Cooperative colorings of trees and of bipartite graphs

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Abstract

Given a system \( G = (G_1, \ldots, G_m) \) of graphs on the same vertex set \( V \), a cooperative coloring for \( G \) is a choice of vertex sets \( I_1, \ldots, I_m \), such that \( I_j \) is independent in \( G_j \) and \( \bigcup_{j=1}^{m} I_j = V \). We give bounds on the minimal \( m \) such that every \( m \) graphs with maximum degree \( d \) have a cooperative coloring, in the cases that (a) the graphs are trees, (b) the graphs are all bipartite.

1 Introduction

A set of vertices in a graph is called independent if no two vertices in it form an edge. A coloring of a graph \( G \) is a covering of \( V(G) \) by independent sets. Given a system \( G = (G_1, \ldots, G_m) \) of graphs on the same vertex set \( V \), a cooperative coloring for \( G \) is a choice of vertex sets \( \{I_j \subset V : j \in [m]\} \) such that \( I_j \) is independent in \( G_j \) and \( \bigcup_{j=1}^{m} I_j = V \). If all \( G_j \)'s are the same graph \( G \), then a cooperative coloring is just a coloring of \( G \) by \( m \) independent sets.

A basic fact about vertex coloring is that every graph \( G \) of maximum degree \( d \) is \((d+1)\)-colorable. It is therefore natural to ask whether \( d+1 \) graphs, each of maximum degree \( d \), always have a cooperative coloring. This was shown to be false:

Theorem 1 (Theorem 5.1 of Aharoni, Holzman, Howard and Sprüssel [AHHS15]). For every \( d \geq 2 \), there exist \( d+1 \) graphs of maximum degree \( d \) that do not have a cooperative coloring.

As a cooperative coloring can be translated to an independent transversal (see [AHHS15, Section 2] for the connection), the fundamental result on independent transversals due to Haxell [Hax01].

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Theorem 2 implies that \(2d\) graphs of maximum degree \(d\) always have a cooperative coloring. Let \(m(d)\) be the minimal \(m\) such that every \(m\) graphs of maximum degree \(d\) have a cooperative coloring. By the above, \(m(1) = 2\) and

\[ d + 2 \leq m(d) \leq 2d, \text{ for every } d \geq 2. \tag{1} \]

The theorem of Loh and Sudakov [LS07, Theorem 4.1] on independent transversals in locally sparse graphs implies that \(m(d) = d + o(d)\). Neither the lower bound nor the upper bound in (1) has been improved for general \(d\); even \(m(3)\) is not known. However, restricting the graphs to specific classes, better upper bounds can be obtained.

**Definition 1.** For a class \(G\) of graphs, denote by \(m_G(d)\) the minimal \(m\) such that every \(m\) graphs belonging to \(G\), each of maximum degree at most \(d\), have a cooperative coloring.

For example, the following was proved:

**Theorem 2** (Corollary 3.3 of Aharoni et al. [ABZ07] and Theorem 6.6 of Aharoni et al. [AHHS15]). Let \(C\) be the class of chordal graphs and let \(P\) be the class of paths. Then \(m_C(d) = d + 1\) for all \(d\), and \(m_P(2) = 3\).

In this paper, we prove some bounds on \(m_G(d)\) for another two classes:

**Theorem 3.** Let \(T\) be the class of trees, and let \(B\) be the class of bipartite graphs. Then for \(d \geq 2\),

\[
\log_2 \log_2 d \leq m_T(d) \leq (1 + o(1)) \log_{4/3} d, \\
\log_2 d \leq m_B(d) \leq (1 + o(1)) \frac{2d}{\ln d}.
\]

**Remark 1.** Let \(F\) be the class of forests. It is evident that \(m_F(d) \geq m_T(d)\) as \(F \supset T\). Conversely, when \(d \geq 2\), given \(m = m_T(d)\) forests \(F_1, \ldots, F_m\) of maximum degree \(d\), we can add a few edges to \(F_i\) to obtain a tree \(F_i'\) of maximum degree \(d\), and the cooperative coloring for \(F_1', \ldots, F_m'\) is also a cooperative coloring for \(F_1, \ldots, F_m\). Therefore \(m_F(d) = m_T(d)\) for \(d \geq 2\).

### 2 Proof of Theorem 2 for trees

**Proof of the lower bound on \(m_T(d)\).** Note that the system \(T_2\), consisting of the following two paths (one in thin red, the other in bold blue) does not have a cooperative coloring.

![Diagram of a tree system](image)

Suppose now that \(S = (F_1, F_2, \ldots, F_m)\) is a system of forests on a vertex set \(V\), not having a cooperative coloring. We shall construct a system \(Q(S)\) of \(m + 1\) new forests \(F_1', F_2', \ldots, F_m', F_{m+1}'\), again not having a cooperative coloring.
The vertex set common to the new forests is $V' = (V \cup \{z\}) \times V$, namely the $|V| + 1$ copies of $V$. For every $u \in V \cup \{z\}$ and every $i \in [m]$, take a copy $F_i'$ of $F_i$ on the vertex set $\{(u, v) : v \in V\}$. Let

$$F_i' := \bigcup_{u \in V \cup \{z\}} F_i^u, \quad \text{for all } i \in [m].$$

To these we add the $(m + 1)$st forest $F_{m+1}'$ obtained by joining $(z, u)$ to $(u, v)$ for all $u, v \in V$. Assume that there is a cooperative coloring $(I_1, I_2, \ldots, I_m, I_{m+1})$ for the system $Q(S)$. Since the forests $F_1', F_2', \ldots, F_m'$ do not have a cooperative coloring, $I_{m+1}$ must contain a vertex from $\{u\} \times V$ for all $u \in V \cup \{z\}$. In particular, $I_{m+1}$ contains a vertex $(z, u) \in I_{k+1}$ for some $u \in V$ and a vertex $(u, v)$ for some $v \in V$. Since $(z, u)$ is connected in $F_{m+1}'$ to $(u, v)$, this is contrary to our assumption that $I_{m+1}$ is independent.

Note that $|V'| = |V|^2 + |V| \leq 2|V|^2$. Note also that the maximum degree of $Q(S)$ is attained in $F_{m+1}'$, and it is equal to $|V|$. Recursively define the system $T_m := Q(T_{m-1})$ consisting of $m$ forests for $m \geq 3$. Because the base $T_2$ has 4 vertices, one can check inductively that $|V(T_m)|$ is at most $2^{3 \cdot 2^{3^{m-2} - 1}}$ using $|V(T_m)| \leq 2|V(T_{m-1})|^2$. Thus the maximum degree of $T_m$ is at most $2^{3 \cdot 2^{3^{m-3} - 1}} \leq 2^{2^{m-1}}$.

Given the maximum degree $d \geq 2$, choose $m := \lceil \log_2 \log_2 d \rceil$. By the choice of $m$, the maximum degree of $T_m$ is at most $2^{2^{m-1}} \leq d$. By adding a few edges between the leaves in each forest of $T_m$, we can obtain a system of $m$ trees of maximum degree $d$ that does not have a cooperative coloring. This means $m_T(d) > m > \log_2 \log_2 d$.

**Proof of the upper bound on $m_T(d)$**. Let $(T_1, T_2, \ldots, T_m)$ be a system of trees of maximum degree $d$. We shall find a cooperative coloring by a random construction if $m \geq (1 + o(1)) \log_{4/3} d$.

Choose arbitrarily for each tree $T_i$ a root so that we can talk about the parent or a sibling of a vertex that is not the root of $T_i$. For each $T_i$, choose independently a random vertex set $S_i$, in which each vertex is included in $S_i$ independently with probability $1/2$. Set

$$R_i := \{v \in S_i : v \text{ is a root or the parent of } v \notin S_i\}.$$ 

Since among any two adjacent vertices in $T_i$ one is the parent of the other, $R_i$ is independent in $T_i$. 

Figure 1: Construction of $Q(S) = (F_1', \ldots, F_m', F_{m+1}')$ from $S = (F_1, \ldots, F_m)$.
We shall show that with positive probability the sets \( R_i \) form a cooperative coloring. For each vertex \( v \), let \( B_v \) be the event that \( v \not\in \bigcup_{i=1}^m R_i \). If \( v \) is the root of \( T_i \), then \( \Pr(v \in R_i) = 1/2 \); otherwise \( \Pr(v \in R_i) = 1/4 \). In any case, \( \Pr(v \not\in R_i) \leq 3/4 \), and so \( \Pr(B_v) \leq (3/4)^m \). Notice that \( B_v \) is only dependent on the events \( B_u \) for \( u \) that is the parent, a sibling or a child of \( v \) in some \( T_i \). Since the degree of \( v \) is at most \( d \), it follows that \( B_v \) is dependent on at most \( m \times 2d \) other events. By the Lovász local lemma, if

\[
\left( \frac{3}{4} \right)^m \times m \times 2d \times e \leq 1, 
\]

then with positive probability no \( B_v \) occurs, meaning that the sets \( R_i \) form a cooperative coloring. The inequality (2) indeed holds under the assumption that \( m \geq (1 + o(1)) \log_{4/3} d \). \( \square \)

3 Proof of Theorem \([3]\) for bipartite graphs

Proof of the lower bound on \( m_{BG}(d) \). Given \( d \), take \( m = \lceil \log_2 d \rceil \). Let the vertex set be \( \{0,1\}^m \), and for \( j \in [m] \) let \( G_j \) be the complete bipartite graph between \( V_j^0 \) and \( V_j^1 \) where

\[
V_j^k = \{ v \in \{0,1\}^m : v_j = k \}, \quad \text{for } k \in \{0,1\}.
\]

Note that the degree of \( G_j \) is \( 2^{m-1} \leq d \).

Suppose that \( I_1, \ldots, I_m \) are independent sets in \( G_1, \ldots, G_m \) respectively. As each \( G_j \) is a complete bipartite graph, \( I_j \subseteq V_j^{k_j} \) for some \( k_j \in \{0,1\} \). Thus \((1-k_1, \ldots, 1-k_m)\) is not in any \( I_j \), and so \( I_1, \ldots, I_m \) do not form a cooperative coloring. This means \( m_{BG}(d) > m \geq \log_2 d \). \( \square \)

Proof of the upper bound on \( m_{BG}(d) \). Let \( G = (G_1, \ldots, G_m) \) be a system of bipartite graphs on the same vertex set \( V \) with maximum degree \( d \). By a semi-random construction, we shall find a cooperative coloring if \( m \geq (1 + \varepsilon) \frac{2d}{m \cdot d} \) for fixed \( \varepsilon > 0 \) and \( d \) sufficiently large. We may assume that \( m = O(d) \) because of (1).

For each \( j \in [m] \), let \( (L_j, R_j) \) be a bipartition of \( G_j \). Define \( J_L(v) := \{ j \in [m] : v \in L_j \} \) and \( J_R(v) := \{ j \in [m] : v \in R_j \} \) for each vertex \( v \in V \), and let \( A := \{ v \in V : |J_L(v)| \geq m/2 \} \). Set \( B := V \setminus A \). Clearly, we have

\[
|J_L(a)| \geq m/2, \quad \text{for all } a \in A; \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (3a)
\]

\[
|J_R(b)| \geq m/2, \quad \text{for all } b \in B. \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (3b)
\]

Consider the following random process.

1. For each \( a \in A \), choose \( j = j(a) \in J_L(a) \) uniformly at random, and put \( a \) in the set \( I_j \).
2. For each \( b \in B \), choose arbitrarily \( j \in J_R(b) \setminus \{ j(a) : a \in A,(a,b) \in E(G_j) \} =: J''_R(b) \) as long as it is possible, and put \( b \) in the set \( I_j \).

For any \( a, a' \in A \cap I_j \), \( a, a' \in L_j \) and so \( (a, a') \not\in G_j \). This means \( A \cap I_j \) is independent, and similarly \( B \cap I_j \) is independent. For any \( b \in B \cap I_j \) and \( (a, b) \in E(G_j) \), by the definition of \( J''_R(b), j(a) \neq j \) and so \( a \not\in I_j \). Therefore \( I_j \) is independent for all \( j \in [m] \).
To prove the existence of a cooperative coloring it suffices to show that $J'_R(b)$ is nonempty for all $b \in B$ with positive probability. For a vertex $b \in B$, let $E_b$ be the contrary event, that is, the event that $J'_R(b)$ is empty.

For a fixed $b \in B$, let us estimate from above the probability of $E_b$. For every $j \in J_R(b)$, let $E^j$ be the event that $j \notin J'_R(b)$, that is the event that $j(a) = j$ for some $a \in A$ that is a neighbor of $b$ in $G_j$. For each $a \in A$ that is a neighbor of $b$ in $G_j$, we have

$$\Pr(j(a) = j) = \frac{1}{|J_L(a)|} \leq \frac{2}{m} \leq \frac{\ln d}{(1 + \varepsilon)d}.$$ 

As there are at most $d$ neighbors of $b$ in $G_j$, we have for sufficiently large $d$ that

$$1 - \Pr(E^j) \geq \left(1 - \frac{\ln d}{(1 + \varepsilon)d}\right)^d \geq \exp\left(-d \frac{\ln d}{m}\right) = \frac{d^{\varepsilon - 1}}{1 - \varepsilon} \geq \frac{8\ln d}{m}. \quad (4)$$

We claim that the events $E^j$, $j \in J_R(b)$, are negatively correlated. This is easier to see with the complementary events $\bar{E}^j$, $j \in J_R(b)$. We have to show that for any choice of indices $j_1, \ldots, j_t \in J_R(b)$ there holds

$$\Pr\left(E^j \mid \bar{E}^{j_1} \cap \bar{E}^{j_2} \cap \cdots \cap \bar{E}^{j_t}\right) \geq \Pr\left(E^j\right).$$

The event $\bar{E}^{j_1} \cap \bar{E}^{j_2} \cap \cdots \cap \bar{E}^{j_t}$ means that for all $a \in A$ if $a$ is a neighbor of $b$ in $G_{j_i}$ then $j(a) \neq j_i$. Then, for any $j \notin \{j_1, \ldots, j_t\}$, for those vertices $a \in A$ that are neighbors of $b$ in $G_j$, knowing that $j(a) \neq j_i$ for certain $i \in [t]$ increases the probability that $j(a) = j$, and therefore increases the probability of $E^j$.

By the claim, the inequality $(4)$ and the fact that $E_b = \bigcap_{j \in J_R(b)} E^j$, we have

$$\Pr(E_b) \leq \prod_{j \in J_R(b)} \Pr(E^j) \leq \left(1 - \frac{8\ln d}{m}\right)^{\frac{m}{2}} \leq \exp\left(-\frac{8\ln d}{m} \cdot \frac{m}{2}\right) = \frac{1}{d^4}.$$ 

The event $E_b$ is dependent on at most $md^2$ other events $E_{b'}$, since for such dependence to exist it is necessary that $b' \in B$ is at distance at most 2 from $b$ in some graph $G_j$. Thus, by the Lovász local lemma, for the positive probability that none of $E_b$ occurs it suffices that

$$\frac{1}{d^4} \times md^2 \times e \leq 1,$$

which indeed holds for $d$ sufficiently large as $m = O(d)$.

\end{proof}

\section{Further directions}

Cooperative colorings of graphs is a special case of a more general concept. Given a family $H_1, \ldots, H_t$ of hypergraphs, all sharing the same vertex set $V$, a cooperative cover is a choice of edges $e_i \in H_i$, such that $\bigcup_{i \leq t} e_i = V$. For a graph $G$ let $\mathcal{I}(G)$ be the independence complex of $G$, namely the set
of independent sets of vertices in $G$. A cooperative coloring of $(G_1, \ldots, G_m)$ is a cooperative cover of the complexes $\mathcal{I}(G_i)$.

Given a hypergraph $H$, let $\beta(H)$ be the minimal number of edges from $H$ whose union is $V(H)$. For a class $\mathcal{H}$ of hypergraphs, let $g_{\mathcal{H}}(b)$ denote the minimal number $t$ such that every family $H_1, \ldots, H_t$ of hypergraphs belonging to $\mathcal{H}$, sharing the same vertex set $V$, and satisfying $\beta(H_i) \leq b$ for all $i \leq t$, has a cooperative cover. Write $g_{\mathcal{H}}(k) = \infty$ if no such $t$ exists.

Let $\mathcal{I}$ be the class of all independence complexes of graphs. The fact that $m_B(d) \geq \log_2(d)$ (see Theorem 3 above) shows that $g_{\mathcal{I}}(2) = \infty$. But there are interesting classes of hypergraphs, in particular, the class of $k$-polymatroids defined in [Edm03], for which $g$ is finite. A $k$-polymatroid is defined via an integer-valued rank function $f$, that is monotone, submodular and satisfying $f(\{v\}) \leq k$ for every vertex $v$. The $k$-polymatroid then consists of all those sets $e$ for which $f(e) = k|e|$.

Classical examples of such hypergraphs are the intersection of $k$ matroids $M_1, \ldots, M_k$, where $f(e) = \sum_{i \leq k} \text{rank}_{M_i}(e)$, and the matching complex of a $k$-uniform hypergraph, where $f(I) = |\bigcup I|$ for every set of edges $I$.

**Theorem 4.** Let $\mathcal{P}_k$ be the class of all $k$-polymatroids. Then $g_{\mathcal{P}_k}(b) \leq kb$ for every $k$ and $b$.

This follows from the topological version of Hall’s marriage theorem (see [AH00]) and an observation of the two first authors that the topological connectivity of a $k$-polymatroid $P$ is at least $\text{rank}(P)/k$, where $\text{rank}(P)$ is the largest size of an edge in $P$. We omit the details. It will be of interest to explore the sharpness of this result. For example, it is possible to show that the result is sharp for $k = 2, m = 2$, namely that $g_{\mathcal{P}_2}(2) = 4$.

**References**


