VERTEX-MINORS AND THE ERDŐS-HAJNAL CONJECTURE

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Abstract. We prove that for every graph H, there exists $\varepsilon > 0$ such that every n-vertex graph with no vertex-minors isomorphic to H has a pair of disjoint sets A, B of vertices such that $|A|, |B| \geq \varepsilon n$ and A is complete or anticomplete to B. We deduce this from recent work of Chudnovsky, Scott, Seymour, and Spirkl (2018). This proves the analog of the Erdős-Hajnal conjecture for vertex-minors.

For a graph G, let $\alpha(G)$ be the maximum size of an independent set, that is a set of pairwise non-adjacent vertices. Let $\omega(G)$ be the maximum size of a clique, that is a set of pairwise adjacent vertices. In 1989, Erdős and Hajnal [4] conjectured that for every graph H, there exists $\varepsilon > 0$ such that if a graph G has no induced subgraph isomorphic to H, then

$$\max(\omega(G), \alpha(G)) \geq |V(G)|^{\varepsilon}.$$

A few years ago, Chudnovsky proposed a weaker question; is it true if we replace "induced subgraphs" by "vertex-minors"?

If a class $\mathcal{G}$ of graphs closed under taking induced subgraphs has some $\varepsilon > 0$ such that every graph in $\mathcal{G}$ has an independent set or a clique of size more than $|V(G)|^{\varepsilon}$, then we say that $\mathcal{G}$ has the *Erdős-Hajnal property*.

We prove that for every graph H, the class of graphs with no vertex-minor isomorphic to H has the Erdős-Hajnal property. In addition, we prove a stronger property that is defined as follows. A set A of vertices is *complete* to a set B of vertices if every vertex in A is adjacent to every vertex of B. A set A of vertices is *anticomplete* to a set B of vertices if every vertex in A is non-adjacent to every vertex of B. If a class $\mathcal{G}$ of graphs closed under taking induced subgraphs has some $\varepsilon > 0$ such

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that every graph in $\mathcal{G}$ has a complete or anticomplete pair of disjoint sets A, B with $|A|, |B| \geq \varepsilon|V(G)|$, then we say that $\mathcal{G}$ has the *strong Erdős-Hajnal property*. It is well known that the strong Erdős-Hajnal property implies the Erdős-Hajnal property, see [1, 5]. We prove that for every graph H, the class of graphs with no vertex-minor isomorphic to H has the strong Erdős-Hajnal property.

Before presenting our theorem, we state the definition of vertex-minors [6]. For a graph G and its vertex v, the *local complementation* at v results in the new graph, denoted by $G*v$, such that $V(G*v) = V(G)$ and two distinct vertices x, y are adjacent in $G*v$ if either

(i) both x and y are neighbors of v in G and x, y are non-adjacent in G, or
(ii) at least one of x or y is non-adjacent to v in G and x, y are adjacent in G.

A graph H is a *vertex-minor* of a graph G if H is an induced subgraph of $G * v_1 * v_2 * \cdots * v_k$ for some sequences of vertices $v_1, v_2, \ldots, v_k$ (not necessarily distinct) with $k \geq 0$.

Now we state our main theorem.

Theorem 1. *For every graph H, there exists $\varepsilon > 0$ such that every n-vertex graph G has a vertex-minor isomorphic to H or has a pair of disjoint sets A, B of of vertices such that A is either complete or anticomplete to B and $|A|, |B| \geq \varepsilon n$.*

As the strong Erdős-Hajnal property implies the Erdős-Hajanl property, we deduce the following.

Corollary 2. *For every graph H, there exists $\varepsilon > 0$ such that if a graph G has no vertex-minor isomorphic to H, then*

$$\max(\alpha(G), \omega(G)) \geq |V(G)|^{\varepsilon}.$$

Now let us present the proof. Our proof is based on the following theorems of Chudnovsky, Scott, Seymour, and Spirkl [3].

Theorem 3 (Chudnovsky et al. [3])**.** *For every graph H, there exists $c > 0$ such that every graph G has an induced subgraph isomorphic to a subdivision of H or the complement of a subdivision of H or has a pair of disjoint sets A, B of of vertices such that A is either complete or anticomplete to B and $|A|, |B| \geq c|V(G)|$.*

Theorem 4 (Chudnovsky et al. [3])**.** *For every graph H, there exists $\delta > 0$ such that every n-vertex graph G with $|E(G)| \leq \delta|V(G)|^2$ has an induced subgraph isomorphic to a subdivision of H or has an anticomplete pair of disjoint sets A, B of of vertices such that $|A|, |B| \geq \delta n$.*

Proof of Theorem 1. Let c, δ be the constants given by Theorems 3 and 4. We claim that $\varepsilon = \min(2c\delta, \delta)$.

If G has an induced subdivision of H, then we can apply local complementations to degree-2 vertices to obtain a vertex-minor isomorphic to H, contradicting our assumption. Thus G has no induced subdivision of H. By the same reason, $G * v$ has no induced subdivision of H for every vertex v.

If every vertex of G has degree at most $2\delta n$, then $|E(G)| \leq \delta n^2$. By Theorem 4, G has an anticomplete pair of disjoint sets A, B with $|A|, |B| \geq \delta n$.

If a vertex v has degree more than $2\delta n$, then let G' be the subgraph of G induced by all neighbors of v. Note that neither G' nor the complement of G' has an induced subdivision of H and therefore by Theorem 3, G' has an anticomplete or complete pair of sets A, B with $|A|, |B| \geq c|V(G')| > 2c\delta n$. □

Remark. There are two major examples of graph classes known to be closed under taking vertex-minors; graphs of rank-width at most k [6] and circle graphs [2]. It is easy to see that the class of graphs of rank-width at most k has the strong Erdős-Hajnal property. To see this, observe that an n-vertex graph G of rank-width at most k has a vertex set X such that the cut-rank of X is at most k and $|X|, |V(G)| - |X| > n/3$. Then one can partition each of X and $V(G) - X$ into at most 2^k subsets such that each part of X is complete or anticomplete to each part of $V(G) - X$. This proves that such a graph has an anticomplete or complete pair of sets A, B such that $|A|, |B| > (n/3)/2^k$. The class of circle graphs has the strong Erdős-Hajnal property, implied by a theorem of Pach and Solymosi [7].

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