3 implies that there exists a component D_3 of $G \setminus \chi(v)$ such that $x_1, x_2 \in N(D_3)$. By the choice of D_2, there exists $x_3 \in N(D_1) \cap N(D_2) \cap Y$ such that $x_3 \notin N(D_3)$. For $\{i, j, k\} = \{1, 2, 3\}$, let P_{ij} be a path from x_i to x_j with interior in D_k.

Let $\mathcal{D}$ be the set of all components D of $G \setminus \chi(v)$ that such that $|N(D) \cap \{x_1, x_2, x_3\}| > 1$. Then $D_1, D_2, D_3 \in \mathcal{D}$.

(6) *We have that $|\mathcal{D} \setminus \{D_1, D_2, D_3\}| \leq 1$.*

Suppose first that there is a component $D \in \mathcal{D} \setminus \{D_1, D_2, D_3\}$ with $N(D) \cap \{x_1, x_2, x_3\} = \{x_1, x_2\}$. Then, we get a theta with ends x_1, x_2 and paths x_1-P_{12}-x_2, x_1-P_{13}-x_3-P_{23}-x_2 as well as a third path with interior in D (using that $x_3 \notin N(D)$). This is a contradiction, and proves that $N(D) = \{x_1, x_2, x_3\}$.

Now suppose that $D, D' \in \mathcal{D} \setminus \{D_1, D_2, D_3\}$ with $D \neq D'$. Then we get a theta with ends x_2, x_3 and paths with interiors in D_1, D, D', respectively, a contradiction. This proves (6).

Thus $|\mathcal{D}| \leq 4$. We may assume that there is a vertex $x \in Y$ such that $x \notin \bigcup_{D \in \mathcal{D}} N(D)$. Since (T, χ) is structured, Theorem 2.3 implies that there exist paths P_1, P_2 where P_i is from x to x_i, and $P_i^* \cap \chi(v) = \emptyset$. It follows that each P_i^* is contained in a component F_i of $G \setminus \chi(v)$. Since $F_i \notin \mathcal{D}$, it follows that $F_1 \neq F_2$, $x_2, x_3 \notin N(F_1)$, and $x_1, x_3 \notin N(F_2)$. Now we get a theta with ends x_1, x_2 and paths x_1-P_{13}-x_3-P_{23}-x_2, x_1-P_{12}-x_2 and x_1-P_1-x-P_2-x_2, a contradiction. This proves Theorem 6.1. ■

Next we show:

Theorem 6.2. *Let $G \in \mathcal{C}_{tt}$, let S be a minimal separator of G, and let $Y \subseteq S \setminus \mathrm{Hub}(G)$ be stable. Let $\tau = \tau(t)$ be as in Theorem 5.6. Then $|Y| \leq \tau$.*

Proof. Suppose $|Y| \geq \tau$. Let D_1, D_2 be distinct full components for X. Apply Theorem 5.6 to $D_1 \cup D_2 \cup Y$, and let H_1, H_2, X be as in the conclusion of 5.6. Then a routine case analysis (whose details we leave to the reader) shows that $H_1 \cup H_2 \cup X$ contains a generalized t-pyramid in G, a contradiction. This proves Theorem 6.2. ■

We can now prove the main result of this section.

Theorem 6.3. *Let $G \in \mathcal{C}_{tt}$, let (T, χ) be a structured tree decomposition of G, and let $v \in T$. Let $Y \subseteq \chi(v) \setminus \mathrm{Hub}(G)$ be stable. Let $\tau = \tau(t)$ be as in Theorem 5.6. Then $|Y| \leq 4\tau$.*

Proof. Suppose $|Y| > 4\tau$. By Theorem 6.1, there exists a component D of $G \setminus \chi(v)$ such that $|N(D) \cap Y| > \tau$. By Theorem 2.4, $N(D) \cap Y$ is a minimal separator of G, contrary to Theorem 6.2. This proves Theorem 6.3. ■

7. Putting everything together

For the remainder of the paper, all logarithms are taken in base 2. We start with the following theorem from [2]:

Theorem 7.1 (Abrishami, Chudnovsky, Hajebi, Spirkl [2])**.** *Let $t \in \mathbb{N}$, and let G be (theta, K_t)-free with $|V(G)| = n$. There exist an integer $d = d(t)$ be and a partition $(S_1, \ldots, S_k)$ of $V(G)$ with the following properties:*

(1) $k \leq \frac{d}{4} \log n$.
(2) S_i is a stable set for every $i \in \{1, \ldots, k\}$.
(3) For every $i \in \{1, \ldots, k\}$ and $v \in S_i$ we have $\deg_{G \setminus \bigcup_{j<i} S_j}(v) \leq d$.

Let $G \in \mathcal{C}_{tt}$ be a graph. A *hub-partition* of G is a partition $S_1, \ldots, S_k$ of $\text{Hub}(G)$ as in Theorem 7.1; we call k the *order* of the partition. We call the *hub-dimension* of G (denoting it by $\text{hdim}(G)$) the smallest k such that G as a hub-partition of order k.

For the remainder of this section, let us fix $t \in \mathbb{N}$. Let $d = d(t)$ as in Theorem 7.1. Let $C_t = \gamma(t) + 1$ with $\gamma(t)$ as in Theorem 3.7. Let $k_t = k(t)$ be as in Theorem 3.6. Let $m = k_t + 2d$. Let $\Psi = 4\tau(t)$ where $\tau(t)$ is as in Theorem 5.6. Let $\Delta = (2m-1)m + C_t$.

The following is a strengthening of Theorem 1.4, which we prove by induction on $\text{hdim}(G)$. By Theorem 7.1, $\text{hdim}(G) \leq \frac{d}{4} \log n$ for every $G \in \mathcal{C}_{tt}$, so Theorem 7.2 immediately implies Theorem 1.4.

Theorem 7.2. *Let $G \in \mathcal{C}_{tt}$ with $|V(G)| = n$. Then $\text{tw}(G) \leq C_t + \Delta\Psi(\log n + \text{hdim}(G))$.*

Proof. The proof is by induction on $\text{hdim}(G)$, and for a fixed value of hdim, by induction on n. If $hdim(G) = 0$, then by Theorem 3.7, we have that $\text{tw}(G) \leq C_t$ and Theorem 7.2 holds as required. Thus we may assume $\text{hdim}(G) > 0$. A special case of Lemma 3.1 from [8] shows that clique cutsets do not affect treewidth; thus we may assume that G does not admit a clique cutset.

(7) *If G has a balanced separator of size at most m, then the theorem holds.*

Let X be a balanced separator of G of size at most m. Let $D_1, \ldots, D_s$ be the components of $G \setminus X$. Since $|D_i| \leq \frac{n}{2}$ for every $i \in \{1, \ldots, s\}$, it follows from our induction on n that $\text{tw}(D_i) \leq C_t + \Delta\Psi(\log n + \text{hdim}(G) - 1)$. Then, by Lemma 2.15, the treewidth of G is at most

$$\begin{aligned}
&\max_{i \in \{1,\ldots,s\}} \text{tw}(D_i) + |X| \\
&\leq C_t + \Delta\Psi(\log n + \text{hdim}(G) - 1) + m \\
&\leq C_t + \Delta\Psi(\log n + \text{hdim}(G)),
\end{aligned}$$

where we use that $\Delta \geq m$. This proves (7).

In view of (7), we may assume that G does not admit a balanced separator of size at most m, and so the results of Section 4 apply.

Let $S_1, \ldots, S_k$ be a hub-partition of G with $k = \text{hdim}(G)$. We now use the terminology from Section 4. If follows from the definition of S_1 that every vertex in S_1 is d-safe. Let (T, χ) be an

m-atomic tree decomposition of G. Let t_0 be a center for T, and let $\beta = \beta(S_1)$ be as in Section 4. Write $\beta_0 = \chi(t_0)$. Our first goal is to prove:

(8) *There is a tree decomposition (T_β, χ_β) of β of such that for every $t \in T_\beta$, we have that $|\chi_\beta(t) \cap \beta_0| \leq C_t + \Delta\Psi(\log n + k - 1) + (\Delta - m)\Psi$.*

We start with:

(9) *If β has a balanced separator of size at most $2m(m-1) + C_t$, then (8) holds.*

To prove (9), we let X be a balanced separator of size at most $2m(m-1) + C_t = \Delta - m$ of β. Let $D_1, \ldots, D_s$ be the components of $\beta \setminus X$. Since $|D_i| \leq \frac{|\beta|}{2} \leq \frac{n}{2}$ for every $i \in \{1, \ldots, s\}$, it follows from our induction on n that $\mathrm{tw}(D_i) \leq C_t + \Delta\Psi(\log n + \mathrm{hdim}(G) - 1)$. Therefore, by Lemma 2.15 applied to β and X, we conclude that:

$$\begin{aligned}\mathrm{tw}(\beta) &\leq \max_{i \in \{1, \ldots, s\}} \mathrm{tw}(D_i) + |X| \\ &\leq C_t + \Delta\Psi(\log|\beta| + \mathrm{hdim}(G) - 1) + \Delta - m \\ &\leq C_t + \Delta\Psi(\log n + \mathrm{hdim}(G) - 1) + (\Delta - m)\Psi.\end{aligned}$$

This proves (9).

Now we may assume that β does not have a balanced separator of size at most $2m(m-1) + C_t$. Therefore $\beta^A(S_1)$ is defined, as in Section 4. By Theorem 4.7(4), we have that $S_1 \cap \mathrm{Hub}(\beta^A(S_1)) = \emptyset$ and $S_2 \cap Hub(\beta^A(S_1)), \ldots, S_k \cap Hub(\beta^A(S_1))$ is a hub-partition of $\beta^A(S_1)$. It follows that $\mathrm{hdim}(\beta^A(S_1)) \leq k - 1$.

Let $D_1, \ldots D_s$ be the components of $\beta \setminus \beta^A(S_1)$. By Theorem 4.7(3), we have that $|D_i| \leq \frac{n}{2}$ for every $i \in \{1, \ldots, s\}$. Moreover, by induction on k, we obtain that

$$\mathrm{tw}(\beta^A(S_1)) \leq C_t + \Delta\Psi(\log n + k - 1).$$

Our induction on n further implies that

$$\mathrm{tw}(D_i) \leq C_t + \Delta\Psi(\log n + k - 1).$$

By Theorem 2.6, the graph $\beta^A(S_1)$ admits a structured tree decomposition (T_0, χ_0) of width $\mathrm{tw}(\beta^A(S_1))$. For every $i \in \{1, \ldots, s\}$, let (T_i, χ_i) be a tree decomposition of D_i of width $\mathrm{tw}(D_i)$. Let (T_β, χ_β) be a tree decomposition of β obtained as in Theorem 4.8. We claim:

(10) *For every $i \in \{0, \ldots, s\}$ and for every $t \in T_i$, we have that $|(\chi_\beta(t) \setminus \chi_i(t)) \cap \beta_0| \leq 2m(m-1)\Psi$.*

Let $i \in \{1, \ldots, s\}$ and let $t \in T_i$. It follows immediately from Theorem 4.8 that $|(\chi_\beta(t) \setminus \chi_i(t)) \cap \beta_0| \leq 2m(m-1)$. Now let $t \in T_0$. Since $S_1 \cap \mathrm{Hub}(\beta^A(S_1)) = \emptyset$, we deduce from Theorem 6.3 applied to $\beta^A(S_1)$ that that $|\chi_0(t) \cap \mathrm{Core}(S_1)| \leq \Psi$. Now again it follows from Theorem 4.8 that $|(\chi_\beta(t) \setminus \chi_0(t)) \cap \beta_0| \leq 2m(m-1)\Psi$. This proves (10).

Now (8) follows immediately from (10), using that $\Delta - m = 2m(m-1) + C_t \geq 2m(m-1)$.

Next we use Theorem 4.9 to turn (T_β, χ_β) into a tree decomposition of G of the required width. Let $D'_1, \ldots, D'_r$ be the components of $G \setminus \beta_0$. In view of Theorem 2.6 and (8), we let (T'_0, χ'_0) be a structured tree decomposition of β such that for every $t \in T'_0$, we have that $|\chi'_0(t) \cap \beta_0| \leq C_t + \Delta\Psi(\log n + k - 1) + (\Delta - m)\Psi$.

Let $i \in \{1, \ldots, r\}$. Since t_0 is a center of T, it follows that $|D'_i| \leq \frac{n}{2}$ for every $i \in \{1, \ldots, r\}$. By induction on n, we have that

$$\mathrm{tw}(D'_i) \leq C_t + \Delta\Psi(\log n + k - 1).$$

Let (T'_i, χ'_i) be a tree decomposition of D'_i of width $\mathrm{tw}(D'_i)$. Let (U, ψ) be a tree decomposition of G obtained as in Theorem 4.9. Recall that for $u \in U$, $\psi(u)$ is defined as follows.

- If $u \in V(T'_0)$, let
$$\psi(u) = (S_1)_{bad} \cup (\chi'_0(u) \cap \beta_0) \cup \bigcup_{t_i \text{ s.t. } \chi'_0(u)\cap(Conn(t_i)\setminus\chi(t_0))\neq\emptyset} (\beta_0 \cap \chi(t_i)).$$
- If $u \in V(T'_i)$ for $i \in \{1, \dots, r\}$, let
$$\psi(u) = (S_1)_{bad} \cup \chi'_i(u) \cup (\beta_0 \cap \chi(t_i)).$$

We claim that
$$\operatorname{width}(U, \psi) \leq C_t + \Delta\Psi(\log n + k).$$
To prove this claim, we let $u \in V(U)$; we will establish an upper bound on $|\psi(u)|$. By Lemma 4.1, we have $|(S_1)_{bad}| \leq 1$. Now let $i \in \{0, \dots, r\}$ be such that $u \in T'_i$.

Suppose first that $i > 0$. Since $|\beta_0 \cap \chi(t_i)| < m$ (because (T, χ) is m-atomic and therefore m-lean by Theorem 2.8), it follows that
$$|\psi(u)| \leq 1 + |\chi'_i(u)| + (m-1) \leq \operatorname{tw}(D'_i) + m \leq C_t + \Delta\Psi(\log n + k),$$
using that $m \leq \Delta$, as required.

Thus we may assume that $i = 0$. By Theorem 4.2(3), we have that $(\chi'_0(u) \setminus \beta_0) \cap \operatorname{Hub}(\beta) = \emptyset$. Since for distinct $t_1, t_2 \in N_T(t_0)$, the set $Conn(t_1) \setminus \beta_0$ is disjoint from and anticomplete to the set $Conn(t_2) \setminus \beta_0$, applying Theorem 6.3 implies that
$$|\{t_i : \chi'_0(u) \cap (Conn(t_i) \setminus \beta_0) \neq \emptyset\}| \leq \Psi.$$
Since by (8), we have $|\chi'_0(u) \cap \beta_0| \leq C_t + \Delta\Psi(\log n + k - 1) + \Psi(\Delta - m)$, and since $|\beta_0 \cap \chi(t_i)| < m$ for every i, we deduce that
$$|\psi(u)| \leq 1 + C_t + \Delta\Psi(\log n + k - 1) + (\Delta - 1)\Psi \leq C_t + \Delta\Psi(\log n + k),$$
as required. This completes the proof of 7.2. ■

8. Algorithmic consequences

We now repeat the main points of the last section of [2] to explain the algorithmic significance of Theorem 1.4. We need the following theorem from [2]:

Theorem 8.1 (Abrishami, Chudnovsky, Hajebi, Spirkl [2])**.** *Let* P *be a problem which admits an algorithm running in time $\mathcal{O}(2^{\mathcal{O}(k)}|V(G)|)$ on graphs G with a given tree decomposition of width at most k. Also, let $\mathcal{G}$ be a class of graphs for which there exists a constant $c = c(\mathcal{G})$ such that $\operatorname{tw}(G) \leq c\log(|V(G)|)$ for all $G \in \mathcal{G}$. Then* P *is polynomial-time solvable in $\mathcal{G}$.*

In view of Theorem 1.4 and Theorem 8.1, we conclude the following.

Theorem 8.2. *Let $t \geq 1$ be fixed and* P *be a problem which admits an algorithm running in time $\mathcal{O}(2^{\mathcal{O}(k)}|V(G)|)$ on graphs G with a given tree decomposition of width at most k. Then* P *is polynomial-time solvable in $\mathcal{C}_{tt}$. In particular,* Stable Set, Vertex Cover, Dominating Set *and r-*Coloring *(with fixed r) are all polynomial-time solvable in $\mathcal{C}_{tt}$.*

Let us now discuss another important problem, and that is Coloring. It is well-known (and also follows immediately from Theorem 7.1), that for every t there exists a number $d(t)$ such that all graphs in $\mathcal{C}_t$ have chromatic number at most $d(t)$. Also, for each fixed r, by Theorem 8.2, r-Coloring is polynomial-time solvable in $\mathcal{C}_{tt}$. Now by solving r-Coloring for every $r \in \{1, \dots, d(t)\}$, we obtain a polynomial-time algorithm for Coloring in $\mathcal{C}_{tt}$.

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