Tournament immersion and cutwidth

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Abstract

A (loopless) digraph H is *strongly immersed* in a digraph G if the vertices of H are mapped to distinct vertices of G, and the edges of H are mapped to directed paths joining the corresponding pairs of vertices of G, in such a way that the paths used are pairwise edge-disjoint, and do not pass through vertices of G that are images of vertices of H. A digraph has *cutwidth* at most k if its vertices can be ordered $\{v_1, \dots, v_n\}$ in such a way that for each j, there are at most k edges uv such that $u \in \{v_1, \dots, v_{j-1}\}$ and $v \in \{v_j, \dots, v_n\}$.

We prove that for every set $\mathcal{S}$ of tournaments, the following are equivalent:

- there is a digraph H such that H cannot be strongly immersed in any member of $\mathcal{S}$
- there exists k such that every member of $\mathcal{S}$ has cutwidth at most k
- there exists k such that every vertex of every member of $\mathcal{S}$ belongs to at most k edge-disjoint directed cycles.

This is a key lemma towards two results that will be presented in later papers: first, that strong immersion is a well-quasi-order for tournaments, and second, that there is a polynomial time algorithm for the k edge-disjoint directed paths problem (for fixed k) in a tournament.

1 Introduction

In this paper, all graphs and digraphs are finite, and may have loops or multiples edges. A digraph is *simple* if it has no loops, and for every pair of distinct vertices u, v there is at most one edge with tail u and head v. A digraph is *semi-complete* if it is simple, and for every pair of distinct vertices u, v, either there is an edge uv (this means an edge with tail u and head v) or an edge vu. A digraph is a *tournament* if it is simple and for every pair of distinct vertices u, v, there is exactly one edge with ends $\{u, v\}$. Thus, every tournament is semi-complete.

Let G, H be digraphs. A *weak immersion* of H in G is a map η such that

- $\eta(v) \in V(G)$ for each $v \in V(H)$
- $\eta(u) \neq \eta(v)$ for distinct $u, v \in V(H)$
- for each non-loop edge $e = uv$ of H (this notation means that e is directed from u to v), $\eta(e)$ is a directed path of G from $\eta(u)$ to $\eta(v)$ (paths do not have "repeated" vertices)
- for each loop e of H incident with $v \in V(H)$, $\eta(e)$ is a directed cycle of G passing through $\eta(v)$
- if $e, f \in E(H)$ are distinct, then $\eta(e), \eta(f)$ have no edges in common, although they may share vertices.

If in addition we add the condition

- if $v \in V(H)$ and $e \in E(H)$, and e is not incident with v in H, then $\eta(v)$ is not a vertex of $\eta(e)$

we call the relation *strong immersion*. Two of us proved the following, which will be presented in another paper [1]:

1.1 *In every infinite set of tournaments there are two tournaments such that one can be strongly immersed in the other.*

The result of the present paper is a key lemma that allows us to prove 1.1. Before its statement we need a few more definitions. If G is a digraph, we define $\lambda(G)$ to be the maximum t such that some vertex of G belongs to t directed cycles that are pairwise edge-disjoint, and $\mu(G)$ the maximum t such that some vertex of G belongs to t directed cycles that are otherwise pairwise vertex-disjoint. If $k \geq 0$ is an integer, an enumeration $(v_1, \dots, v_n)$ of the vertex set of a digraph has *cutwidth* at most k if for all $j \in \{2, \dots, n\}$, there are at most k edges uv such that $u \in \{v_1, \dots, v_{j-1}\}$ and $v \in \{v_j, \dots, v_n\}$; and a digraph has *cutwidth* at most k if there is an enumeration of its vertex set with cutwidth at most k. Two vertices u, v are *k-edge-connected* if there are k pairwise edge-disjoint directed paths from u to v, and k pairwise edge-disjoint directed paths from v to u. We say u, v are *strongly k-vertex-connected* if there are k directed paths from u to v, each with an internal vertex and pairwise vertex-disjoint except for u, v, and there are k directed paths from v to u, each with an internal vertex and pairwise vertex-disjoint except for u, v.

Let T be a digraph, let $k \geq 0$ be an integer, and let $u, v \in V(T)$ be distinct. We say that (u, v) is a

- *k-pair of the first type* if there is a set A of k vertices in T each adjacent to u and adjacent from v, and there is a set B of k vertices each adjacent from u and adjacent to v, with $A \cup B = \emptyset$

- *k-pair of the second type* if there is a set C of k vertices in T each adjacent to u and not from u, and each adjacent from v and not to v, and there is a set of k edges $\{a_1b_1, \ldots, a_kb_k\}$ such that $a_1, \ldots, a_k, b_1, \ldots, b_k$ are all distinct and not in C, and $a_1, \ldots, a_k$ are adjacent from u and not to u, and $b_1, \ldots, b_k$ are adjacent to v and not from v.

Our main theorem is the following.

1.2 *For every set $\mathcal{S}$ of semi-complete digraphs, the following are equivalent:*

1. *there exists k such that every member of $\mathcal{S}$ has cutwidth at most k*
2. *there exists k such that $\lambda(T) \leq k$ for every $T \in \mathcal{S}$*
3. *there exists k such that for each $T \in \mathcal{S}$, no two vertices of T are k-edge-connected*
4. *there exists k such that for each $T \in \mathcal{S}$, there do not exist k vertices of T that are pairwise k-edge-connected*
5. *there is a digraph H such that H cannot be weakly immersed in any member of $\mathcal{S}$*
6. *there is a digraph H such that H cannot be strongly immersed in any member of $\mathcal{S}$*
7. *there exists k such that $\mu(T) \leq k$ for every $T \in \mathcal{S}$*
8. *there exists k such that for each $T \in \mathcal{S}$, no two vertices of T are strongly k-vertex-connected*
9. *there exists k such that for each $T \in \mathcal{S}$, no pair of vertices is a k-pair of either the first or second type.*

The proof is given in the next section. Incidentally, here are a couple more statements that are NOT equivalent to the statements of 1.2:

- there exists k such that for each $T \in \mathcal{S}$, there do not exist k vertices of T that are pairwise strongly k-vertex-connected
- there is a digraph H such that no subdivision of H is a subgraph of any member of $\mathcal{S}$. (A *subdivision* of a digraph H is obtained by repeatedly deleting an edge uv, and adding a new vertex w, and adding two new edges uw and wv.)

To see the non-equivalence, take a tournament T with $2k+1$ vertices $v_0, v_1, \ldots, v_{2k}$, in which v_i is adjacent to v_j for $1 \leq i < j \leq 2k$, and v_0 is adjacent to $v_1, \ldots, v_k$ and from $v_{k+1}, \ldots, v_{2k}$. Then $\mu(T) = k$, and yet no three vertices are strongly 2-vertex-connected. Moreover, if H is the digraph obtained from a directed cycle of length three by adding a new edge parallel to each of the three original edges, then no subdivision of H is a subgraph of T.

There are also at least two algorithmic consequences of 1.2. In the final section we show that for every fixed digraph H there is an algorithm to test whether H can be strongly immersed in a semi-complete digraph T, with running time polynomial in the size of T; and also such an algorithm for weak immersion. Secondly, two of us proved, using 1.2, that for all fixed k there is an algorithm which, given a tournament T and k pairs $s_1, t_1, \ldots, s_k, t_k$ of vertices, tests in polynomial time whether there are k edge-disjoint directed paths of T where the ith path is from s_i to t_i for $1 \leq i \leq k$. This will be presented in a later paper [3].

2 The main proof

We begin by studying the semi-complete digraphs that do not have any k-pair of the first type. Let T be a digraph. For every enumeration $(v_1, \dots, v_n)$ of the vertex set of T, we define the *converse-degree* of this enumeration to be the maximum over all $j \in \{1, \dots, n\}$ of the larger of

- the number of edges with head v_j and tail in $\{v_1, \dots, v_{j-1}\}$
- the number of edges with tail v_j and head in $\{v_{j+1}, \dots, v_n\}$.

We define the *converse-degree* of T to be the smallest k such that some enumeration of $V(T)$ has converse-degree k. Thus, the converse-degree of T is at most the cutwidth of T. We first prove:

2.1 *Let T be a digraph, and let $k \geq 0$ be an integer.*

- *If some pair of vertices is a k-pair of the first type then the converse-degree of T is at least $k/2$.*
- *If T is semi-complete and no pair of vertices is a k-pair of the first type then the converse-degree of T is at most $4k$.*

Proof. For the first assertion, we assume that some pair of vertices of T is a k-pair of the first type. Let b be the converse-degree of T, and let $\{v_1, \dots, v_n\}$ be an enumeration of $V(T)$ with converse-degree b. Let (v_i, v_j) be a k-pair of the first type. Since also (v_j, v_i) is a k-pair of the first type, we may assume that $i < j$. Let X be a set of k vertices adjacent from v_i and to v_j. For every $v_h \in X$, either $h > i$ or $h < j$, since $i < j$; but there at most b values of h with $v_h \in X$ such that $h > i$, since the enumeration has converse-degree b and each such v_h is adjacent from v_i; and similarly there are at most b values of h such that $v_h \in X$ and $h < j$. Consequently $|X| \leq 2b$. Since $|X| = k$, we deduce that $b \geq k/2$. This proves the first assertion of the theorem.

For the second, we assume that T is semi-complete and no pair of vertices of T is a k-pair of the first type. For distinct $u, v \in V(T)$, let us write $u \Rightarrow v$ if there are at least $2k$ vertices that are adjacent from u and adjacent to v.

(1) *There is no sequence $x_1, \dots, x_t$ of vertices such that*

$$x_1 \Rightarrow x_2 \Rightarrow x_3 \Rightarrow \cdots \Rightarrow x_t \Rightarrow x_1.$$

For suppose that $x_1, \dots, x_t$ is such a sequence; thus $t \geq 2$. For $1 \leq i \leq t$, let A_i be a set of $2k$ vertices that are adjacent from x_i and to x_{i+1} (where x_{t+1} means x_1). Now x_1 is adjacent to at least k members of A_1 (indeed, to all $2k \geq k$ members of A_1), and so we may choose i with $1 \leq i \leq t-1$ maximum such that x_1 is adjacent to at least k members of A_i. Choose $A \subseteq A_i$ with $|A| = k$ such that x_1 is adjacent to every vertex in A. If $i = t-1$, then there exists $B \subseteq A_t$ with $|B| = k$ and $A \cap B = \emptyset$, since $|A_t| = 2k$, and so (x_1, x_{t-1}) is a k-pair of the first type, a contradiction. Thus $i < t-1$; and from the maximality of i, we deduce that there is a set $B \subseteq A_{i+1}$ with $|B| = k$ such that x_1 is not adjacent to any member of B. In particular, $A \cap B = \emptyset$, and since T is semi-complete it follows that x_1 is adjacent from every member of B, and so (x_1, x_{i+1}) is a k-pair of the first type, a contradiction. This proves (1).

From (1) we may write $V(T) = \{v_1, \dots, v_n\}$ such that for all distinct $i, j \in \{1, \dots, n\}$, if $v_i \Rightarrow v_j$ then $j < i$. We claim this enumeration has converse-degree at most $4k$. For let $1 \le j \le n$, and let

$$X = \{v_i : 1 \le i < j, v_i \text{ is adjacent to } v_j\},\ Y = \{v_i : j < i \le n, v_i \text{ is adjacent from } v_j\}.$$

We claim that $|X| \le 4k$. Thus we may assume that $X \ne \emptyset$, and so, since T is semi-complete, some vertex $v_i \in X$ is adjacent to at least half of the other members of X, that is, to at least $(|X| - 1)/2$ other members of X. Since $i < j$ (because $v_i \in X$), it follows from the choice of the enumeration that $v_i \not\Rightarrow v_j$, and so $(|X|-1)/2 < 2k$, that is, $|X| \le 4k$. Similarly $|Y| \le 4k$, and so T has converse-degree at most $4k$. This proves 2.1. ∎

The second part of 2.1 is easily converted to an algorithm; we have:

2.2 *There is an algorithm with running time $O(n^3)$, which, given as input a semi-complete digraph with n vertices and an integer $k \ge 0$, outputs a k-pair of the first type if one exists, and otherwise outputs an enumeration of $V(T)$ with converse-degree at most $4k$.*

Proof. For every pair of distinct vertices u, v, we find the set of all vertices adjacent from u and to v. (This takes time $O(n^3)$.) From this information we read off whether some pair is a k-pair of the first type, and if so we output it and stop. If there is no k-pair of the first type, we find all pairs u, v such that $u \Rightarrow v$ (defined as in the proof of 2.1); and it follows that statement (1) in the proof of 2.1 holds. Construct the enumeration $(v_1, \dots, v_n)$ as in the proof of 2.1 (to do so, repeatedly choose a vertex u such that there is no v satisfying $v \Rightarrow u$, and then delete u; the order in which vertices are chosen is the desired enumeration); this takes time $O(n^2)$. Then output this enumeration. This proves 2.2. ∎

We use 2.1 for part of 1.2, the following.

2.3 *Let T be a semi-complete digraph and let $k \ge 0$ be an integer. Suppose that no pair of vertices of T is a k-pair of the first or second type. Then the cutwidth of T is at most $72k^2 + 8k$; and indeed every enumeration of $V(T)$ with converse-degree at most $4k$ has cutwidth at most $72k^2 + 8k$.*

Proof. Since there is no k-pair of the first type, there is an enumeration of $V(T)$ with converse-degree at most $4k$, by 2.1. Take some such enumeration $(v_1, \dots, v_n)$. We claim that this enumeration has cutwidth at most $72k^2 + 8k$. For let $2 \le j \le n$; let $A = \{v_j, v_{j+1}, \dots, v_n\}$ and $B = \{v_1, \dots, v_{j-1}\}$. We must show that there are at most $72k^2 + 8k$ edges with tail in B and head in A. Let F be the set of all such edges. Since the enumeration has converse-degree at most $4k$, we have immediately

(1) *Every vertex of T is incident with at most $4k$ edges in F.*

Consequently $|F| \le 4k|A|$, and so we may assume that $|A| > 18k + 2$; and in particular $j + 9k + 1 \le n$. Let $m = j + 9k + 1$, and let $C = \{v_j, v_{j+1}, \dots, v_m\}$ and $D = \{v_{m+1}, \dots, v_n\}$.

(2) *There are fewer than $36k^2$ edges in F from B to D.*

For suppose that there are at least $36k^2$ such edges. These edges form the edge set of a bipartite graph (with bipartition (B, D)) with maximum degree at most $4k$; and so every set of vertices that meets every edge of this bipartite graph has cardinality at least $36k^2/(4k) = 9k$. By König's theorem it follows that this bipartite graph has a matching of cardinality $9k$; and so there exist distinct $a_1, \dots, a_{9k} \in D$ and distinct $b_1, \dots, b_{9k} \in B$ such that b_i is adjacent in T to a_i for $1 \le i \le 9k$. Since the enumeration has converse-degree at most $4k$, it follows that there are most $4k$ values of $i \in \{1, \dots, 9k\}$ such that v_j is adjacent from b_i; and at most $4k$ values of $i \in \{1, \dots, 9k\}$ such that v_m is adjacent to a_i. Consequently there are at least k values of $i \in \{1, \dots, 9k\}$ such that v_j is not adjacent from b_i, and v_m is not adjacent to a_i. Moreover, since the enumeration has converse-degree at most $4k$, there are at most $4k$ values of $i \in \{j+1, \dots, m-1\}$ such that v_j is adjacent to v_i, and at most $4k$ such that v_m is adjacent from v_i, and so at least k such that v_i is not adjacent from v_j and not adjacent to v_m. But then (v_j, v_m) is a k-pair of the second type, a contradiction. This proves (2).

Now $C \cup D = A$, and every edge in F is either from B to C or from B to D. Since $|C| = 9k + 2$, (1) implies that there are at most $4k(9k + 2)$ edges from B to C; and so by (2) it follows that $|F| \le 72k^2 + 8k$. This proves 2.3. ∎

Consequently, we have:

2.4 *There is an algorithm with running time $O(n^4)$, which, given as input a semi-complete digraph T with n vertices and an integer $k \ge 0$, outputs a k-pair of the first or second type if one exists, and otherwise outputs an enumeration of $V(T)$ with cutwidth at most $72k^2 + 8k$. There is also an algorithm with running time $O(n^3)$, which with the same input, outputs a k-pair of the first type if one exists, and otherwise outputs either a k-pair of the second type, or an enumeration of $V(T)$ with cutwidth at most $72k^2 + 8k$.*

Proof. To test whether a given pair (u, v) is a k-pair of the second type takes time $O(n^2)$ (we find the set A of out-neighbours of u, and the set B of in-neighbours of v, duplicating any vertex that belongs to both sets; and then run a bipartite matching algorithm on the graph formed by the edges of T from A to B). Thus we can output a k-pair of the second type (if one exists) in time $O(n^4)$, by trying all pairs (u, v). If there is no such pair, we run 2.2. If this provides a k-pair of the first type, we output it. Otherwise it provides an enumeration of $V(T)$ with converse-degree at most $4k$, and by 2.3 this has cutwidth at most $72k^2 + 8k$; we output it. This proves the first assertion.

For the second, we begin by running 2.2. If it give us a k-pair of the first type, we output it, and if not then we are given an enumeration $(v_1, \dots, v_n)$ with converse-degree at most $4k$. We test its cutwidth. If its cutwidth is at most $72k^2 + 8k$ then we output the enumeration and stop. Otherwise we find some j such that $|F| > 72k^2 + 8k$, with notation as in the proof of 2.3. Defining A, B, C, D as in that proof, it follows that there are at most $4k(9k+2)$ edges from B to C, and so at least $36k^2$ edges from B to D. By running a bipartite matching algorithm in the corresponding bipartite graph, we find a $9k$-edge matching of edges from B to D; and as in the proof of step (2) of 2.3, we convert this to a k-pair of the second type. (This takes time $O(n^3)$.) This proves the second assertion, and so completes the proof of 2.4. ∎

We use $\setminus$ to denote deletion; thus, $G \setminus X$ is the graph obtained from G by deleting X. (Here X may be a vertex or an edge, or a set of vertices or edges.)
Proof of 1.2. By 2.3 it follows that 1.2.9 implies 1.2.1. We prove the remaining implications in order (except for two).

(1) *If T is a loopless digraph of cutwidth at most k then $\lambda(T) \le 2k$. In particular 1.2.1 implies 1.2.2.*

For let $(v_1, \dots, v_n)$ be an enumeration of $V(T)$ of cutwidth at most k. Let $1 \le j \le n$. Let A be the set of edges from $\{v_i : 1 \le i < j\}$ to $\{v_i : j \le i \le n\}$, and let B be the set of edges from $\{v_i : 1 \le i \le j\}$ to $\{v_i : j < i \le n\}$. Since the enumeration has cutwidth at most k it follows that $|A|, |B| \le k$. Suppose that $C_1, \dots, C_t$ are edge-disjoint directed cycles, all containing v_j. Let $1 \le h \le t$. We claim that some edge of C_h belongs to $A \cup B$. For if some vertex of C_h is in $\{v_i : 1 \le i < j\}$ then some edge of C_h is in A, and if some vertex of C_h is in $\{v_i : j < i \le n\}$ then some edge of C_h is in B; and if neither of these happens then $V(C_h) = \{v_j\}$, which is impossible since T is loopless. This proves that some edge of C_h belongs to $A \cup B$. Since $|A \cup B| \le 2k$ and $C_1, \dots, C_t$ are pairwise edge-disjoint, it follows that $t \le 2k$ and so $\lambda(T) \le 2k$. This proves (1).

(2) *If T is a loopless digraph with $\lambda(T) \le k$, then there do not exist two vertices u, v that are $(k+1)$-edge-connected to each other. In particular, 1.2.2 implies 1.2.3.*

For suppose that u, v are $(k+1)$-edge-connected to each other. Let H be the digraph obtained from T by deleting u and adding two new vertices u_1, u_2, where the edges incident with u_1, u_2 are as follows. If e is an edge of T with tail u and head x say, then in H let e be an edge with tail u_1 and head x; and if e has head u and tail x in T, then in H let e have head u_2 and tail x. We claim that there are $k+1$ directed paths of H from u_1 to u_2, pairwise edge-disjoint. For suppose not; then by Menger's theorem there exists $X \subseteq V(H)$ with $u_1 \in X$ and $u_2 \notin X$ such that there are at most k edges of H with tail in X and head in $V(H) \setminus X$. Since u, v are $(k+1)$-edge-connected to each other, Menger's theorem applied to T implies that there are $k+1$ edge-disjoint directed paths of T from u to v; and hence there are $(k+1)$ edge-disjoint paths in H from u_1 to v. Since there are at most k edges in H from X to $V(H) \setminus X$, one of these paths uses no such edge, and so, since $u_1 \in X$, it follows that $v \in X$. But similarly since there are $k+1$ edge-disjoint directed paths in T from v to u, it follows that $v \in V(H) \setminus X$, a contradiction. Thus there are no such u, v. This proves (2).

It is clear that 1.2.3 implies 1.2.4; also 1.2.4 implies 1.2.5 (take H to be the digraph obtained from a directed cycle of length k by replacing each edge by k parallel edges; if H can be weakly immersed in T then the k images of vertices of H are pairwise k-edge-connected). Also, trivially 1.2.5 implies 1.2.6.

(3) *For every digraph H there exists an integer $k \ge 0$ such that there is a strong immersion of H in every tournament with a k-pair of either the first or second type. In particular 1.2.6 implies 1.2.9.*

Let H' be the digraph obtained by subdividing twice every edge of H (that is, replacing each edge by a directed three-edge path joining the same pair of vertices, so that these paths have pairwise disjoint interiors). Every tournament that admits a strong immersion of H' also admits a strong immersion of H, and so it suffices to prove the result for H'. Thus we may assume that H is a subdigraph of a tournament; and indeed, by adding any missing edges, we may assume that H is a tournament. Let $|V(H)| = t$ and let $k = 2^{t(t+2)}$. We claim that this choice of k satisfies (3). For let T be a tournament, and let (u, v) be a k-pair of either the first or second type. Thus there is a set $X \subseteq V(T)$ with $|X| = k$ such that every vertex in X is adjacent to u and from v; and since $|E(H)| \leq k$, there is a set $\{P_e : e \in E(H)\}$ of directed paths from u to v, all of length two or all of length three, and pairwise vertex-disjoint except for their common ends u, v, and each containing no vertex in X. Now every tournament with 2^n vertices contains a transitive tournament with n vertices. (This is easy to prove by induction on n; let v be one vertex, and choose N be either the set of all out-neighbours of v, or the set of all in-neighbours of v, whichever is larger; then the result follows by induction applied to N.) Thus we may assume that there exist $x_1, x_2, \ldots, x_{t^2+2t} \in X$, such that x_i is adjacent to x_j for $1 \leq i < j \leq t^2 + 2t$. Let $V(H) = \{h_1, \ldots, h_t\}$, and for $1 \leq i \leq t$ define $\eta(h_i) = x_{i(t+1)}$. For each edge $e = h_i h_j$ of H, we define $\eta(e)$ as follows. Let $p = i(t+1)$ and $q = j(t+1)$. Then (in the obvious notation) $\eta(e)$ is the directed path

$$\eta(h_i) = x_p\text{-}x_{p+j}\text{-}u\text{-}P_e\text{-}v\text{-}x_{q-i}\text{-}x_q = \eta(h_j).$$

It is easy to check that η is a strong immersion of H in T. This proves (3).

Thus 1.2.1,…,1.2.6 and 1.2.9 are all equivalent. But 1.2.2 implies 1.2.7, and 1.2.7 implies 1.2.9 (because if (u, v) is a k-pair of either type, one of u, v is in at least $k/2$ directed cycles that are otherwise vertex-disjoint); and 1.2.3 implies 1.2.8, and the latter implies 1.2.9. This completes the proof of 1.2. ∎

3 Testing for immersion

In this section we use 1.2 to give a polynomial-time algorithm to test whether a fixed digraph H can be strongly (or weakly) immersed in a given semi-complete digraph G. We remark first that it is important that G is semi-complete; for general digraphs G the analogous problem is NP-complete. To see this, let H be the digraph with two vertices h_1, h_2 and four edges, namely a loop at h_1, a loop at h_2, and edges h_1h_2, h_2h_1.

3.1 *It is NP-hard to test whether H can be strongly immersed in a digraph G; and the same holds for weak immersion.*

Proof. Fortune, Hopcroft and Wyllie [2] (FHW) showed that it is NP-complete to decide whether two given vertices x_1, x_2 of a digraph are in a directed cycle; and we may assume that x_1, x_2 both have indegree one and outdegree one. But given a hard instance G of FHW's question, if some vertex v has indegree at least two, let e_1, e_2 be edges with head v and with tails u_1, u_2 say; then we may delete e_1, e_2 from G and add a new vertex v' and three new edges $u_1v', u_2v', v'v$, and in this new digraph the answer to FHW's question is the same as in G. By repeating this it follows that FHW's question is NP-hard even for digraphs G in which every vertex has indegree at most one and outdegree at most two, or outdegree at most one and indegree at most two. For such a digraph G add a loop at x_1 and a loop at x_2, forming G'; then there is a strong (or weak) immersion of H in G' if and only if there is a directed cycle of G containing x_1, x_2. This proves 3.1. ∎

The idea of our algorithm is: choose k as in 1.2 such that there is a strong immersion of H in every semi-complete digraph with a k-pair of either the first or second type. Now, given the input a semi-complete digraph G, run 2.4 on G with this value of k. If we get a k-pair we convert it to a strong immersion of H and we are done. Otherwise we get an enumeration of $V(G)$ with cutwidth at most $72k^2 + 8k$; and now we use this enumeration to test for a strong or weak immersion of H using dynamic programming. We need to explain the dynamic programming in more detail, and that is the main content of this section.

Throughout the following, H is a fixed digraph and $k \geq 0$ is a fixed integer; we will describe an algorithm to test whether an input semi-complete digraph G with an enumeration of cutwidth at most k contains a strong or weak immersion of H. Thus, let $(v_1, \ldots, v_n)$ be an enumeration of $V(G)$ with cutwidth at most k.

First we prove a couple of theorems, and later we shall show how to use them to make an algorithm. Let $0 \leq i \leq n$, and let $S_i = \{v_1, \ldots, v_i\}$ and $T_i = \{v_{i+1}, \ldots, v_n\}$.

Let J be a digraph (not necessarily semicomplete, and not necessarily a subdigraph of G) such that

- $T_i \subseteq V(J) \subseteq V(G)$
- for all $u, v \in V(J)$ with not both $u, v \in S_i$, there is an edge from u to v in J if and only if there is such an edge in G, and there is at most one edge from u to v, and none if $u = v$
- for all distinct $u, v \in V(J) \cap S_i$, there are at most $|E(H)|$ edges of J from u to v, and for each $v \in V(J) \cap S_i$ there are at most $|E(H)|$ loops incident with v.

We say that J is an *i-extension*. Let $\mathcal{C}_i^{p,q}$ denote the set of all pairs (J, X) such that

- J is an i-extension, and $J \setminus T_i$ has at most p vertices and at most q edges
- $X \subseteq V(J) \cap S_i$
- there is a strong immersion η of H in J such that $X = \eta(V(H)) \cap S_i$, where $\eta(V(H))$ denotes $\{\eta(v) : v \in V(H)\}$.

3.2 *Let $0 \leq i \leq n$ as above. Let $p \geq |V(H)| + |E(H)| + 2k + 1$ and let $q \geq 0$. Let J be an i-extension such that $J \setminus T_i$ has at most p vertices and at most q edges, and let $X \subseteq V(J) \cap S_i$. Then $(J, X) \in \mathcal{C}_i^{p,q}$ if and only if either*

- *$(J, X) \in \mathcal{C}_i^{p-1,q}$, or*
- *there exists $v \in V(J) \cap (S_i \setminus X)$ such that $(J \setminus v, X) \in \mathcal{C}_i^{p-1,q}$, or*
- *there exists $e \in E(J)$ with both ends in S_i, such that $(J \setminus e, X) \in \mathcal{C}_i^{p,q-1}$, or*
- *there exist vertices $u, v, w \in V(J) \cap S_i$, with $u, w \neq v$ and $v \notin X$, and edges $e = uv$ and $f = vw$ of J, such that $(J', X) \in \mathcal{C}_i^{p,q-1}$, where J' denotes the digraph obtained from J by deleting e and f and adding a new edge from u to w.*

Proof. The "if" part if clear, and holds for all p. For "only if", suppose that $(J, X) \in \mathcal{C}_i^{p,q}$, and let η be a strong immersion of H in J such that $X = \eta(V(H)) \cap S_i$. Let K be the minimal subdigraph of J such that η is a strong immersion in K; thus, K is formed by the union of the vertices $\eta(v)$ $(v \in V(H))$

and all the subgraphs $\eta(e)$ $(e \in E(H))$. It follows that every vertex $u \in V(K)$ has outdegree at most $|E(H)|$ in K, since each $\eta(e)(e \in E(H))$ uses at most one edge with tail u.

Let $F, F' \subseteq E(J)$ be the set of edges of J from S_i to T_i, and from T_i to S_i, respectively. For each edge e of H, $\eta(e)$ is a path or cycle, and between any two members of F' in $\eta(e)$ there is a member of F, and consequently $|E(\eta(e)) \cap F'| \leq |E(\eta(e)) \cap F| + 1$. Since $|F| \leq k$, by summing over all $e \in E(H)$ we deduce that $|E(K) \cap F'| \leq k + |E(H)|$. Consequently there are at most $2k + |E(H)|$ vertices in S_i that are adjacent in K to or from a vertex in T_i. Moreover $|\eta(V(H) \cap S_i| \leq |V(H)|$. Now we may assume that $|S_i| = p$, for otherwise the first assertion of the theorem holds. Thus $|S_i| = p > |V(H)| + |E(H)| + 2k$, and it follows that there exists $v \in S_i$ such that v is not adjacent in K with any member of T_i, and $v \notin \eta(V(H))$. Thus $v \notin X$. Now we may assume that $v \in V(K)$, since otherwise the second assertion of the theorem holds. From the minimality of K it follows that there is an edge $g = ab \in E(H)$ such that v belongs to $\eta(g)$; and $v \neq \eta(a), \eta(b)$ from our choice of v. Let u, w be the vertices of $\eta(g)$ such that $e = uv$ and $f = vw$ are edges of $\eta(e)$; then $u, w \in S_i$, since v is not adjacent in K with any member of T_i. Let J' be obtained from J by deleting e, f and adding a new edge from u to w; then there is a strong immersion η' of H in J' with $X = \eta'(V(H)) \cap S_i$. If J' is an i-extension then the fourth assertion of the theorem holds, so we may assume not; and therefore there are more than $|E(H)|$ edges of J' from u to w. Consequently there are more than $|E(H)|$ edges of J with tail u and head in S (namely, at least $|E(H)|$ with head v, and one with head w). Since u has outdegree at most $|E(H)|$ in K, as we saw earlier, it follows that there is an edge of J with tail u and head in S_i that is not an edge of K. But then the third statement of the theorem holds. This proves 3.2. ∎

Let $\mathcal{C}_i^p$ be the union of the sets $\mathcal{C}_i^{p,q}$ over all $q \in \{0, 1, \ldots, p^2|E(H)|\}$. (Note that if J is an i-extension then $J \setminus T_i$ has at most $p^2|E(H)|$ edges.) Two members $(J, X), (J', X') \in \mathcal{C}_i^p$ are *equivalent* if there is an isomorphism between J, J' taking X to X' and fixing each of $v_{i+1}, \ldots, v_n$. For all p, the number of equivalence classes of members of $\mathcal{C}_i^p$ depends only on p and k, since there at most k edges of G from S_i to T_i, and since G is semicomplete. (Note that this step depends very strongly on G being semicomplete; for general digraphs the proof breaks down here.) Since the set $\mathcal{C}_i^p$ is the union of its equivalence classes, we handle this set in the algorithms that follow by listing its equivalence classes. For simplicity we speak of "a knowledge of $\mathcal{C}_i^p$" when what we mean is "a knowledge of the equivalence classes that have union $\mathcal{C}_i^p$", and so on.

3.3 *Let $0 \leq i < n$. Let $p = |V(H)| + |E(H)| + 2k$. Then $\mathcal{C}_i^p$ can be computed from a knowledge of $\mathcal{C}_{i+1}^p$ in time that depends only on k, H.*

Proof. Starting from a knowledge of $\mathcal{C}_{i+1}^p$, we shall first compute $\mathcal{C}_{i+1}^{p+1}$, and then use this to compute $\mathcal{C}_i^p$, as follows.

To compute $\mathcal{C}_{i+1}^{p+1}$ from a knowledge of $\mathcal{C}_{i+1}^p$. From 3.2 we can compute $\mathcal{C}_{i+1}^{p+1,q}$ from a knowledge of $\mathcal{C}_{i+1}^p$ and of $\mathcal{C}_{i+1}^{p+1,q-1}$ in time that depends only on k, H; and by repeating for $q = 1, \ldots, p^2|E(H)|$ we compute $\mathcal{C}_{i+1}^{p+1}$.

To compute $\mathcal{C}_i^p$ from a knowledge of $\mathcal{C}_{i+1}^{p+1}$. Let J be an i-extension such that $J \setminus V(T_i)$ has at most p vertices, and let $X \subseteq V(J) \setminus V(T_i)$. We need to determine whether $(J, X) \in \mathcal{C}_i^p$. But J is an $(i+1)$-extension, and therefore $(J, X) \in \mathcal{C}_i^p$ if and only if $(J, X) \in \mathcal{C}_{i+1}^{p+1}$ or $(J, X \cup \{v_{i+1}\}) \in \mathcal{C}_{i+1}^{p+1}$.

This proves 3.3. ∎

Now we can describe the algorithm.

3.4 *For each digraph H and each integer k, there is an algorithm with running time $O(n)$, which, with input a semi-complete digraph G with n vertices and an enumeration $(v_1, \ldots, v_n)$ of $V(G)$ with cutwidth at most k, outputs whether there is a strong immersion of H in G.*

Proof. Let $p = |V(H)| + |E(H)| + 2k$. Now $\mathcal{C}_n^p$ can be computed in constant time, since all n-extensions have at most p vertices and at most $p^2|E(H)|$ edges (so we just check them all, up to equivalence). By n applications of 3.3, we can determine $\mathcal{C}_0^p$ in time $O(n)$. But the only 0-extension is G itself (because of the condition that $V(J) \subseteq V(G)$ for i-extensions), and so there is a strong immersion of H in G if and only if $\mathcal{C}_0^p \neq \emptyset$. This proves 3.4. ∎

This can easily be modified to do weak immersion (just change strong to weak in the definition of $\mathcal{C}_i^{p,q}$ above); and also can be modified to output an immersion if one exists, rather than just a yes/no answer (for each equivalence class in $\mathcal{C}_i^{p,q}$, we store one member, and a corresponding immersion of H.). We omit these details.

Consequently, as explained at the start of this section, we have:

3.5 *For every digraph H there is an algorithm, with running time $O(n^3)$, which, with input a semi-complete digraph G with n vertices, outputs whether there is a strong or weak immersion of H in G.*

Once again, this can be modified to output the immersion if one exists.

We remark that, if we permit parallel edges in the input digraph (so for every pair of distinct vertices u, v there is at least one edge between them, either from u to v or from v to u, but there might be many such edges), then our algorithm does not work any more (it was crucial that $\mathcal{C}_i^{p,q}$ was the union of only constantly many equivalence classes, and this is no longer true). In another paper [3] two of us give a algorithm for this problem, with running time at most an^b where b is independent of H.

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