INDUCED SUBGRAPHS AND TREE DECOMPOSITIONS XI. LOCAL STRUCTURE IN EVEN-HOLE-FREE GRAPH OF LARGE TREEWIDTH

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Abstract. Sintiari and Trotignon showed that for every $h \geq 1$, there are (even-hole, K_4)-free graphs of arbitrarily large treewidth in which every h-vertex induced subgraph is chordal. We prove the converse: given a graph H, every (even-hole, K_4)-free graph of large enough treewidth contains an induced subgraph isomorphic to H, if and only if H is chordal (and K_4-free).

As an immediate corollary, the above result settles a conjecture of Sintiari and Trotignon, asserting that every (even-hole, K_4)-free graph of sufficiently large treewidth contains an induced subgraph isomorphic to the graph obtained from the two-edge path by adding a universal vertex (also known as the *diamond*).

We further prove yet another extension of their conjecture with "K_4" replaced by an arbitrary complete graph and the "two-edge path" replaced by an arbitrary forest. This turns out to characterize forests: given a graph F, for every $t \geq 1$, every (even-hole, K_t)-free graph of sufficiently large treewidth contains an induced subgraph isomorphic to the graph obtained from F by adding a universal vertex, if and only if F is a forest.

1. Introduction

1.1. **Background and the main results.** Graphs in this paper have finite vertex sets, no loops and no parallel edges. Let $G = (V(G), E(G))$ be a graph. For $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X, and by $G \setminus X$ the induced subgraph of G obtained by removing X. We use induced subgraphs and their vertex sets interchangeably. For graphs G and H, we say G *contains* H if G has an induced subgraph isomorphic to H, and we say G is H*-free* if G does not contain H. For a family $\mathcal{H}$ of graphs, we say G is $\mathcal{H}$*-free* if G is H-free for every $H \in \mathcal{H}$. A class of graphs is *hereditary* if it is closed under isomorphism and taking induced subgraphs, or equivalently, if it is the class of all $\mathcal{H}$-free graphs for some other family $\mathcal{H}$ of graphs.

The *treewidth* of a graph G (denoted by $\text{tw}(G)$) is the smallest integer $w \geq 1$ for which one may choose a tree T as well as an assignment $(T_v : v \in V(G))$ of non-empty subtrees of T to the vertices of G, with the following specifications.

(T1) For every edge $uv \in E(G)$, T_u and T_v share at least one vertex.
(T2) For every $x \in V(T)$, there are at most $w+1$ vertices $v \in V(G)$ for which $x \in V(T_v)$.

In more intuitive terms, the treewidth of G is the optimized "load" imposed on each node of an underlying tree T in a representation of G as (a subgraph of) the intersection graph of subtrees of T.

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[II] Supported by NSF-EPSRC Grant DMS-2120644 and by AFOSR grant FA9550-22-1-0083.
[¶] Supported by DMS-EPSRC Grant EP/V002813/1.
[‖] We acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC), [funding reference number RGPIN-2020-03912]. Cette recherche a été financée par le Conseil de recherches en sciences naturelles et en génie du Canada (CRSNG), [numéro de référence RGPIN-2020-03912]. This project was funded in part by the Government of Ontario.
Date: September 20, 2023.

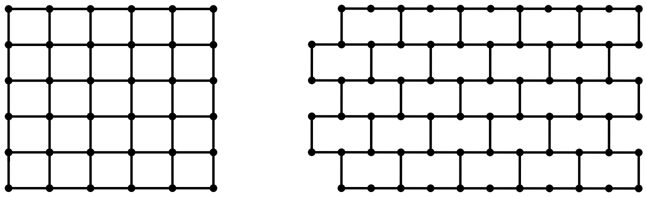

FIGURE 1. The 6-by-6 square grid (left) and the 6-by-6 wall $W_{6\times 6}$ (right).

Treewidth is widely recognized as a fundamental graph invariant. Initially a part of Robertson and Seymour's graph minors project, the notion of treewidth has gained independent significance over time, partly due to the convenient structural [17] and algorithmic [6] properties of graphs with small treewidth.

This also motivates exploring the opposite perspective: What characteristics prevent a graph from having small treewidth? The typical answer to this question, as well as its analogues for other width parameters, is to certify large treewidth by a local obstruction which is structurally less complex than the host graph, yet its treewidth is still relatively large. The prototypical example of this approach is the *Grid Theorem* of Robertson and Seymour [17], Theorem 1.1 below, which says that in minor-closed (and subgraph-closed) graph classes, the only obstructions to bounded treewidth are the "basic" ones: the t-by-t square grid for minors and subdivisions of the t-by-t hexagonal grid for subgraphs. The t-by-t hexagonal grid is also referred to as the *t-by-t wall*, denoted by $W_{t\times t}$ (see Figure 1, and also [3] for full definitions).

Theorem 1.1 (Robertson and Seymour [17])**.** *For every integer $t \geq 1$, every graph of sufficiently large treewidth contains the t-by-t square grid as a minor, or equivalently, a subdivision of $W_{t\times t}$ as a subgraph.*

For hereditary classes, the basic obstructions come in a larger variety: complete graphs, complete bipartite graphs, subdivided walls, and line-graphs of subdivided walls. Nevertheless, there are many well-known hereditary classes for which even the above list of obstructions is far from comprehensive [7, 9, 15, 18]. Let us be more precise. Given an integer $t \geq 1$, we say a graph H is a *t-basic obstruction* if G is isomorphic to one the following: the complete graph K_t, the complete bipartite graph $K_{t,t}$, a subdivision of $W_{t\times t}$, or the line-graph of a subdivision of $W_{t\times t}$, where the *line-graph* $L(F)$ of a graph F is the graph with vertex set $E(F)$, such that two vertices of $L(F)$ are adjacent if the corresponding edges of F share an end (see Figure 2). We say a graph G is *t-clean* if G does not contain a t-basic obstruction (as an induced subgraph). A graph class $\mathcal{G}$ is called *clean* if for every integer $t \geq 1$, there exists an integer integer $w(t) \geq 1$ (depending on $\mathcal{G}$) for which every t-clean graph in $\mathcal{G}$ has treewidth at most $w(t)$.

The basic obstructions can have arbitrarily large treewidth: K_{t+1}, $K_{t,t}$, all subdivisions of $W_{t\times t}$ and line-graphs of all subdivisions of $W_{t\times t}$ are known to have treewidth t. As a result, an exhaustive list of induced subgraph obstructions to bounded treewidth must include (an induced subgraph of) a basic obstruction of each type. One may then speculate that induced subgraphs admit the neatest possible analogue of Theorem 1.1: *every hereditary class is clean.* As mentioned above, however, this is far from true. Sintiari and Trotignon [18] were the first to identify a non-clean hereditary class, and their find is none other than the much-studied class of "even-hole-free" graphs. The full statement of their result appears in Theorem 1.2 below. A *hole* is an induced cycle on at least four vertices, the *length* of a hole is its number of vertices (or edges), and an *even hole* is a hole of even length.

Theorem 1.2 (Sintiari and Trotignon [18])**.** *For all integers $h, w \geq 1$, there exists an (even-hole, K_4)-free graph $G_{h,w}$ of treewidth more than w and with no hole of length at most h.*

One may observe that for every $t \geq 3$:

- the complete bipartite graph $K_{t,t}$ as well as all subdivisions of $W_{t\times t}$ contain "thetas;"

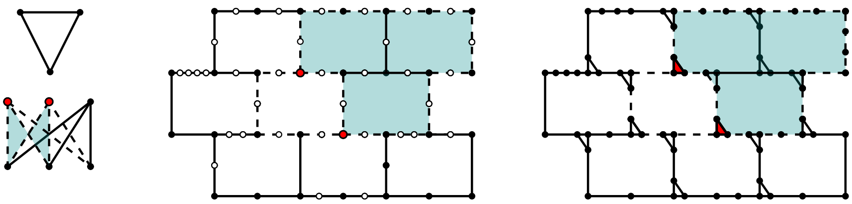

FIGURE 2. The 3-basic obstructions. A theta in $K_{3,3}$ (left), a theta in a subdivision of $W_{3\times 3}$ (middle) and a prism in the line-graph of the same subdivision of $W_{3\times 3}$ (right) are all depicted with dashed lines. An even hole in each theta and prism is also highlighted.

- line-graphs of all subdivisions of $W_{t\times t}$ contain line-graphs of thetas, also known as "prisms;" and
- thetas and prisms contain even holes.

See Figure 2 (and also the next section for the definition of a theta and a prism). In particular, for every $t \geq 1$, an even-hole-free graph is t-clean if and only if it is K_t-free. It follows from Theorem 1.2 that the class of (even-hole, K_4)-free graphs is a hereditary class of 4-clean graphs with unbounded treewidth. In fact, for a while, Theorem 1.2 (along with its counterpart in [18] concerning theta-free graphs) were the only known reasons why the class of all graphs is not clean; other (and less complicated) constructions were (re-)discovered later [7, 9, 15].

One would then desire to adjust a non-clean class to make it clean. For instance, Korhonen [13] proved that every graph class of bounded maximum degree is clean. In the context of hereditary classes, we pursue this line of inquiry through forbidding (more) induced subgraphs. Specifically, on may ask the following question.

Question 1.3. *Let $\mathcal{G}$ be a hereditary graph class. For which graphs H is it true that the class of all H-free graphs in $\mathcal{G}$ is clean?*

The class of all graphs then would be the first non-clean class to investigate from this point of view, which we did in recent joint work with Abrishami:

Theorem 1.4 (Abrishami, Alecu, Chudnovsky, Hajebi and Spirkl [1])**.** *Let H be a graph. Then the class of all H-free graphs is clean if and only if H is a subdivided star forest.*

The next candidate, suggested by Theorem 1.2, is the class of (even-hole, K_4)-free graphs. Considering Question 1.3 for the class $\mathcal{G}$ of (even-hole, K_4)-free graphs, a necessary condition for graphs H in the answer is provided by Theorem 1.2 and stated as Observation 1.5 below. Recall that a graph is *chordal* if it contains no hole; see Figure 3.

Observation 1.5. *Let H be graph such that every (even-hole, K_4)-free graph of sufficiently large treewidth contains H. Then H is a K_4-free chordal graph.*

In an earlier paper of this series [2] (joint with Abrishami), we approached the converse of Observation 1.5 by showing that if H is a K_3-free chordal graph, that is, a forest, then every (even-hole, K_4)-free graph of sufficiently large treewidth contains H. In fact, we proved:

Theorem 1.6 (Abrishami, Alecu, Chudnovsky, Hajebi and Spirkl [2])**.** *Let H be a graph. Then for every $t \geq 1$, every (theta, prism, K_t)-free graphs of sufficiently large treewidth contains H, if and only if H is a forest.*

Our main result in this paper is a full converse to Observation 1.5:

Theorem 1.7. *Given a graph H, every (even-hole, K_4)-free graph of sufficiently large treewidth contains H if and only if H is a K_4-free chordal graph.*

It is worth noting that, until now, essentially the only chordal graph of clique number three known to satisfy this result was the complete graph K_3 [8]. Particularly, the case where H is a

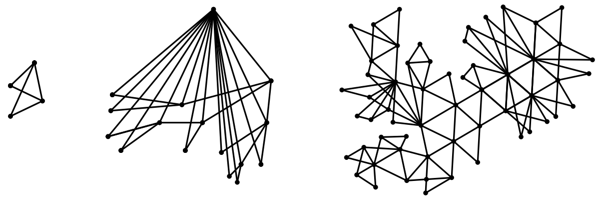

FIGURE 3. An assortment of K_4-free chordal graphs: the diamond (left), an extension of the diamonds as in Theorems 1.9 and 1.13 (middle), and an arbitrary one (right). All these graphs are in fact 2-trees.

"diamond" (see Figure 3) was of special interest [18]. Due to the prevalence of induced diamonds in the graphs $G_{h,w}$ from Theorem 1.2, this was posed by Sintiari and Trotignon as a conjecture:

Conjecture 1.8 (Sintiari and Trotignon [18])**.** *Every (even-hole, K_4)-free of sufficiently large treewidth contains the graph obtained from the two-edge path by adding a universal vertex, also known as the diamond.*

Since a diamond is K_4-free and chordal, Theorem 1.7 immediately implies Conjecture 1.8. Furthermore, our approach in this paper yields another extension of Conjecture 1.8 in which "K_4" is replaced by "K_t" for arbitrary $t \geq 1$ and the "two-edge path" is replaced by an arbitrary forest. Surprisingly, in view of Theorem 1.2, this is tight. For a graph F, let cone(F) be the graph obtained from F by adding a vertex adjacent to all vertices in $V(F)$.

Theorem 1.9. *Let F be graph. Then for every $t \geq 1$, every (even-hole, K_t)-free graph of sufficiently large treewidth contains* cone(F)*, if and only if F is forest.*

Theorem 1.9 in particular shows that Conjecture 1.8 holds for even-hole-free graph of bounded clique number in general, which in turn extends the main result of [5]:

Corollary 1.10. *For every integer $t \geq 1$, (even-hole, diamond, K_t)-free graphs have bounded treewidth.*

1.2. **Reduction to 2-trees.** Note that the "only if" implications in both Theorems 1.7 and 1.9 follow directly from Theorem 1.2. For the "if" implication, we prove the corresponding results in the slightly more general class $\mathcal{E}$ of (C_4, theta, prism, even wheel)-free graphs (where C_4 denotes the 4-vertex cycle, and a definition of "even wheels" will appear in the next section). At the same time, as a convenient technical step in our proof, we will reduce the "if" implication in Theorems 1.7 and 1.9 to the case where the excluded chordal graph is "maximal" with respect to its clique number.

Let us elaborate. For an integer n, we write $[n]$ for the set of all positive integers less than or equal to n (so $[n] = \emptyset$ if $n \leq 0$). A well-known characterization of chordal graphs due to Dirac [10] shows that for all integers $h, k \geq 1$, an h-vertex graph H is a K_{k+2}-free chordal graph if and only if there exists a bijection $\pi : V(H) \to [h]$ such that for every $i \in [h-1]$, the neighborhood of $\pi(i)$ in $V(H) \setminus \pi([i])$ is a clique of cardinality at most k. This inspires the following definition: a *k-tree* is a graph ∇ which is either a k-vertex complete graph, or we have $|V(\nabla)| = h > k$ and there exists a bijection $\varpi_\nabla : V(\nabla) \to [h]$ such that for every $i \in [h-k]$, the set of neighbors of $\varpi_\nabla(i)$ in $V(\nabla) \setminus \varpi_\nabla([i])$, which we refer to as the *forward neighbors of $\varpi_\nabla(i)$ in ∇*, is a clique of cardinality *exactly* k in ∇. For instance, 1-trees are exactly trees. It follows that every k-tree is a connected, K_{k+2}-free chordal graph. More importantly, a partial converse holds, too (the proof is straightforward, yet we include it to keep the paper self-contained).

Theorem 1.11. *For every $k \geq 1$ and every K_{k+2}-free chordal graph H, there exists a k-tree ∇ such that H is an induced subgraph of ∇.*

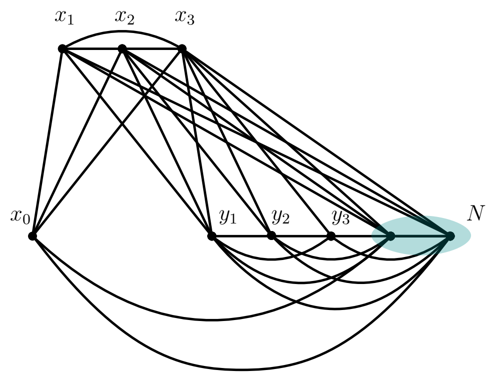

FIGURE 4. Proof of Theorem 1.11 (when $k = 5$ and $|N| = 2$).

Proof. Note that every K_{k+2}-free chordal graph H is an induced subgraph of a connected K_{k+2}-free chordal graph (which, for instance, may be obtained from H by adding a new vertex with exactly one neighbor in each component of H). Therefore, we only need to show that for every connected K_{k+2}-free chordal graph H, there exists a k-tree ∇ such that H is an induced subgraph of ∇. We prove this by induction on $|V(H)| = h$. The case $h = 1$ is trivial, so assume that $h > 1$.

Let $\pi : V(H) \to [h]$ be the bijection obtained from Dirac's result. Thus, for every $i \in [h-1]$, the set of neighbors of $\pi(i)$ in $V(H) \setminus \pi([i])$ is a clique on at most k vertices in H. Let $\pi(1) = x_0$, let $H^- = H \setminus \{x_0\}$, and let N be the set of neighbors x_0 in H. Then N is a non-empty clique on at most k vertices in H, which in turn implies that H^- is a connected K_{k+2}-free chordal graph on $h-1$ vertices and N is a non-empty clique on at most k vertices in H^-. By the induction hypothesis, there is a k-tree ∇^- such that H^- is an induced subgraph of ∇^-. In particular, N is a non-empty clique on at most k vertices in ∇^-. We deduce:

(1) *There is a clique K of cardinality k in ∇^- such that $N \subseteq K$.*

This is immediate if ∇^- is a complete graph. So we may assume that ∇^- is a k-tree that is not complete. Choose $x \in N$ with $\varpi^{-1}_{\nabla^-}(x) \in [h-1]$ as small as possible. Let M be the set of forward neighbors of x in ∇^- . Then $M \cup \{x\}$ is a clique of cardinality $k+1$ in ∇^- which contains the clique N of cardinality at most k. This proves (1).

Let K be as in (1). Fix an enumeration $\{y_i : i \in [k - |N|]\}$ of the elements of $K \setminus N$. We define ∇ as follows. Let

$$V(\nabla) = V(\nabla^-) \cup \{x_i : i \in \{0, \dots, k - |N|\}\};$$

$$E(\nabla) = E(\nabla^-) \cup \left(\bigcup_{i=0}^{k-|N|} (\{x_i y : y \in N\} \cup \{x_i y_j : j \in [i]\} \cup \{x_i x_j : j \in [k - |N|] \setminus [i]\}) \right).$$

Let $\varpi_\nabla(x_i) = i + 1$ for all $i \in \{0, \dots, k - |N|\}$ and let $\varpi_\nabla(z) = \varpi_{\nabla^-}(v) + k - |N| + 1$ for all $v \in V(\nabla^-)$ (see Figure 4). One may check that ∇ is k-tree and H is an induced subgraph of ∇; we omit the details. ■

For every $t \geq 1$, we denote by $\mathcal{E}_t$ the class of all K_t-free graphs in $\mathcal{E}$. In view of Theorem 1.11, Theorems 1.7 and 1.9 follow, respectively, from Theorems 1.12 and 1.13 below. We will prove both results in the last section.

Theorem 1.12. *For every 2-tree ∇, there exists an integer $\Upsilon = \Upsilon(\nabla) \geq 1$ such that every graph $G \in \mathcal{E}_4$ with $\mathrm{tw}(G) > \Upsilon$ contains ∇.*

Theorem 1.13. *For every integer $t \geq 1$ and every tree T, there exists an integer $\Gamma = \Gamma(t, T) \geq 1$ such that every $G \in \mathcal{E}_t$ with $\mathrm{tw}(G) > \Gamma$ contains $\mathrm{cone}(T)$.*

1.3. **Outline.** We conclude the introduction with a rough account of our main ideas. Due to the fact that adding universal vertices to a tree results in a 2-tree, the proofs of Theorems 1.12 and 1.13 follow a similar outline, which we give below.

Let $G \in \mathcal{E}_t$ be a graph of huge treewidth. We will show that G contains a copy of a given 2-tree ∇ as a subgraph, not necessarily induced, but close enough so that the outcomes of Theorems 1.12 and 1.13 would follow immediately from known results. Our method is to grow this copy of ∇ in G through an inductive process, adding one vertex at a time with respect to the ordering imposed on $V(\nabla)$ by ϖ_∇, reversed (so $\varpi_\nabla(1)$ is the last vertex to be added).

In order to grow our 2-tree, we use an auxiliary structure that we call a "kaleidoscope," consisting of many holes sharing a three-vertex path and otherwise vertex-disjoint. We then define a notion of "mirroring," whereby, roughly speaking, a set of vertices is "d-mirrored" by a kaleidoscope if every vertex in the set has at least d neighbors in each of the kaleidoscope's holes, outside of the shared three-vertex path.

The main ingredients to our argument are then as follows:

- In Section 4 (and in particular in Theorem 4.1), we show that given an integer d, if a vertex is 1-mirrored by a suitably large kaleidoscope, then it is in fact d-mirrored by a "sub-kaleidoscope" of prescribed size. In particular, when $d \geq 3$, it is a wheel center for all of the holes in the sub-kaleidoscope.
- To begin the growing process, we show in Theorem 5.1 that, assuming our graph has large treewidth (or more specifically, that it has two vertices connected by many disjoint paths), we can produce, using the previous bullet point and results from an earlier paper in the series, a clique of size two which is 3-mirrored by a large kaleidoscope. In general, our induction hypothesis will be that we can find a large kaleidoscope 3-mirroring the 2-tree we have constructed so far.
- Given two adjacent wheel centers with the same rim, Theorem 3.2 provides some constraints on their neighborhoods along the rim. This allows us to find common neighbors of adjacent vertices in our 2-tree on each of the holes of the large kaleidoscope 3-mirroring it. These common neighbours are candidates for extending our 2-tree.
- Theorem 5.5 is the core of our induction step. In it, we show that, by choosing the candidate carefully, we are able to guarantee that it has three neighbors on many of the other holes of the kaleidoscope. In other words, we are able to find a large sub-kaleidoscope 3-mirroring the extended 2-tree, thus maintaining the property we need for the induction.
- Finding that right candidate requires a novel idea, presented in Section 3, which is totally different from the material developed in the earlier papers in this series: we show that taking specific minors of G keeps us in the class $\mathcal{E}$. This enables us to essentially "pretend" certain edges in G do not exist, which in turn allows us to use the machinery from an earlier paper in order to find the right candidate for extending the 2-tree.

In Section 2, we cover the terminology and the results from earlier papers in this series to be used in subsequent sections. We complete the proofs of Theorems 1.12 and 1.13 in Section 6.

2. Preliminaries

In this section, we set up our notation and terminology. We also mention a few results from the earlier papers in this series [1, 2].

Let $G = (V(G), E(G))$ be a graph and let $x \in V(G)$. We denote by $N_G(x)$ the set of all neighbors of x in G, and write $N_G[x] = N_G(x) \cup \{x\}$. For an induced subgraph H of G (not necessarily containing x), we define $N_H(x) = N_G(x) \cap H$ and $N_H[x] = N_H(x) \cup \{x\}$. Also, for $X \subseteq G$, we denote by $N_G(X)$ the set of all vertices in $G \setminus X$ with at least one neighbor in X, and define $N_G[X] = N_G(X) \cup X$. Let $X, Y \subseteq G$ be disjoint. We say X is *complete* to Y if every vertex in X is adjacent to every vertex in Y in G, and X is *anticomplete* to Y if there is no edge in G with an end in X and an end in Y.

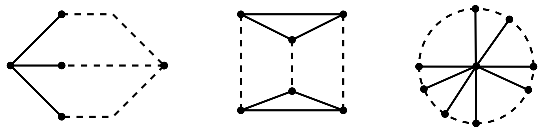

FIGURE 5. From left to right, a theta, a prism and an even wheel. Dashed lines represent paths of length at least one.

A *path in* G is an induced subgraph of G that is a path. If P is a path in G, we write $P = p_1\text{-}\cdots\text{-}p_k$ meaning $V(P) = \{p_1, \dots, p_k\}$ and p_i is adjacent to p_j if and only if $|i - j| = 1$. We call the vertices p_1 and p_k the *ends of* P, and say that P is *from* p_1 *to* p_k. The *interior of* P, denoted by P^*, is the set $P \setminus \{p_1, p_k\}$. The *length* of a path is its number of edges (so a path of length at most one has empty interior). Similarly, if C is a cycle, we write $C = c_1\text{-}\cdots\text{-}c_k\text{-}c_1$ to mean that $V(C) = \{c_1, \dots, c_k\}$ and c_i is adjacent to c_j if and only if $|i-j| \in \{1, k-1\}$. The *length* of a cycle is its number of vertices (or edges).

A *theta* is a graph Θ consisting of two non-adjacent vertices a, b, called the *ends of* Θ, and three pairwise internally disjoint paths P_1, P_2, P_3 from a to b of length at least two, called the *paths of* Θ, such that P_1^*, P_2^*, P_3^* are pairwise anticomplete to each other. For a graph G, by a *theta in* G we mean an induced subgraph of G which is a theta.

A *prism* is a graph Π consisting of two disjoint triangles $\{a_1, a_2, a_3\}$and $\{b_1, b_2, b_3\}$, called the *triangles of* Π, and three pairwise disjoint paths P_1, P_2, P_3 called the *paths of* Π, where P_i has ends a_i, b_i for each $i \in \{1, 2, 3\}$, and for distinct $i, j \in \{1, 2, 3\}$, a_ia_j and b_ib_j are the only edges between P_i and P_j. For a graph G, by a *prism in* G we mean an induced subgraph of G which is a prism.

A *wheel* in a graph G is a pair (C, v) where C is a hole in G and $v \in G \setminus C$ is a vertex with at least three neighbors in C. An *even wheel* in G is a wheel (C, v) in G where v has an even number of neighbors in H. We say G is *even-wheel-free* if there is no even wheel in G.

See Figure 5 for a depiction of a theta, a prism, a pyramid and an even wheel. It is straightforward to check that the class $\mathcal{E}$ of all (C_4, theta, prism, even wheel)-free graphs contains the class of all even-hole-free graphs.

We now mention a few results from two previous papers in the current series [1, 2]. Let k be a positive integer and let G be a graph. A *strong* k*-block* in G is a set B of at least k vertices in G such that for every 2-subset $\{x, y\}$ of B, there exists a collection $\mathcal{P}_{\{x,y\}}$ of at least k distinct and pairwise internally disjoint paths in G from x to y, where for every two distinct 2-subsets $\{x, y\}, \{x', y'\} \subseteq B$ of G, and every choice of paths $P \in \mathcal{P}_{\{x,y\}}$ and $P' \in \mathcal{P}_{\{x',y'\}}$, we have $P \cap P' = \{x, y\} \cap \{x', y'\}$. In [1], it is proved that:

Theorem 2.1 (Abrishami, Alecu, Chudnovsky, Hajebi and Spirkl [1])**.** *For every integer $k \geq 1$, the class of all graphs with no strong k-block is clean.*

As discussed in Subsection 1.1, for every $t \geq 1$, a (theta, prism)-free graph is t-clean if and only if it is K_t-free. So the following is immediate from Theorem 2.1:

Corollary 2.2. *For all integers $k, t \geq 1$, there exists an integer $\beta = \beta(k, t)$ such that every (theta, prism, K_t)-free graph with no strong k-block has treewidth at most $\beta(k, t)$.*

Moreover, Theorem 2.3 below reveals further information about the adjacency between different paths joining two vertices in a strong block. This was a major ingredient in our proof of Theorem 1.7 in [2].

Theorem 2.3 (Abrishami, Alecu, Chudnovsky, Hajebi and Spirkl [2])**.** *For all integers $t, \nu \geq 1$, there exists an integer $\psi = \psi(t, \nu) \geq 1$ with the following property. Let G be a (theta, prism, K_t)-free graph, let $a, b \in V(G)$ be distinct and non-adjacent and let $\mathcal{P}$ be a collection of pairwise internally disjoint paths in G from a to b with $|\mathcal{P}| \geq \psi$. For each $P \in \mathcal{P}$, let x_P be the neighbor of a in P (so $x_P \neq b$). Then there exist $P_1, \dots, P_\nu \in \mathcal{P}$ such that:*

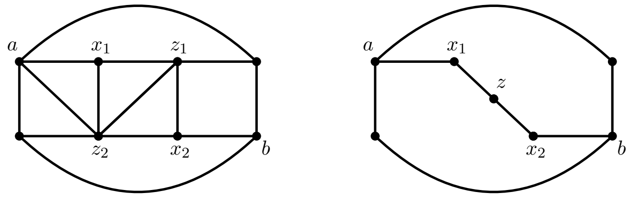

Figure 6. Left: a (theta, prism, even wheel)-free graph G containing a clique $\{z_1, z_2\}$ for which $N_G(z_1) \cap N_G(z_2) = \{x_1, x_2\}$ is a stable set of vertices of degree three in G (observe that G does contain C_4). Right: the graph $G \lhd^{z_1}_{z_2}$ which is a theta with ends a, b.

- *$\{x_{P_1}, \ldots, x_{P_\nu}, b\}$ is a stable set in G; and*
- *for all $i, j \in [\nu]$ with $i < j$, x_{P_i} has a neighbor in $P^*_j \setminus \{x_{P_j}\}$.*

We also need a quantified version of Ramsey's classical theorem, which has appeared in several references; see, for instance, [11].

Theorem 2.4 (Ramsey [16], see also [11])**.** *For all integers $c, s \geq 1$, every graph G on at least c^s vertices contains either a clique of cardinality c or a stable set of cardinality s.*

Finally, we include the following well-known lemma which follows directly from Theorem 2.4 combined with Lemma 2 in [14].

Lemma 2.5. *For all integers $q, r, s, t \geq 1$ there exists an integer $o = o(q, r, s, t) \geq 1$ with the following property. Let G be a $(K_{s,s}, K_t)$-free graph. Let $\mathcal{X}$ be a collection of pairwise disjoint subsets of $V(G)$, each of cardinality at most r, with $|\mathcal{X}| \geq o$. Then there are q distinct sets $X_1, \ldots, X_q \in \mathcal{X}$ which are pairwise anticomplete in G.*

3. Class-preserving minors in $\mathcal{E}$

In this section we develop the vital constituent of our procedure for growing 2-trees by iteratively obtaining common neighbors of prescribed pairs of adjacent vertices. The main result is Theorem 3.1, which shows that, although $\mathcal{E}$ is a hereditary class, certain minors of certain graphs in $\mathcal{E}$ belong to $\mathcal{E}$. Specifically, for a graph G and two adjacent vertices $z_1, z_2 \in V(G)$, we define $G \lhd^{z_1}_{z_2}$ to be the graph with the following specifications:

- $V(G \lhd^{z_1}_{z_2}) = (V(G) \setminus \{z_1, z_2\}) \cup \{z\}$;
- $G \lhd^{z_1}_{z_2}[V(G) \setminus \{z_1, z_2\}] = G \setminus \{z_1, z_2\}$; and
- $N_{G \lhd^{z_1}_{z_2}}(z) = N_G(z_1) \cap N_G(z_2)$.

See Figure 6. In other words, $G \lhd^{z_1}_{z_2}$ is the minor of G (without parallel edges) obtained by first contracting the edge z_1z_2 into a new vertex z, and then removing every edge in the resulting graph between z and a vertex in $(N_G(z_1) \setminus N_G(z_2)) \cup (N_G(z_2) \setminus N_G(z_1))$. Our goal in this section is to prove the following:

Theorem 3.1. *Let $G \in \mathcal{E}$ be a graph and let $z_1, z_2 \in V(G)$ be distinct and adjacent such that $N_G(z_1) \cap N_G(z_2)$ is a stable set of vertices of degree at most three in G. Then we have $G \lhd^{z_1}_{z_2} \in \mathcal{E}$.*

Two remarks: first, as far as our application of Theorem 3.1 is concerned, it suffices to show that $G \lhd^{z_1}_{z_2}$ is (theta, prism)-free (under the same hypotheses). This is thanks to Theorem 2.3 holding true for the larger class of (theta, prism)-free graphs rather than just $\mathcal{E}$. Second, the proof of Theorem 3.1 (and so its application in the proof of Theorem 5.4) is the only place in this paper where we use the assumption that G is C_4-free. Nevertheless, as unfortunate as it may appear, excluding C_4 is necessary even if we ask for $G \lhd^{z_1}_{z_2}$ not to "be" a theta; see Figure 6.

We now plunge into the proof of Theorem 3.1, beginning with the following definition. Let G be a graph, let H be an induced subgraph of G and let $v \in G \setminus H$. We say that:

(G) v is *H-good* if $|N_H(v)| = 1$;
(B) v is *H-bad* if $N_H(v)$ is a clique in H with at least two vertices; and
(U) v is *H-ugly* if $N_H(v)$ is not a clique in H.

So every vertex in $N_G(H) \subseteq G \setminus H$ is exactly one of H-good, H-bad, or H-ugly. The next result is an important ingredient for the proof of Theorem 3.1. Similar results have also appeared in [4, 5].

Theorem 3.2. *Let G be a (theta, prism, even wheel)-free graph, let C be a hole in G and let $z_1, z_2 \in G \setminus C$ be distinct and adjacent, each with at least one neighbor in C. Assume that z_1 and z_2 have no common neighbor in C. Then either both z_1 and z_2 are C-good and their (unique) neighbors in C are distinct and adjacent, or exactly one of z_1 and z_2 is C-bad. Consequently, if $G \in \mathcal{E}$, then exactly one of z_1 and z_2 is C-bad.*

Proof. Note that if both z_1 and z_2 are C-bad, then since z_1 and z_2 have no common neighbor in C, it follows that $C \cup \{z_1, z_2\}$ is a prism in G, a contradiction. So we may assume without loss of generality that z_1 is either C-good or C-ugly. If z_2 is C-bad, then we are done. So we can consider the case that z_2 is also either C-good or C-ugly; in particular, since neither (C, z_1) nor (C, z_2) is an even wheel in G, it follows that for every $i \in \{1, 2\}$, $|N_C(z_i)|$ is an odd integer. Assume first that both z_1 and z_2 are C-good, say $N_C(z_i) = \{x_i\}$ for $i \in \{1, 2\}$. Then since z_1 and z_2 have no common neighbor in C, and $C \cup \{z_1, z_2\}$ is not a theta in G, it follows that x_1 and x_2 are distinct and adjacent in G, as required.

This leaves the case where one of z_1 and z_2, say the former, is C-ugly. Since z_1 and z_2 have no common neighbor in C, it follows that $N_C(z_2) \subseteq C \setminus N_C(z_1)$. Note that every component of $C \setminus N_C(z_1)$ is a path in C (and so in G). Moreover, for every component P of $C \setminus N_C(z_1)$, $C_P = N_C[P] \cup \{z_1\}$ is a hole in G. Since $C_P \cup \{z_2\}$ is not a theta in G, and (C_P, z_2) is not even wheel in G, and z_1 and z_2 have no common neighbor in C, it follows that z_2 has an even number of neighbors in P. In conclusion, we have shown that z_2 has an even number of neighbors in each component of $C \setminus N_C(z_1)$. But then z_2 has an even number of neighbors in C, a contradiction. We conclude that either both z_1 and z_2 are C-good and their neighbors in C are distinct and adjacent, or exactly one of z_1 and z_2 is C-bad. In addition, if $G \in \mathcal{E}$, then the first outcome does not hold, as otherwise $G[N_C[z_1] \cup N_C[z_2]]$ is isomorphic to C_4, a contradiction. This completes the proof of Theorem 3.2. ■

We also need the following lemma.

Lemma 3.3. *Let $G \in \mathcal{E}$ be a graph and let $z_1, z_2 \in V(G)$ be distinct and adjacent such that $N_G(z_1) \cap N_G(z_2)$ is a stable set of vertices of degree at most three in G. Let $z \in V(G \lhd^{z_1}_{z_2})$ be as in the definition of $G \lhd^{z_1}_{z_2}$ and let W be an induced subgraph of $G \lhd^{z_1}_{z_2}$ which is either a theta, or a prism, or an even wheel. Then there is a path P in W with ends a, b for which the following hold.*

(a) *We have $z \in P \setminus (N_W[a] \cup N_W[b])$, and so $W \setminus P^* \subseteq G \setminus (N_G[z_1] \cap N_G[z_2])$.*
(b) *The vertices in P^* (including z) have degree two in W, and a, b both have degree three in W.*
(c) *In the graph G, both z_1 and z_2 have a neighbor in $W \setminus P$.*

Proof. First, assume that there is no path P in W satisfying 3.3(a) and 3.3(b). Since $G \in \mathcal{E}$, it follows that $z \in W$. Also, since $N_G(z_1) \cap N_G(z_2)$ is a stable set of vertices of degree at most three in G, it follows that $N_{G \lhd^{z_1}_{z_2}}(z)$ is a stable set of vertices of degree at most two in $G \lhd^{z_1}_{z_2}$. In particular, there is no wheel (C, v) in $G \lhd^{z_1}_{z_2}$ where $z \in N_C[v]$, and z does not belong to a triangle of a prism in $G \lhd^{z_1}_{z_2}$. Moreover, from the assumption that there is no path in W satisfying 3.3(a) and 3.3(b), it follows that there is no wheel (C, v) in $G \lhd^{z_1}_{z_2}$ where $z \in C \setminus N_C(v)$, and z does not belong the interior of a path of a theta or a prism in $G \lhd^{z_1}_{z_2}$. We deduce that W is a theta in $G \lhd^{z_1}_{z_2}$ and z is an end of W. Let $z' \in V(G \lhd^{z_1}_{z_2}) \setminus N_{G \lhd^{z_1}_{z_2}}[z] = V(G) \setminus (N_G[z_1] \cap N_G[z_2])$ be the other end of W and let P_1, P_2, P_3 be the paths of W. Then for every $i \in [3]$, P_i has ends z, z', and for some $j \in \{1,2\}$, z_j is not adjacent to z' in G. On the other hand, for every $i \in [3]$, we have $N_{P_i}(z) \subseteq N_G(z_j) \cap (P_i \setminus \{z\})$. Thus, traversing $P_i \setminus \{z\}$ starting at z', we may choose x_i to be the first vertex in $N_G(z_j) \cap (P_i \setminus \{z\})$; it follows that $x_i \in P_i^*$. But then there is a theta in G with ends z_j, z' and paths z_j-x_i-P_i-z' for $i \in [3]$, which violates $G \in \mathcal{E}$. This proves that there exists a path P in W with ends a, b for which 3.3(a) and 3.3(b) hold.

It remains to show that P satisfies 3.3(c). Suppose for a contradiction, and without loss of generality, that z_1 is anticomplete to $W \setminus P$ in G. Then $U = (P \setminus \{z\}) \cup \{z_1\}$ is a connected induced subgraph of G with $a, b \in U$ such that $U \setminus \{a, b\} \subseteq P^* \cup \{z_1\}$ is anticomplete to $W \setminus P$ in G. Consequently, there exists a path P_1 in U from a to b where P_1^* is anticomplete to $W \setminus P$ in G. But now $(W \setminus P) \cup P_1$ is a theta, a prism or an even wheel in G (depending on whether W is a theta, a prism or an even wheel in G, respectively), which is impossible because $G \in \mathcal{E}$. This completes the proof of Lemma 3.3. ■

The next two lemmas, in turn, show that under the assumptions of Theorem 3.1, the graph $G \lhd_{z_2}^{z_1}$ is theta-free and prism-free.

Lemma 3.4. *Let $G \in \mathcal{E}$ be a graph and let $z_1, z_2 \in V(G)$ be distinct and adjacent such that $N_G(z_1) \cap N_G(z_2)$ is a stable set of vertices of degree at most three in G. Then $G \lhd_{z_2}^{z_1}$ is theta-free.*

Proof. Suppose for a contradiction that there is a theta W in $G \lhd_{z_2}^{z_1}$. Let $z \in V(G \lhd_{z_2}^{z_1})$ be as in the definition of $G \lhd_{z_2}^{z_1}$. Let P be the path in W with ends a, b satisfying Lemma 3.3. It follows from Lemma 3.3(a) and (b) that a, b are the ends of W, P is a path of W, and we have $z \in P \setminus (N_P[a] \cup N_P[b])$. Let Q_1, Q_2 be the paths of W distinct from P; so Q_1 and Q_2 both have ends a, b, as well. Let $C = Q_1 \cup Q_2$. Then C is a hole in $G \setminus \{z_1, z_2\}$ and we have $C = W \setminus P^*$.

From the definition of $G \lhd_{z_2}^{z_1}$, it follows that $W \setminus \{z\} \subseteq G \setminus \{z_1, z_2\}$ and $\{z_1, z_2\}$ is complete to $N_P(z)$. As a result, for every $i \in \{1,2\}$, there are two paths $P_{a,i}, P_{b,i}$ in $(P \setminus \{z\}) \cup \{z_i\}$ from a to z_i and from b to z_i, respectively, such that $P_{a,i} \setminus \{z_i\}$ and $P_{b,i} \setminus \{z_i\}$ are disjoint and anticomplete in G, and both $P_{a,i}^*$ and $P_{b,i}^*$ are disjoint from and anticomplete to $C \setminus \{a, b\}$ in G. We claim that:

(2) *Let $i \in \{1,2\}$. Then either a is adjacent to z_i in G, or $N_C(z_i) \subseteq N_{Q_j}[b]$ for some $j \in \{1,2\}$. Similarly, either b is adjacent to z_i in G, or $N_C(z_i) \subseteq N_{Q_j}[a]$ for some $j \in \{1,2\}$.*

We only need to show that for every $i \in \{1,2\}$, either a is adjacent to z_i in G, or $N_C(z_i) \subseteq N_{Q_j}[b]$ for some $j \in \{1,2\}$. Suppose not. Then we may assume without loss of generality, that a is a not adjacent to z_1 in G, and there is a vertex in $Q_1^* \setminus N_{Q_1}(b)$ which is adjacent to z_1 in G. It follows that $P_{a,1}$ has length at least two, and that there is a path R of length at least two in G from a to z_1 such that $R^* \subseteq Q_1^* \setminus N_{Q_1}(b)$. Also, since $P_{b,1} \cup Q_2$ is a connected induced subgraph of G containing the two non-adjacent vertices a and z_1, it follows that there exists a path S of length at least two in $P_{b,1} \cup Q_2$ from a to z_1. But then there is theta in G with ends a, z_1 and paths $P_{a,1}, R, S$, contrary to the fact that $G \in \mathcal{E}$. This proves (2).

From (2), it follows immediately that:

(3) *Let $i \in \{1,2\}$ such that z_i has a neighbor in C. Then the following hold.*
- *If z_i is C-good, then we have $N_C(z_i) \subseteq (N_C(a) \cap N_C(b)) \cup \{a, b\}$.*
- *If z_i is C-bad, then for some $j \in \{1,2\}$, either $N_C(z_i) = N_{Q_j}[a]$ or $N_C(z_i) = N_{Q_j}[b]$.*
- *If z_i is C-ugly, then we have $a, b \in N_C(z_i)$.*

Now, since $C \setminus \{a, b\} = W \setminus P$ and $W \setminus P \subseteq W \setminus P^* = C$, from Lemma 3.3(a) and the definition of $G \lhd_{z_2}^{z_1}$ it follows that z_1 and z_2 have no common neighbor in C, and from Lemma 3.3(c), it follows that z_1 and z_2 each have at least one neighbor in $C \setminus \{a, b\}$. Consequently, by Theorem 3.2 and without loss of generality, we may assume that z_1 is C-bad and z_2 is either C-good or C-ugly. It follows from the second bullet of (3) that for some $j \in \{1,2\}$, we have either $N_C(z_1) = N_{Q_j}[a]$ or $N_C(z_1) = N_{Q_j}[b]$. We may exploit the symmetry between a, b and between Q_1, Q_2, and assume that $N_C(z_1) = N_{Q_1}[a]$. Since $a \in V(H)$ is not a common neighbor of z_1 and z_2, we deduce from the third bullet of (3) that z_2 is C-good. This, together with the first bullet of (3), the fact that z_2 has a neighbor in $C \setminus \{a, b\}$ and the fact that z_1 and z_2 have no common neighbor in C, implies that Q_2 has length two, say $Q_2 = a\text{-}q\text{-}b$, and we have $N_C(z_2) = \{q\}$. But then $G[\{a, q, z_1, z_2\}]$ is isomorphic to C_4, contrary to the fact that $G \in \mathcal{E}$. This completes the proof of Lemma 3.4. ■

Lemma 3.5. *Let $G \in \mathcal{E}$ be a graph and let $z_1, z_2 \in V(G)$ be distinct and adjacent such that $N_G(z_1) \cap N_G(z_2)$ is a stable set of vertices of degree at most three in G. Then $G \triangleleft^{z_1}_{z_2}$ is prism-free.*

Proof. Suppose for a contradiction that there is a prism W in $G \triangleleft^{z_1}_{z_2}$. Let $z \in V(G \triangleleft^{z_1}_{z_2})$ be as in the definition of $G \triangleleft^{z_1}_{z_2}$. Let P be the path in W with ends a, b satisfying Lemma 3.3. It follows from Lemma 3.3(a) and (b) that P is a path of W, a and b belong to distinct triangles of W and we have $z \in P \setminus (N_P[a] \cup N_P[b])$. Let aa_1a_2 and bb_1b_2 be the triangles of W and let Q_1, Q_2 be the paths of W distinct from P such that Q_i has ends a_i, b_i for $i \in \{1, 2\}$. Let $C = Q_1 \cup Q_2$. Then C is a hole in $G \setminus \{z_1, z_2\}$ and we have $C = W \setminus P$.

From the definition of $G \triangleleft^{z_1}_{z_2}$, it follows that $W \setminus \{z\} \subseteq G \setminus \{z_1, z_2\}$ and $\{z_1, z_2\}$ is complete to $N_P(z)$. As a result, for every $i \in \{1, 2\}$, there are two paths $P_{a,i}, P_{b,i}$ in $(P \setminus \{z\}) \cup \{z_i\}$ from a to z_i and from b to z_i, respectively, such that $P_{a,i} \setminus \{z_i\}$ and $P_{b,i} \setminus \{z_i\}$ are disjoint and anticomplete in G, and both $P^*_{a,i}$ and $P^*_{b,i}$ are disjoint from and anticomplete to C in G.

Now, since $C = W \setminus P \subseteq W \setminus P^*$, it follows from Lemma 3.3(a) and the definition of $G \triangleleft^{z_1}_{z_2}$ that z_1 and z_2 have no common neighbor in C, and it follows from Lemma 3.3(c) that z_1 and z_2 each have at least one neighbor in $C \setminus \{a, b\}$, Consequently, by Theorem 3.2, one of z_1 and z_2 is C-bad; say z_1 is C-bad. Let us write $N_C(z_1) = \{q_1, q_2\}$ where q_1 and q_2 are adjacent. Due to the symmetry between $\{a_1, a_2\}$ and $\{b_1, b_2\}$, we may assume, without loss of generality, that $\{a_1, a_2\} \cap \{q_1, q_2\} \subseteq \{a_1\}$, and there are disjoint paths R_1 and R_2 in C from a_1 to q_1 and from a_2 to q_2, respectively. It follows that either $\{a_1, a_2\} \cap \{q_1, q_2\} = \emptyset$ or $\{a_1, a_2\} \cap \{q_1, q_2\} = \{a_1\} = \{q_1\}$. In the former case, there is a prism in G with triangles $aa_1a_2, z_1q_1q_2$ and paths $P_{a,1}, R_1$ and R_2. Also, in the latter case, $C' = a\text{-}P_{a,1}\text{-}z_1\text{-}q_2\text{-}R_2\text{-}a_2\text{-}a$ is a hole in G and $a_1 = q_1 \in G \setminus C'$ has exactly four neighbors in C', namely a, a_2, q_2 and z_1. But then (C', z_1) is an even wheel in G. Both latter conclusions violate the assumption that $G \in \mathcal{E}$, hence completing the proof of Lemma 3.5. ■

We can now give a proof of Theorem 3.1:

Proof of Theorem 3.1. Suppose not. Let $z \in V(G \triangleleft^{z_1}_{z_2})$ be as in the definition of $G \triangleleft^{z_1}_{z_2}$. First, we show that :

(4) *$G \triangleleft^{z_1}_{z_2}$ is C_4-free.*

To see this, suppose there is a hole C of length four in $G \triangleleft^{z_1}_{z_2}$. Since $G \in \mathcal{E}$, it follows that $z \in C$. So we have $N_C(z) = C \cap N_G(z_1) \cap N_G(z_2)$ and there exists exactly one vertex z' in C with $z' \in V(G \triangleleft^{z_1}_{z_2}) \setminus N_{G \triangleleft^{z_1}_{z_2}}[z] = V(G) \setminus (N_G[z_1] \cap N_G[z_2])$. As a result, for some $j \in \{1, 2\}$, z_j is not adjacent to z' in G. But now $(C \setminus \{z\}) \cup \{z_j\}$ is a hole of length four in G, contrary to the fact that $G \in \mathcal{E}$. This proves (4).

From (4) and Lemmas 3.4 and 3.5, we deduce that there exists an even wheel (C, v) in $G \triangleleft^{z_1}_{z_2}$. Let $W = G[V(C) \cup \{v\}]$ and let P be the path in W with ends a, b satisfying Lemma 3.3. It follows from Lemma 3.3(a) and (b) that $z \in P \setminus (N_P[a] \cup N_P[b])$ and P is a path of length at least four in C such that $a, b \in N_C(v) \subseteq N_G(v)$ and v is anticomplete to P^* in $G \triangleleft^{z_1}_{z_2}$. Let $Q = C \setminus P^*$. Then Q is a path in G from a to b. Let a' and b' be the neighbors of a and b in Q, respectively. Since C is an even wheel in $G \triangleleft^{z_1}_{z_2}$, it follows that Q has length at least three, and so a, a', b, b' are all distinct. In addition, we have $W \setminus P^* = Q \cup \{v\}$, $W \setminus P = Q^* \cup \{v\}$, and $|N_Q(v)| = |N_C(v)| \geq 4$ is an even integer.

From the definition of $G \lhd^{z_1}_{z_2}$, it follows that $W \setminus \{z\} \subseteq G \setminus \{z_1, z_2\}$ and $\{z_1, z_2\}$ is complete to $N_P(z)$. As a result, for every $i \in \{1, 2\}$, there are two paths $P_{a,i}, P_{b,i}$ in $(P \setminus \{z\}) \cup \{z_i\}$ from a to z_i and from b to z_i, respectively, such that $P_{a,i} \setminus \{z_i\}$ and $P_{b,i} \setminus \{z_i\}$ are disjoint and anticomplete in G, and both $P^*_{a,i}$ and $P^*_{b,i}$ are disjoint from and anticomplete to $Q^* \cup \{v\}$ in G. We claim that:

(5) *The vertex v has a neighbor in $Q \setminus \{a, a', b, b'\}$.*

Suppose not. Then we have $N_Q(v) = \{a, a', b, b'\}$. Then $C' = Q^* \cup \{v\} = W \setminus P$ is a hole in G. By Lemma 3.3(a) and the definition of $G \lhd^{z_1}_{z_2}$, z_1 and z_2 have no common neighbor in C', and by Lemma 3.3(c), z_1 and z_2 each have at least one neighbor in C'. Therefore, by Theorem 3.2, one of z_1 and z_2, say the former, is C'-bad in G. Let $N_{C'}(z_1) = \{q, q'\}$ where q and q' are adjacent. The symmetry between a' and b' and between q and q' allows us to assume that $|\{a', v\} \cap \{q, q'\}| \leq 1$, and there are disjoint paths R and R' in C' from v to q and from a' to q', respectively. It follows that either

- $\{a', v\} \cap \{q, q'\} = \emptyset$; or
- $\{a', v\} \cap \{q, q'\} = \{a'\} = \{q'\}$; or
- $\{a', v\} \cap \{q, q'\} = \{v\} = \{q\}$.

If the first bullet above holds, then there is a prism in G with triangles $aa'v, qq'z_1$ and paths $P_{a,1}, R$ and R'. Also, if the second bullet above holds, then $C'' = a\text{-}P_{a,1}\text{-}z_1\text{-}q\text{-}R\text{-}v\text{-}a$ is a hole in G and $a' = q' \in G \setminus C''$ has exactly four neighbors in C'', namely a, v, q and z_1, which in turn implies that (C'', a') is an even wheel in G. Similarly, if the third bullet above holds, then $C'' = a\text{-}P_{a,1}\text{-}z_1\text{-}q'\text{-}R'\text{-}a'\text{-}a$ is a hole in G and $v = q \in G \setminus C''$ has exactly four neighbors in C'', namely a, a', q' and z_1. It follows that (C'', v) is an even wheel in G. Each of the last three conclusions goes against the assumption that $G \in \mathcal{E}$. This proves (5).

(6) *Let $i \in \{1, 2\}$ such that v is not adjacent to z_i in G. Then $N_{W \setminus P}(z_i)$ is a non-empty subset of $\{a', b'\}$.*

Suppose not. By Lemma 3.3(c), z_1 and z_2 each have at least one neighbor in $W \setminus P = Q^* \cup \{v\}$, and so $N_{W\setminus P}(z_i) \neq \emptyset$. It follows that there exists a vertex $q \in Q^* \setminus \{a', b'\} = Q \setminus (N_G[a] \cup N_G[b])$ which is adjacent to z_i in G. This, together with (5), implies that $(Q^* \setminus \{a', b'\}) \cup \{v, z_i\}$ is connected, and so there exists a path S of length at least two in $(Q^* \setminus \{a', b'\}) \cup \{v, z_i\}$ from v to z_i. But now there is a theta in G with ends v, z_i and path $v\text{-}a\text{-}P_{a,i}\text{-}z_i$, $v\text{-}b\text{-}P_{b,i}\text{-}z_i$ and S, contrary to the fact that $G \in \mathcal{E}$. This proves (6).

(7) *There exists $i \in \{1, 2\}$ for which v is adjacent to z_i in G.*

Suppose for a contradiction that v is anticomplete $\{z_1, z_2\}$. By (6), both $N_{W\setminus P}(z_1)$ and $N_{W\setminus P}(z_2)$ are non-empty subset of $\{a', b'\}$. Also, by Lemma 3.3(a) and the definition of $G \lhd^{z_1}_{z_2}$, in the graph G, z_1 and z_2 do not have a common neighbor in $W \setminus P^* = Q \cup \{v\}$. Therefore, due to the symmetry between z_1 and z_2 and between a' and b', it is safe to assume that $N_{Q^*\cup\{v\}}(z_1) = \{a'\}$ and $N_{Q^*\cup\{v\}}(z_2) = \{b'\}$. In particular, $D = a'\text{-}z_1\text{-}z_2\text{-}b'\text{-}Q\text{-}a'$ is a hole in G and $N_D(v) = N_Q(v) \setminus \{a, b\}$. Recall that $|N_Q(v)|$ is an even integer which is at least four, and so $|N_D(v)|$ is a non-zero even integer. Since (D, v) is not even wheel in G and $D \cup \{v\}$ is not a theta in G, it follows that v is D-bad. From this combined with (5), and without loss of generality, we may assume that v is not adjacent to a' and there is a path R in G from a' to v with $R^* \subseteq Q^* \setminus \{a', b'\}$. Moreover, again by Lemma 3.3(a) and the definition of $G \lhd^{z_1}_{z_2}$, in the graph G, z_1 and z_2 do not have a common neighbor in $W \setminus P^* = Q \cup \{v\}$. Specifically, there exists $i \in \{1, 2\}$ for which z_i is no adjacent to a. But now there is a theta in G with ends a, z_i and paths $a\text{-}a'\text{-}z_i$, $P_{a,i}$ and $a\text{-}v\text{-}b\text{-}P_{b,i}\text{-}z_i$, a contradiction. This proves (7).

(8) *Let $i \in \{1,2\}$ such that v is not adjacent to z_i in G. Then either z_i is anticomplete to $\{a,b\}$ in G, or v is anticomplete to $\{z_1, z_2\}$ in G.*

Suppose not. By Lemma 3.3(a), z_1 and z_2 do not have a common neighbor in $\{a,b,v\} \subseteq Q \cup \{v\} = W \setminus P^*$. But now either either $G[\{a,v,z_1,z_2\}]$ or $G[\{b,v,z_1,z_2\}]$ is isomorphic to C_4, contrary to the fact that $G \in \mathcal{E}$. This proves (8).

Let us now finish the proof. In view of (7), we may assume, without loss of generality, that v is adjacent to z_2 in G. Thus, by (8), z_1 is anticomplete to $\{a,b\}$ in G. Since $N_{W\setminus P}(z_1)$ is a non-empty subset of $\{a',b'\}$, we may assume that a' is adjacent to z_1 in G. Moreover, since $G[\{a',v,z_1,z_2\}]$ is not isomorphic to C_4, it follows that a' and v are not adjacent in G. But then there is a theta in G with ends a, z_1 and paths a-a'-z_1, $P_{a,1}$ and a-v-b-$P_{b,1}$-z_1, contrary to the fact that $G \in \mathcal{E}$. This completes the proof of Theorem 3.1. ■

4. Third time is the charm

This section supplies the technical foundation for the proofs of Theorems 1.12 and 1.13. Despite its crucial role, the proof of Theorem 4.1 below as the main result revolves around three successive applications of Theorem 2.3. Perhaps more importantly, it highlights the significance of not giving up after the first two rounds.

In essence, Theorem 4.1 asserts that in a (theta, prism, even wheel)-free graph of bounded clique number with a configuration of sufficiently many "put-together" holes, a vertex z with at least one private neighbor in each hole will have *several* neighbors in most of these holes. This result, combined with Theorem 3.2, will be used in Section 5 to demonstrate that adjacent pairs of vertices with neighbors in these holes must indeed have common neighbors within them.

To make this precise, we begin with the definition of said configuration of holes. Given a graph G and an integer $w \geq 1$, a *w-kaleidoscope* in G is a 4-tuple $(a,x,y,\mathcal{W})$ where:

- (K1) $a,x,y \in V(G)$, and x-a-y is a path in G (so x and y are distinct and non-adjacent);
- (K2) $\mathcal{W}$ is a set of w pairwise internally disjoint paths in $G \setminus a$ from x to y; and
- (K3) for every $W \in \mathcal{W}$, a is anticomplete to W^* in G.

Furthermore, given a subset $Z \subseteq V(G)$ and an integer $d \geq 1$, we say that Z *is d-mirrored by* $(a,x,y,\mathcal{W})$ if:

- (M1) Z is disjoint from $(\bigcup_{W \in \mathcal{W}} V(W)) \cup \{a\}$;
- (M2) the vertex a has at most one neighbor in Z; and
- (M3) for every $z \in Z$ and every $W \in \mathcal{W}$, z is anticomplete to $N_W[x] \cup N_W[y]$, and z has at least d distinct neighbors in W. In particular, z is anticomplete to $\{x,y\}$.

We also say a vertex $z \in V(G)$ is *d-mirrored by* $(a,x,y,\mathcal{W})$ if $\{z\}$ is d-mirrored by $(a,x,y,\mathcal{W})$. Our goal in this section is to show that:

Theorem 4.1. *For all integers $d,t,w \geq 1$, there exists an integer $\kappa = \kappa(d,t,w) \geq 1$ with the following property. Let G be a (theta, prism, even wheel, K_t)-free graph, let $(a,x,y,\mathcal{W})$ be a κ-kaleidoscope in G, and let $z \in V(G)$ be 1-mirrored by $(a,x,y,\mathcal{W})$. Then there exists $\mathcal{W}' \subseteq \mathcal{W}$ with $|\mathcal{W}'| = w$ such that z is d-mirrored by $(a,x,y,\mathcal{W}')$.*

We start with a number of further definitions and lemmas. Let G be a graph, and let $s,l \geq 1$ be integers. An *(s,l)-palanquin* in G is a triple $(a,S,\mathcal{L})$ where:

- (P1) $a \in V(G)$, $S \subseteq N_G(a)$ is a stable set of cardinality s in G, and $\mathcal{L}$ is a collection of l pairwise disjoint paths in $G \setminus (S \cup \{a\})$; and
- (P2) for every $L \in \mathcal{L}$, a is anticomplete to L, and every vertex in S has a neighbor in L.

For instance, given a w-kaleidoscope $(a,x,y,\mathcal{W})$ in G for some $w \geq 1$, one may easily observe that $(a,\{x,y\},\{W^* : W \in \mathcal{W}\})$ is a $(2,w)$-palanquin in G. Next we have two lemmas about palanquins, with short and self-contained proofs (and, although less efficiently, these lemmas can also be deduced from appropriate result in earlier papers of this series).

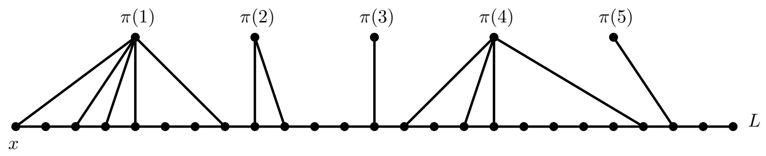

FIGURE 7. A 5-alignment, where $\pi(3)$ and $\pi(5)$ are L-good, $\pi(2)$ is L-bad and $\pi(1)$ and $\pi(4)$ are L-ugly.

Lemma 4.2. *Let $p, q, t \geq 1$ be integers. Let G be a (theta, K_t)-free graph and let $(a, S, \mathcal{L})$ be a $(2^q p + 2q, (2^q p + 2q)^2 t^3 + q)$-palanquin in G. Then there exists $S_1 \subseteq S$ with $|S_1| = p$ and $\mathcal{L}_1 \subseteq \mathcal{L}$ with $|\mathcal{L}_1| = q$ such that for every $L \in \mathcal{L}_1$, no two vertices in S_1 have a common neighbor in L, and either x is L-bad for all $x \in S_1$, or x is L-ugly for all $x \in S_1$.*

Proof. We first show that:

(9) *There exists $\mathcal{L}_0 \subseteq \mathcal{L}$ with $|\mathcal{L}_0| = q$ such that for every $L \in \mathcal{L}_0$, no two vertices in S have a common neighbor in L.*

Suppose not. Since $|\mathcal{L}| = (2^q p + 2q)^2 t^3 + q$, it follows that there exists $\mathcal{L}'_1 \subseteq \mathcal{L}$ with $|\mathcal{L}'_1| = (2^q p + 2q)^2 t^3$ such that for every $L \in \mathcal{L}'_1$, there exist two distinct vertices in S with a common neighbor in L. Since $|S| = 2^q p + 2q$, this in turn implies that there are two vertices $x, x' \in S$ as well as a subset $\mathcal{L}''_1$ of $\mathcal{L}'_1$ with $|\mathcal{L}''_1| = t^3$ such that for every $L \in \mathcal{L}''_1$, the vertices x, x' have a common neighbor y_L in L. Thus, we have $|\{y_L : L \in \mathcal{L}''_1\}| = t^3$, which along with Theorem 2.4 and the assumption that G is K_t-free implies that there are three distinct paths $L_1, L_2, L_3 \in \mathcal{L}''_1$ for which $\{y_{L_1}, y_{L_2}, y_{L_3}\}$ is a stable set in G. But now there is a theta in G with ends x, x' and paths x-y_{L_1}-x', x-y_{L_2}-x' and x-y_{L_3}-x', a contradiction. This proves (9).

Henceforth, let $\mathcal{L}_0$ be as in (9).

(10) *There exists $S_0 \subseteq S$ with $|S_0| = 2^q p$ such that for every $x \in S_0$ and every $L \in \mathcal{L}_0$, x is either L-bad or L-ugly.*

Note that every vertex in S has a neighbor in every path in $\mathcal{L}_0 \subseteq \mathcal{L}$. Therefore, since $|S| = 2^q p + 2q$ and $|\mathcal{L}_0| = q$, in order to prove (10), it suffices to show that for every $L \in \mathcal{L}_0$, there at most two vertices in S which are L-good. Suppose for a contradiction that there exists a 3-subset $\{x_1, x_2, x_3\}$ of S and a path $L \in \mathcal{L}_0$ such that x_1, x_2, x_3 are all L-good. For each $i \in [3]$, let y_i be the unique neighbor of x_i in L. We may assume without loss of generality that the path in L from y_1 to y_3 contains y_2. Recall also that a is complete to $\{x_1, x_2, x_3\} \subseteq S$ and anticomplete to $L \in \mathcal{L}_0 \subseteq \mathcal{L}$. But now there is a theta in G with ends a, y_2 and paths a-x_1-y_1-L-y_2, a-x_2-y_2 and a-x_3-y_3-L-y_2, a contradiction. This proves (10).

Let S_0 be as in (10). Fix an enumeration $L_1, \dots, L_q$ of the elements of $\mathcal{L}_0$. In view of (10), one may construct a sequence $X_0 \supset X_1 \supset \cdots \supset X_q$ of sets such that:

- $X_0 = S_0$ and for every $i \in [q]$, we have $|X_i| = |X_{i-1}|/2$; and
- for every $i \in [q]$, either x is L_i-bad for all $x \in X_i$ or x is L_i-ugly for all $x \in X_i$.

Let $S_1 = X_q$. Then we have $|S_1| = |X_q| = 2^{-q}|X_0| = 2^{-q}|S_0| = p$, and for every $i \in [q]$, either x is L_i-bad for all $x \in S_1$ or x is L_i-ugly for all $x \in S_1$. But then by (9), S_1 and $\mathcal{L}_1$ satisfy Lemma 4.2, as required. ■

Let G be a graph and let $s \geq 1$ be an integer. An *s-alignment* is a quadruple (S, L, x, π) where:

(A1) $S \subseteq V(G)$ is stable with $|S| = s$ and L is a path in $G \setminus S$ and x is an end of L;
(A2) every vertex in S has a neighbor in L; and
(A3) $\pi : [s] \to S$ is a bijection such that for all $i, j \in [s]$ with $i < j$, traversing L starting at x, all neighbors of $\pi(i)$ appear strictly before all neighbors of $\pi(j)$ in L.

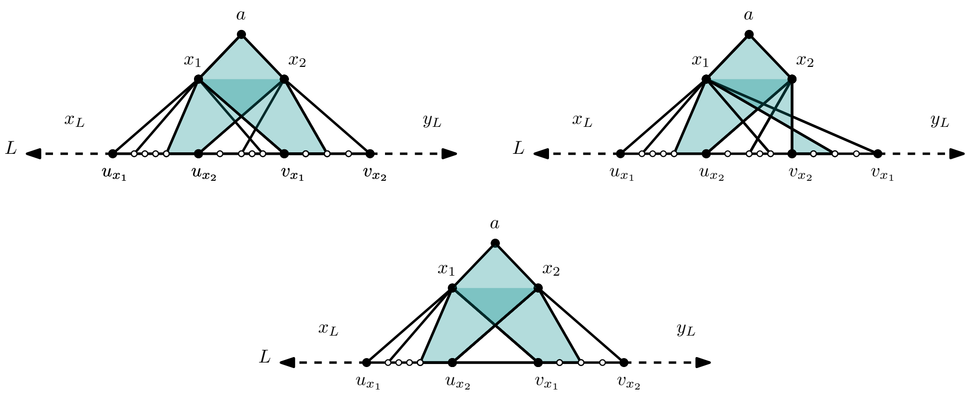

FIGURE 8. Proof of Lemma 4.3. Top left: the case $i = 1$. Top right: the case $i = 2$.

See Figure 7. It follows in particular from (A3) that no two vertices in S have a common neighbor in L.

Lemma 4.3. *Let $s \geq 1$ be an integer. Let G be a theta-free graph, let $(a, S, \{L\})$ be an $(s, 1)$-palanquin in G and let x_L be an end of L. Assume that no two vertices in S have a common neighbor in L. Assume also that either x is L-bad for all $x \in S$, or x is L-ugly for all $x \in S$. Then there exists a bijection $\pi : S \to [s]$ such that (S, L, x_L, π) is an s-alignment in G.*

Proof. Since no two vertices in S have a common neighbor in L, the result is trivial if every vertex in S is L-bad. So we may assume that all vertices S are L-ugly. For every $x \in S$, traversing L starting at x_L, let u_x, v_x be the first and the last neighbor of x in L, respectively; thus, u_x and v_x are distinct. In order to prove Lemma 4.3, it suffices to show that for every two vertices $x_1, x_2 \in S$, the paths u_{x_1}-L-v_{x_1} and u_{x_2}-L-v_{x_2} have no vertex in common. Suppose this is violated by $x_1, x_2 \in S$. Since x_1 and x_2 have no common neighbor in L, it follows that $u_{x_1}, u_{x_2}, v_{x_1}, v_{x_2}$ are pairwise distinct, x_1 is not adjacent to u_{x_2}, v_{x_2}, and x_2 is not adjacent to u_{x_1}, v_{x_1}. Since both x_1 and x_2 are L-ugly, we may assume without loss of generality that for some $\{i, j\} = \{1, 2\}$, L traverses the vertices $x_L, u_{x_1}, u_{x_2}, v_{x_i}, v_{x_j}$ in this order (where x_L and u_{x_1} might be the same). It follows that there are two paths P and Q in G from x_1 to x_2 such that P^* contains u_{x_2} and is contained in u_{x_1}-L-u_{x_2}, and Q^* contains v_{x_i} and is contained in v_{x_1}-L-v_{x_2}. In particular, P and Q are internally disjoint. Recall also that a is complete to $\{x_1, x_2\} \subseteq S$ and anticomplete to L. Now, if the path u_{x_2}-L-v_{x_i} has non-empty interior, then P^* and Q^* are anticomplete, and so there is a theta in G with ends x_1, x_2 and paths x_1-a-x_2, P and Q, a contradiction (see Figure 8 top). Otherwise, since x_2 is L-ugly, we have $i = 1$, and so there is a theta in G with ends x_1, u_{x_2} and paths x_1-a-x_2-u_{x_2}, x_1-P-u_{x_2} and x_1-v_{x_1}-u_{x_2}, again a contradiction (see Figure 8 bottom). This completes the proof of Lemma 4.3. ■

We apply Lemmas 4.2 and 4.3 to take the main step in the proof of Theorem 4.1. This is where two of the three applications of Theorem 2.3 show up (and the third one will appear at the beginning of the proof of Theorem 4.1).

Lemma 4.4. *For all integers $s, l, t \geq 1$, there exist integers $\sigma = \sigma(l, s, t) \geq 1$ and $\lambda = \lambda(l, s, t) \geq 1$ with the following property. Let G be a (theta, prism, even wheel, K_t)-free graph. Let $(a, S, \mathcal{L})$ be an (σ, λ)-palanquin in G. For every $L \in \mathcal{L}$, fix an end x_L of L. Then there exist $S' \subseteq S$ with $|S'| = s$, an l-subset $\mathcal{L}'$ of $\mathcal{L}$ and a bijection $\pi : S' \to [s]$ such that the following hold.*

(a) *For every $L \in \mathcal{L}'$, the quadruple (S', L, x_L, π) is an s-alignment in G.*
(b) *For every $x \in S'$ and every $L \in \mathcal{L}'$, the vertex x is L-ugly.*

Proof. We begin with defining the values of σ and λ. Let $\psi(\cdot, \cdot)$ be as in Theorem 2.3 and let $o(\cdot, \cdot, \cdot, \cdot)$ be as in Lemma 2.5. The two successive applications of Theorem 2.3 are already signaled by the two nested appearances of $\psi(\cdot,\cdot)$ below. Let

$$\psi_1 = \psi_1(t) = \psi(t, 2);$$

$$o_1 = o_1(t) = o(\psi_1, 4, 3, t);$$

$$\psi_2 = \psi_2(t) = \psi(t, o_1 + 1);$$

$$o_2 = o_2(t) = o(\psi_2, 2, 3, t);$$

Also, let $p = p(s) = s + 2$ and let $q = q(l, s, t) = (l + o_2)(s + 2)!$. We claim that $\sigma = \sigma(l, s, t) = 2^q p + 2q$ and $\lambda = \lambda(l, s, t) = q(2^q p + 2q)^2 t^3$ satisfy Lemma 4.4.

Suppose not. Due to the choice of σ and λ, we can apply Lemma 4.2 to $(a, S, \mathcal{L})$, obtaining $S_1 \subseteq S$ with $|S_1| = s + 2$ and $\mathcal{L}_1 \subseteq \mathcal{L}$ with $|\mathcal{L}_1| = (l + o_2)(s + 2)!$ such that for every $L \in \mathcal{L}_1$, no two vertices in S_1 have a common neighbor in L, and either x is L-bad for all $x \in S_1$, or x is L-ugly for all $x \in S_1$. This, combined with Lemma 4.3, implies that for every $L \in \mathcal{L}_1$, there exists a bijection $\pi_L : S_1 \to [s + 2]$ such that (S_1, L, x_L, π_L) is an $(s + 2)$-alignment in G. We deduce that:

(11) *There exists $\mathcal{L}_2 \subseteq \mathcal{L}_1$ with $|\mathcal{L}_2| = o_2(s + 2)!$ such that for every $x \in S_1$ and every $L \in \mathcal{L}_1$, x is L-bad.*

Suppose not. From $|\mathcal{L}_1| = (l+o_2)(s+2)!$, it follows that there exists $\mathcal{L}'_2 \subseteq \mathcal{L}_1$ with $\mathcal{L}'_2 = l(s+2)!$ such that for every $x \in S_1$ and every $L \in \mathcal{L}'_2$, x is L-ugly. Since $|S_2| = s+2$, this in turn implies that there exists $\mathcal{L}' \subseteq \mathcal{L}'_2$ with $|\mathcal{L}'| = l$ as well as a bijection $\pi' : S_2 \to [s + 2]$, such that $\pi_L = \pi'$ for all $L \in \mathcal{L}'$. Let $S' = \pi'([s])$ and let $\pi = \pi'|_{[s]}$. Then for every $L \in \mathcal{L}'$, (S', L, x_L, π) is an s-alignment in G, and so $S', \mathcal{L}'$ satisfy 4.4(a). Also, since $S' \subseteq S_1$ and $\mathcal{L}' \subseteq \mathcal{L}'_2$, it follows that $S', \mathcal{L}'$ satisfy 4.4(b). This violates the assumption that Lemma 4.4 fails to hold for our chosen values of σ and λ, hence proving (11).

Let $\mathcal{L}_2$ be as in (11). Since $|S_1| = s + 2$ and $|\mathcal{L}_2| = o_2(s + 2)!$, it follows that there exists $\mathcal{L}_3 \subseteq \mathcal{L}_2$ with $|\mathcal{L}_3| = o_2$ as well as a bijection $\pi : S_1 \to [s + 2]$, such that $\pi_L = \pi$ for all $L \in \mathcal{L}_3$. Let us write $x = \pi(1), z = \pi(2)$ and $y = \pi(3)$ (this is possible because $s + 2 \geq 3$).

For every $L \in \mathcal{L}_3$, traversing L starting at x_L, let u_L be the last neighbor of x in L, let z^1_L, z^2_L be the first and the last neighbor of z in L, respectively, and let v_L be first neighbor of y in L. By (11), u_L, z^1_L, z^2_L and v_L are all distinct, appearing on L in this order, and $N_L(z) = \{z^1_L, z^2_L\}$ is a clique in G. Since G is $(K_{3,3}, K_t)$-free and from the choice of o_2, it follows that we can apply Lemma 2.5 to the sets $\{\{z^1_L, z^2_L\} : L \in \mathcal{L}_3\}$ and show that:

(12) *There exists $\mathcal{L}'_3 \subseteq \mathcal{L}_3$ with $|\mathcal{L}'_3| = \psi_2$ such that for all distinct $L, L' \in \mathcal{L}'_3$, $\{z^1_L, z^2_L\}$ is anticomplete to $\{z^1_{L'}, z^2_{L'}\}$.*

Next, we launch the first application of Theorem 2.3. Note that $\{z\text{-}z^2_L\text{-}L\text{-}v_L\text{-}y : L \in \mathcal{L}'_3\}$ is a collection of ψ_2 pairwise internally disjoint paths in G between non-adjacent vertices z and y. Consequently, due to the choice of ψ_2, we can apply Theorem 2.3 to this collection, and deduce that there exists $L_3 \in \mathcal{L}'_3$ as well as $\mathcal{L}_4 \subseteq \mathcal{L}'_3 \setminus \{L_3\}$ with $|\mathcal{L}_4| = o_1$ such that:

- $\{z^2_L : L \in \mathcal{L}_4\} \cup \{z^2_{L_3}, y\}$ is a stable set in G (though this is already guaranteed by (12)); and
- for all $L \in \mathcal{L}_4$, $z^2_{L_3}$ has a neighbor in the interior of $z^2_L\text{-}L\text{-}v_L\text{-}y$.

For each $L \in \mathcal{L}_4$, traversing $z^2_L\text{-}L\text{-}v_L\text{-}y$ from z^2_L to y, let w^1_L, w^2_L be the first and the last neighbors of $z^2_{L_3}$ in $z^2_L\text{-}L\text{-}v_L\text{-}y$, respectively; it follows that $\{w^1_L, w^2_L\} \cap \{z^2_L, y\} = \emptyset$ and $z^1_L, z^2_L, w^1_L, w^2_L$ appear on L in this order. For every $L \in \mathcal{L}_4$, let C_L denote the hole $a\text{-}x\text{-}u_L\text{-}L\text{-}v_L\text{-}y\text{-}a$ in G. We deduce that:

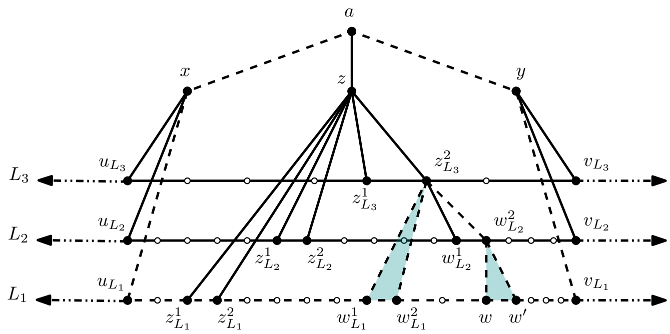

FIGURE 9. Proof of Lemma 4.4.

(13) *For every $L \in \mathcal{L}_4$, $z^2_{L_3}$ is C_L-bad. More explicitly, w^1_L and w^2_L are distinct and adjacent, and we have $N_{C_L}(z^2_{L_3}) = \{w^1_L, w^2_L\}$.*

To see this, note that z and $z^2_{L_3}$ are two adjacent vertices in $G \setminus C_L$, each with at least one neighbor in C_L. In fact, we have $N_{C_L}(z) = \{a, z^1_L, z^2_L\}$, and so z is C_L-ugly. Since a is anticomplete to L_3 and from (12), it follows that $z^2_{L_3}$ is anticomplete to $\{a, z^1_L, z^2_L\} = N_{C_L}(z)$. Thus, z and $z^2_{L_3}$ have no common neighbor in C_L. So by Theorem 3.2, $z^2_{L_3}$ is C_L-bad. This proves (13).

Furthermore, since G is $(K_{3,3}, K_t)$-free and from the choice of o_1, we can apply Lemma 2.5 to the sets $\{\{w^1_L, w^2_L, z^1_L, z^2_L\} : L \in \mathcal{L}_4\}$, and deduce that:

(14) *There exists $\mathcal{L}'_4 \subseteq \mathcal{L}_4$ with $|\mathcal{L}'_4| = \psi_1$ such that for all distinct $L, L' \in \mathcal{L}'_4$, $\{w^1_L, w^2_L, z^1_L, z^2_L\}$ is anticomplete to $\{w^1_{L'}, w^2_{L'}, z^1_{L'}, z^2_{L'}\}$.*

Now, note that $\{z^2_{L_3}\text{-}w^2_L\text{-}L\text{-}v_L\text{-}y : L \in \mathcal{L}'_4\}$ is a collection of ψ_1 pairwise internally disjoint paths in G between non-adjacent vertices $z^2_{L_3}$ and y. Together with the choice of ψ_1, this allows an application of Theorem 2.3 to $\{z^2_{L_3}\text{-}w^2_L\text{-}L\text{-}v_L\text{-}y : L \in \mathcal{L}'_4\}$. We obtain two distinct paths $L_1, L_2 \in \mathcal{L}'_4$ such that

- $\{w^2_{L_1}, w^2_{L_2}, y\}$ is a stable set in G (though this already follows from (14)); and
- the vertex $w^2_{L_2}$ has a neighbor in the interior of $w^2_{L_1}\text{-}L_1\text{-}v_L\text{-}y$.

Traversing $w^2_{L_1}\text{-}L_1\text{-}v_L\text{-}y$ from $w^2_{L_1}$ to y, let w, w' be the first and the last neighbors of $w^2_{L_2}$ in $w^2_{L_1}\text{-}L_1\text{-}v_L\text{-}y$, respectively; it follows that $\{w, w'\} \cap \{w^2_{L_1}, y\} = \emptyset$ and $w^1_{L_1}, w^2_{L_1}, w, w'$ appear on L_1 in this order. In addition, we have:

(15) *The vertex $w^2_{L_2}$ is C_{L_1}-bad. More precisely, w and w' are distinct and adjacent, and we have $N_{C_{L_1}}(w^2_{L_2}) = \{w, w'\}$.*

Let $C_1 = a\text{-}z\text{-}z^2_{L_1}\text{-}L_1\text{-}v_L\text{-}y\text{-}a$; then C_1 is a hole in G. Note that $w^2_{L_2}$ and $z^2_{L_3}$ are two adjacent vertices in $G \setminus C_1$, each with at least one neighbor in C_1. In fact, we have $N_{C_1}(z^2_{L_3}) = \{z, w^1_{L_1}, w^2_{L_1}\}$, and so $z^2_{L_3}$ is C_1-ugly. This, along with (14), implies that $w^2_{L_2}$ is anticomplete to $\{z, w^1_{L_1}, w^2_{L_1}\} = N_{C_1}(z^2_{L_3})$. It follows that $w^2_{L_2}$ and $z^2_{L_3}$ have no common neighbor in C_1. But then by Theorem 3.2, $w^2_{L_2}$ is C_1-bad. More precisely, w and w' are distinct and adjacent, and we have $N_{C_1}(w^2_{L_2}) = \{w, w'\}$. Since $C_{L_1} \setminus C_1 = x\text{-}u_{L_1}\text{-}L_1\text{-}z^1_{L_1}$, it remains to show that $w^2_{L_2}$ is anticomplete to $u_{L_1}\text{-}L_1\text{-}z^1_{L_1}$. Suppose not. Recall that by (14), a and $w^2_{L_2}$ are not adjacent in G. Consequently, there is a path Q of length at least two in G from a to $w^2_{L_2}$ such that Q^* is contained in the interior of $a\text{-}x\text{-}u_{L_1}\text{-}L_1\text{-}z^1_{L_1}$. But then in view of (13), there is a theta in G with ends $a, w^2_{L_2}$ and paths $a\text{-}z\text{-}z^2_{L_3}\text{-}w^2_{L_2}$, $a\text{-}y\text{-}v_{L_1}\text{-}L_1\text{-}w'\text{-}w^2_{L_2}$ and Q, a contradiction. This proves (15).

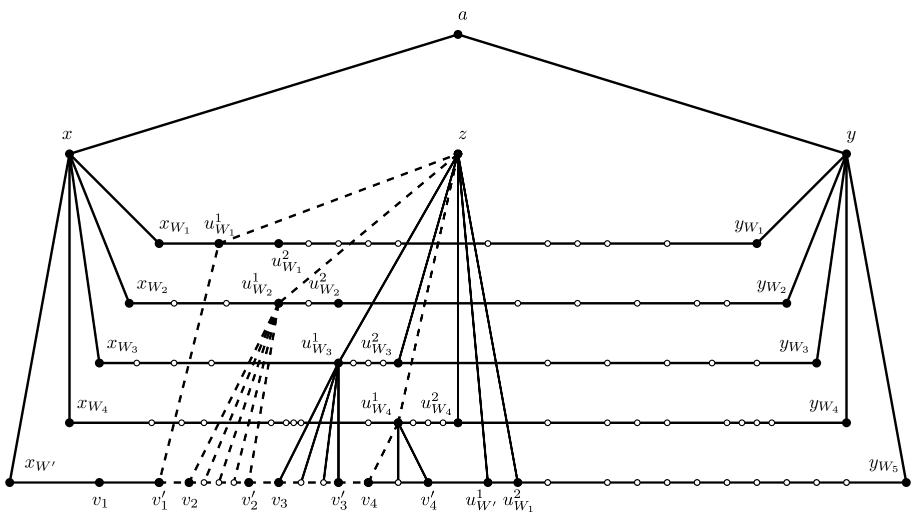

FIGURE 10. Proof of (17).

Finally, by, (13) and (15), there is a prism in G with triangles $z^2_{L_3}w^1_{L_1}w^2_{L_1}$, $w^2_{L_2}ww'$ and paths $z^2_{L_3}$-$w^2_{L_2}$, $w^2_{L_1}$-L_1-w and $w^1_{L_1}$-L_1-u_{L_1}-x-a-y-v_{L_1}-L_1-w' (see Figure 9), a contradiction. This completes the proof of Lemma 4.4. ■

We are now in a position to prove Theorem 4.1:

Proof of Theorem 4.1. Let $\sigma = \sigma(1, 4, t)$ and $\lambda = \lambda(1, 4, t)$ be as in Lemma 4.4. Let $\psi = \psi(t, \sigma + \lambda)$ be as in Theorem 2.3 and let $o = o(\psi, d, 3, t)$ be as in Lemma 2.5. Our goal is to show that $\kappa = \kappa(d, t, w) = o + w$ satisfies 4.1.

Suppose not. Let G be a (theta, prism, even wheel, K_t)-free graph, let $(a, x, y, \mathcal{W})$ be a κ-kaleidoscope in G, and let $z \in V(G)$ be 1-mirrored by $(a, x, y, \mathcal{W})$. Let $\mathcal{W}' \subseteq \mathcal{W}$ be the set of all paths $W \in \mathcal{W}$ for which z has at least d neighbors in W. It follows that $|\mathcal{W}'| < w$, and so there exists $\mathcal{W}_0 \subseteq \mathcal{W}$ with $|\mathcal{W}_0| = o$ such that for every $W \in \mathcal{W}_0$, z has less than d neighbors in W.

For every $W \in \mathcal{W}_0$, traversing W from x to y, let x_W be the neighbor of x in W, let u^1_W, u^2_W be the first and the last neighbor of z in W, respectively, and let y_W be first neighbor of y in W. It follows that the vertices $x, x_W, u^1_W, u^2_W, y_W, y$ appear on W in this order, and u^1_W, u^2_W are the only two vertices among them which may be the same. Since G is $(K_{3,3}, K_t)$-free and from the choice of o, it follows that we can apply Lemma 2.5 to the sets $\{N_W(z) : W \in \mathcal{W}_0\}$, and show that:

(16) *There exists $\mathcal{W}_1 \subseteq \mathcal{W}_0$ with $|\mathcal{W}_1| = \psi$ such that for all distinct $W, W' \in \mathcal{W}_0$, $N_W(z)$ is anticomplete to $N_{W'}(z)$.*

Next, note that $\{z\text{-}u^1_W\text{-}W\text{-}x_W\text{-}x : W \in \mathcal{W}_1\}$ is a collection of ψ pairwise internally disjoint paths in G between non-adjacent vertices z and x. Consequently, by the choice of ψ, we can apply Theorem 2.3 to this collection, and deduce that there exist two disjoint subsets $\mathcal{W}_2$ and $\mathcal{W}_3$ of $\mathcal{W}_1$ with $|\mathcal{W}_2| = \sigma$ and $|\mathcal{W}_3| = \lambda$, such that:

- $\{u^1_W : W \in \mathcal{W}_2 \cup \mathcal{W}_3\} \cup \{x\}$ is a stable set in G (though this is already guaranteed by (16) and (M3) as z is 1-mirrored by $(a, x, y, \mathcal{W})$); and
- for every $W \in \mathcal{W}_2$ and every $W' \in \mathcal{W}_3$, u^1_W has a neighbor in the interior of $L_{W'} = u^1_{W'}\text{-}W'\text{-}x_{W'}\text{-}x$.

Let $S = \{u^1_W : W \in \mathcal{W}_2\}$ and let $\mathcal{L} = \{L^*_{W'} : W' \in \mathcal{W}_3\}$. Then $(z, S, \mathcal{L})$ is a (σ, λ)-palanquin in G. This, together with the choices of σ and λ, allows for an application of Lemma 4.4. We deduce that there exist $W_1, W_2, W_3, W_4 \in \mathcal{W}_2$ and $W' \in \mathcal{W}_3$ such that the following hold.

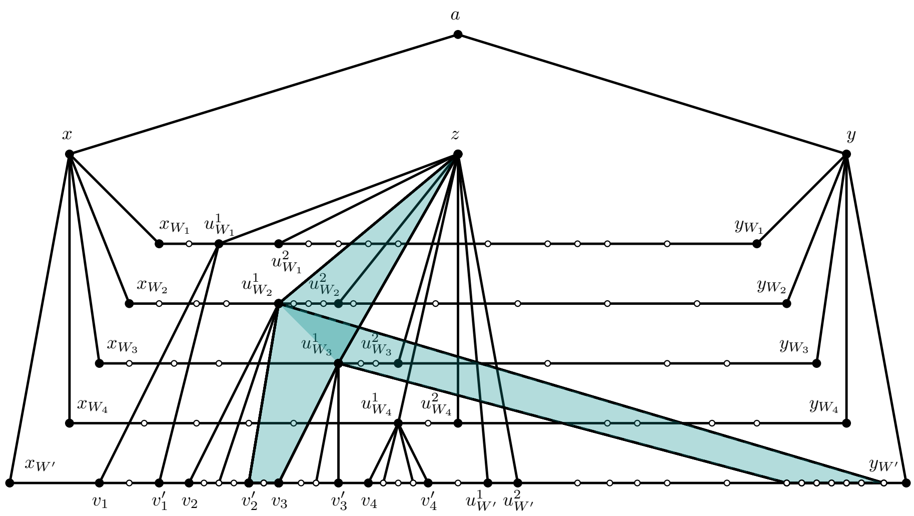

FIGURE 11. Proof of Theorem 4.1.

- $(\{u^1_{W_i} : i \in [4]\}, L^*_{W'}, x_{W'}, \pi)$ is a 4-alignment in G, where $\pi(u^1_{W_i}) = i$ for all $i \in [4]$.
- For every $i \in [4]$, $u^1_{W_i}$ is $L_{W'}$-ugly.

Now, for each $i \in [4]$, traversing $L^*_{W'}$ starting at $x_{W'}$, let v_i, v'_i be the first and the last neighbors of $u^1_{W_i}$ in $L^*_{W'}$, respectively; it follows that $\{v_i, v'_i\} \cap \{x_{W'}, u^1_{W'}\} = \emptyset$ and the vertices $x_{W'}, v_1, v'_1, v_2, v'_2, v_3, v'_3, v_4, v'_4, u^1_{W'}, u^2_{W'}, y_{W'}$ appear on W' in this order. Let $C = a\text{-}x\text{-}x_{W'}\text{-}W'\text{-}y_{W'}\text{-}y\text{-}a$. Then C is a hole in G and $u^1_{W_1}$ and z are two adjacent vertices in $G \setminus C$, each with a neighbor in C. Also, $u^1_{W_1}$ is $L_{W'}$-ugly, and so C-ugly, and by (16), $u^1_{W_1}$ and z have no common neighbor in C. This, combined with Theorem 3.2, implies that z is C-bad. More precisely, a and z are not adjacent (though we do not use this), and $N_C(z) = N_{W'}(z) = \{u^1_{W'}, u^2_{W'}\}$ is a two-vertex clique in G. We further deduce that:

(17) *For every $i \in \{2,3\}$, $u^1_{W_i}$ has a neighbor in the interior of $u^2_{W'}\text{-}W'\text{-}y_{W'}\text{-}y$.*

Suppose not. Then by (16), $u^1_{W_i}$ is anticomplete to $u^1_{W'}\text{-}u^2_{W'}\text{-}W'\text{-}y_{W'}$. Let C' denote the hole $z\text{-}u^1_{W_1}\text{-}v'_1\text{-}W'\text{-}v_4\text{-}u^1_{W_4}\text{-}z$. Then we have $N_{C'}(u^1_{W_i}) = N_{L_{W'}}(u^1_{W_i}) \cup \{z\} = N_C(u^1_{W_i}) \cup \{z\}$. Also, $u^1_{W_i}$ is C-ugly, as it is $L_{W'}$-ugly. This, along with the fact that $C \cup \{u^1_{W_i}\}$ is not a theta in G and $(C, u^1_{W_i})$ is not an even wheel in G, implies that $|N_C(u^1_{W_i})|$ is an odd integer which is at least three. But then $(C', u^1_{W_i})$ is an even wheel in G, a contradiction (see Figure 10). This proves (17).

To finish the proof, note that by (17), there exists a path P in G from $u^1_{W_2}$ to $u^1_{W_3}$ such that P^* is contained in the interior of $u^2_{W'}\text{-}W'\text{-}y_{W'}\text{-}y$. But now there is a theta in G with ends $u^1_{W_2}, u^1_{W_3}$ and paths $u^1_{W_2}\text{-}z\text{-}u^1_{W_3}$, $u^1_{W_2}\text{-}v'_2\text{-}W'\text{-}v_3\text{-}u^1_{W_3}$ and P, a contradiction (see Figure 11). This completes the proof of Theorem 4.1. ■

5. BLURRY 2-TREES

Here we develop the inductive procedure discussed in Subsection 1.3 for growing a 2-tree while maintaining its vertex set, all along, mirrored enough by a kaleidoscope of proportionate size. In particular, this process starts with a pair of adjacent vertices and then repeatedly adds carefully-chose common neighbors. Let us first ensure that a valid choice of the two initial vertices is available under the right circumstances:

Theorem 5.1. *For all integers $d, t, w \geq 1$, there exists $\zeta = \zeta(d,t,w) \geq 1$ with the following property. Let G be a (theta, prism, K_t)-free graph, let $a, b \in V(G)$ be distinct and non-adjacent* *and let $\mathcal{P}$ be a collection of pairwise internally disjoint paths in G from a to b with $|\mathcal{P}| \geq \zeta$. Then there exists a w-kaleidoscope $(a, x, y, \mathcal{W})$ in G as well as a clique Z_0 in G with $|Z_0| = 2$, such that:*

(a) *Z_0 is d-mirrored by $(a, x, y, \mathcal{W})$; and*
(b) *some vertex in Z_0 is adjacent to a.*

Proof. Let $\psi(\cdot, \cdot)$ be as in Theorem 2.3 and let $\kappa = \kappa(\cdot, \cdot, \cdot)$ be as in Theorem 4.1. Define $\kappa_1 = \kappa(d, t, w)$, $\psi_1 = \psi(t, t^3 + \kappa_1 + 1)$ and $\kappa_2 = \kappa(d, t, \psi_1)$. Let $\sigma = \sigma(\kappa_2, 3, t)$ and $\lambda = \lambda(\kappa_2, 3, t)$ be as in be as in Lemma 4.4. Define $\zeta = \zeta(t, w) = \psi(t, \sigma + \lambda)$. We prove that this value of ζ satisfies 5.1. Let G be a (theta, prism, K_t)-free graph, let $a, b \in V(G)$ be distinct and non-adjacent and let $\mathcal{P}$ be a collection of pairwise internally disjoint paths in G from a to b with $|\mathcal{P}| \geq \zeta = \psi(t, \sigma + \lambda)$. For each $P \in \mathcal{P}$, let x_P be the neighbor of a in P (so $x_P \neq b$). From Theorem 2.3 applied to a, b and $\mathcal{P}$, we deduce that there exist disjoint subsets $\mathcal{Q}$ and $\mathcal{R}$ of $\mathcal{P}$ with $|\mathcal{Q}| = \sigma$ and $|\mathcal{R}| = \lambda$, such that:

- $\{x_P : P \in \mathcal{Q} \cup \mathcal{R}\} \cup \{b\}$ is a stable set in G; and
- for every $Q \in \mathcal{Q}$ and every $R \in \mathcal{R}$, x_Q has a neighbor in $R^* \setminus \{x_R\}$.

For every $R \in \mathcal{R}$, let $L_R = R^* \setminus \{x_R\}$, and let x'_{L_R} be the unique neighbor of x_R in L_R. Then x'_{L_R} is an end of L_R, and the vertices $\{x'_{L_R} : R \in \mathcal{R}\}$ are pairwise distinct.

We define $S = \{x_Q : Q \in \mathcal{Q}\}$ and $\mathcal{L} = \{L_R : R \in \mathcal{L}_R\}$. Note that a is complete to the stable set $S \subseteq \{x_P : P \in \mathcal{P}\}$ and anticomplete to $P^* \setminus x_P$ for every $P \in \mathcal{P}$. Therefore, the triple $(a, S, \mathcal{L})$ is a (σ, λ)-palanquin in G. For every path $L \in \mathcal{L}$, fix the end x'_L of L, chosen as above.

Since G is (theta, K_t)-free and due to the choices of σ and λ, we can apply Lemma 4.4 to $(a, S, \mathcal{L})$ together with $\{x'_L : L \in \mathcal{L}\}$, and obtain a stable set $S' \subseteq S$ with $|S'| = 3$, $\mathcal{L}' \subseteq \mathcal{L}$ with $|\mathcal{L}'| = \kappa$, and a bijection $\pi : S' \to [3]$, such that for every $L \in \mathcal{L}'$, (S', L, x'_L, π) is a 3-alignment in G.

Let us write $x = \pi(1), z_1 = \pi(2)$ and $y = \pi(3)$. For every $L \in \mathcal{L}'$, traversing L starting at x'_L, let u_L be the last neighbor of x in L, let z_L be the last neighbor of z_1 in L and let v_L be the first neighbor of y in L. Let $W_L = x\text{-}u_L\text{-}L\text{-}v_L\text{-}y$. Then W_L is a path in G from x to y and we have $z_L \in W_L \setminus (N_{W_L}[x] \cup N_{W_L}[y])$. In particular:

(18) *For every $L \in \mathcal{L}'$, z_1 is anticomplete to $N_{W_L}[x] \cup N_{W_L}[y]$ and z_1 has a neighbor in W_L (namely z_L).*

From (18), it follows that $(a, x, y, \{W_L : L \in \mathcal{L}'\})$ is a κ_2-kaleidoscope in G by which z_1 is 1-mirrored. This, along with the choice of κ_2 and Theorem 4.1, implies that there exists $\mathcal{L}_1 \subseteq \mathcal{L}'$ with $|\mathcal{L}_1| = \psi_1$ such that z_1 is d-mirrored by the ψ_1-kaleidoscope $(a, x, y, \{W_L : L \in \mathcal{L}_1\})$.

Next, for every path $L \in \mathcal{L}_1$, let $P_L = z_1\text{-}z_L\text{-}L\text{-}v_L\text{-}y$. Then $\mathcal{P}' = \{P_L : L \in \mathcal{L}_1\}$ is a collection of ψ_1 pairwise internally disjoint paths in G between the two non-adjacent vertices z_1 and y. By Theorem 2.3, this time applied to z_1, y and $\mathcal{P}'$, there exist $L_0, L_1, \ldots, L_{t^3+\kappa} \in \mathcal{L}'$ such that:

- $\{z_{L_0}, z_{L_1}, \ldots, z_{L_{t^3+\kappa_1}}, y\}$ is a stable set in G; and
- for all $j \in [t^3 + \kappa_1]$, z_{L_0} has a neighbor in $P^*_{L_j} \setminus \{z_{L_j}\}$.

Let $z_2 = z_{L_0}$. Then z_2 is anticomplete to $\{a, x, y\}$. In addition, we claim that:

(19) *The following hold.*

- *The vertex z_2 is anticomplete to $\{u_{L_j} : j \in [t^3 + \kappa_1]\}$.*
- *There exists $I \subseteq [t^3 + \kappa_1]$ with $|I| = \kappa_1$ for which z_2 is anticomplete to $\{v_{L_j} : j \in I\}$.*

Consequently, for every $j \in I$, z_2 is anticomplete to $\{a\} \cup N_{W_{L_j}}[x] \cup N_{W_{L_j}}[y]$, and z_2 has a neighbor in W_{L_j}.

To see the first assertion, note that for all $j \in [t^3 + \kappa_1]$, there is a path Q_j in G from z_2 to y with $Q^*_j \subseteq P^*_{L_j} \setminus \{z_{L_j}\}$; in particular, u_{L_j} is anticomplete to Q^*_j. Therefore, if z_2 is adjacent to

u_{L_j} for some $j \in [t^3+\kappa_1]$, then there is a theta in G with ends a, z_2 and paths a-x-u_{L_j}-z_2, a-z_1-z_2 and a-y-Q_j-z_2, a contradiction. To prove the second assertion, suppose for a contradiction that there exists $I' \subseteq [t^3+\kappa_1]$ with $|I'| = t^3$ such that z_2 is complete to $V' = \{v_{L_j} : j \in I'\}$. Since G is K_t-free, it follows from Theorem 2.4 applied to $G[V']$ that there exists $\{v, v', v''\} \subseteq V'$ which is a stable set in G. But this yields a theta in G with ends z_2, y and paths z_2-v-y, z_2-v'-y and z_2-v''-y, a contradiction. This proves (19).

Let I be as in (19). It follows that $(a, x, y, \{W_{L_j} : j \in I\})$ is a κ_1-kaleidoscope in G by which z_1 is d-mirrored and z_2 is 1-mirrored. From the choice of κ_1 and Theorem 4.1 applied to $(a, x, y, \{W_{L_j} : j \in I\})$ and z_2, we conclude that there exists $\mathcal{W} \subseteq \{W_{L_j} : j \in I\}$ with $|\mathcal{W}| = w$ such that $Z_0 = \{z_1, z_2\}$ is d-mirrored by the w-kaleidoscope $(a, x, y, \mathcal{W})$. Also, a is adjacent to $z_1 \in Z_0$ and non-adjacent to $z_2 \in Z_0$. This completes the proof of Theorem 5.1. ■

Incidentally, there is an immediate corollary of Theorem 5.1 which may be of independent interest. Let G be a graph and let $d \geq 1$ be an integer. We say vertex $v \in V(G)$ is *d-substantial* if there is a hole C in $G \setminus \{v\}$ such that v has at least $d+1$ neighbors in C and $C \setminus N_C(v)$ is not connected.

Corollary 5.2. *For all integers $d, t \geq 1$, there exists an integer $k = k(d,t)$ with the following property. Let G be a (theta, prism, even wheel, K_t)-free graph and let $a, b \in V(G)$ be distinct and non-adjacent. Assume that no vertex in $N_G(a)$ is d-substantial in G. Then there do not exist k pairwise internally disjoint paths in G from a to b.*

Proof. We show that $k(d,t) = \zeta(d,t,1)$ satisfies 5.2, where $\zeta(\cdot,\cdot,\cdot)$ comes from Theorem 5.1. Suppose for a contradiction that there is a collection $\mathcal{P}$ of pairwise internally disjoint paths in G from a to b with $|\mathcal{P}| \geq k$. By Theorem 5.1, there exists a 1-kaleidoscope $(a, x, y, \{W\})$ in G as well as a clique Z_0 in G with $|Z_0| = 2$ such that Z_0 is d-mirrored by $(a, x, y, \{W\})$, and some vertex $z_1 \in Z_0$ is adjacent to a. Let $C = a$-x-W-y-a. Then C is a hole in G and z_1 has at least $d+1$ neighbors in C. Also, the vertices x, y belong to distinct components of $C \setminus N_C(z_1)$. But now z_1 is neighbor of a in G which is d-substantial, a contradiction. ■

Back to our main theme, the 2-trees we are about to obtain are in fact subgraphs of their host graphs, falling short of being induced by only an (annoying) notch: for a graph G and a 2-tree ∇ with $|V(\nabla)| = h \geq 2$, by a *blurry copy of ∇ in G* we mean an induced subgraph Z of G which in turn contains a spanning subgraph Y with the following specifications.

(B1) Y is a 2-tree isomorphic to ∇.
(B2) Let $i, j \in [|Y|] = [|Z|] = [|V(\nabla)|]$ with $i < j$ for which $\varpi_Y(i)$ and $\varpi_Y(j)$ are adjacent in G (and so in Z) but not in Y. Then $\varpi_Y(j)$ is adjacent in G to both forward neighbors of $\varpi_Y(i)$ in Y.

In particular, it follows that:

Observation 5.3. *Let G be a graph and let ∇ be a 2-tree such that G there is a blurry copy of ∇ in G. Then G has a subgraph isomorphic to ∇. Moreover, if G is K_4-free, then G has an induced subgraph isomorphic to ∇.*

Next we demonstrate, through the following lemma, the inductive step of our 2-tree growing process. Given a 2-tree ∇ on $h+2$ vertices for $h \geq 1$ and an integer $i \in [h]$, we denote by ∇/i the 2-tree $\nabla \setminus (\varpi_\nabla[i])$ on $h-i+2$ vertices where $\varpi_{\nabla/i}(j) = \varpi_\nabla(i+j)$ for all $j \in [h-i+2]$. It is also useful to define $\nabla/0 = \nabla$.

Lemma 5.4. *For all integers $h, t, w \geq 1$, there exists an integer $\xi(h,t,w) \geq 1$ with the following property. Let $G \in \mathcal{E}_t$ be a graph and let ∇ be a 2-tree on $h+2$ vertices. Assume that there is a blurry copy Z' of $\nabla/1$ in G which is 3-mirrored by a ξ-kaleidoscope in G. Then there is a blurry copy Z of ∇ in G which is 3-mirrored by a w-kaleidoscope in G.*

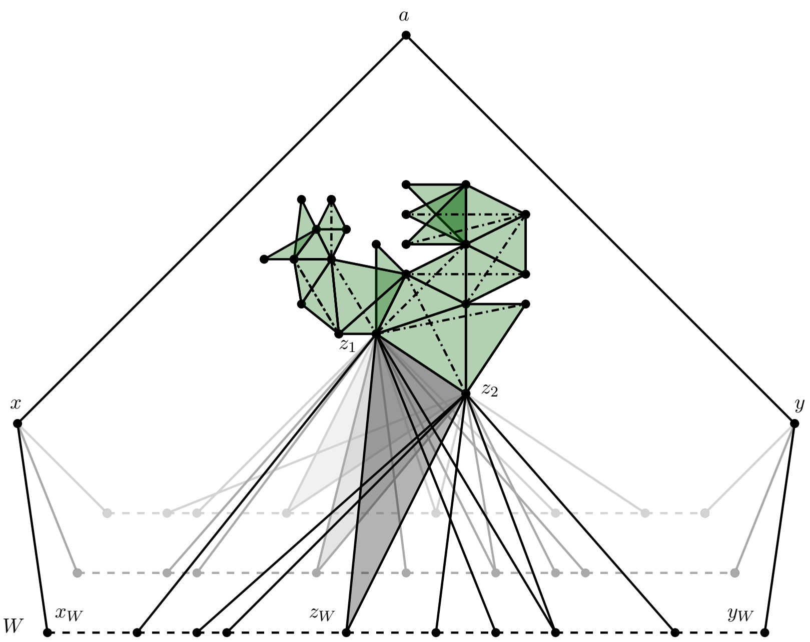

FIGURE 12. Proof of Lemma 5.4 (dashed lines represent paths of undetermined lengths, and dash-dotted lines depict possible edges in the blurry copy).

Proof. Let $\psi = \psi(t, 2)$ be as in Theorem 2.3. Let $\kappa = \kappa(3, t, w)$ be as in Theorem 4.1. Let $o = o(\cdot, \cdot, \cdot, \cdot)$ be as in Lemma 2.5. We claim that $\xi = \xi(h, t, w) = o(3h + 2\psi\kappa, 3, 3, t)$ satisfies 5.4. Let $G \in \mathcal{E}_t$ be a graph and let ∇ be 2-tree on $h + 2$ vertices. Let Z' be a blurry copy of $\nabla/1$ in G and let $(a, x, y, \mathcal{W}')$ be a ξ-kaleidoscope in G by which Z' is 3-mirrored. Let Y' be the spanning subgraph of Z' such that Y', Z' and $\nabla/1$ satisfy (B1) and (B2). In particular, there is an isomorphism $f : V(\nabla/1) \to V(Y')$ between the 2-trees $\nabla/1$ and Y'. Let u_1, u_2 be the two forward neighbors of $\varpi_\nabla(1)$ in ∇ and let $z_i = f(u_i)$ for $i \in \{1, 2\}$. Then we have $z_1, z_2 \in V(Y') = Z'$. Since Z' is a blurry copy of $\nabla/1$ which is 3-mirrored by $(a, x, y, \mathcal{W}')$, and from the definition of a blurry copy and a mirrored set, it follows immediately that:

(20) *Let $z \in G \setminus Z'$ and $\mathcal{W} \subseteq \mathcal{W}'$ such that:*

- *a is not adjacent to z;*
- *we have $\{z_1, z_2\} \subseteq N_{Z'}(z) \subseteq N_{Z'}[z_1] \cap N_{Z'}[z_2]$; and*
- *we have $|\mathcal{W}| = w$ and z is 3-mirrored by the w-kaleidoscope $(a, x, y, \mathcal{W})$ in G.*

Then $Z = Z' \cup \{z\}$ is a blurry copy of ∇ in G which is 3-mirrored by the w-kaleidoscope $(a, x, y, \mathcal{W})$ in G.

Therefore, in order to proved Lemma 5.4, it suffices to argue the existence of a vertex $z \in G \setminus Z'$ as well as a subset $\mathcal{W}$ of $\mathcal{W}'$ for which the three bullets points of (20) hold. We devote the rest of the proof to this goal.

For every $W \in \mathcal{W}'$, let $C_W = a\text{-}x\text{-}W\text{-}y\text{-}a$. Then C_W is a hole in G. Also, since Z' is 3-mirrored by $(a, x, y, \mathcal{W}')$, it follows from (M2) and (M3) that a is not a common neighbor of z_1 and z_2, and that both z_1 and z_2 are C_W-ugly. This, combined with Theorem 3.2, implies that:

(21) *For every $W \in \mathcal{W}'$, the vertices z_1 and z_2 have a common neighbor in W.*

Consequently, for every $W \in \mathcal{W}'$, traversing W from x to y, we may choose z_W to be the first common neighbor of z_1 and z_2 in W. Note also that, by (M3), $\{z_1, z_2\}$ is anticomplete to $N_W[x] \cup N_W[y]$, which in turn shows that $z_W \in W \setminus (N_W[x] \cup N_W[y])$. For every $W \in \mathcal{W}'$, let x_W, y_W be the neighbors of x and y in W, respectively (see Figure 12). We now deduce that:

(22) *There exists* $\mathcal{W}_0 \subseteq \mathcal{W}'$ *with* $|\mathcal{W}_0| = 2\psi\kappa$ *such that:*

- *for all distinct* $W, W' \in \mathcal{W}_0$*, the sets* $\{x_W, z_W, y_W\}$ *and* $\{x_{W'}, z_{W'}, y_{W'}\}$ *are anticomplete in* G*; and*
- *for every* $W \in \mathcal{W}_0$*, we have* $\{z_1, z_2\} \subseteq N_{Z'}(z_W) \subseteq N_{Z'}[z_1] \cap N_{Z'}[z_2]$.

To see this, note that since G is $(K_{3,3}, K_t)$-free and from choice of $|\mathcal{W}'| = \xi$, we can apply Lemma 2.5 to the sets $\{\{x_W, z_W, y_W\} : W \in \mathcal{W}'\}$, and obtain a subset $\mathcal{W}'_0$ of $\mathcal{W}'$ with $|\mathcal{W}'_0| = 3h + 2\psi\kappa$ such that for all distinct $W, W' \in \mathcal{W}'_0$, the sets $\{x_W, z_W, y_W\}$ and $\{x_{W'}, z_{W'}, y_{W'}\}$ are anticomplete in G. It remains to show that there exists a subset $\mathcal{W}_0$ of $\mathcal{W}'_0$ with $|\mathcal{W}_0| = 2\psi\kappa$ such that for every $W \in \mathcal{W}_0$, we have $\{z_1, z_2\} \subseteq N_{Z'}(z_W) \subseteq N_{Z'}[z_1] \cap N_{Z'}[z_2]$. Suppose not. Since z_W is a common neighbor of z_1 and z_2 for all $W \in \mathcal{W}'$, it follows that there exists $\mathcal{W}''_0 \subseteq \mathcal{W}'_0$ with $|\mathcal{W}''_0| = 3h$ such that for every $W \in \mathcal{W}''_0$, z_W has a neighbor $z'_W \in Z' \setminus \{z_1, z_2\}$ which is adjacent to at most one of z_1 and z_2. From this and the fact that $|Z' \setminus \{z_1, z_2\}| < |V(\nabla)| - 2 = h$, we deduce that there are three distinct paths $W_1, W_2, W_3 \in \mathcal{W}''_0$ such that for some vertex $z' \in Z' \setminus \{z_1, z_2\}$, we have $z'_{W_1} = z'_{W_2} = z'_{W_3} = z'$. Recall also that $z_{W_1}, z_{W_2}, z_{W_3}$ are pairwise non-adjacent because $W_1, W_2, W_3 \in \mathcal{W}''_0 \subseteq \mathcal{W}'_0$. But now for some $i \in \{1, 2\}$, there is a theta in G with ends z_i, z' and paths z_i-z_{W_1}-z', z_i-z_{W_2}-z' and z_i-z_{W_3}-z', a contradiction. This proves (22).

Let $\mathcal{W}_0$ be as in (22). The following captures the bulk of the difficulty in this proof, also involving our application of Theorem 3.1. Intuitively, the motivation is to apply Theorem 2.3 to the "paths in $\mathcal{W}_0$" from z_1 to x. But we cannot; those paths are not induced as z_1 may have neighbours in them. So we pass to the minor offered by Theorem 3.1 precisely to surmount this complication.

(23) *There exist* $W_0 \in \mathcal{W}_0$ *and* $\mathcal{W}_1 \subseteq \mathcal{W}_0 \setminus \{W_0\}$ *with* $|\mathcal{W}_1| = \kappa$*, such that for every* $W \in \mathcal{W}_1$*,* z_{W_0} *has a neighbor in* W^*.

Let D be the digraph with vertex set $\mathcal{W}_0$ where for distinct paths $W_1, W_2 \in \mathcal{W}_0$, (W_1, W_2) is an arc in D if and only if z_{W_1} has a neighbor in W_2^*. Note that in order to prove (23), it is enough to show that D has a vertex of out-degree at least κ. Suppose not. Then every vertex in D has out-degree less than κ, which in turn implies that every vertex in every "subdigraph" of D has out-degree less than κ. It follows that every "subdigraph" of D has a vertex of in-degree less than κ. Let $D^\natural$ be underlying undirected graph of D (which may have pairs of parallel edges). Then every subgraph of $D^\natural$ has a vertex of degree less than 2κ, and so $D^\natural$ has chromatic number at most 2κ. Consequently, there exists a stable set $\mathcal{W}'_1 \subseteq \mathcal{W}_0 = V(D^\natural)$ in $D^\natural$ with $|\mathcal{W}'_1| = \lceil |\mathcal{W}_0|/2\kappa \rceil = \psi$. From the definition of D, it follows that for all distinct $W_1, W_2 \in \mathcal{W}'_1$, z_{W_1} is anticomplete to W_2^*. Let $G_1 = G[(\bigcup_{W \in \mathcal{W}'_1} V(z_W\text{-}W\text{-}x)) \cup \{z_1, z_2\}]$. Then we have $G_1 \in \mathcal{E}$, $z_1, z_2 \in V(G_1)$ are distinct and adjacent, and $N_{G_1}(z_1) \cap N_{G_1}(z_2) = \{z_W : W \in \mathcal{W}'_1\}$ is a stable set of vertices of degree three in G_1. We are now prepared to apply Theorem 3.1 and deduce that $G_1 \triangleleft^{z_1}_{z_2} \in \mathcal{E}$. Let $z \in V(G_1 \triangleleft^{z_1}_{z_2})$ be as in the definition of $G_1 \triangleleft^{z_1}_{z_2} \in \mathcal{E}$. Then we have $V(G_1 \triangleleft^{z_1}_{z_2}) = (\bigcup_{W \in \mathcal{W}'_1} V(z_W\text{-}W\text{-}x)) \cup \{z\}$ and $N_{G_1 \triangleleft^{z_1}_{z_2}}(z) = N_{G_1}(z_1) \cap N_{G_1}(z_2) = \{z_W : W \in \mathcal{W}'_1\}$ is a stable set of vertices of degree two in $G_1 \triangleleft^{z_1}_{z_2}$. Also, $\{z\text{-}z_W\text{-}W\text{-}x : W \in \mathcal{W}'_1\}$ is a collection of ψ pairwise internally disjoint paths in $G_1 \triangleleft^{z_1}_{z_2}$ between non-adjacent vertices z and x. Hence, by the choice of ψ, we can apply Theorem 2.3 to this collection, and obtain $W_1, W_2 \in \mathcal{W}'_1$ such that z_{W_1} has a neighbor in the interior of z_{W_2}-W_2-x in $G_1 \triangleleft^{z_1}_{z_2}$. But then z_{W_1} has degree at least three in $G_1 \triangleleft^{z_1}_{z_2}$, a contradiction. This proves (23).

We can now finish the proof. Let W_0 and $\mathcal{W}_1$ be as in (23), and write $z = z_{W_0}$. Then $z \in G \setminus Z'$ and a is not adjacent to z because $z \in W_0^*$, which implies that a satisfies the first bullet of (20). Also, the second bullet of (22) implies that z satisfies the first second of (20). In addition, from the first bullet of (22) combined with (23) and the fact that $z \in W_0 \setminus (N_{W_0}[x] \cup N_{W_0}[y])$, it follows that z is 1-mirrored by the κ-kaleidoscope $(a, x, y, \mathcal{W}_1)$. Due to the choice of κ, we may apply Theorem 4.1 and deduce that there exists $\mathcal{W} \subseteq \mathcal{W}_1 \subseteq \mathcal{W}_0 \subseteq \mathcal{W}'$ with $|\mathcal{W}| = w$ such that z is 3-mirrored by the w-kaleidoscope $(a, x, y, \mathcal{W})$. Hence, z and $\mathcal{W}$ satisfy the third bullet of (20). This completes the proof of Lemma 5.4. ■

We now use Lemma 5.4 to prove the main result of this section:

Theorem 5.5. *For all integers $h \geq 0$ and $t, w \geq 1$, there exists an integer $\Xi(h, t, w) \geq 1$ with the following property. Let $G \in \mathcal{E}_t$ be a graph and let ∇ be a 2-tree on $h + 2$ vertices. Assume that there is a clique Z_0 in G with $|Z_0| = 2$ which is 3-mirrored by a Ξ-kaleidoscope in G. Then there is a blurry copy Z of ∇ in G which is 3-mirrored by a w-kaleidoscope in G.*

Proof. We begin with defining a sequence $\{\Xi_i\}_{i=0}^{h}$ using a backward recursion. Let $\Xi_h = w$. For every $0 \leq i < h$, let $\Xi_i = \xi(i + 1, t, \Xi_{i+1})$, where $\xi(\cdot, \cdot, \cdot)$ is as in Lemma 5.4. We claim that $\Xi = \Xi(h, t, w) = \Xi_0$ satisfies 5.5. In fact, we prove a slightly stronger statement which is tailored to our inductive argument:

(24) *Let $i \in \{0, \dots, h\}$. Assume that there is a clique Z_0 in G with $|Z_0| = 2$ which is 3-mirrored by a Ξ_0-kaleidoscope in G. Then there is a blurry copy Z_i of $\nabla/(h-i)$ in G which is 3-mirrored by a Ξ_i-kaleidoscope in G.*

We induct on i. The case $i = 0$ is trivial as ∇/h is a 2-vertex complete graph. Suppose $h \geq 1$ and $i \in [h]$. Assume that there is a clique Z_0 in G with $|Z_0| = 2$ which is 3-mirrored by a Ξ_0-kaleidoscope in G. Then $\nabla/(h - i)$ is a 2-tree on $i + 2$ vertices, and by the induction hypothesis, there is a blurry copy Z_{i-1} of $\nabla/(h-i+1) = (\nabla/(h-i))/1$ in G which is 3-mirrored by a Ξ_{i-1}-kaleidoscope in G, where $\Xi_{i-1} = \xi(i, t, \Xi_i)$. Hence, it follows from Lemma 5.4 that there is a blurry copy Z_i of $\nabla/h - i$ in G which is 3-mirrored by a Ξ_i-kaleidoscope in G. This proves (24).

Now the result follows from (24) for $i = h$. This completes the proof of Theorem 5.5. ■

6. The coda

With Theorems 5.1 and 5.5 in our arsenal, we are ready to prove Theorems 1.12 and 1.13. In fact, both results follow directly from the one below:

Theorem 6.1. *For all integers $h \geq 0$ and $t \geq 1$, there exists an integer $\Omega = \Omega(h, t)$ such that for every 2-tree ∇ on h vertices and every graph $G \in \mathcal{E}_t$ with $\mathrm{tw}(G) > \Omega$, there is a blurry copy of ∇ in G.*

Proof. Let $\Xi = \Xi(h, t, 1)$ be as in Theorem 5.5 and let $\zeta = \zeta(3, t, \Xi)$ be as in Theorem 5.1. We show that $\Omega = \Omega(h, t) = \beta(\max\{\zeta, t\}, t)$ satisfies 6.1, where $\beta(\cdot, \cdot)$ is as in Corollary 2.2. Let ∇ be a 2-tree on h-vertices and let $G \in \mathcal{E}_t$ be a graph with $\mathrm{tw}(G) > \Omega$. From Corollary 2.2, it follows that G has a strong $\max\{\zeta, t\}$-block. Since G is K_t-free, it follows that there are non-adjacent vertices $a, b \in V(G)$ for which there exists a collection $\mathcal{P}$ of ζ pairwise internally disjoint paths in G from a to b. By Theorem 5.1, there exists a clique Z_0 in G with $|Z_0| = 2$ which is 3-mirrored by a Ξ-kaleidoscope in G. This, along with Theorem 5.5, implies that there is a blurry copy of ∇ in G (which is 3-mirrored by a 1-kaleidoscope in G), as desired. ■

Theorem 1.12 is now immediate:

Theorem 1.12. *For every 2-tree ∇, there exists an integer $\Upsilon = \Upsilon(\nabla) \geq 1$ such that every graph $G \in \mathcal{E}_4$ with $\mathrm{tw}(G) > \Upsilon$ contains ∇.*

Proof. Let $\Upsilon = \Upsilon(\nabla) = \Omega(|V(\nabla)|, 4)$. Then the result follows from Theorem 6.1 combined with Observation 5.3. ■

To prove Theorem 1.13, we also need a result from [12]. For integers $d, r \geq 0$, let T_d^r denote the rooted tree in which every leaf is at distance r from the root, the root has degree d, and every vertex that is neither a leaf nor the root has degree $d + 1$.

Theorem 6.2 (Kierstead and Penrice [12]). *For all integers $d, r \geq 0$ and $s, t \geq 1$, there exists an integer $f = f(d, r, s, t) \geq 1$ such that if a graph G contains T_f^f as a subgraph, then G contains one of $K_{s,s}$, K_t or T_d^r as an induced subgraph.*

Finally, we prove Theorem 1.13:

Theorem 1.13. *For every integer $t \geq 1$ and every tree T, there exists an integer $\Gamma = \Gamma(t, T) \geq 1$ such that every graph $G \in \mathcal{E}_t$ with $\text{tw}(G) > \Gamma$ contains $\text{cone}(T)$.*

Proof. Let d and r be the maximum degree and the radius of T, respectively. It follows that T_d^r contains T as an induced subgraph. Let $f = f(d, r, 3, t)$ be as in Theorem 6.2 and let $T^+ = \text{cone}(T)$. We claim that $\Gamma = \Gamma(t, T) = \Omega(|V(T^+)|, t)$ satisfies 1.13, where $\Omega(\cdot, \cdot)$ comes from Theorem 6.1. Let $G \in \mathcal{E}_t$ be a graph of treewidth more than Γ. From Theorem 6.1 and Observation 5.3, we deduce that there exists $X \subseteq V(G)$ such that $G[X]$ has a spanning subgraph isomorphic to the 2-tree T^+. As a result, there exists a vertex $x \in X$ complete to $X \setminus \{x\}$ such that $G[X \setminus \{x\}]$ has a spanning subgraph isomorphic to T_f^f. Since G is $\{K_{3,3}, K_t\}$-free, it follows from Theorem 6.2 that $G[X \setminus \{x\}]$ contains T_d^r (as an induced subgraph). Hence, $G[X \setminus \{x\}]$ contains T, and so G contains $\text{cone}(T)$, as required. ■

7. Acknowledgement

Tara Abrishami worked with us on our initial attempts at Conjecture 1.8 using a different method (which led to a weakening [5]), and Rose McCarty worked with us when we were suspicious of Theorem 1.7 and briefly tried to find a counterexample. We thank them both.

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