Cooperative colorings of trees and of bipartite graphs

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Abstract

Given a system $\mathcal{G} = (G_1, \ldots, G_m)$ of graphs on the same vertex set V, a cooperative coloring for $\mathcal{G}$ is a choice of vertex sets $I_1, \ldots, I_m$, such that I_j is independent in G_j and $\bigcup_{j=1}^m I_j = V$. We give bounds on the minimal m such that every m graphs with maximum degree d have a cooperative coloring, in the cases that (a) the graphs are trees, (b) the graphs are all bipartite.

1 Introduction

A set of vertices in a graph is called *independent* if no two vertices in it form an edge. A *coloring* of a graph G is a covering of $V(G)$ by independent sets. Given a system $\mathcal{G} = (G_1, \ldots, G_m)$ of graphs on the same vertex set V, a *cooperative coloring* for $\mathcal{G}$ is a choice of vertex sets $\{I_j \subset V : j \in [m]\}$ such that I_j is independent in G_j and $\bigcup_{j=1}^m I_j = V$. If all G_j's are the same graph G, then a cooperative coloring is just a coloring of G by m independent sets.

A basic fact about vertex coloring is that every graph G of maximum degree d is $(d+1)$-colorable. It is therefore natural to ask whether $d+1$ graphs, each of maximum degree d, always have a cooperative coloring. This was shown to be false:

Theorem 1 (Theorem 5.1 of Aharoni, Holzman, Howard and Sprüssel [AHHS15]). *For every $d \geq 2$, there exist $d + 1$ graphs of maximum degree d that do not have a cooperative coloring.*

As a cooperative coloring can be translated to an independent transversal (see [AHHS15, Section 2] for the connection), the fundamental result on independent transversals due to Haxell [Hax01,

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Theorem 2] implies that $2d$ graphs of maximum degree d always have a cooperative coloring. Let $m(d)$ be the minimal m such that every m graphs of maximum degree d have a cooperative coloring. By the above, $m(1) = 2$ and

$$d + 2 \leq m(d) \leq 2d, \text{ for every } d \geq 2. \tag{1}$$

The theorem of Loh and Sudakov [LS07, Theorem 4.1] on independent transversals in locally sparse graphs implies that $m(d) = d + o(d)$. Neither the lower bound nor the upper bound in (1) has been improved for general d; even $m(3)$ is not known. However, restricting the graphs to specific classes, better upper bounds can be obtained.

Definition 1. For a class $\mathcal{G}$ of graphs, denote by $m_{\mathcal{G}}(d)$ the minimal m such that every m graphs belonging to $\mathcal{G}$, each of maximum degree at most d, have a cooperative coloring.

For example, the following was proved:

Theorem 2 (Corollary 3.3 of Aharoni et al. [ABZ07] and Theorem 6.6 of Aharoni et al. [AHHS15])**.** *Let $\mathcal{C}$ be the class of chordal graphs and let $\mathcal{P}$ be the class of paths. Then $m_{\mathcal{C}}(d) = d + 1$ for all d, and $m_{\mathcal{P}}(2) = 3$.*

In this paper, we prove some bounds on $m_{\mathcal{G}}(d)$ for another two classes:

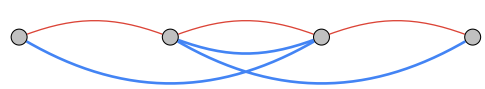
Theorem 3. *Let $\mathcal{T}$ be the class of trees, and let $\mathcal{B}$ be the class of bipartite graphs. Then for $d \geq 2$,*

$$\log_2 \log_2 d \leq m_{\mathcal{T}}(d) \leq (1 + o(1)) \log_{4/3} d,$$
$$\log_2 d \leq m_{\mathcal{B}}(d) \leq (1 + o(1)) \frac{2d}{\ln d}.$$

Remark 1. Let $\mathcal{F}$ be the class of forests. It is evident that $m_{\mathcal{F}}(d) \geq m_{\mathcal{T}}(d)$ as $\mathcal{F} \supset \mathcal{T}$. Conversely, when $d \geq 2$, given $m = m_{\mathcal{T}}(d)$ forests $F_1, \ldots, F_m$ of maximum degree d, we can add a few edges to F_i to obtain a tree F'_i of maximum degree d, and the cooperative coloring for $F'_1, \ldots, F'_m$ is also a cooperative coloring for $F_1, \ldots, F_m$. Therefore $m_{\mathcal{F}}(d) = m_{\mathcal{T}}(d)$ for $d \geq 2$.

2 Proof of Theorem 3 for trees

Proof of the lower bound on $m_{\mathcal{T}}(d)$. Note that the system $\mathcal{T}_2$, consisting of the following two paths (one in thin red, the other in bold blue) does not have a cooperative coloring.



Suppose now that $\mathcal{S} = (F_1, F_2, \ldots, F_m)$ is a system of forests on a vertex set V, not having a cooperative coloring. We shall construct a system $Q(\mathcal{S})$ of $m + 1$ new forests $F'_1, F'_2, \ldots, F'_m, F'_{m+1}$, again not having a cooperative coloring.

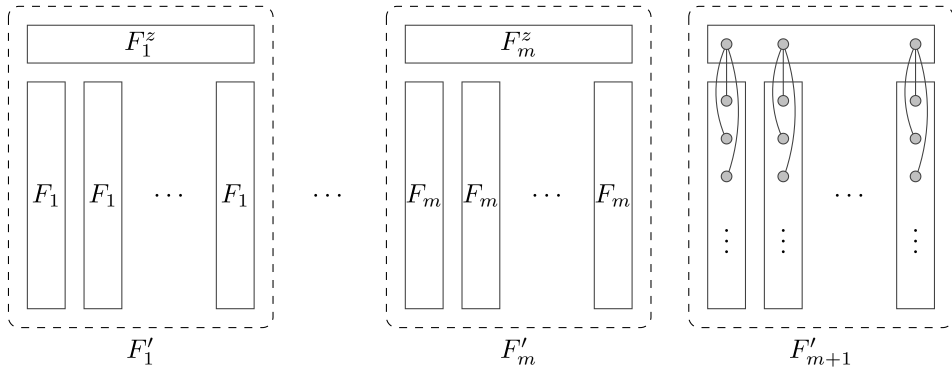


Figure 1: Construction of $Q(\mathcal{S}) = (F_1', \dots, F_m', F_{m+1}')$ from $\mathcal{S} = (F_1, \dots, F_m)$.

The vertex set common to the new forests is $V' = (V \cup \{z\}) \times V$, namely the $|V| + 1$ copies of V. For every $u \in V \cup \{z\}$ and every $i \in [m]$, take a copy F_i^u of F_i on the vertex set $\{(u, v) : v \in V\}$. Let

$$F_i' := \bigcup_{u \in V \cup \{z\}} F_i^u, \qquad \text{for all } i \in [m].$$

To these we add the $(m+1)$st forest F_{m+1}' obtained by joining (z, u) to (u, v) for all $u, v \in V$. Assume that there is a cooperative coloring $(I_1, I_2, \dots, I_m, I_{m+1})$ for the system $Q(\mathcal{S})$. Since the forests $F_1^u, F_2^u, \dots, F_m^u$ do not have a cooperative coloring, I_{m+1} must contain a vertex from $\{u\} \times V$ for all $u \in V \cup \{z\}$. In particular, I_{m+1} contains a vertex $(z, u) \in I_{k+1}$ for some $u \in V$ and a vertex (u, v) for some $v \in V$. Since (z, u) is connected in F_{m+1}' to (u, v), this is contrary to our assumption that I_{m+1} is independent.

Note that $|V'| = |V|^2 + |V| \le 2|V|^2$. Note also that the maximum degree of $Q(\mathcal{S})$ is attained in F_{m+1}', and it is equal to $|V|$. Recursively define the system $\mathcal{T}_m := Q(\mathcal{T}_{m-1})$ consisting of m forets for $m \ge 3$. Because the base $\mathcal{T}_2$ has 4 vertices, one can check inductively that $|V(\mathcal{T}_m)|$ is at most $2^{3 \cdot 2^{m-2} - 1}$ using $|V(\mathcal{T}_m)| \le 2|V(\mathcal{T}_{m-1})|^2$. Thus the maximum degree of $\mathcal{T}_m$ is at most $2^{3 \cdot 2^{m-3} - 1} \le 2^{2^{m-1}}$.

Given the maximum degree $d \ge 2$, choose $m := \lceil \log_2 \log_2 d \rceil$. By the choice of m, the maximum degree of $\mathcal{T}_m$ is at most $2^{2^{m-1}} \le d$. By adding a few edges between the leaves in each forest of $\mathcal{T}_m$, we can obtain a system of m trees of maximum degree d that does not have a cooperative coloring. This means $m_{\mathcal{T}}(d) > m > \log_2 \log_2 d$. □

Proof of the upper bound on $m_{\mathcal{T}}(d)$. Let $(T_1, T_2, \dots, T_m)$ be a system of trees of maximum degree d. We shall find a cooperative coloring by a random construction if $m \ge (1 + o(1)) \log_{4/3} d$.

Choose arbitrarily for each tree T_i a root so that we can talk about the parent or a sibling of a vertex that is not the root of T_i. For each T_i, choose independently a random vertex set S_i, in which each vertex is included in S_i independently with probability 1/2. Set

$$R_i := \{v \in S_i : v \text{ is a root or the parent of } v \notin S_i\}.$$

Since among any two adjacent vertices in T_i one is the parent of the other, R_i is independent in T_i.

We shall show that with positive probability the sets R_i form a cooperative coloring. For each vertex v, let B_v be the event that $v \notin \bigcup_{i=1}^m R_i$. If v is the root of T_i, then $\Pr(v \in R_i) = 1/2$; otherwise $\Pr(v \in R_i) = 1/4$. In any case, $\Pr(v \notin R_i) \leq 3/4$, and so $\Pr(B_v) \leq (3/4)^m$. Notice that B_v is only dependent on the events B_u for u that is the parent, a sibling or a child of v in some T_i. Since the degree of v is at most d, it follows that B_v is dependent on at most $m \times 2d$ other events. By the Lovász local lemma, if

$$\left(\frac{3}{4}\right)^m \times m \times 2d \times e \leq 1, \tag{2}$$

then with positive probability no B_v occurs, meaning that the sets R_i form a cooperative coloring. The inequality (2) indeed holds under the assumption that $m \geq (1 + o(1)) \log_{4/3} d$. □

3 Proof of Theorem 3 for bipartite graphs

Proof of the lower bound on $m_{\mathcal{B}}(d)$. Given d, take $m = \lceil \log_2 d \rceil$. Let the vertex set be $\{0,1\}^m$, and for $j \in [m]$ let G_j be the complete bipartite graph between V_j^0 and V_j^1 where

$$V_j^k = \{v \in \{0,1\}^m : v_j = k\}, \quad \text{for } k \in \{0,1\}.$$

Note that the degree of G_j is $2^{m-1} \leq d$.

Suppose that $I_1, \ldots, I_m$ are independent sets in $G_1, \ldots, G_m$ respectively. As each G_j is a complete bipartite graph, $I_j \subseteq V_j^{k_j}$ for some $k_j \in \{0,1\}$. Thus $(1 - k_1, \ldots, 1 - k_m)$ is not in any I_j, and so $I_1, \ldots, I_m$ do not form a cooperative coloring. This means $m_{\mathcal{B}}(d) > m \geq \log_2 d$. □

Proof of the upper bound on $m_{\mathcal{B}}(d)$. Let $\mathcal{G} = (G_1, \ldots, G_m)$ be a system of bipartite graphs on the same vertex set V with maximum degree d. By a semi-random construction, we shall find a cooperative coloring if $m \geq (1 + \varepsilon)\frac{2d}{\ln d}$ for fixed $\varepsilon > 0$ and d sufficiently large. We may assume that $m = O(d)$ because of (1).

For each $j \in [m]$, let (L_j, R_j) be a bipartition of G_j. Define $J_L(v) := \{j \in [m] : v \in L_j\}$ and $J_R(v) := \{j \in [m] : v \in R_j\}$ for each vertex $v \in V$, and let $A := \{v \in V : |J_L(v)| \geq m/2\}$. Set $B := V \setminus A$. Clearly, we have

$$|J_L(a)| \geq m/2, \quad \text{for all } a \in A; \tag{3a}$$

$$|J_R(b)| \geq m/2, \quad \text{for all } b \in B. \tag{3b}$$

Consider the following random process.

1. For each $a \in A$, choose $j = j(a) \in J_L(a)$ uniformly at random, and put a in the set I_j.
2. For each $b \in B$, choose arbitrarily $j \in J_R(b) \setminus \{j(a) : a \in A, (a,b) \in E(G_j)\} =: J'_R(b)$ as long as it is possible, and put b in the set I_j.

For any $a, a' \in A \cap I_j$, $a, a' \in L_j$ and so $(a, a') \notin G_j$. This means $A \cap I_j$ is independent, and similarly $B \cap I_j$ is independent. For any $b \in B \cap I_j$ and $(a,b) \in E(G_j)$, by the definition of $J'_R(b)$, $j(a) \neq j$ and so $a \notin I_j$. Therefore I_j is independent for all $j \in [m]$.

To prove the existence of a cooperative coloring it suffices to show that $J'_R(b)$ is nonempty for all $b \in B$ with positive probability. For a vertex $b \in B$, let E_b be the contrary event, that is, the event that $J'_R(b)$ is empty.

For a fixed $b \in B$, let us estimate from above the probability of E_b. For every $j \in J_R(b)$, let E^j be the event that $j \notin J'_R(b)$, that is the vent that $j(a) = j$ for some $a \in A$ that is a neighbor of b in G_j. For each $a \in A$ that is a neighbor of b in G_j, we have

$$\Pr(j(a) = j) = \frac{1}{|J_L(a)|} \overset{(3a)}{\leq} \frac{2}{m} \leq \frac{\ln d}{(1+\varepsilon)d}.$$

As there are at most d neighbors of b in G_j, we have for sufficiently large d that

$$1 - \Pr\left(E^j\right) \geq \left(1 - \frac{\ln d}{(1+\varepsilon)d}\right)^d \geq \exp\left(-(1-\varepsilon)\ln d\right) = d^{\varepsilon - 1} \geq \frac{8\ln d}{m}. \tag{4}$$

We claim that the events E^j, $j \in J_R(b)$, are negatively correlated. This is easier to see with the complementary events $\bar{E}^j$, $j \in J_R(b)$. We have to show that for any choice of indices $j_1, \dots, j_t \in J_R(b)$ there holds

$$\Pr\left(E^j \mid \bar{E}^{j_1} \cap \bar{E}^{j_2} \cap \ldots \cap \bar{E}^{j_t}\right) \geq \Pr\left(E^j\right).$$

The event $\bar{E}^{j_1} \cap \bar{E}^{j_2} \cap \cdots \cap \bar{E}^{j_t}$ means that for all $a \in A$ if a is a neighbor of b in G_{j_i} then $j(a) \neq j_i$. Then, for any $j \notin \{j_1, \dots, j_t\}$, for those vertices $a \in A$ that are neighbors of b in G_j, knowing that $j(a) \neq j_i$ for certain $i \in [t]$ increases the probability that $j(a) = j$, and therefore increases the probability of E^j.

By the claim, the inequality (4) and the fact that $E_b = \bigcap_{j \in J_R(b)} E^j$, we have

$$\Pr(E_b) \leq \prod_{j \in J_R(b)} \Pr\left(E^j\right) \overset{(3b)}{\leq} \left(1 - \frac{8\ln d}{m}\right)^{\frac{m}{2}} \leq \exp\left(-\frac{8\ln d}{m} \cdot \frac{m}{2}\right) = \frac{1}{d^4}.$$

The event E_b is dependent on at most md^2 other events $E_{b'}$, since for such dependence to exist it is necessary that $b' \in B$ is at distance at most 2 from b in some graph G_j. Thus, by the Lovász local lemma, for the positive probability that none of E_b occurs it suffices that

$$\frac{1}{d^4} \times md^2 \times e \leq 1,$$

which indeed holds for d sufficiently large as $m = O(d)$. □

4 Further directions

Cooperative colorings of graphs is a special case of a more general concept. Given a family $H_1, \dots, H_t$ of hypergraphs, all sharing the same vertex set V, a *cooperative cover* is a choice of edges $e_i \in H_i$, such that $\bigcup_{i \leq t} e_i = V$. For a graph G let $\mathcal{I}(G)$ be the independence complex of G, namely the set of independent sets of vertices in G. A cooperative coloring of $(G_1, \ldots, G_m)$ is a cooperative cover of the complexes $\mathcal{I}(G_i)$.

Given a hypergraph H, let $\beta(H)$ be the minimal number of edges from H whose union is $V(H)$. For a class $\mathcal{H}$ of hypergraphs, let $g_{\mathcal{H}}(b)$ denote the minimal number t such that every family $H_1, \ldots, H_t$ of hypergraphs belonging to $\mathcal{H}$, sharing the same vertex set V, and satisfying $\beta(H_i) \leq b$ for all $i \leq t$, has a cooperative cover. Write $g_{\mathcal{H}}(k) = \infty$ if no such t exists.

Let $\mathcal{I}$ be the class of all independence complexes of graphs. The fact that $m_{\mathcal{B}}(d) \geq \log_2(d)$ (see Theorem 3 above) shows that $g_{\mathcal{I}}(2) = \infty$. But there are interesting classes of hypergraphs, in particular, the class of k-polymatroids defined in [Edm03], for which g is finite. A k-polymatroid is defined via an integer-valued rank function f, that is monotone, submodular and satisfying $f(\{v\}) \leq k$ for every vertex v. The k-polymatroid then consists of all those sets e for which $f(e) = k|e|$.

Classical examples of such hypergraphs are the intersection of k matroids $M_1, \ldots, M_k$, where $f(e) = \sum_{i \leq k} \text{rank}_{M_i}(e)$, and the matching complex of a k-uniform hypergraph, where $f(I) = |\bigcup I|$ for every set of edges I.

Theorem 4. *Let $\mathcal{P}_k$ be the class of all k-polymatroids. Then $g_{\mathcal{P}_k}(b) \leq kb$ for every k and b.*

This follows from the topological version of Hall's marraige theorem (see [AH00]) and an observation of the two first authors that the topological connectivity of a k-polymatroid P is at least $\text{rank}(P)/k$, where $\text{rank}(P)$ is the largest size of an edge in P. We omit the details. It will be of interest to explore the sharpness of this result. For example, it is possible to show that the result is sharp for $k = 2, m = 2$, namely that $g_{\mathcal{P}_2}(2) = 4$.

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