Abstract. We prove that for every graph $H$, there exists $\varepsilon > 0$ such that every $n$-vertex graph with no vertex-minors isomorphic to $H$ has a pair of disjoint sets $A$, $B$ of vertices such that $|A|, |B| \geq \varepsilon n$ and $A$ is complete or anticomplete to $B$. We deduce this from a recent work of Chudnovsky, Scott, Seymour, and Spirkl (2018). This proves a variant of the Erdős-Hajnal conjecture.

For a graph $G$, let $\alpha(G)$ be the maximum size of an independent set, that is a set of pairwise non-adjacent vertices. Let $\omega(G)$ be the maximum size of a clique, that is a set of pairwise adjacent vertices. In 1989, Erdős and Hajnal conjectured that for every graph $H$, there exists $\varepsilon > 0$ such that if a graph $G$ has no induced subgraph isomorphic to $H$, then

$$\max(\omega(G), \alpha(G)) \geq |V(G)|^\varepsilon.$$  

A few years ago, Chudnovsky proposed a weaker question; is it true if we replace “induced subgraphs” by “vertex-minors”? 

If a class $\mathcal{G}$ of graphs closed under taking induced subgraphs has some $\varepsilon > 0$ such that every graph in $\mathcal{G}$ has an independent set or a clique of size more than $|V(G)|^\varepsilon$, then we say that $\mathcal{G}$ has the Erdős-Hajnal property.

We prove that for every graph $H$, the class of graphs with no vertex-minor isomorphic to $H$ has the Erdős-Hajnal property. In addition, we prove a stronger property that is defined as follows. A set $A$ of vertices is complete to a set $B$ of vertices if every vertex in $A$ is adjacent to all vertices of $B$. A set $A$ of vertices is anticomplete to a set $B$ of vertices if every vertex in $A$ is non-adjacent to all vertices of $B$. If a class $\mathcal{G}$ of graphs closed under taking induced subgraphs has some $\varepsilon > 0$ such
that every graph in $G$ has a complete or anticomplete pair of disjoint sets $A, B$ with $|A|, |B| \geq \varepsilon |V(G)|$, then we say that $G$ has the strong Erdős-Hajnal property. It is well known that the strongly Erdős-Hajnal property implies the Erdős-Hajnal property, see [1, 5]. We are going to prove that for every graph $H$, the class of graphs with no vertex-minor isomorphic to $H$ has the strong Erdős-Hajnal property.

Before presenting our theorem, we state the definition of vertex-minors [6]. For a graph $G$ and its vertex $v$, the local complementation at $v$ is an operation to obtain a new graph, denoted by $G^* v$, such that $V(G^* v) = V(G)$ and two distinct vertices $x, y$ are adjacent in $G^* v$ if either

(i) both $x$ and $y$ are neighbors of $v$ in $G$ and $x, y$ are non-adjacent in $G$, or
(ii) at least one of $x$ or $y$ is non-adjacent to $v$ in $G$ and $x, y$ are adjacent in $G$.

A graph $H$ is a vertex-minor of a graph $G$ if $H$ is an induced subgraph of $G^* v_1^* v_2^* \cdots v_k^*$ for some sequences of vertices $v_1, v_2, \ldots, v_k$ (not necessarily distinct) with $k \geq 0$.

Now we state our main theorem.

**Theorem 1.** For every graph $H$, there exists $\varepsilon > 0$ such that every $n$-vertex graph $G$ has a vertex-minor isomorphic to $H$ or has a pair of disjoint sets $A, B$ of vertices such that $A$ is either complete or anticomplete to $B$ and $|A|, |B| \geq \varepsilon n$.

As the strong Erdős-Hajnal property implies the Erdős-Hajnal property, we deduce the following.

**Corollary 2.** For every graph $H$, there exists $\varepsilon > 0$ such that if a graph $G$ has no vertex-minor isomorphic to $H$, then

$$\max(\alpha(G), \omega(G)) \geq |V(G)|^\varepsilon.$$ 

Now let us present the proof. Our proof is based on the following theorems of Chudnovsky, Scott, Seymour, and Spirkl [3].

**Theorem 3** (Chudnovsky et al. [3]). For every graph $H$, there exists $c > 0$ such that every graph $G$ has an induced subgraph isomorphic to a subdivision of $H$ or the complement of a subdivision of $H$ or has a pair of disjoint sets $A, B$ of vertices such that $A$ is either complete or anticomplete to $B$ and $|A|, |B| \geq c |V(G)|$.

**Theorem 4** (Chudnovsky et al. [3]). For every graph $H$, there exists $\delta > 0$ such that every $n$-vertex graph $G$ with $|E(G)| \leq \delta |V(G)|^2$ has an induced subgraph isomorphic to a subdivision of $H$ or has an anticomplete pair of disjoint sets $A, B$ of vertices such that $|A|, |B| \geq \delta n$. 
**Proof of Theorem 1.** Let $c$, $\delta$ be the constants given by Theorems 3 and 4. We claim that $\epsilon = \min(2c\delta, \delta)$. 

If $G$ has an induced subdivision of $H$, then we can apply local complementations to degree-2 vertices to obtain a vertex-minor isomorphic to $H$, contradicting our assumption. Thus $G$ has no induced subdivision of $H$. By the same reason, $G*v$ has no induced subdivision of $H$ for every vertex $v$.

If every vertex of $G$ has degree at most $2\delta n$, then $|E(G)| \leq \delta n^2$. By Theorem 4, $G$ has an anticomplete pair of disjoint sets $A, B$ with $|A|, |B| \geq \delta n$.

If a vertex $v$ has degree more than $2\delta n$, then let $G'$ be the subgraph of $G$ induced by all neighbors of $v$. Note that neither $G'$ nor the complement of $G'$ has an induced subdivision of $H$ and therefore by Theorem 3 $G'$ has an anticomplete or complete pair of sets $A, B$ with $|A|, |B| \geq c|V(G')| > 2c\delta n$. □

**Remark.** There are two major examples of graph classes known to be closed under taking vertex-minors; graphs of rank-width at most $k$ [6] and circle graphs [2]. It is easy to see that the class of graphs of rank-width at most $k$ has the strong Erdős-Hajnal property. To see this, observe that an $n$-vertex graph $G$ of rank-width at most $k$ has a vertex set $X$ such that the cut-rank of $X$ is at most $k$ and $|X|, |V(G)| - |X| > n/3$. Then one can partition each of $X$ and $V(G) - X$ into at most $2^k$ subsets such that each part of $X$ is complete or anticomplete to each part of $V(G) - X$. This proves that such a graph has an anticomplete or complete pair of sets $A, B$ such that $|A|, |B| > (n/3)/2^k$. The class of circle graphs has the strong Erdős-Hajnal property, implied by a theorem of Pach and Solymosi [7].

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